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Convolution equations and mean-value theorems for solutions of linear elliptic equations with constant coefficients in the complex plane

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Abstract. In terms of the Bessel functions, we characterize smooth solutions of some convolution equations in the complex plane and prove a two-radius theorem for solutions of homogeneous linear elliptic equations with constant coefficients whose left-hand sides are representable in the form of a product of some non-negative integer powers of the complex differentiation operators ∂ and $\bar{\partial}$.

Keywords. Convolution equation, mean-value theorem, Bessel function, distribution, spherical transformation.

Introduction

The convolution equations generated by distributions with compact supports and the corresponding mean-value theorems were investigated by many authors (see, e.g., [1, 2]). In particular, Volchkov [2, Part 3, Chapter 2] described a wide class of radial distributions with compact supports such that the solutions of the corresponding convolution equations in open Euclidean balls can be efficiently characterized in terms of the Bessel functions and proved the general uniqueness and two-radius theorems for solutions of these equations that go back to the classical results by John [3, Chapter 6] and Delsarte [4, Part 3] about spherical means, respectively.

Let $0 < r < R \leq +\infty$, $m \in \mathbb{N} := \{1, 2, \dots\}$, and $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $s \leq m$. In the present paper, we study smooth functions $f(z)$ defined in the disk $B_R := \{z \in \mathbb{C} : |z| < R\}$ that satisfy the convolution equation

$$\frac{1}{2\pi} \iint\limits_{|\zeta - z| \leq r} f(\zeta)(\zeta - z)^s d\xi\, d\eta = \sum_{p=s}^{m-1} \frac{r^{2p+2}}{(2p+2)(p-s)!p!} \partial^{p-s} \bar{\partial}^p f(z) \tag{1}$$

for all $z \in B_{R-r}$, where i is the imaginary unit, $z = x + iy$, $\zeta = \xi + i\eta$ $(x, y, \xi, \eta \in \mathbb{R})$,

$$\partial f = \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

and the right-hand side of (1) is zero for $s = m$, i.e., in this case Eq. (1) has the form

$$\iint\limits_{|\zeta - z| \leq r} f(\zeta)(\zeta - z)^s d\xi\, d\eta = 0.$$

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One of the main results of the paper is Theorem 1 that describes functions $f \in C^\infty(B_R)$ satisfying Eq. (1) in terms of the representation of the Fourier coefficients of the function $\partial^{m-s}\bar{\partial}^m f$ by series of special functions. A similar but more complicated description of such functions f was obtained by the author [5] under the condition $s < m$ in terms of the representation of the Fourier coefficients of the function f. Theorems 2 and 3 are a sharp uniqueness theorem for solutions of the convolution equation (1) and a two-radius theorem characterizing solutions of the elliptic equation

$$\partial^{m-s}\bar{\partial}^m f = 0, \tag{2}$$

respectively.

A remarkable feature of the convolution equation (1) is that, for $s > 0$, this equation is generated by non-radial distributions. The use of the function $\partial^{m-s}\bar{\partial}^m f$ in Theorem 1 instead of the function f in [5] reduces the general case of this theorem to the investigation of some specific radial distributions with compact supports and their spherical transformations, which justifies the validity of the application of Volchkov's results [2, Part 3, Chapter 2] on the representation of solutions of convolution equations generated by radial distributions with compact supports.

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1. Main results

To each function $f \in C(B_R)$, we assign its Fourier series

$$f(z) \sim \sum_{k=-\infty}^{+\infty} f_k(\rho) e^{ik\varphi}, \tag{3}$$

where $z = \rho e^{i\varphi}$ is the trigonometric form of z,

$$f_k(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho e^{it}) e^{-itk} dt \qquad (z \neq 0, \quad k \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}).$$

For $z = 0$, we define the Fourier coefficients by continuity, i.e., $f_0(0) = f(0)$, $f_k(0) = 0$ for all integer $k \neq 0$. In the sequel, we shall use the following well-known property of the Fourier coefficients [2, Part 1, §5.1]: if $f \in C^\infty(B_R)$, then, for any $k \in \mathbb{Z}$, the function $f_k(\rho)e^{ik\varphi}$ is infinitely differentiable with respect to x and y ($z = x + iy = \rho e^{i\varphi}$) and the Fourier series (3) converges to the function f in the space $\mathcal{E}(B_R)$, i.e., converges uniformly together with its all partial derivatives of any order on each compact subset of the ball B_R.

Let

$$J_{s+1}(z) := \left(\frac{z}{2}\right)^{s+1} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(s+p+2)} \left(\frac{z}{2}\right)^{2p} \qquad (z \in \mathbb{C})$$

be the Bessel function of order $s+1$. Denote, by $Z(g_r)$, the set of all zeroes of the entire function

$$g_r(z) := \frac{J_{s+1}(zr)}{(zr)^{s+1}} - \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}} \ \text{ for } \ s < m,$$

$$g_r(z) := \frac{J_{s+1}(zr)}{(zr)^{s+1}} \ \text{ for } \ s = m.$$

To each $k \in \mathbb{Z}$, $\lambda \in Z_r := Z(g_r) \setminus \{0\}$, and $\eta \in \{0, ..., n_\lambda - 1\}$, where n_λ is the multiplicity of the zero λ, we assign the function

$$\Phi_{\lambda,\eta,k}(\rho) = \left(\frac{d}{dz}\right)^{\eta} (J_k(z\rho))\,|_{z=\lambda} \qquad (\rho > 0).$$

Using the introduced notation and the above assumptions, we can formulate our main results as follows.

Theorem 1. *A function $f \in C^\infty(B_R)$ satisfies Eq.* (1) *for all $z \in B_{R-r}$ iff the Fourier coefficients $f_k^*(\rho)$ of the function $f^*(z) := \partial^{m-s}\bar{\partial}^m f(z)$ are represented in the form*

$$f_k^*(\rho) = \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta,k} \Phi_{\lambda,\eta,k}(\rho) \qquad (0 < \rho < R, \quad k \in \mathbb{Z}), \tag{4}$$

where, for any $\alpha > 0$ and $k \in \mathbb{Z}$,

$$\max_{\eta=0,\ldots,n_\lambda-1} |c_{\lambda,\eta,k}| = O(|\lambda|^{-\alpha}) \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in Z_r. \tag{5}$$

Under condition (5), *series* (4) *converges in $\mathcal{E}(B_R)$ and all the coefficients $c_{\lambda,\eta,k}$ are defined uniquely with respect to f^*.*

It follows from Theorem 1 that the solutions of the elliptic equation (2) in B_R satisfy the convolution equation (1) for all $r \in (0,\ R)$ and $z \in B_{R-r}$. Note that $f \in C^\infty(B_R)$ yields $f^* \in C^\infty(B_R)$. Therefore, all the terms $f_k^*(\rho)e^{ik\varphi}$ of the Fourier series of the function f^* are infinitely differentiable functions with respect to x and y and this series converges to the function f^* in the space $\mathcal{E}(B_R)$. On the other hand, we will justify in Lemmas 1 and 2 that all zeroes λ of the function $g_r(z)$ with sufficiently large $|\lambda|$ are simple. Hence, we have from (5) that, for each $k \in \mathbb{Z}$, all the terms of the series

$$\sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta,k} \Phi_{\lambda,\eta,k}(\rho) e^{ik\varphi}$$

are infinitely differentiable functions with respect to x and y and this series converges to the kth term of the Fourier series of the function f^* in $\mathcal{E}(B_R)$.

Theorem 2. *The following assertions are true:*

(a) if $0 < r < R$, $f \in C^\infty(B_R)$, Eq. (1) *holds for any $z \in B_{R-r}$, and $f(z) = 0$ for all $z \in B_r$, then $f(z) \equiv 0$ in B_R;*

(b) for any $r > 0$ and $\varepsilon \in (0,\ r)$, there exists a function $f \in C^\infty(\mathbb{C})$ such that Eq. (1) *is satisfied everywhere in $\mathbb{C}$, $f(z) = 0$ for all $z \in B_{r-\varepsilon}$, and $f(z) \not\equiv 0$.*

Theorem 3. *Let r_1 and r_2 be real positive numbers. Then the following assertions are valid:*

(a) if $R \geq r_1 + r_2$, $Z_{r_1} \cap Z_{r_2} = \varnothing$, $f \in C^\infty(B_R)$, and Eq. (1) *holds for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, then f is a real analytic function that satisfies the elliptic equation* (2) *in B_R;*

(b) if $\max\{r_1, r_2\} < R < r_1 + r_2$ or $Z_{r_1} \cap Z_{r_2} \neq \varnothing$, then there exists a function $f \in C^\infty(B_R)$ satisfying Eq. (1) *for all $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$ and such that $\partial^{m-s}\bar{\partial}^m f \not\equiv 0$ in B_R.*

Corollary 1. *If $s = m$, r_1 and r_2 are real positive numbers such that $R > r_1 + r_2$ and r_1/r_2 is not a ratio of two distinct zeroes of the function J_{s+1}, $f \in C(B_R)$, and Eq.* (1) *holds for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, then f is an m-analytic function in B_R.*

Proof. Suppose that the conditions of Corollary 1 are satisfied. Since all zeros of the Bessel function J_{s+1} are real, the condition that r_1/r_2 is not a ratio of two distinct zeroes of the function J_{s+1} is equivalent to the condition $Z_{r_1} \cap Z_{r_2} = \varnothing$. Fix a function $\varphi \in C_0^\infty(\mathbb{C})$ supported in the closed unit disk $\overline{B}_1$ such that $\iint_{\mathbb{C}} \varphi(z)\, dx\, dy = 1$ $(z = x + iy)$. Then, for any $\varepsilon \in (0,\ R - r_1 - r_2)$, the function

$$f_\varepsilon(z) := \varepsilon^{-n} \iint_{\overline{B}_\varepsilon} f(z-\zeta)\varphi(\zeta/\varepsilon)\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta)$$

is defined in the disk $B_{R-\varepsilon}$, belongs to the class $C^\infty(B_{R-\varepsilon})$, and satisfies the convolution equation (1) for all $r = \{r_1, r_2\}$ and $z \in B_{R-r-\varepsilon}$. It follows from Theorem 3 that, for each $z \in B_{R-\varepsilon}$, we have the equality $\bar{\partial}^m f_\varepsilon(z) = 0$. Since $f_\varepsilon(z)$ converges to $f(z)$ as $\varepsilon \to +0$ uniformly on compact subsets of B_R, f is a distributional solution of the elliptic equation $\bar{\partial}^m f = 0$ in B_R. Hence, f is an m-analytic in B_R. □

The following corollary of Theorem 3 can be proved similarly.

Corollary 2. *If $s < m$, r_1 and r_2 are real positive numbers such that $r_1 + r_2 < R$ and $Z_{r_1} \cap Z_{r_2} = \varnothing$, $f \in C^{2m-s-2}(B_R)$, and Eq.* (1) *holds for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, then f is a real analytic function that satisfies the elliptic equation* (2) *in B_R.*

For $s < m$, Theorem 2 was proved in [6]. Corollary 1 for $R = +\infty$ was obtained by Zalcman [7]. Volchkov [8] proved Theorem 3 for $s = m$. On the other hand, the case $m = 1$ and $s = 0$ of Theorem 3 is the case $n = 2$ of the local version of the classical Delsarte's two-radius theorem characterizing harmonic functions in $\mathbb{R}^n$ [2, Part 5, §5.1]. Corollary 2 was obtained in [5].

2. Auxiliary results and constructions

Throughout this paper, we consider linear spaces of functions and distributions over the field of complex numbers $\mathbb{C}$. Recall that a distribution $T \in \mathcal{D}'(\mathbb{C})$ is said to be radial if, for any function $\varphi(w) \in C_0^\infty(\mathbb{C})$ and any $\alpha \in \mathbb{R}$, we have $T(\varphi(w)) = T(\varphi(e^{i\alpha}w))$, where T acts on the function $\varphi(e^{i\alpha}w)$ in w.

Let T be a radial distribution with compact support in $\mathbb{C}$. Then the function $\widetilde{T}(z) := T(J_0(z|w|))$ $(z \in \mathbb{C})$ is called the spherical transformation of the distribution T. The function $\widetilde{T}(z)$ is an even entire function of the exponential type and characterized by the following Paley–Wiener–Schwartz theorem for the spherical transformation [2, Part 1, §6.2, Theorem 6.5]: the even entire function $f(z)$ is a spherical transformation of some radial distribution supported in the closed disk $\overline{B}_r := \{z \in \mathbb{C} : |z| \le r\}$ iff there are $c > 0$ and $N > 0$ such that, for any $z \in \mathbb{C}$, we have $|f(z)| \le c(1 + |z|^N)e^{r|\operatorname{Im} z|}$.

As usual, Δ and δ denote the Laplace operator and the Dirac delta-function. The following properties of the spherical transformation are easily deduced from the definition:

$$\widetilde{\Delta T}(z) \equiv -z^2\widetilde{T}(z), \qquad \widetilde{\delta}(z) \equiv 1, \qquad \widetilde{\Delta^k \delta}(z) \equiv (-z^2)^k \quad (k \in \mathbb{N}_0). \tag{6}$$

Consider the following radial distributions supported in $\overline{B}_r$:

$$U_r(\varphi) := \frac{1}{2\pi} \iint_{|w| \le r} \partial^s \varphi(w) w^s du\, dv \qquad (\varphi \in C_0^\infty(\mathbb{C}), \quad w = u + iv),$$

$$S_r := \sum_{p=s}^{m-1} \frac{r^{2p+2}}{2(p+1)(p-s)!p!} \left(\frac{\Delta}{4}\right)^p \delta \qquad (S_r := 0 \text{ for } s = m),$$

$$T_r := U_r - S_r.$$

From (6), we get consequently:

$$\begin{aligned}
\widetilde{S_r}(z) &= \sum_{p=s}^{m-1} \frac{r^{2p+2}(-1)^p}{2(p+1)(p-s)!p!4^p} z^{2p} \\
&= (-1)^s 2^{-2s} z^{2s} \sum_{p=s}^{m-1} \frac{r^{2p+2}(-1)^{p-s}}{2(p+1)(p-s)!p!2^{2(p-s)}} z^{2(p-s)} \\
&= (-1)^s 2^{-2s} z^{2s} r^{2s+2} \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-2s+1}} \\
&= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} \sum_{p=s}^{m-1} \frac{(zr)^{2p-s}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-2s+1}} \\
&= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \sum_{p=s}^{m-1} \frac{(zr)^{2p-s}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}} \ (\widetilde{S_r}(z) \equiv 0 \text{ for } s = m),
\end{aligned}$$

$$\begin{aligned}
\widetilde{U_r}(z) &= \frac{1}{2\pi} \iint_{|w|\le r} w^s \partial^s \left(\sum_{p=0}^{\infty} \frac{(-1)^p}{p!p!} \left(\frac{z}{2}\right)^{2p} w^p \bar{w}^p \right) du\, dv \\
&= \frac{1}{2\pi} \iint_{|w|\le r} \left(\sum_{p=s}^{\infty} \frac{(-1)^p}{p!(p-s)!} \left(\frac{z}{2}\right)^{2p} |w|^{2p} \right) du\, dv \\
&= \sum_{q=0}^{\infty} \frac{(-1)^{q+s} r^{2(q+s+1)}}{2(q+s+1)!q!} \left(\frac{z}{2}\right)^{2(q+s)} \\
&= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} \sum_{q=0}^{\infty} \frac{(-1)^q}{\Gamma(q+s+2)q!} \left(\frac{zr}{2}\right)^{2q} \\
&= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \frac{J_{s+1}(zr)}{(zr)^{s+1}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{U_r}(z) - \widetilde{S_r}(z) &= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \frac{J_{s+1}(zr)}{(zr)^{s+1}} - (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}} \\
&= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} 2^s \left(\frac{J_{s+1}(zr)}{(zr)^{s+1}} - \sum_{p=s}^{m-1} \frac{(zr)^{2(p-s)}(-1)^{p-s}}{(p+1)!(p-s)!2^{2p-s+1}} \right) \\
&= (-1)^s r^2 \left(\frac{zr}{2}\right)^{2s} g_r(z) = r^{2s+2} \left(-\frac{z^2}{4}\right)^s g_r(z),
\end{aligned}$$

and

$$\widetilde{T_r}(z) \equiv r^{2s+2} \left(-\frac{z^2}{4}\right)^s g_r(z).$$

Applying the Paley–Wiener–Schwartz theorem for the spherical transformation, we conclude that there exists a unique radial distribution V_r supported in $\overline{B}_r$ such that $\widetilde{V}(z) \equiv 2^{-2s} r^{2s+2} g_r(z)$ and

$T_r(\varphi) = \Delta^s V_r(\varphi)$ for all $\varphi \in C^\infty(\mathbb{C})$. Since the function $g_r(z)$ has zero of order $2(m-s)$ at the point $z=0$, the function $\widetilde{T_r}(z)$ has zero of order $2m$ at this point. Hence, by the mentioned Paley–Wiener–Schwartz theorem, there exists a unique radial distribution W_r supported in $\overline{B}_r$ and such that

$$\widetilde{T_r}(z) \equiv z^{2m}\widetilde{W_r}(z).$$

This implies that $\widetilde{W_r}(0) \neq 0$ and

$$\widetilde{W_r}(z) \equiv z^{-2m} r^{2s+2}\left(-\frac{z^2}{4}\right)^s g_r(z).$$

In the sequel, we shall use the following properties of the Bessel functions.

Lemma 1. *Let $s \in \mathbb{N}_0$ and let $r > 0$. Then the following assertions are valid:*
(a) for any $\varepsilon \in (0,\ \pi)$, the asymptotic expansion

$$J_{s+1}(z) = (2/(\pi z))^{1/2}[\cos(z-(s+1)\pi/2-\pi/4) + O(z^{-1}e^{|\operatorname{Im} z|})]$$

holds as $z \to \infty$, $|\arg z| \leq \pi - \varepsilon$;
(b) all zeroes of the function $J_{s+1}(\lambda r)/(\lambda r)^{s+1}$ are real;
(c) there exists a constant $c > 0$ such that

$$\left|\frac{d}{d\lambda}\left(\frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}}\right)\right| \geq \frac{c}{|\lambda|^{s+3/2}} \quad \text{as} \quad \lambda \to \infty,\ \ J_{s+1}(\lambda r) = 0. \tag{7}$$

Proof. Assertions (a) and (b) are well-known (see, e.g., [10, §12.2] and [9, §23], respectively). The proof of assertion (c) is based on the following differentiation formula for the Bessel functions [9, §6]:

$$\frac{d}{dz}\left(\frac{J_{s+1}(z)}{z^{s+1}}\right) = -\frac{J_{s+2}(z)}{z^{s+1}}. \tag{8}$$

Since the function $J_{s+1}(\lambda r)/(\lambda r)^{s+1}$ is even, we can assume without loss of generality that λ is a real positive number in (7) and, consequently, that $\operatorname{Im}(\lambda r) = 0$. Applying this formula, we have

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}}\right) &= -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r = -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}} r + i\frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}} r \\
&= -\frac{r}{(\lambda r)^{s+1}}(J_{s+2}(\lambda r) - iJ_{s+1}(\lambda r)) \\
&= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2}[\cos(\lambda r-(s+2)\pi/2-\pi/4) \\
&- \ i\cos(\lambda r-(s+1)\pi/2-\pi/4) + O((\lambda r)^{-1}e^{|\operatorname{Im}\lambda r|})] \\
&= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2}[\cos(\lambda r-(s+2)\pi/2-\pi/4) \\
&+ \ i\sin(\lambda r-(s+2)\pi/2-\pi/4) + O((\lambda r)^{-1})] \\
&= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2}[\exp i(\lambda r-(s+2)\pi/2-\pi/4) + O((\lambda r)^{-1})]
\end{aligned}$$

as $\lambda \to \infty,\ J_{s+1}(\lambda r) = 0,\ \lambda > 0$.

This yields the asymptotic equality

$$\left|\frac{d}{d\lambda}\left(\frac{J_{s+1}(\lambda r)}{(\lambda r)^{s+1}}\right)\right| = \sqrt{\frac{2}{\pi}} \cdot \frac{r}{|\lambda r|^{s+3/2}}(1+o(1)) \text{ as } \lambda \to \infty, \ J_{s+1}(\lambda r) = 0,$$

whence we obtain (7) with $c = (2\pi)^{-1} r^{-s-1/2}$. □

The above arguments show that $g_r(z)$ is the spherical transformation of some radial distribution supported in a disk $\overline{B}_r$. This implies that the set Z_r is symmetric with respect to the origin and infinite. Indeed, the symmetry of Z_r follows from the evenness of the function $g_r(z)$. Suppose that the set Z_r is finite, i.e., $Z_r = \{z_1, -z_1, \dots, z_n, -z_n\}$ for some $n \in \mathbb{N}$. Then we have

$$g_r(z) = z^{2(m-s)} \prod_{j=1}^{n} \left(1 - \left(\frac{z}{z_j}\right)^2\right) e^{F(z)},$$

where $F(z)$ is an entire function such that $e^{F(z)} \equiv e^{F(-z)}$. Therefore, $e^{F(z)-F(-z)} \equiv 1$. From whence, we obtain $F(z) - F(-z) \equiv 2k\pi i$ for some integer k. Since $\lim_{z\to 0}(F(z) - F(-z)) = 0$, we have $k = 0$. This means that the entire function $F(z)$ is even.

The Paley–Wiener–Schwartz theorem for the spherical transformation yields the existence of a constant $C > 0$ such that the inequality $|F(z)| \le C \log(1+|z|)|\mathrm{Im}\, z|$ holds everywhere in $\mathbb{C}$. Therefore, $|F(z)| = o(|z|^2)$ as $z \to \infty$, whence the passage to the limit as $R \to \infty$ in the Cauchy inequalities $|a_{2k}| \le R^{-2k} \max_{\overline{B}_R} |F(z)|$ for the Taylor coefficients of the function $F(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k}$ shows that $F(z) \equiv a_0$. Hence, the function $g_r(z)$ must be a polynomial. This contradiction justifies that the set Z_r is infinite.

The following lemma plays the key role in the present work.

Lemma 2. *Let $s < m$. Then there are positive constants c_1, c_2, and c_3 such that, for any $\lambda \in Z_r \cap \{z \in \mathbb{C} : |z| \ge c_3\}$, we have the estimations*

$$|\mathrm{Im}\,\lambda| \le c_1 \log(2 + |\lambda|) \tag{9}$$

and

$$|g_r'(\lambda)| \ge c_2 |\lambda|^{2m-s-1}. \tag{10}$$

In particular, any $\lambda \in Z_r \cap \{z \in \mathbb{C} : |z| \ge c_3\}$ is a simple zero of the function $g_r(z)$.

Proof. Define the polynomial

$$P(z) := z^{s+1} \sum_{p=s}^{m-1} \frac{z^{2p-s+1}(-1)^{p-s}}{(p-s)!(p+1)!2^{2p-s+1}} \qquad (z \in \mathbb{C}).$$

Since the set Z_r is symmetric with respect to the origin, it is sufficient to prove Lemma 2 under the assumption $|\arg \lambda r| < 3\pi/4$. In this case, applying the first assertion of Lemma 1, rewriting the condition $g_r(\lambda) = 0$ in the form

$$J_{s+1}(\lambda r) = P(\lambda r), \tag{11}$$

using the formula $\cos z = (e^{iz} + e^{-iz})/2$, and taking into account that $P(z)$ is a polynomial of degree $2m$, we have

$$J_{s+1}(\lambda r) = (2/(\pi\lambda r))^{1/2}[\cos(\lambda r - (s+1)\pi/2 - \pi/4) + O((\lambda r)^{-1} e^{|\mathrm{Im}\,(\lambda r)|})] \qquad \text{as } \lambda \to \infty, \ \lambda \in Z_r, \tag{12}$$

$$
\begin{aligned}
\cos(\lambda r-(s+1)\pi/2-\pi/4) &= \frac{1}{2}[\exp(i\lambda r-i(s+1)\pi/2-i\pi/4)+\exp(-i\lambda r+i(s+1)\pi/2+i\pi/4)]\\
&= \frac{1}{2}[\exp(-\mathrm{Im}(\lambda r)+i\mathrm{Re}(\lambda r)-i(s+1)\pi/2-i\pi/4)\\
&\quad+\exp(\mathrm{Im}(\lambda r)-i\mathrm{Re}(\lambda r)+i(s+1)\pi/2+i\pi/4)]. \qquad (13)
\end{aligned}
$$

This implies that $|\mathrm{Im}(\lambda r)|\to\infty$ as $\lambda\to\infty$, $\lambda\in Z_r$. Otherwise, the asymptotics $J_{s+1}(\lambda r)\to 0$ as $\lambda\to\infty$ for $\lambda\in Z_r$ follows from relations (12) and (13), which contradicts (11). Therefore relations(11), (12), and (13) yield the asymptotic formulas

$$
|J_{s+1}(\lambda r)|=(2/(\pi|\lambda|r))^{1/2}\frac{e^{|\mathrm{Im}(\lambda r)|}}{2}(1+o(1)) \text{ as } \lambda\to\infty,\ \lambda\in Z_r
$$

and

$$
\frac{e^{|\mathrm{Im}(\lambda r)|}}{2}=(\pi|\lambda|r/2)^{1/2}|P(\lambda r)|(1+o(1)) \text{ as } \lambda\to\infty,\ \lambda\in Z_r. \qquad (14)
$$

From whence, we obtain (9).

In order to prove (10), we shall use the differentiation formula (8) and the first assertion of Lemma 1. Then the following chain of relations is valid as $\lambda\to\infty$, $\lambda\in Z_r$, $0\le\arg\lambda\le 3\pi/4$:

$$
\begin{aligned}
g_r'(\lambda) &= \frac{d}{d\lambda}\left(\frac{J_{s+1}(\lambda r)-P(\lambda r)}{(\lambda r)^{s+1}}\right)=-\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}}r-\frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right)\\
&= -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}}r-\frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right)+irg_r(\lambda)\\
&- \frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}}r-\frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right)+ir\frac{J_{s+1}(\lambda r)-P(\lambda r)}{(\lambda r)^{s+1}}\\
&= -\frac{J_{s+2}(\lambda r)-iJ_{s+1}(\lambda r)}{(\lambda r)^{s+1}}r-\frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right)-ir\frac{P(\lambda r)}{(\lambda r)^{s+1}}\\
&= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2}[\cos(\lambda r-(s+2)\pi/2-\pi/4)\\
&- i\cos(\lambda r-(s+1)\pi/2-\pi/4)+O(|\lambda r|^{-1}e^{|\mathrm{Im}\,\lambda r|})]\\
&- \frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right)-ir\frac{P(\lambda r)}{(\lambda r)^{s+1}}\\
&= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2}[\cos(\lambda r-(s+2)\pi/2-\pi/4)\\
&+ i\sin(\lambda r-(s+2)\pi/2-\pi/4)+O(|\lambda r|^{-1}e^{|\mathrm{Im}\,\lambda r|})]\\
&- \frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right)-ir\frac{P(\lambda r)}{(\lambda r)^{s+1}}\\
&= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2}\\
&\times [\exp\left(i(\lambda r-(s+2)\pi/2-\pi/4)\right)+O(|\lambda r|^{-1}e^{|\mathrm{Im}\,\lambda r|})]\\
&- \frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right)-ir\frac{P(\lambda r)}{(\lambda r)^{s+1}}.
\end{aligned}
$$

Using (14) and the estimate

$$
|\exp\left(i(\lambda r-(s+2)\pi/2-\pi/4)\right|=\exp\left(-\mathrm{Im}\,(\lambda r)\right)\le 1,
$$

which holds for $\operatorname{Im}(\lambda r) \geq 0$, we can continue this chain of relations as follows:

$$\begin{aligned}
g_r'(\lambda) &= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2}\exp\left(i(\lambda r-(s+2)\pi/2-\pi/4)\right) \\
&+ O(|\lambda r|^{-s-2}|P(\lambda r)|) \\
&- \frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right) - ir\frac{P(\lambda r)}{(\lambda r)^{s+1}} = -\frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right) - ir\frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
&+ O(|\lambda r|^{-s-3/2}|\exp\left(i(\lambda r-(s+2)\pi/2-\pi/4)\right)| + O(|\lambda r|^{-s-2}|P(\lambda r)|) \\
&= -\frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right) - ir\frac{P(\lambda r)}{(\lambda r)^{s+1}} + O(|\lambda r|^{-s-3/2}) + O(|\lambda r|^{-s-2}|P(\lambda r)|) \\
&= -ir\frac{P(\lambda r)}{(\lambda r)^{s+1}}(1+o(1)) = -irc(\lambda r)^{2m-s-1}(1+o(1))
\end{aligned}$$

$$= -icr^{2m-s}\lambda^{2m-s-1}(1+o(1)) \text{ as } \lambda\to\infty,\ \lambda\in Z_r,\ 0\leq\arg\lambda\leq 3\pi/4, \tag{15}$$

where c is the leading coefficient of the polynomial $P(z)$. Similarly, we have

$$\begin{aligned}
g_r'(\lambda) &= -\frac{J_{s+2}(\lambda r)}{(\lambda r)^{s+1}}r - \frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right) - irg_r(\lambda) \\
&= -\frac{J_{s+2}(\lambda r)+iJ_{s+1}(\lambda r)}{(\lambda r)^{s+1}}r - \frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right) + ir\frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
&= -\frac{r}{(\lambda r)^{s+1}}(2/(\pi\lambda r))^{1/2} \\
&\times \left[\exp\left(-i(\lambda r-(s+2)\pi/2-\pi/4)\right) + O(|\lambda r|^{-1}e^{|\operatorname{Im}\lambda r|})\right] \\
&- \frac{d}{d\lambda}\left(\frac{P(\lambda r)}{(\lambda r)^{s+1}}\right) + ir\frac{P(\lambda r)}{(\lambda r)^{s+1}} \\
&= ir\frac{P(\lambda r)}{(\lambda r)^{s+1}}(1+o(1)) = icr^{2m-s}\lambda^{2m-s-1}(1+o(1))
\end{aligned}$$

$$\text{as } \lambda\to\infty,\ \lambda\in Z_r,\ -3\pi/4\leq\arg\lambda<0. \tag{16}$$

Since $c\neq 0$ and the function $g_r(z)$ is even, relations (15) and (16) justify the validity of estimate (10) with $c_2 = 2^{-1}|c|r^{2m-s}$. □

Let φ be a radial distribution with compact support in $\mathbb{C}$, let $r(\varphi) := \inf\{r>0 : \operatorname{supp}\varphi\subset B_r\}$, and let $R\in(r(\varphi),\ +\infty]$. As usual, we say that a function $f\in C^\infty(B_R)$ satisfies the convolution equation $f*\varphi=0$ in B_R if the equality $\varphi(f(x-y))=0$ holds for all $x\in B_{R-r(\varphi)}$, where the distribution φ acts on the function $f(x-y)$ in y. Denote, by $Z(\widetilde{\varphi})$, the set of all zeroes of the spherical transformation $\widetilde{\varphi}(z)$ of the distribution φ. Let $\mathcal{R}$ be the set of all radial distributions φ with compact supports in $\mathbb{C}$ satisfying the following conditions: (a) $r(\varphi)>0$; (b) there are constants $c_1\geq 0$ and $c_2\geq 0$ such that, for any $\lambda\in Z(\widetilde{\varphi})$, we have the estimates $|\operatorname{Im}\lambda|\leq c_1\log(2+|\lambda|)$ and $|\partial^{n_\lambda}\widetilde{\varphi}(\lambda)|\geq(2+|\lambda|)^{n_\lambda-c_2}$, where n_λ denotes the multiplicity of the zero λ.

Combining assertion (c) of Lemma 1, Lemma 2, and the equality

$$\widetilde{W_r'}(\lambda) = \lambda^{-2m}r^{2s+2}\left(-\frac{\lambda^2}{4}\right)^s g_r'(\lambda),$$

which is valid for all $\lambda\in Z_r$, we obtain the following lemma.

Lemma 3. *For any $r > 0$, the radial distribution W_r belongs to the class $\mathcal{R}$.*

The following four lemmas are special cases of the Volchkov results [2, Part 3, §2.4, Theorem 2.3], [2, Part 3, §2.3, Theorem 2.1], [2, Part 3, §4.2, Theorem 4.8], and [2, Part 3, §4.2, Theorem 4.9], respectively.

Lemma 4. *Let $\varphi \in \mathcal{R}$, $\widetilde{\varphi}(0) \neq 0$, and let $R \in (r(\varphi),\ +\infty]$. Then a function $f \in C^\infty(B_R)$ is a solution of the convolution equation $f * \varphi = 0$ in B_R if the Fourier coefficients $f_k(\rho)$ of the function f can be represented in the form*

$$f_k(\rho) = \sum_{\lambda \in Z_r} \sum_{\eta=0}^{n_\lambda - 1} c_{\lambda,\eta,k} \Phi_{\lambda,\eta,k}(\rho) \qquad (0 < \rho < R, \quad k \in \mathbb{Z}), \tag{17}$$

where, for any $\alpha > 0$ and $k \in \mathbb{Z}$,

$$\max_{\eta=0,\dots,n_\lambda-1} |c_{\lambda,\eta,k}| = O(|\lambda|^{-\alpha}) \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in Z_r. \tag{18}$$

Under condition (18), *series* (17) *converges in $\mathcal{E}(B_R)$ and all the coefficients $c_{\lambda,\eta,k}$ are defined uniquely with respect to f.*

Lemma 5. *Let φ be a radial distribution with compact support in $\mathbb{C}$ such that $r(\varphi) > 0$ and let $R \in (r(\varphi),\ +\infty]$. Then the following statements hold:*

(a) if f is a function from $C^\infty(B_R)$ satisfying the convolution equation $f * \varphi = 0$ in B_R and $f(z) = 0$ everywhere in $B_{r(\varphi)}$, then $f(z) \equiv 0$ in B_R;

(b) if $\varphi \in \mathcal{R}$, then, for any $\varepsilon \in (0,\ r(\varphi))$, there exists a nonzero radial function $f \in C^\infty(\mathbb{R}^n)$ that satisfies the convolution equation $f * \varphi = 0$ in $\mathbb{R}^n$ and equals zero in $B_{r(\varphi)-\varepsilon}$.

Lemma 6. *Let $\varphi_1, \varphi_2 \in \mathcal{R}$, $Z(\widetilde{\varphi_1}) \cap Z(\widetilde{\varphi_2}) = \varnothing$, $R \in [r(\varphi_1) + r(\varphi_2),\ +\infty]$, and let f be a function from $C^\infty(B_R)$ that satisfies the convolution equations $f * \varphi_1 = 0$ and $f * \varphi_2 = 0$ in B_R. Then $f \equiv 0$ in B_R.*

Lemma 7. *Let $\varphi_1, \varphi_2 \in \mathcal{R}$ and let $R \in (\max\{r(\varphi_1), r(\varphi_2)\},\ r(\varphi_1) + r(\varphi_2))$. Then there exists a nonzero function $f \in C^\infty(B_R)$ that satisfies the convolution equations $f * \varphi_1 = 0$ and $f * \varphi_2 = 0$ in B_R.*

3. Proof of Theorem 1

Proof. Let a function $f \in C^\infty(B_R)$ satisfy Eq. (1) for all $z \in B_{R-r}$ and let

$$G(z,\zeta) := \log\left|\frac{R^2 - \zeta\bar{z}}{R(\zeta - z)}\right|, \qquad z, \zeta \in B_R, \quad z \neq \zeta,$$

be the Green function for the Laplace operator in a disk B_R. Then the function

$$Lf(z) := \int_{B_R} G(z,\zeta) f(\zeta)\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta) \tag{19}$$

belongs to the class $C^\infty(B_R)$, vanishes on the boundary of B_R, and satisfies the Poisson equation $\Delta(Lf) = f$ in the disk B_R [11, §1.6].

Define the function $d(z) := 4^s\bar{\partial}^s(L^s f)(z)$, $z \in B_R$. Then this function belongs to the class $C^\infty(B_R)$ and, for any $z \in B_R$, we have

$$\partial^s d(z) \;=\; 4^s\partial^s\bar{\partial}^s(L^s f)(z) \;=\; \Delta^s(L^s f)(z) \;=\; \Delta^{s-1}(L^{s-1} f)(z) \;=\; \dots \;=\; \Delta(Lf)(z) \;=\; f(z).$$

It was shown in Section 2 that the function $f(z)$ satisfies condition (1) for all $z \in B_{R-r}$ iff the function $d(z)$ is a solution of the convolution equation $d * T_r = 0$ in B_R. On the other hand, by the Paley–Wiener–Schwartz theorem for the spherical transformation, condition (1) holds for all $z \in B_{R-r}$ iff the function $d(z)$ is a solution of the convolution equation $(\Delta^m d) * W_r = 0$ in the disk B_R, where the radial distribution W_r is defined in Section 2. By Lemma 3 and Lemma 4, this is equivalent to that the Fourier coefficients of the function $f^*(z) = 4^{-m}\Delta^m d(z)$ are represented in the form (4), where the coefficients $c_{\lambda,\eta,k}$ are defined uniquely with respect to f^* and satisfy condition (5). Since $f^*(z) = \partial^{m-s}\bar{\partial}^m f(z)$ for all $z \in B_R$, we obtain the assertion of Theorem 1. $\square$

4. Proof of Theorem 2

Suppose that $0 < r < R$, $f \in C^\infty(B_R)$, Eq. (1) is valid for all $z \in B_{R-r}$, and the condition $f(z) = 0$ holds everywhere in B_r. Using the notation from the proof of Theorem 1, we consider the function $f^*(z) := \partial^{m-s}\bar{\partial}^m f(z)$. This function belongs to the class $C^\infty(B_R)$ and satisfies the convolution equation $f^* * W_r = 0$ in B_R. Since $f(z) = 0$ holds everywhere in B_r, we have $f^*(z) = 0$ for all $z \in B_r$. The radial distribution W_r satisfies the conditions of Lemma 5 [assertion (a)]. From whence, we have $f^*(z) \equiv 0$ in B_R. This means that the function f satisfies the equation $\partial^{m-s}\bar{\partial}^m f = 0$ in B_R. Since solutions of this equation are real analytic functions, the first assertion of Theorem 2 follows from the uniqueness theorem for real analytic functions.

The second assertion of Theorem 2 is deduced from assertion (b) of Lemma 5 applied to the radial distribution T_r defined in Section 2, which implies that there is a function $g \in C^\infty(\mathbb{C})$ such that the convolution equation $g * T_r = 0$ is valid everywhere in $\mathbb{C}$, $g(z) \not\equiv 0$, and $g(z) = 0$ for all $B_{r-\varepsilon}$. Therefore, the function $f(z) := \partial^s g(z)$ equals zero everywhere in $B_{r-\varepsilon}$ and, by the uniqueness theorem for real analytic functions, we have $f(z) \not\equiv 0$. On the other hand, it follows from the definition of the distribution T_r that the function $f(z)$ satisfies Eq. (1) in the whole $\mathbb{C}$.

5. Proof of Theorem 3

Suppose that $R \geq r_1 + r_2$, $Z_{r_1} \cap Z_{r_2} = \varnothing$, $f \in C^\infty(B_R)$, and Eq. (1) holds for all $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$. It was shown in the proof of Theorem 1 that the function $f^*(z) := \partial^{m-s}\bar{\partial}^m f(z)$, $z \in B_{R-\varepsilon}$, satisfies the convolution equations $f^* * W_{r_1} = 0$ and $f^* * W_{r_2} = 0$ in $B_{R-\varepsilon}$. Applying Lemma 6, we obtain the first assertion of Theorem 3.

Prove the second assertion. If $Z_{r_1} \cap Z_{r_2} \neq \varnothing$, then there is a $\lambda \in Z_{r_1} \cap Z_{r_2}$, $\lambda \neq 0$. Theorem 1 implies that each solution of the equation

$$\partial^{m-s}\bar{\partial}^m f = J_0(\lambda|z|)$$

in the disk B_R belongs to the class $C^\infty(B_R)$ and satisfies Eq. (1) for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$.

Consider the case $Z_{r_1} \cap Z_{r_2} = \varnothing$ and $\max\{r_1, r_2\} < R < r_1 + r_2$. Then the function $\widetilde{W_{r_1}}(z)/\widetilde{W_{r_2}}(z)$ is entire and has no zeroes in $\mathbb{C}$. Therefore, each of the functions $\widetilde{W_{r_1}}(z)$ and $\widetilde{W_{r_2}}(z)$ has no zeroes in $\mathbb{C}$. It follows from Lemma 7 that there is a nonzero function $g \in C^\infty(B_R)$ that satisfies the convolution equations $g * W_{r_1} = 0$ and $g * W_{r_2} = 0$ in B_R. Then the function $f(z) := \partial^s L^m g(z)$, where the operator

$L : C^\infty(B_R) \to C^\infty(B_R)$ is defined by (19), belongs to the class $C^\infty(B_R)$ and satisfies Eq. (1) for any $r \in \{r_1, r_2\}$ and $z \in B_{R-r}$, but $\partial^{m-s}\bar{\partial}^m f(z) \not\equiv 0$ in B_R (otherwise, we have $g(z) \equiv 0$ in B_R).

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