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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF A THIRD-ORDER NONLINEAR DIFFERENTIAL EQUATION

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The asymptotic properties of the solutions of some third-order differential equation are examined. Sufficient conditions for the square integrability and oscillation of the solutions are established.

1. Introduction

The aim of the present paper is to study the qualitative behavior of solutions of a class of nonlinear differential equations of the third order. In particular, we give sufficient conditions for the existence of square integrable solutions of the nonlinear equation

$$(x''(t) + p(t)x(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{1.1}$$

where we assume that $p, q\colon [t_0, \infty) \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ are continuous, $q(t) > 0$, and $xf(x) > 0$ for $x \neq 0$. Some necessary and sufficient relationships between the square integrability of the first and second derivatives of solutions are also presented.

The problem of obtaining sufficient conditions to ensure that all solutions of certain classes of third-order nonlinear differential equations are oscillatory or nonoscillatory is an important problem of the qualitative theory of ordinary differential equations. This problem has been a subject of extensive study in the last three decades, and we refer the reader to the monographs by Greguš [6] and Kiguradze and Chanturia [10], as well as to the papers [2–5, 7–9, 11–16] and the references contained therein. By a *solution* of Eq. (1.1), we mean a function

$$x \in C^3([T_x, \infty)), \quad T_x \geq t_0,$$

that satisfies Eq. (1.1) on $[T_x, \infty)$. We only consider the solutions $x(t)$ of (1.1) that are continuable and nontrivial, i.e., satisfying

$$\sup\{|x(t)|\colon t \geq T_x\} > 0$$

for all $T_x \geq t_0$. We assume that Eq. (1.1) possesses a solution of this kind. A nontrivial solution of Eq. (1.1) is said to be *oscillatory* if it has a sequence of zeros tending to infinity, and it is called *nonoscillatory* otherwise. An equation is said to be oscillatory if all its solutions are oscillatory.

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2. Preliminary Results

First, we give some lemmas used in the proofs of our main results.

Lemma 2.1. *Let $y(t)$ be a continuous and twice differentiable function on the interval $[t_0, +\infty)$ such that $y(t) > 0$ for $t \geq t_0$. If*

$$\lim_{t\to\infty} \frac{y'(t)}{y(t)} = -\infty,$$

then

$$\lim_{t\to\infty} y(t) = 0.$$

If, in addition, $y''(t) > 0$, then

$$\lim_{t\to\infty} y'(t) = 0.$$

Proof. Since $y(t) > 0$, we conclude that $y'(t) < 0$ for large t and, hence, $\lim_{t\to\infty} y(t)$ exists and $y(t)$ is eventually decreasing. Suppose that

$$\lim_{t\to\infty} y(t) = \lambda > 0.$$

Let $\delta < 0$ be a number satisfying

$$\frac{y'(t)}{y(t)} < \delta$$

on $[t_1, +\infty)$ for some $t_1 \geq t_0$. Then

$$y'(t) < \delta y(t) < \delta\lambda < 0.$$

This implies that $y(t) < 0$ for large t, which is a contradiction. If we have $y''(t) > 0$, then

$$\lim_{t\to\infty} y'(t) = c < 0,$$

which also gives a contradiction.

Lemma 2.2. *Let $x(t)$ be a solution of (1.1). Then*

$$F[x(t)] = x(t)\left[x''(t) + p(t)x(t)\right] - \frac{1}{2}x'^2(t) \tag{2.1}$$

is nonincreasing on $[T, +\infty)$ for some $T \geq t_0$.

Proof. Differentiating and using the fact that $x(t)$ satisfies (1.1), we see that

$$F'[x(t)] = -q(t)x(t)f(x(t)) \leq 0. \tag{2.2}$$

Thus, $F[x(t)]$ is nonincreasing for large t.

Further, we define two classes of solutions of Eq. (1.1) as follows:

Definition 2.1. *We say that a solution $x(t)$ of Eq. (1.1) belongs to Class I if $F[x(t)] \geq 0$ on $[T_x, +\infty)$ for some $T_x \geq t_0$.*

Definition 2.2. *We say that a solution $x(t)$ of Eq. (1.1) belongs to Class II if $F[x(T)] < 0$ for some $T > T_x$.*

3. Main Results I

In this section, we study asymptotic properties of the solutions of Eq. (1.1) that belong to the Class I. Note that, while we have assumed that $q(t) > 0$ for $t \geq t_0$, the function $p(t)$ need not be a function of constant sign. We also assume that the following conditions hold:

(H_1) there are constants α and β such that

$$-\infty < \alpha \leq \inf p(t) \leq \sup p(t) \leq \beta < \infty \quad \text{and} \quad \omega = \max\{|\alpha|, |\beta|\};$$

(H_2) $\displaystyle\int_{t_0}^{\infty} p(s)ds = +\infty;$

(H_3) there exist constants $N \geq M > 0$ such that

$$0 < M \leq \frac{f(x)}{x} \leq N \quad \text{for all} \quad x \neq 0;$$

(H_4) there exists $\lambda > 0$ such that

$$\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} q(s)ds = \lambda < \infty;$$

(H_5) there exists $\mu > 0$ such that $|p'(t)| \leq \mu < \infty$ for all $t \geq t_0$.

We are now ready to prove our first result.

Theorem 3.1. *Let $x(t)$ be a solution of equation (1.1) from Class I. Assume that $q'(t) \geq 0$ and conditions (H_1) and (H_3)–(H_5) are satisfied. Then:*

(i) $\displaystyle\int^{\infty} x^2(s)ds < \infty;$

(ii) $\displaystyle\int^{\infty} x'^2(s)ds < \infty;$

(iii) $\displaystyle\int^{\infty} x''^2(s)ds < \infty.$

Proof. (i) Let $x(t)$ be a solution from Class I. We choose $t_1 \geq t_0$ such that $F(x(t)) \geq 0$ for $t \geq t_1$. Multiplying (1.1) by $x(t)$, we get

$$x(t)\left[x''(t) + p(t)x(t)\right]' + p(t)x(t)x'(t) + q(t)x(t)f(x(t)) = 0.$$

Further, integrating by parts from t_1 to t, we obtain

$$F[x(t)] - F[x(t_1)] + \int_{t_1}^{t} q(s)x(s)f(x(s))ds = 0. \tag{3.1}$$

Now

$$x(t)f(x(t)) \geq Mx^2(t)$$

by (H_3). Since $x(t)$ belongs to Class I, it follows from (3.1) that there is a positive constant c such that

$$\int_{t_1}^{t} x^2(s)ds \leq c < \infty$$

for all $t \geq t_1$. Hence, (i) holds.

(ii) Suppose that

$$\int_{t_1}^{\infty} x'^2(t) = \infty.$$

Since x belongs to Class I,

$$x(t)\left[x''(t) + p(t)x(t)\right] \geq \frac{1}{2}x'^2(t)$$

for $t \geq t_1$. Integrating from t_1 to t, we find

$$\frac{3}{2}\int_{t_1}^{t} x'^2(s)ds \leq k_0 + x(t)x'(t) + \int_{t_1}^{t} p(s)x^2(s)ds,$$

where $k_0 = x(a)x'(a)$. Thus, from (H_1), we obtain

$$\frac{3}{2}\int_{t_1}^{t} x'^2(s)ds \leq k_1 + x(t)x'(t)$$

where $k_1 = k_0 + \omega c$. Hence, the equality

$$\lim_{t\to\infty}\int_{t_1}^{t} x'^2(s)ds = \infty$$

implies that, for any

$$A > \frac{2k_1}{3},$$

there exist $t_2 \geq t_1$ such that

$$0 < \frac{3}{2}A - k_1 \leq x(t)x'(t) \quad \text{for} \quad t \geq t_2.$$

However,

$$\int_{t_2}^{t} x(s)x'(s)ds = \frac{1}{2}x^2(t) - \frac{1}{2}x^2(t_2) > \left(\frac{3}{2}A - k_1\right)(t - t_2) \quad \text{for} \quad t \geq t_2.$$

This implies that

$$\lim_{t\to\infty} \int_{t_2}^{t} x^2(s)ds = \infty,$$

which contradicts part (i). Hence, there exists $d > 0$ such that

$$\int_{t_1}^{t} x'^2(s)ds \leq d < \infty$$

for all $t \geq t_1$ and, therefore, (ii) holds.

(iii) We now assume that

$$\int_{t_1}^{t} x''^2(s)ds = \infty.$$

Multiplying (1.1) by $x'(t)$ and integrating from t_1 to t, we obtain

$$\int_{t_1}^{t} x'(s)\left(x''(s) + p(s)x(s)\right)' ds = -\int_{t_1}^{t} p(s)x'^2(s)ds - \int_{t_1}^{t} q(s)x'(s)f(x(s))ds.$$

Integrating the term on the left-hand side by parts, we get

$$\int_{t_1}^{t} x'(s)\left(x''(s) + p(s)x(s)\right)' ds = x'(t)\left(x''(t) + p(t)x(t)\right)$$

$$-\int_{t_1}^{t} x''^2(s)ds - \int_{t_1}^{t} p(s)x(s)x''(s)ds - C_0,$$

where

$$C_0 = x'(t_1)\left(x''(t_1) + p(t_1)x(t_1)\right).$$

Thus,

$$\int_{t_1}^{t} x''^2(s)ds = -\int_{t_1}^{t} p(s)x(s)x''(s)ds + \int_{t_1}^{t} p(s)x'^2(s)ds + x'(t)x''(t)$$

$$+ p(t)x'(t)x(t) + \int_{t_1}^{t} q(s)x'(s)f(x(s))ds - C_0. \tag{3.2}$$

Integrating the first term on the right-hand side by parts, applying the conditions (H_1), (H_3), and (H_5), as well as parts (i) and (ii) in the proof, we find

$$\int_{t_1}^{t} x''^2(s)ds = C_1 p(t_1) + 2\omega d + \int_{t_1}^{t} p'(s)x(s)x'(s)ds + x'(t)x''(t)$$

$$+ \int_{t_1}^{t} q(s)x'(s)f(x(s))ds - C_0 \le C_1 p(t_1) + 2\omega d - C_0$$

$$+ \frac{1}{2}\int_{t_1}^{t} \left|p'(s)\right| \left(x^2(s) + x'^2(s)\right) ds + x'(t)x''(t) + \int_{t_1}^{t} q(s)x'(s)f(x(s))ds$$

$$\le C_1 p(t_1) + 2\omega d - C_0 + \frac{1}{2}\mu(c+d) + x'(t)x''(t)$$

$$+ \frac{1}{2}\int_{t_1}^{t} q(s)\left[x'^2(s) + f^2(x(s))\right] ds \le C_1 p(t_1) + 2\omega d - C_0$$

$$+ \frac{1}{2}\mu(c+d) + x'(t)x''(t) + \frac{1}{2}q(t)\left[d + N^2 c\right]$$

$$\le K_0 + x'(t)x''(t) + K_1 q(t), \tag{3.3}$$

since $q(t)$ is increasing, where

$$C_1 = x(t_1)x'(t_1), \quad K_0 = C_1 p(t_1) + 2\omega d - C_0 + \frac{1}{2}\mu(c+d),$$

and

$$K_1 = \frac{1}{2}[d + N^2 c].$$

Since

$$\int_{t_1}^{\infty} x''^2(s)ds = \infty,$$

we can find $C > K_0 + k_1\lambda > 0$ and $t_2 \geq t_1$ such that

$$0 < C - K_0 \leq x'(t)x''(t) + K_1 q(t) \tag{3.4}$$

for $t \geq t_2$. Further, integrating (3.4) from t_2 to t, we obtain

$$(C - K_0)(t - t_2) - K_1 \int_{t_2}^{t} q(s)ds \leq \frac{1}{2} x'^2(t),$$

or

$$t\left[(C - K_0) - \frac{t_2}{t}(C - K_0) - \frac{K_1}{t}\int_{t_2}^{t} q(s)ds)\right] \leq \frac{1}{2} x'^2(t).$$

In view of our choice of C and the condition (H_4), we see that $x'^2(t) \to \infty$ as $t \to \infty$. This contradicts part (ii) of the theorem and completes the proof.

In the next theorem, we replace the condition (H_4) by the condition that the function q must be bounded above.

Theorem 3.2. *Let $x(t)$ be a solution of equation (1.1) from Class I. Assume that (H_1), (H_3), and (H_5) hold and there exists $q_1 > 0$ such that $q(t) \leq q_1$ for all $t \geq t_0$. Then the conclusion of Theorem 3.1 is true.*

Proof. The proofs of cases (i) and (ii) coincide with the proof of the previous theorem. In case (iii), from (3.3), we obtain

$$\int_{t_1}^{t} x''^2(s)ds \leq K_0 + x'(t)x''(t) + K_1 q_1.$$

By analogy with the proof of Theorem 3.1, there exist $C_1 > 0$ and $t_3 \geq t_0$ such that

$$0 < C_1 \leq x'(t)x''(t).$$

The integration again contradicts part (ii) and completes the proof.

4. Main Result II

In this section of the paper, we consider the solutions of Eq. (1.1) that belong to Class II. In our first result, we give sufficient conditions for a solution to be either oscillatory or to converge to zero.

Theorem 4.1. *Assume that the condition (H_2) is satisfied and that $x(t)$ is a solution of Eq. (1.1) from Class II. Then either $x(t)$ is oscillatory or $x(t) \to 0$ as $t \to \infty$.*

Proof. Let x be a nonoscillatory solution of (1.1), say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. (The proof in the case where $x(t) < 0$ for $t \geq t_1$ is similar.) To prove the theorem, we need to show that $x(t) \to 0$ as $t \to \infty$. In view of Lemma 2.2, $F[x(t)] < 0$ for $t \geq t_2$ for some $t_2 \geq t_1$. We define

$$R[x(t)] = \frac{x'(t)}{x(t)} + \int_{t_2}^{t} p(s)ds.$$

Then

$$R'[x(t)] = \frac{F[x(t)]}{x^2(t)} - \frac{1}{2}\left[\frac{x'(t)}{x(t)}\right]^2 < 0,$$

and, hence, $R[x(t)]$ is decreasing on $[t_2, \infty)$. Together with the condition (H_2), this implies

$$\lim_{t\to\infty} \frac{x'(t)}{x(t)} = -\infty.$$

By Lemma 2.1, $x(t) \to 0$ as $t \to \infty$, which proves the theorem.

Theorem 4 immediately yields the following corollary:

Corollary 4.1. *Assume that (H_2) holds. Then, any solution $x(t)$ of Eq. (1.1) that has one zero is either oscillatory or converges to* 0 *as $t \to \infty$.*

Proof. Suppose that $x(t)$ is a solution of Eq. (1.1) with one zero for some $t_1 \geq t_0$. Since

$$F[x(t_1)] = -\frac{1}{2}x'^2(t_1) \leq 0,$$

$x(t)$ belongs to Class II, and the conclusion follows from the theorem.

Further, we consider equation (1.1) with:

(H_6) there exists a constant σ such that $p(t) \geq \sigma > 0$;

(H_7) $q'(t) \leq 0$.

Theorem 4.2. *Assume that the conditions (H_2) and (H_5)–(H_7) hold and that $x(t)$ is a solution of equation (1.1) from Class II. Then:*

(iv) $\displaystyle\int^{\infty} x'^2(s)ds = \infty;$

(v) $\displaystyle\int^{\infty} x''^2(s)ds = \infty.$

Proof. Let $x(t)$ be a solution of Eq. (1.1) from Class II. Then

$$F\,[x(t)] \leq F\,[x(t_1)] < 0$$

for $t \geq t_1$ for some $t_1 \geq t_0$. We define

$$J[x(t)] = x(t)x'(t) - \frac{3}{2}\int_{t_1}^{t} x'^2(s)ds. \tag{4.1}$$

Thus, by (4.1),

$$J'[x(t)] = F[x(t)] - p(t)x^2(t) < F[x(t)] \leq F[x(t_1)] < 0 \tag{4.2}$$

for $t \geq t_1$. Integrating (4.2) from t_1 to t, we find

$$J[x(t)] < F[x(t_1)](t - t_1) - J[x(t_1)].$$

Hence,

$$J[x(t)] = x(t)x'(t) - \frac{3}{2}\int_{t_1}^{t} x'^2(s)ds \to -\infty \quad \text{as} \quad t \to \infty.$$

If

$$\int_{0}^{\infty} x'^2(s)ds < \infty,$$

then $x(t)x'(t) \to -\infty$ as $t \to \infty$. Thus, there exist $B > 0$ and $t_2 \geq t_1$ such that $x(t)x'(t) \leq -B < 0$ for $t \geq t_2$. As a result of integration, we obtain

$$x^2(t)/2 \leq -B(t - t_2) + x^2(t_2)$$

for $t \geq t_2$, which is impossible. Thus, (iv) is true.

To prove (v), note that

$$F[x(t)] = x(t)\left[x''(t) + p(t)x(t)\right] - \frac{1}{2}x'^2(t) \leq 0. \tag{4.3}$$

Hence, from (3.2), (4.3), and (H_6), we obtain

$$-\int_{T}^{t} q(s)x'(s)f(x(s))ds + V_1(t) + C_0 + \int_{T}^{t} x''^2(s)ds$$

$$= -\left[\int_{T}^{t} p(s)\left(x(s)x''(s) - \frac{1}{2}x'^2(s)\right)ds\right] + \frac{1}{2}\int_{T}^{t} p(s)x'^2(s)ds$$

$$\geq \frac{1}{2}\int_T^t p(s)x'^2(s)ds \geq \frac{\sigma}{2}\int_T^t x'^2(s)ds,$$

where

$$V_1(t) = -x'(t)(x''(t) + p(t)x(t)).$$

Furthermore,

$$-\int_T^t q(s)x'(s)f(x(s))ds = -q(t)H(x(t)) + \int_T^t q'(t)H(x(s))ds + C_1,$$

where

$$H(x) = \int_0^x f(u)du \geq 0$$

and $C_1 = q(T)H(x(T))$. Since $q'(t) \leq 0$, we get

$$C_1 + C_0 + V_1(t) + \int_T^t x''^2(s)ds \geq \frac{\sigma}{2}\int_T^t x'^2(s)ds. \tag{4.4}$$

By Theorem 4.1, either $x(t)$ oscillates or $x(t) \to 0$ as $t \to \infty$. Assume that

$$\int_{t_0}^t x''^2(s)ds < \infty.$$

If $x(t)$ is oscillatory, then $x'(t)$ is oscillatory. Hence, we can choose an increasing sequence $\{t_n\}$ of zeros of $x'(t)$. In view of (4.4), we must have $V_1(t) \to \infty$ as $t \to \infty$. However, $V_1(t_n) = 0$ for $t = t_n$, $n = 1, 2, \ldots,$ which contradicts the convergence $V_1(t) \to \infty$ as $t \to \infty$.

If $x(t) \to 0$ as $t \to \infty$, then there exists $\epsilon > 0$ such that $|x(t)| \leq \epsilon$ for large t, say, for $t \geq T_1$ for some $T_1 \geq T$. Since $V_1(t) \to \infty$ as $t \to \infty$, there exist $D > \mu\epsilon^2$ and $T_2 \geq T_1$ such that

$$D < -x'(t)x''(t) - p(t)x(t)x'(t)$$

for $t \geq T_2$. Integrating, we obtain

$$D(t - T_2) < -\int_{T_2}^t x'(s)x''(s)ds - \int_{T_2}^t p(s)x(s)x'(s)ds$$

$$\leq -\frac{1}{2}x'^2(t) - \frac{1}{2}p(t)x^2(t) + k + \int_{T_2}^t \left|p'(s)\right| x^2(s)ds,$$

where

$$k = \frac{1}{2}x'^2(T_2) + \frac{1}{2}p(T_2)x^2(T_2).$$

It follows from (H$_5$) that

$$(D - \mu\epsilon^2)t < k + DT_2$$

for $t \geq T_2$, which is impossible. This contradiction shows that part (v) holds and completes the proof of the theorem.

The following result is an immediate consequence of Theorems 3.1 and 4.1:

Theorem 4.3. *Assume that $q'(t) \geq 0$ and the conditions (H_1)–(H_5) are satisfied. If $x(t)$ is a nonoscillatory solution of Eq. (1.1), then*

$$\liminf_{t\to\infty} \|x(t)\| = 0.$$

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). If $x(t)$ belongs to Class II, then the conclusion follows from Theorem 4.1. If $x(t)$ is in Class I and

$$\liminf_{t\to\infty} |x(t)| > 0,$$

then there exists $\lambda > 0$ such that

$$|x(t)| \geq \lambda > 0.$$

This implies that

$$\lim_{t\to\infty} \int_{t_1}^{t} x^2(s)ds = \infty,$$

which contradicts part (i) of Theorem 3.1. This proves the theorem.

Prior to presenting our final result in this paper, we make the following remark: It follows from Theorems 3.1 and 4.2 that if x is a solution from Class I, then

$$\int^{\infty} x'^2(s)ds < \infty,$$

and if

$$\int^{\infty} x'^2(s)ds < \infty,$$

then x is not in Class II and, hence, it must be in Class I. Therefore, we get the following necessary and sufficient relationships:

Theorem 4.4. *Assume that (H_1)–(H_3) and (H_5)–(H_7) hold. Let $x(t)$ be a solution of (1.1). Then:*

$$x(t) \quad \textit{belongs to Class I} \quad \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x'^2(s)ds < \infty \right\} \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x''^2(s)ds < \infty \right\},$$

$$x(t) \quad \textit{belongs to Class II} \quad \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x'^2(s)ds = \infty \right\} \Longleftrightarrow \left\{ \int_{t_1}^{\infty} x''^2(s)ds = \infty \right\}.$$

Proof. As soon as we note that the condition (H_7) implies that $q(t)$ is bounded from above, the required conclusions follow from Theorems 3.2 and 4.2.

Concluding Remarks

In view of the results presented above, it would be reasonable to ask what, if anything, can be said about the square integrability of the solutions from Class II. In this case, we know that $F[x(t)]$ is negative and decreasing for $t \geq T$ and T is sufficiently large; therefore,

$$F[x(t)] \to F_0 \geq -\infty.$$

If we assume for the moment that $0 < q_0 \leq q(t) \leq q_1$, then it follows from ($H_3$) and (2.2) that

$$q_0 M \int_T^{\infty} x^2(s)ds \leq -F[x(t)] + F[x(T)] \leq q_1 N \int_T^{\infty} x^2(s)ds. \tag{4.5}$$

If $F_0 = -\infty$, then

$$\int_T^{\infty} x^2(s)ds = \infty$$

as one might suspect for a solution from Class II. However, if $F_0 > -\infty$, then, from the left-hand side of (4.5), we see that

$$\int_T^{\infty} x^2(s)ds < \infty.$$

In order to exclude this situation, a condition implying that $q(t) \to 0$ at $t \to \infty$ might be needed. This is an open question at present. Another interesting problem is to establish sufficient conditions for the solutions of equation (1.1) to belong to Class I or II.

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