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SMALL DEVIATION PROBABILITIES OF A SUM OF INDEPENDENT POSITIVE RANDOM VARIABLES, THE COMMON DISTRIBUTION OF WHICH DECREASES AT ZERO NOT FASTER THAN EXPONENTIAL FUNCTION

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We investigate small deviation probabilities of the cumulative sum of independent positive random variables, the common distribution of which decreases at zero not faster than exponential function. Bibliography: 8 titles.

1. Introduction and results. Let X denote a positive random variable with distribution function $V(x)$, and let $\{X_i\}_{i\geq 1}$ be independent copies of X.

The purpose of the present paper is first of all to generalize results from [1], where small deviation probabilities of the sum $X_1 + \cdots + X_n$ as $n \to \infty$ were studied in the case missing in the literature; namely, when the function $V(x)$ is slowly varying at zero. More exactly, it was assumed in [1] that

$$\nu(y) = \frac{1}{y} \int\limits_0^y u\, dV(u) \sim l(y), \quad y \to +0, \tag{1.1}$$

where the function $l(y)$ is slowly varying at zero. This implies that $l(+0) = 0$ and $V(y) \sim \widetilde{l}(y) = \int\limits_0^y l(u)/u\ du$, $y \to +0$, and also the function $\widetilde{l}(y)$ is slowly varying at zero.

In what follows, assumption (1.1) is replaced by the following essentially weaker condition (**R**) introduced in [2]:

there exist constants $b \in (0,1)$, $c_1 > b$, $c_2 > 1$, *and* $\varepsilon > 0$ *such that for every* $r \leq \varepsilon$,

$$c_1\, \nu(b\, r) \leq \nu(r) \leq c_2\, \nu(b\, r). \tag{1.2}$$

It was also shown in [8] that (**R**) is preferable to the known condition (**L**) from [1]. Namely (**L**) $\Longleftrightarrow$ $(\mathbf{R})\Big|_{c_1>1}$ and, in addition, (**R**) allows the function $V(r)$ to decrease at zero more slowly than any power of r and, in particular, to satisfy (1.1).

To present the results, we need some additional notation.

Let $\{\lambda_j\}$ be some positive numbers.

Set $S_n = \sum\limits_{j=1}^n \lambda_j X_j$ for $n \geq 1$, $\quad L(h) = \mathbf{E}e^{-h\,X}$, for $h \geq 0$, and

$$\begin{gathered} \Lambda_n(\gamma) = \prod_{j=1}^n L(\gamma\, \lambda_j), \quad m_n(\gamma) = -\left(\ln \Lambda_n(\gamma)\right)', \quad \sigma_n^2(\gamma) = \left(\ln \Lambda_n(\gamma)\right)'', \\ \widetilde{\nu}(y) = \int\limits_0^y \nu(u)\, du/u, \quad \kappa(y) = \nu(y)/\widetilde{\nu}(y), \quad y > 0. \end{gathered} \tag{1.3}$$

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Theorem 1. *If the condition* (**R**) *is satisfied, then for all positive* r, s, δ, *and* $\gamma > \delta / \min_{1\le j\le n} \lambda_j$,

$$\mathbf{P}(r-s < S_n \le r) = e^{-Q_n(\gamma)}\,\frac{1-e^{-\gamma s}}{\tau\sqrt{2\pi}}\Big(e^{-\beta^2/2} + \theta\Big(\tau^{-1} + \Big(\frac{\ln(1+\tau)}{\tau^2}\Big)^{1/\alpha}(1+(\gamma s)^{-1})\Big)\Big),$$

where

$$Q_n(\gamma) = -\ln\Lambda_n(\gamma) - \gamma\, r, \quad \beta = (r - m_n(\gamma))/\sigma_n(\gamma), \quad \tau = \gamma\,\sigma_n(\gamma), \tag{1.4}$$

$\alpha = \ln c_2/|\ln b|$ *(see* (1.2)*), and* $|\theta|$ *is bounded by a constant depending only on* V *and* δ.

Theorem 1 is a generalization of a result from [3, Remark 2] (the condition (**L**) is replaced by the condition (**R**)).

When formulating the subsequent results of this section, we assume that the condition (**R**) holds and the weights $\{\Lambda_j\}$ satisfy the condition

$$\delta \le \lambda_j \le 1/\delta, \quad j \ge 1, \tag{1.5}$$

for some $\delta > 0$, i.e., they are uniformly bounded away from zero and infinity (for example, are equal to one).

Corollary 1. *Let* ε *be an arbitrary positive constant. Then in the notation of Theorem* 1,

$$\mathbf{P}(r-s < S_n \le r) = e^{-Q_n(\gamma)}\,\frac{1-e^{-\gamma s}}{\tau\sqrt{2\pi}}\left(e^{-\beta^2/2} + O\left(1/\sqrt{\lambda} + (|\ln\lambda|/\lambda)^{1/\alpha}\big(1+(\gamma s)^{-1}\big)\right)\right), \quad n\to\infty$$

uniformly in $\gamma > \varepsilon$, $r > 0$, *and* $s > 0$, *where* $\lambda = n\kappa(1/\gamma)$ *(see* (1.3)*).*

Under the condition (**L**), the parameter λ is of order n uniformly in $\gamma > \varepsilon$ (see [1] or [3, (4.4.b)]. Therefore, Corollary 1 is a generalization of the main result in [2] and also implies statement (1) of Theorem 2 in [7].

Clearly, Corollary 1 is of main interest when the parameter γ is chosen in such a way that as n tends to infinity, λ and β tend to infinity and zero, respectively. In particular, Corollary 1 implies the following assertion.

Corollary 2. *If a sequence* $h_n \le \infty$ *tends to infinity so that*

$$n \inf_{1\le\gamma<\gamma_n} \kappa(1/\gamma) \to \infty, \tag{1.6}$$

then

$$\mathbf{P}(S_n < r) = \frac{\Lambda_n(\gamma)\,e^{\gamma r}}{\tau\sqrt{2\pi}}\,\big(1+o(1)\big), \quad n\to\infty, \tag{1.7}$$

uniformly in $r \in (n\,\kappa(1/\gamma_n)/\gamma_n,\ \mu\, n\,\bar\lambda_n)$, *where* $\mu < \mathbf{E}X$ *is a constant,* $\bar\lambda_n = (\lambda_1 + \cdots + \lambda_n)/n$, *and the function* γ *satisfies the equation*

$$m_n(\gamma) = r. \tag{1.8}$$

Note that if the condition (**L**) is satisfied, then condition (1.6) is satisfied for $\gamma_n = \infty$. Therefore, formula (1.7) is valid for *all* $0 < r \le \mu\, n\,\bar\lambda_n$. And if (**L**) is violated (say, condition (1.1) holds and hence $\kappa(+0) = 0$), then condition (1.6) becomes nontrivial and asymptotics (1.7) is no longer valid for r small enough. To confirm what has been said, we give some results.

Theorem 2. *Let* $\gamma \ge 1 \ge \gamma r > 0$. *Then*

$$\mathbf{P}(S_n < r) = \Lambda_n(\gamma)\,\big(1 + \theta\,(\gamma r + \lambda/(\gamma r)^{1-\varepsilon})\big), \quad |\theta| \le A, \tag{1.9}$$

where $\varepsilon \in (0,1)$ *and* A *are some constants not depending of* n, γ, *and* r.

Theorem 2 is a generalization and a refinement of statement (3) of Theorem 2 in [7].

Note that equality (1.9) is nontrivial only if $\lambda(= n\kappa(1/\gamma)) \to 0$.

Corollary 3. *If $\kappa(+0) = 0$ (see (1.3)), then*

$$\mathbf{P}(S_n < r) = \Lambda_n(\gamma)\bigl(1 + o(1)\bigr), \quad n \to \infty, \tag{1.10}$$

uniformly in $r \to +0$ such that λ (or γr) $\to 0$ as $n \to \infty$, where γ satisfies equation (1.8) *or, more generally, $\lambda \asymp \gamma r$.*

We note that under conditions of Corollary 3, the denominator on the right-hand side of (1.6) tends to infinity (see (2.4)), that is, for $\lambda \to \infty$ and for $\lambda \to 0$, the probability $\mathbf{P}(S_n < r)$ has *different* asymptotic representations.

In conclusion, we give one more result which is valid under the condition (**R**) (and (3.5)) and can be useful if the condition $\lambda \to 0/\infty$ is violated.

Theorem 3. *Let $r \in (0, \mu\, n\, \bar{\lambda}_n)$, where $\mu < \mathbf{E}X$ is a constant. Then*

$$-\ln \mathbf{P}(S_n < r) = Q_n(\gamma) + 0.5\, \ln(1 + \gamma\, r) + \theta, \quad |\theta| \le A, \tag{1.11}$$

where γ satisfies equation (1.8), *$Q_n(\gamma)$ and $\bar{\lambda}_n$ are defined in* (1.4) *and Corollary* 2, *respectively, and the parameter A does not depend on n and r.*

We note (see (1.3)) that

$$0 < c < \frac{Q_n(u)}{n\,|\ln \widetilde{\nu}(1/u)|} < 1/c < \infty, \quad u > u_0 > 0, \tag{1.12}$$

and c is independent of n and u.

Corollary 4. *Let $\mu < \mathbf{E}X$ be a constant. Then*

$$\mathbf{P}(S_n < r) \asymp \frac{e^{-Q_n(u)}}{\sqrt{1 + u\, r}}, \quad n \to \infty, \tag{1.13}$$

uniformly in $r \in (0, \mu\, \delta\, n)$ (see (1.5)*), where $u = u_n(r)$ is a positive number satisfying the conditions*

$$m_n(u) \asymp r, \qquad \frac{u\,(m_n(u) - r)}{\sqrt{u\, r}} = O\,(1), \quad n \to \infty. \tag{1.14}$$

Note that if $u\, r$ is large enough, then the first condition in (1.14) follows from the second one.

2. Proofs.

Theorem 1 is verified in the same way as Theorem 2 in [8] (with notable simplifications). In this case, formula (4.15) from [2] is involved.

Let us prove Theorem 2. Set

$$L(h) = \mathbf{E}e^{-hX}, \quad m(h) = -\bigl(\log L(h)\bigr)', \quad \sigma^2(h) = \bigl(\log L(h)\bigr)'', \quad h \ge 0.$$

For h, $u > 0$, we have

$$\log L(u) - \log L(h) = -\int_h^u m(t)\, dt, \quad \log \frac{m(u)}{m(h)} = -\int_h^u \frac{\sigma^2(t)}{m(t)}\, dt. \tag{3.1}$$

From [8, Lemma 2], it follows that under the condition (**R**),

$$c_1\,\kappa(1/h) \le h\,m(h), \quad h^2\,\sigma^2(h) \le c_2\,\kappa(1/h) \tag{3.2}$$

for all $h \ge h_0 > 0$, where h_0 is an arbitrary fixed number and the parameters $c_1, c_2 > 0$ do not depend on h. For $c = c_1/c_2$ and u, $h \ge h_0$, this and (3.1) imply that

$$\frac{m(u)}{m(h)} = \left(\frac{u}{h}\right)^{-\theta}, \qquad c \le \theta = \theta(u,h,V) \le 1/c. \tag{3.3}$$

By [4, Theorem 2 and (1.8)] for r, γ, $u > 0$,

$$\Lambda_n(\gamma)\,e^{\gamma\,r} \ge \mathbf{P}(S_n < r) \ge \Lambda_n(u)\,(1 - m_n(u)/r). \tag{3.4}$$

Let $K \ge 1$ be a parameter to be chosen later, and $u = K\,\gamma$. Taking into account (3.1) and the fact that the function $m(h)$ decreases, we have

$$\log\frac{\Lambda_n(u)}{\Lambda_n(\gamma)} = \sum_{j=1}^{n}(\log L(\lambda_j u) - \log L(\lambda_j\gamma)) \ge -\sum_{j=1}^{n}\lambda_j(u-\gamma)\,m(\lambda_j\gamma).$$

Using (3.1)–(3.3) and (1.5), under the assumption $\delta\gamma \ge h_0$ we obtain

$$\lambda_j(u-\gamma)\,m(\lambda_j\gamma) < \frac{u\,m(\delta\gamma)}{\delta m(\gamma)}\;m(\gamma) \le \frac{K}{\delta}\,\left(\frac{1}{\delta}\right)^{1/c}\gamma\,m(\gamma) \le c_2\,K\,\delta^{-(1+1/c)}\,\kappa(1/\gamma).$$

Therefore for $\lambda = n\kappa(1/\gamma)$,

$$\Lambda_n(u) \ge \Lambda_n(\gamma)\,e^{-A\,k\,\lambda}, \quad A = c_2\,\delta^{-(1+1/c)}. \tag{3.5}$$

Similarly,

$$m_n(u) = \sum_{j=1}^{n}\lambda_j\,m(\lambda_j u) \le c_2\frac{\lambda}{\delta\gamma}\,\frac{m(\delta\gamma)}{m(\gamma)} \le A\,k^{-c}\,\lambda/\gamma. \tag{3.6}$$

From (3.4)–(3.6), it follows that if $\delta\gamma \ge h_0$, then

$$\begin{aligned}\Lambda_n(\gamma)\,e^{\gamma\,r} &\ge \mathbf{P}(S_n < r) \ge \Lambda_n(\gamma)\,e^{-A\,k\,\lambda}\,(1 - A\,k^{-c}\,\lambda/(\gamma\,r)) \\ &\ge \Lambda_n(\gamma)\,\big(1 - A\,(k\,\lambda + k^{-c}\,\lambda/(\gamma\,r)\big).\end{aligned} \tag{3.7}$$

Estimate (1.9) follows from (3.7) with $h_0 = \delta$, $k = (\gamma\,r)^{-1/(1+c)}$, and $\varepsilon = c/(1+c)$. Theorem 2 is proved.

The proofs of Theorem 3 and Corollary 4 are quite similar to the proofs of Theorem 3 and Remark 3 in [5] (see also [6]).

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