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QUASIELLIPTIC OPERATORS AND EQUATIONS NOT SOLVABLE WITH RESPECT TO THE HIGHER ORDER DERIVATIVE

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We consider a class of quasielliptic operators in R^n and establish the isomorphism property in special weighted Sobolev spaces. In more general weighted spaces, we obtain the unique solvability conditions for quasielliptic equations and prove estimates for solutions. Based on the obtained results, we study the solvability of the initial problem for equations that are not solvable with respect to the higher order derivative. Bibliography: 22 *titles.*

In this paper, we consider a class of quasielliptic operators $\mathscr{L}(D_x)$ in the space R^n. For such operators we establish the isomorphism property in special scales of weighted Sobolev spaces and study the solvability of quasielliptic equations

$$\mathscr{L}(D_x)u = f(x), \quad x \in R^n, \tag{1}$$

in larger spaces. We also obtain estimates for the solutions. The obtained results can be used to prove the solvability of the initial problem for some classes of equations that are not solvable with respect to the higher derivative:

$$\mathscr{L}(D_x)D_t^m u + \sum_{k=0}^{m-1} \mathscr{L}_{m-k}(D_x)D_t^k u = F(t,x). \tag{2}$$

Isomorphism theorems for differential operators admit various applications to the theory of partial differential equations. However, in many cases, the formulations of such theorems are not obvious. For example, the differential operator

$$\triangle - \varepsilon I : \ W_p^2(R^n) \longrightarrow L_p(R^n), \quad 1 < p < \infty,$$

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for $\varepsilon > 0$ establishes an isomorphism, but for the Laplace operator

$$\triangle : \; W_p^2(R^n) \longrightarrow L_p(R^n), \quad 1 < p < \infty,$$

this is not the case. A similar situation holds for the polyharmonic operator (cf., for example, [1, Chapter 12])

$$\triangle^m : \; W_p^{2m}(R^n) \longrightarrow L_p(R^n), \quad 1 < p < \infty.$$

We note that the first isomorphism theorems for the Laplace operator $\triangle$ in R^n were proved in [2, 3]. The isomorphism theorems for homogeneous elliptic operators appear in the literature in the 70-80s of the last century (cf., for example, [4]–[9]). The first isomorphism theorem for homogeneous quasielliptic operators in R^n was proved by the author [10]. Properties of operators were further studied in [11]. The isomorphism theorems for more general classes of matrix quasielliptic operators in R^n can be found in [12, 13].

In this paper, we continue the study started in [10, 12, 13].

1 The Main Results

We first recall the definition of a quasielliptic operator $L(D_x)$ with homogeneous symbol $L(i\xi)$ with respect to the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, $1/\alpha_j \in \mathbb{N}$, $j = 1, \ldots, n$, i.e.,

$$L(c^\alpha i\xi) = cL(i\xi), \quad c > 0.$$

Definition 1. A differential operator $L(D_x)$ is *quasielliptic* if the equality

$$L(i\xi) = 0, \quad \xi \in R^n,$$

takes place if and only if $\xi = 0$.

Quasielliptic operators $L(D_x)$, with homogeneous symbols $L(i\xi)$ with respect to some vector α are called *quasielliptic operators without lower terms*. Such operators are represented as

$$L(D_x) = \sum_{\beta\alpha=1} a_\beta D_x^\beta. \tag{3}$$

Examples of such operators are presented by homogeneous elliptic operators, $2b$-parabolic operators without lower terms etc.

We note that the symbol of a quasielliptic operator without lower terms satisfies the estimate

$$c_1\langle\xi\rangle \leqslant |L(i\xi)| \leqslant c_2\langle\xi\rangle, \quad \langle\xi\rangle^2 = \sum_{j=1}^n \xi_j^{2/\alpha_j}, \quad \xi \in \mathbb{R}^n,$$

where $c_1, c_2 > 0$ are constants.

Now, we consider differential operators $\mathscr{L}(D_x)$ of the form

$$\mathscr{L}(D_x) = L(D_x) + \sum_{\beta\alpha<1} a_\beta D_x^\beta, \tag{4}$$

where $L(D_x)$ is a quasielliptic operator of the form (3). Operators of the form (4) are referred to as *quasielliptic operators with lower terms*.

The differential operator corresponding to lower order terms is denoted by

$$L'(D_x) = \sum_{\beta\alpha<1} a_\beta D_x^\beta.$$

In what follows, we assume that the symbol of the differential operator (4) satisfies the inequalities

$$c_3(\langle\xi\rangle + \langle\xi\rangle^q) \leqslant |L(i\xi) + L'(i\xi)| \leqslant c_4(\langle\xi\rangle + \langle\xi\rangle^q), \quad \xi \in \mathbb{R}^n, \tag{5}$$

where $0 \leqslant q < 1$, $c_3, c_4 > 0$ are constants.

In [10]–[13], the isomorphism theorems were proved for classes of matrix quasielliptic operators with quasihomogeneous symbols. For this purpose special weighted Sobolev spaces $W^l_{p,\sigma}(R^n)$, $l = (1/\alpha_1, \dots, 1/\alpha_n)$, $1 < p < \infty$, $\sigma \geqslant 0$, introduced in [14] were used. To prove the isomorphism property of the operators (4), we introduce another class of weighted Sobolev spaces $W^l_{p,q,\sigma}(R^n)$, $l = (1/\alpha_1, \dots, 1/\alpha_n)$, $1 < p < \infty$, $0 \leqslant q \leqslant 1$, $\sigma \geqslant 0$.

Definition 2. We say that a locally integrable function u belongs to the weighted Sobolev space $W^l_{p,q,\sigma}(R^n)$ if u has the generalized derivatives $D^\nu_x u$, $\nu\alpha \leqslant 1$, in R^n; moreover, $D^\nu_x u \in L_p(R^n)$ if $q \leqslant \nu\alpha \leqslant 1$ and

$$\|(1+\langle x\rangle)^{-\sigma(q-\nu\alpha)} D^\nu_x u(x), L_p(R^n)\| < \infty, \quad \langle x\rangle^2 = \sum_{j=1}^n x_j^{2/\alpha_j} \quad \text{if } 0 \leqslant \nu\alpha < q.$$

The $W^l_{p,q,\sigma}(R^n)$-norm is defined by

$$\begin{aligned} \big\|u, W^l_{p,q,\sigma}(R^n)\big\| = &\sum_{q\leqslant\nu\alpha\leqslant1} \|D^\nu_x u(x), L_p(R^n)\| \\ &+ \sum_{0\leqslant\nu\alpha<q} \big\|(1+\langle x\rangle)^{-\sigma(q-\nu\alpha)} D^\nu_x u(x), L_p(R^n)\big\|. \end{aligned} \tag{6}$$

As in [14], it is possible to show that the set of functions in $C_0^\infty(R^n)$ is everywhere dense in $W^l_{p,q,\sigma}(R^n)$ for $\sigma \leqslant 1$. In what follows, we assume that $0 \leqslant \sigma \leqslant 1$.

For some parameters l, q, σ the above-introduced spaces coincide with the well known ones. We give some examples.

By definition, for $\sigma = 0$ the power weight in the norm (6) vanishes and, consequently, the above-introduced space coincides with the Sobolev space $W^l_p(R^n)$. For $q = 0$ this space also coincides with $W^l_p(R^n)$. In the case $q = 1$ the above space coincides with $W^l_{p,\sigma}(R^n)$ since

$$\|u, W^l_{p,\sigma}(R^n)\| = \sum_{0\leqslant\nu\alpha\leqslant1} \|(1+\langle x\rangle)^{-\sigma(1-\nu\alpha)} D^\nu_x u(x), L_p(R^n)\|$$

by definition [14]. In the isotropic case, where $1/\alpha_1 = \ldots = 1/\alpha_n = \bar{l}$, for $q = \sigma = 1$ the norm (6) is equivalent to the following:

$$\sum_{0\leqslant|\beta|\leqslant\bar{l}} \|(1+|x|)^{-(\bar{l}-|\beta|)} D^\beta_x u(x), L_p(R^n)\|.$$

Therefore, for $p > n$ the space $W^l_{p,1,1}(R^n)$ coincides with the Kudryavtsev space [15]: $W^{\bar{l}}_{p,\Box}(R^n)$, $\Box = \{x \in R^n :\ |x_j| < 1,\ j = 1, \ldots, n\}$, equipped with the norm

$$\|u, W^{\bar{l}}_{p,\Box}(R^n)\| = \int\limits_{\Box} |u(x)|dx + \sum_{|\beta|=\bar{l}} \|D^\beta_x u(x), L_p(R^n)\|.$$

In the isotropic case ($1/\alpha_1 = \ldots = 1/\alpha_n = \bar{l}$), for $q = \sigma = 1$ and any $p > 1$ the above-introduced space coincides with the Nirenberg–Walker–Cantor space $M^p_{\bar{l},m}(R^n)$ with the norm (cf. [16, 17])

$$\|u, M^p_{\bar{l},m}(R^n)\| = \sum_{0 \leqslant |\beta| \leqslant \bar{l}} \|(1 + |x|)^{m+|\beta|} D^\beta_x u(x), L_p(R^n)\|$$

provided that $m = -\bar{l}$.

We denote by $L_{p,\gamma}(R^n)$ the space of integrable functions equipped with the norm

$$\|u, L_{p,\gamma}(R^n)\| = \|(1 + \langle x\rangle)^{-\gamma} u(x), L_p(R^n)\|.$$

To prove the isomorphism theorem for the quasielliptic operator $\mathscr{L}(D_x)$, we need to establish the unique solvability of the quasielliptic equation (1) in the weighted Sobolev spaces $W^l_{p,q,\sigma}(R^n)$. We set $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$.

Theorem 1. *Assume that* $|\alpha| > q$ *and* $|\alpha|/p > \sigma q > |\alpha|/p - (|\alpha| - q)$. *Then for any* $f \in L_{p,(\sigma-1)q}(R^n)$ *there exists a unique solution* $u \in W^l_{p,q,\sigma}(R^n)$ *to Equation* (1) *and*

$$\|u, W^l_{p,q,\sigma}(R^n)\| \leqslant c\|f, L_{p,(\sigma-1)q}(R^n)\|, \tag{7}$$

where $c > 0$ *is a constant independent of* f.

We formulate the isomorphism theorem.

Theorem 2. *Let* $|\alpha|/p > q$. *Then the quasielliptic operator* $\mathscr{L}(D_x) :\ W^l_{p,q,\sigma}(R^n) \longrightarrow L_p(R^n)$, $1 < p < \infty$, $\sigma = 1$, *generates an isomorphism.*

Remark 1. Theorems 1 and 2 are counterparts of the corresponding results of [10, 12, 13] in the case of quasielliptic operators without lower terms.

For an example we consider the differential operator

$$\mathscr{L}(D_x) = \triangle^m + \varepsilon(-1)^{m-k}\triangle^k, \quad m \geqslant k, \quad \varepsilon > 0. \tag{8}$$

In our notation,

$$L(D_x) = \triangle^m, \quad L'(D_x) = \varepsilon(-1)^{m-k}\triangle^k, \quad \alpha_1 = \ldots = \alpha_n = 1/(2m).$$

Therefore, the condition (5) is satisfied for $q = k/m$. By Theorem 2, the operator

$$\mathscr{L}(D_x) :\ W^l_{p,q,1}(R^n) \longrightarrow L_p(R^n), \quad l = (2m, \ldots, 2m), \quad n > 2k, \tag{9}$$

generates an isomorphism for $p \in (1, \frac{n}{2k})$.

We consider the extreme cases in (8): $k = 0$ and $k = m$.

For $k = 0$, since $\triangle^0 = I$, $q = 0$, and $W^l_{p,0,1}(R^n) = W^l_p(R^n)$, we can write (9) as

$$\triangle^m + \varepsilon(-1)^m I: \; W^l_p(R^n) \longrightarrow L_p(R^n).$$

In this case, the isomorphism theorem is a classical results.

For $k = m$, since $q = 1$ and $W^l_{p,1,1}(R^n) = W^l_{p,1}(R^n)$, we can write (9) as

$$(1+\varepsilon)\triangle^m: \; W^l_{p,1}(R^n) \longrightarrow L_p(R^n), \quad n > 2m.$$

In this case, the isomorphism theorem follows from [6].

We show how to use the isomorphism theorems for the initial problem for a class of equations that are not solvable with respect to higher time-derivative (2):

$$\begin{aligned} &\mathscr{L}(D_x)D_t^m u + \sum_{k=0}^{m-1} \mathscr{L}_{m-k}(D_x)D_t^k u = F(t,x), \quad t > 0, \\ &D_t^k u|_{t=0} = \varphi_{k+1}(x), \quad k = 0, \ldots, m-1. \end{aligned} \tag{10}$$

Equations of the form (2) are often referred to as *Sobolev type equations* since the systematical study of such equations was started by Sobolev (cf. [18]).

Assume that the differential operators $\mathscr{L}_k(D_x)$, $k = 1, \ldots, m$, have constants coefficients and take the form:

$$\mathscr{L}_k(D_x) = \sum_{q \leqslant \beta\alpha \leqslant 1} a^k_\beta D^\beta_x. \tag{11}$$

We formulate the result about unconditional solvability of the initial problem (10). For the sake of simplicity, we assume that the initial data are zero.

Theorem 3. *Assume that $|\alpha|/p > q$, $\varphi_k(x) \equiv 0$, $k = 1, \ldots, m$. Then for any $F(t,x) \in C([0,T]; L_p(R^n))$ the problem (10) has a unique solution $u(t,x) \in C^m([0,T]; W^l_{p,q,1}(R^n))$ and*

$$\|u(t,x), C^m([0,T]; W^l_{p,q,1}(R^n))\| \leqslant c(T)\|F(t,x), C([0,T]; L_p(R^n))\|, \tag{12}$$

where $c(T)$ is a constant independent of $F(t,x)$.

2 Approximate Solutions to Quasielliptic Equation

To prove Theorem 1, we apply the method for constructing approximate solutions, described in [19] in detail. The method is based on the integral representation due to Uspenskii [20] for integrable functions (cf. also [19, Chapter 1])

$$\varphi(x) = \lim_{h \to 0} (2\pi)^{-n} \int_h^{h^{-1}} v^{-|\alpha|-1} \int_{R^n}\int_{R^n} \exp\Big(i\frac{x-y}{v^\alpha}\xi\Big) G(\xi)\varphi(y)\,d\xi\,dy\,dv, \tag{13}$$

where

$$G(\xi) = 2m\langle\xi\rangle^{2m}\exp(-\langle\xi\rangle^{2m}), \quad \langle\xi\rangle^2 = \sum_{i=1}^n \xi_i^{2/\alpha_i}, \quad m, \; 1/\alpha_i \in N. \tag{14}$$

We first assume that the function $f \in L_p(R^n)$ on the right-hand side of (7) is compactly supported. We consider the family of integral operators P_h, $0 < h < 1$, defined by

$$P_h f(x) = (2\pi)^{-n} \int\limits_h^{h^{-1}} v^{-1} \int\limits_{R^n}\int\limits_{R^n} \exp\left(i(x-y)\xi\right) G(\xi v^{\alpha}) \left(\mathscr{L}(i\xi)\right)^{-1} f(y) d\xi dy dv. \qquad (15)$$

By the definition (15), the functions $P_h f$ are infinitely differentiable and

$$\mathscr{L}(D_x) P_h f(x) = f_h(x),$$

where

$$f_h(x) = (2\pi)^{-n} \int\limits_h^{h^{-1}} v^{-|\alpha|-1} \int\limits_{R^n}\int\limits_{R^n} \exp\left(i\frac{x-y}{v^{\alpha}}\xi\right) G(\xi) f(y) d\xi dy dv.$$

By the representation (13), we have

$$\|f_h - f, L_p(R^n)\| \to 0, \quad h \to 0.$$

Consequently, the function $u_h = P_h f$ can be regarded as an approximate solution to the quasielliptic equation (1).

It is obvious that there is a natural number m_1 such that for $m \geqslant m_1$ in (14) the functions $P_h f$ are integrable with any power $p \geqslant 1$ (cf. [12]). We will assume that $m \geqslant m_1$ in (14).

The following three lemmas yield estimates for the functions $u_h = P_h f$ in the $W^l_{p,q,\sigma}(R^n)$-norm, which implies the convergence of the sequence $\{u_h\}$ as $h \to 0$ in the space $W^l_{p,q,\sigma}(R^n)$.

We begin with estimates for the norms of the higher order derivatives of the function (15).

Lemma 1. *Let* $\beta = (\beta_1, \ldots, \beta_n)$, $q \leqslant \beta\alpha \leqslant 1$. *Then*

$$\|D_x^{\beta} u_h, L_p(R^n)\| \leqslant c_{\beta} \|f, L_p(R^n)\|, \qquad (16)$$

where $c_{\beta} > 0$ *is a constant independent of* f *and* h; *moreover,*

$$\|D_x^{\beta} u_{h_1} - D_x^{\beta} u_{h_2}, L_p(R^n)\| \to 0, \quad h_1, h_2 \to 0. \qquad (17)$$

Proof. It suffices to obtain the estimate (16) for $f \in C_0^{\infty}(R^n)$. By the definition (15),

$$D_x^{\beta} u_h(x) = (2\pi)^{-n} \int\limits_h^{h^{-1}} v^{-1} \int\limits_{R^n}\int\limits_{R^n} \exp(i(x-y)\xi) G(\xi v^{\alpha}) (i\xi)^{\beta} \left(\mathscr{L}(i\xi)\right)^{-1} f(y) d\xi dy dv. \qquad (18)$$

Using properties of the Fourier transform, we can write this formula in the form

$$D_x^{\beta} u_h(x) = (2\pi)^{-n} \int\limits_h^{h^{-1}} v^{-1} \int\limits_{R^n} \Big(\int\limits_{R^n} \exp(i(x-y)s) G(s v^{\alpha}) ds\Big) F_{\beta}(y) dy dv, \qquad (19)$$

where

$$F_{\beta}(y) = (2\pi)^{-n/2} \int\limits_{R^n} \exp(iy\xi)(i\xi)^{\beta} \left(\mathscr{L}(i\xi)\right)^{-1} \widehat{f}(\xi) d\xi.$$

The multiindex β such that $q \leqslant \beta\alpha \leqslant 1$. By (5), the function

$$\mu_\beta(\xi) = (i\xi)^\beta \left(\mathscr{L}(i\xi)\right)^{-1}$$

satisfies the assumptions of the multiplier theorem [21] which implies the inequality

$$\|F_\beta, L_p(R^n)\| \leqslant c_\beta \|f, L_p(R^n)\|,$$

where c_β is a constant independent of f. Consequently, using properties of the integral representation (13) (cf. [19, Chapter 1]), from (19) we obtain the estimate (16). The proof of (17) is similar. The lemma is proved. □

To estimate the derivatives $D_x^\beta u_h$ for $0 \leqslant \beta\alpha < q$, we need to estimate the following integrals:

$$\mathscr{K}_{\beta,h}(x) = (2\pi)^{-n} \int_h^{h^{-1}} v^{-1} \int_{R^n} \exp(ix\xi) G(\xi v^\alpha) \frac{(i\xi)^\beta}{\mathscr{L}(i\xi)} d\xi dv, \quad 0 < h < 1. \tag{20}$$

Lemma 2. *Let $|\alpha| + \beta\alpha > q > \beta\alpha$. Then there exists $m_2 \geqslant m_1$ such that if $m \geqslant m_2$ in the definition* (14) *of $G(\xi)$, then the following estimate holds::*

$$\langle x\rangle^{|\alpha|+\beta\alpha-q} |\mathscr{K}_{\beta,h}(x)| \leqslant c, \quad x \in R^n, \tag{21}$$

where $c > 0$ is a constant independent of h.

Proof. We consider the inner integral in (20)

$$k_{\beta,v}(x) = \int_{R^n} \exp(ix\xi) G(\xi v^\alpha) \frac{(i\xi)^\beta}{\mathscr{L}(i\xi)} d\xi$$

and write it in the form

$$k_{\beta,v}(x) = v^{-|\alpha|-\beta\alpha} \int_{R^n} \exp\left(i\frac{xs}{v^\alpha}\right) G(s) \frac{(is)^\beta}{\mathscr{L}(i\frac{s}{v^\alpha})} ds.$$

Taking into account the definition of $G(\xi)$ and condition (5), it is easy to show that there exists $m_2 \in N$ such that if $m \geqslant m_2$ in (14), then

$$|k_{\beta,v}(x)| \leqslant c v^{-|\alpha|-\beta\alpha+q} \left(1 + \langle \frac{x}{v^\alpha}\rangle\right)^{-|\alpha|}, \quad x \in R^n.$$

Using this inequality, from (20) we find

$$\langle x\rangle^{|\alpha|+\beta\alpha-q} |\mathscr{K}_{\beta,h}(x)| \leqslant c \langle x\rangle^{|\alpha|+\beta\alpha-q} \int_h^{h^{-1}} v^{-|\alpha|-\beta\alpha+q-1} \Big(1 + \frac{\langle x\rangle}{v}\Big)^{-|\alpha|} dv.$$

Making the change of variables $v = \langle x\rangle\omega$, we get

$$\langle x\rangle^{|\alpha|+\beta\alpha-q} |\mathscr{K}_{\beta,h}(x)| \leqslant c \int_{h\langle x\rangle^{-1}}^{h^{-1}\langle x\rangle^{-1}} \omega^{-|\alpha|-\beta\alpha+q-1} \Big(1 + \frac{1}{\omega}\Big)^{-|\alpha|} d\omega \leqslant$$

$$\leqslant c \int_0^\infty \omega^{-|\alpha|-\beta\alpha+q-1} \Big(1 + \frac{1}{\omega}\Big)^{-|\alpha|} d\omega \leqslant c \int_0^1 \omega^{-\beta\alpha+q-1} d\omega + c \int_1^\infty \omega^{-|\alpha|-\beta\alpha+q-1} d\omega.$$

It is obvious that the inequality (21) holds. The lemma is proved. □

We will assume that $m \geqslant \max\{m_1, m_2\}$ in (14).

Let us estimate the norms of the derivatives $D_x^\beta u_h$ for $0 \leqslant \beta\alpha < q$.

Lemma 3. *Assume that* $\beta = (\beta_1, \dots, \beta_n)$*,* $q > \beta\alpha \geqslant 0$*, and*

$$\frac{|\alpha|}{p} > \sigma(q - \beta\alpha) > q - \beta\alpha - \frac{|\alpha|}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{22}$$

Then

$$\|\langle x\rangle^{-\sigma(q-\beta\alpha)} D_x^\beta u_h(x), L_p(R^n)\| \leqslant c\|\langle x\rangle^{(1-\sigma)(q-\beta\alpha)} f(x), L_p(R^n)\|, \tag{23}$$

where $c > 0$ *is a constant independent of* f *and* h*; moreover,*

$$\left\|\langle x\rangle^{-\sigma(q-\beta\alpha)}\Big(D_x^\beta u_{h_1}(x) - D_x^\beta u_{h_2}(x)\Big), L_p(R^n)\right\| \to 0, \quad h_1, h_2 \to 0. \tag{24}$$

Proof. It is obvious that it suffices to obtain the estimate (23) for $f \in C_0^\infty(R^n)$. Taking into account (18) and (20), we write the derivative $D_x^\beta u_h$ in the form

$$D_x^\beta u_h(x) = \int\limits_{R^n} \mathscr{K}_{\beta,h}(x-y) f(y)dy.$$

By Lemma 2,

$$\left\|\langle x\rangle^{-\sigma(q-\beta\alpha)} D_x^\beta u_h(x), L_p(R^n)\right\| \leqslant c\|\langle x\rangle^{-\sigma(q-\beta\alpha)} \int\limits_{R^n} \langle x-y\rangle^{-(|\alpha|+\beta\alpha-q)}|f(y)|dy, L_p(R^n)\|.$$

From the assumptions of the lemma it follows that $|\alpha| + \beta\alpha - q > 0$, $q - \beta\alpha > 0$. Therefore,

$$\begin{aligned}
&\|\langle x\rangle^{-\sigma(q-\beta\alpha)} D_x^\beta u_h(x), L_p(R^n)\| \leqslant \\
&\quad\leqslant c_1\| \int\limits_{R^n} \prod_{i=1}^n |x_i|^{-\sigma(q-\beta\alpha)/|\alpha|}|x_i - y_i|^{(q-\beta\alpha)/|\alpha|-1}|y_i|^{-(1-\sigma)(q-\beta\alpha)/|\alpha|} \times \\
&\qquad\qquad \times \langle y\rangle^{(1-\sigma)(q-\beta\alpha)}|f(y)|dy, L_p(R^n)\|.
\end{aligned}$$

Taking into account the condition (22) and applying the Hardy–Littlewood inequality [22], we obtain the estimate (23). The convergence (24) is obtained in a similar way. The lemma is proved. □

3 Proof of Theorems

Proof of Theorem 1. Under the assumptions of the theorem, from Lemmas 1 and 3 we obtain the following estimate for $u_h = P_h f$:

$$\|u_h, W^l_{p,q,\sigma}(R^n)\| \leqslant c\|f, L_{p,(\sigma-1)q}(R^n)\|, \quad 0 < h < 1, \tag{25}$$

where $c > 0$ is a constant independent of h and f. We recall that the integral operators P_h in (15) were defined on compactly supported functions $f \in L_p(R^n)$. These lemmas also imply the convergence

$$\|P_{h_1} f(x) - P_{h_2} f(x), W^l_{p,q,\sigma}(R^n)\| \to 0, \quad h_1, h_2 \to 0.$$

Since the space $W^l_{p,q,\sigma}(R^n)$ is complete, there exists a linear continuous operator

$$P: \ L_{p,(\sigma-1)q}(R^n) \longrightarrow W^l_{p,q,\sigma}(R^n)$$

defined on compactly supported functions f by

$$Pf = \lim_{h\to 0} P_h f.$$

Since the set of compactly supported infinitely differentiable functions is dense in $L_{p,(\sigma-1)q}(R^n)$, the operator P can be uniquely extended to the space $L_{p,(\sigma-1)q}(R^n)$ with the same norm. We use the same notation P for the extended operator.

By (25), the linear operator

$$P_h: \ L_{p,(\sigma-1)q}(R^n) \longrightarrow W^l_{p,q,\sigma}(R^n)$$

is continuous and the set of norms $\{\|P_h\|\}$ is bounded: $\|P_h\| \leqslant c$. Consequently, by the Banach–Steinhaus theorem,

$$\|P_h f - Pf, W^l_{p,q,\sigma}(R^n)\| \to 0, \quad h \to 0,$$

for any $f \in L_{p,(\sigma-1)q}(R^n)$.

The above arguments imply that for any $f \in L_{p,(\sigma-1)q}(R^n)$ Equation (1) has a solution $u \in W^l_{p,q,\sigma}(R^n)$ and the estimate (7) holds.

Since $C_0^\infty(R^n)$ is dense in $W^l_{p,q,\sigma}(R^n)$ for $0 \leqslant \sigma \leqslant 1$, the uniqueness of a solution to Equation (1) is proved in the same way as in [10]. Theorem 1 is proved. □

Proof of Theorem 2. For $|\alpha|/p > q$, $\sigma = 1$ the assumptions of Theorem 1 are satisfied. Consequently, for any $f \in L_p(R^n)$ Equation (1) has a unique solution $u = Pf \in W^l_{p,q,\sigma}(R^n)$; moreover,

$$\|u, W^l_{p,q,1}(R^n)\| \leqslant c\|f, L_p(R^n)\|.$$

From the form (4) of the operator $\mathscr{L}(D_x)$ and the conditions (5) it follows that the operator

$$\mathscr{L}(D_x): \ W^l_{p,q,1}(R^n) \longrightarrow L_p(R^n) \tag{26}$$

is linear and continuous. By the aforesaid, the range of this operator coincides with the entire space $L_p(R^n)$; moreover,

$$\|u, W^l_{p,q,1}(R^n)\| \leqslant c\|\mathscr{L}(D_x)u, L_p(R^n)\|.$$

Consequently, the inverse operator $(\mathscr{L}(D_x))^{-1}: \ L_p(R^n) \longrightarrow W^l_{p,q,1}(R^n)$ exists, is linear and continuous. Therefore, the quasielliptic operator (26) establishes an isomorphism. Theorem 2 is proved. □

Proof of Theorem 3. By the isomorphism theorem and the conditions (11), the linear operators

$$(\mathscr{L}(D_x))^{-1}\mathscr{L}_{m-k}(D_x): \ W^l_{p,q,1}(R^n) \longrightarrow W^l_{p,q,1}(R^n), \quad k = 0, \dots, m-1,$$

with $|\alpha|/p > q$ are continuous. Consequently, for the zero initial data the problem (10) is equivalent to the Cauchy problem for differential equation with bounded operator coefficients

$$D_t^m u + \sum_{k=0}^{m-1} (\mathscr{L}(D_x))^{-1} \mathscr{L}_{m-k}(D_x) D_t^k u = (\mathscr{L}(D_x))^{-1} F(t,x),$$
$$D_t^k u|_{t=0} = 0, \quad k = 0, \dots, m-1.$$

Since $(\mathscr{L}(D_x))^{-1}F(t,x) \in C([0,T]; W^l_{p,q,1}(R^n))$, from the general theory of the Cauchy problem it follows that there exists a unique solution $u(t,x) \in C^m([0,T]; W^l_{p,q,1}(R^n))$ to the initial problem and the estimate (12) holds. Theorem 3 is proved. □

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