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Anscombe-type theorem and moderate deviations for trajectories of a compound renewal process

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Abstract. An Anscombe-type theorem for the large deviations principle for trajectories of a random process is proved. As a consequence, the moderate deviations principle for the compound renewal processes is obtained.

Keywords. Anscombe theorem, the large deviations principle, moderate deviations principle, compound renewal processes, Cramer condition, function of deviations.

1. Introduction

The theorem by F. Anscombe [1] proved in 1952 is a convenient tool for the construction of various limit theorems (central limit theorem, law of large numbers, law of the iterated logarithm; see [2–7]), where the index, by which the limiting transition is carried out, is a sequence of random variables.

It is natural to expect that, for the large deviations principle (l.d.p.), the result analogous to the Anscombe theorem would take place. Such theorem will be proved in Section 2 below. The convenience of its application will be demonstrated by the example of the proof of the moderate deviations principle (m.d.p.) for trajectories of a compound renewal process in Section 3.

We denote an arbitrary metric space (m.s.) by $\mathbf{X}_\rho$, the Borel σ-algebra of its subsets by $\mathfrak{B}_{\mathbf{X}_\rho}$, and the complement, closure, and interior of the set B by $\overline{B}$, $[B]$, and (B), respectively.

Recall the necessary definitions (see, e.g., [8–15]).

Definition 1.1. A family of random processes s_T satisfies l.d.p. in m.s. $\mathbf{X}_\rho$ with a functional of deviations (f.d.) $I = I(f) : \mathbf{X} \to [0, \infty]$ and the normalizing function (n.f.) $\psi(T) : \lim\limits_{T\to\infty} \psi(T) = \infty$, if, for any $c \geq 0$, the set $\{f \in \mathbf{X} : I(f) \leq c\}$ is a compact set in m.s. $\mathbf{X}_\rho$, and, for any set $B \in \mathfrak{B}_{\mathbf{X}_\rho}$, the following inequalities hold:

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\, s_T \in B \,) \leq -I([B]),$$

$$\liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\, s_T \in B \,) \geq -I((B)),$$

where $I(B) = \inf\limits_{y\in B} I(y)$ for $B \in \mathfrak{B}_{\mathbf{X}_\rho}$, $I(\varnothing) = \infty$.

In what follows, the words "the family of random processes s_T satisfies $(I, \psi(T), \mathbf{X}_\rho)$-l.d.p." means that the family of random processes s_T satisfies l.d.p. in m.s. $\mathbf{X}_\rho$ with f.d. $I = I(f)$ and n.f. $\psi = \psi(T)$.

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Definition 1.2. A family of random processes s_T is called exponentially tight (e.t.) in m.s. $\mathbf{X}_\rho$ with n.f. $\psi(T)$, if, for any $N > 0$, there exists a compact set $K_N \subseteq \mathbf{X}$ such that

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\, s_T \in \overline{K_N}\,) \le -N.$$

We use the following notations: $\mathbb{C}^d[0,c]$, $d \in \mathbb{N}$ is the space of d-dimensional functions with the uniform metric $\rho(f,g) = \sup\limits_{t\in[0,c]} \|f(t) - g(t)\|$, where $\|\cdot\|$ is the Euclidean norm, which are continuous on the segment $[0,c]$; $\mathbb{D}^d[0,c]$ and $\mathbb{D}^d_S[0,c]$ are the spaces of d-dimensional functions, which are continuous on the right and have limits on the left on the segment $[0,c]$, with a uniform metric and the Skorokhod metric, respectively; $\mathbb{AC}^d_0[0,c]$ is a set of d-dimensional functions, which are absolutely continuous on the segment $[0,c]$ and start from zero. A scalar product in the space $\mathbb{R}^2$ is denoted by $\langle\cdot,\cdot\rangle$.

The present work is written under the impression from the report by A. A. Borovkov "A generalization of the Anscombe theorem to random processes. Convergence of compound renewal processes" (March 30, 2017, Institute of Mathematics of the SD of the RAS). In this report, A. A. Borovkov proposed a version of the Anscombe theorem for random processes and, as a consequence, obtained the principle of invariance for the compound random processes (see Remark 3.3 for more details).

The following part of the present work includes three sections. In Section 2, we propose a version of the Anscombe theorem of large deviations of the trajectories of random processes; in Section 3, we will establish m.d.p. for the compound renewal processes; Section 4 is devoted to some auxiliary assertions.

2. Main result

Here, we will prove the Anscombe-type theorem of large deviations, which is the main result of the present work.

In what follows, we consider that all random elements participating in the statements of propositions are set on some probabilistic space $(\Omega, \mathfrak{F}, \mathbf{P})$. The expectation and variance relative to the measure $\mathbf{P}$ are denoted by $\mathbf{E}$ and $\mathbf{D}$, respectively.

Theorem 2.1. *Let, for a fixed $c > 0$, the following conditions hold:*

1) there exists $\Delta > 0$ such that the family of continuous random processes $s_T(t)$, $t \ge 0$ is e.t. in m.s. $\mathbb{C}^d[0, c+\Delta]$, $d \ge 1$ with n.f. $\psi(T)$;

2) stochastically continuous random process $\eta_T(t) \in \mathbb{D}[0,1]$ is nonnegative, and, for any $\delta > 0$,

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}\left(\sup_{t\in[0,1]} |\eta_T(t) - ct| > \delta \right) = -\infty.$$

Then, for any $\varepsilon > 0$,

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}\left(\sup_{t\in[0,1]} \|s_T(ct) - s_T(\eta_T(t))\| > \varepsilon \right) = -\infty.$$

Proof. By virtue of the fact that the family of processes $s_T(t)$ is e.t., for any $N > 0$, there exists a compact set $K_N \subseteq \mathbb{C}^d[0, c+\Delta]$ such that, for sufficiently large T,

$$\mathbf{P}(\, s_T \in \overline{K_N}\,) \le \exp\{-N\psi(T)\}\,. \tag{2.1}$$

By virtue of the Ascoli–Arzelá theorem, there exists $\delta \in [0, \Delta]$ such that, for any function $f \in K_N$, the inequality

$$\sup_{\substack{0 \le t,s \le 1+\Delta/c \\ c|t-s| \le \delta}} \|f(ct) - f(cs)\| < \frac{\varepsilon}{2} \tag{2.2}$$

holds. Denoting $A_\delta = \{\omega : \sup_{t\in[0,1]} |\eta_T(t) - ct| \le \delta\}$, we have

$$\begin{aligned} \mathbf{P}\left(\sup_{t\in[0,1]} \|s_T(ct) - s_T(\eta_T(t))\| > \varepsilon\right) &\le \mathbf{P}\left(\sup_{t\in[0,1]} \|s_T(ct) - s_T(\eta_T(t))\| > \varepsilon, A_\delta, s_T \in K_N\right) \\ &+ \mathbf{P}(\overline{A_\delta}) + \mathbf{P}(\, s_T \in \overline{K_N}\,) =: \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3. \end{aligned} \tag{2.3}$$

Denote

$$\Theta := \left\{\theta \in \mathbb{D}[0,1] : \sup_{t\in[0,1]} |\theta(t)| \le 1\right\}.$$

Inequality (2.2) yields

$$\begin{aligned} &\left\{\omega : \sup_{t\in[0,1]} \|s_T(ct) - s_T(\eta_T(t))\| > \varepsilon, A_\delta, s_T \in K_N\right\} \\ = &\left\{\omega : \sup_{t\in[0,1]} \left\|s_T(ct) - s_T\left(ct + \delta\left(\frac{\eta_T(t) - ct}{\delta}\right)\right)\right\| > \varepsilon, A_\delta, s_T \in K_N\right\} \\ \subseteq &\left\{\omega : \sup_{\substack{t\in[0,1] \\ \theta\in\Theta}} \|s_T(ct) - s_T(ct + \delta\theta(t))\| > \varepsilon, s_T \in K_N\right\} = \varnothing. \end{aligned}$$

Here, we used the fact that the trajectories of the random process $\frac{\eta_T(t)-ct}{\delta}$ on the event A_δ belong to the set of functions Θ.

Hence, $\mathbf{P}_1 = 0$.

Using inequalities (2.1) and (2.3) and condition 2) for any $\varepsilon > 0$, we get

$$\begin{aligned} &\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}\left(\sup_{t\in[0,1]} \|s_T(ct) - s_T(\eta_T(t))\| > \varepsilon\right) \\ &\qquad = \limsup_{T\to\infty} \frac{1}{\psi(T)} \ln(\mathbf{P}_2 + \mathbf{P}_3) \le \limsup_{T\to\infty} \frac{1}{\psi(T)} \ln(2\max\{\mathbf{P}_2, \mathbf{P}_3\}) \le -N. \end{aligned}$$

The limiting transition $N \to \infty$ completes the proof of the theorem. □

Remark 2.2. Lemma 4.9 in [15] contains a similar result, but it was required there that the families of processes $s_T(t)$ and $\eta_T(t)$ be independent, the constant $c = 1$, and the n.f. take the form $\psi(T) = T$.

We will be interested in l.d.p. in the space $\mathbb{D}^d[0, c]$ with a uniform metric. But, due to the inseparability, the Borel σ-algebra constructed by sets that are open relative to this metric will contain sets nonmeasurable for the probabilistic measure $\mathbf{P}$, see [16, §18]. Therefore, in what follows, we will consider the measure $\mathbf{P}$ on sets from the σ-algebra constructed by open cylindrical subsets of the space $\mathbb{D}^d[0, c]$.

Definition 2.3. The families of random processes $v_T(t)$ and $s_T(t)$, whose trajectories belong to m.s. $\mathbf{X}_\rho$, are equivalent from the viewpoint of l.d.p. ($v_T \overset{L.D.}{\sim} s_T$), if, for any $\varepsilon > 0$,

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}\left(\rho(v_T, s_T) > \varepsilon\right) = -\infty.$$

It is easy to prove that if $v_T \overset{L.D.}{\sim} s_T$, m.s. $\mathbf{X}_\rho$ is complete, and one of the families of processes satisfies l.d.p., then the second family satisfies the same l.d.p. (see, e.g., [8, Theorem 4.2.13]).

Definition 2.4. The family of stochastically continuous random processes $v_T(t)$ satisfies $\mathbb{C}$–$(I, \psi(T), \mathbb{D}^d[0,c])$-l.d.p., if there exists a family of continuous random processes $s_T(t)$ that is equivalent to it from the viewpoint of l.d.p. and satisfies $(I, \psi(T), \mathbb{C}^d[0,c])$-l.d.p.

Remark 2.5. If the family of processes $v_T(t)$ satisfies $\mathbb{C}$–$(I, \psi(T), \mathbb{D}^d[0,c])$-l.d.p. and if, in this case, $\mathbf{P}(v_T \in \mathbb{C}^d[0,c]) = 1$, then it satisfies $(I, \psi(T), \mathbb{C}^d[0,c])$-l.d.p.

Corollary 2.6. *Let the conditions of Theorem 2.1 be satisfied. Then*

$$s_T(\eta_T(t)) \overset{L.D.}{\sim} s_T(ct).$$

Hence, if the family of processes $s_T(ct)$ satisfies $(I, \psi(T), \mathbb{C}^d[0,1])$-l.d.p., then the family of processes $s_T(\eta_T(t))$ will satisfy $\mathbb{C}$–$(I, \psi(T), \mathbb{D}^d[0,1])$-l.d.p.

Remark 2.7. It is obvious that $\mathbb{C}$–$(I, \psi(T), \mathbb{D}^d[0,c])$-l.d.p. yields $(I, \psi(T), \mathbb{D}^d_S[0,c])$-l.d.p.

We now give a simple example of applications of Theorem 2.1.

Let a Wiener process $w(t)$ and a Poisson process $\nu(t)$ with parameter $\mathbf{E}\nu(t) = ct$ be given on the probabilistic space. Consider the family of random processes

$$w_T(t) = \frac{1}{x(T)} w(tT),$$

where the function $x(T)$ satisfies the conditions

$$\lim_{T\to\infty} \frac{x(T)}{\sqrt{T}} = \infty, \quad \lim_{T\to\infty} \frac{x(T)}{T} = 0.$$

We are interested in m.d.p. for the family of processes

$$w_{T,\nu}(t) := \frac{1}{x(T)} w(\nu(tT)).$$

Theorem 5.3.2 [18] and Lemma 4.1 (i) (see Section 4) imply that, for any fixed $c > 0$ and $\Delta \ge 0$, the family of random processes $w_T(t)$ satisfies $\left(I, \frac{x^2(T)}{T}, \mathbb{C}[0, c+\Delta]\right)$-l.d.p., where

$$I(f) = \begin{cases} \frac{1}{2}\int_0^{c+\Delta} (f'(t))^2 dt, & \text{if } f \in \mathbb{AC}_0[0, c+\Delta], \\ \infty, & \text{otherwise}. \end{cases}$$

Therefore, the Pukhalskii theorem [19] implies that the family of processes $w_T(t)$ is e.t. in m.s. $\mathbb{C}[0, c+\Delta]$. Hence, condition 1) of Theorem 2.1 is satisfied.

Let us verify condition 2) of Theorem 2.1. Consider the random process $\eta_T(t) = \frac{\nu(tT)}{T}$. Denote $\tilde{\nu}(t) = \nu(t) - ct$.

Using the Doob inequality (see, e.g., [17, Chapt. 2, Theorem 1.7]), for any $r>0$, we have

$$\mathbf{P}\left(\sup_{t\in[0,1]}|\eta_T(tT)-ct|>\delta\right)\le\mathbf{P}\left(\sup_{t\in[0,1]}r\tilde{\nu}(tT)>r\delta T\right)$$

$$+\mathbf{P}\left(\sup_{t\in[0,1]}-r\tilde{\nu}(tT)>r\delta T\right)\le\frac{\mathbf{E}e^{r\tilde{\nu}(tT)}}{e^{r\delta T}}+\frac{\mathbf{E}e^{-r\tilde{\nu}(tT)}}{e^{r\delta T}}$$

$$=\exp\{(e^r-1-r)cT-r\delta T\}+\exp\{(e^{-r}-1+r)cT-r\delta T\}.$$

Let us estimate each term on the right-hand side. We note that,for $r>0$,

$$e^r-1-r\le r^2e^r.$$

Therefore, by choosing $r=\frac{x(T)}{T}$, we have

$$(e^r-1-r)cT-r\delta T\le x(T)\left[\frac{cx(T)}{T}e^{\frac{x(T)}{T}}-\delta\right].$$

For sufficiently large T, the first term in square brackets is less than $\frac{\delta}{2}$. Therefore, for sufficiently large T, we get

$$\exp\{(e^r-1-r)cT-r\delta T\}\le\exp\left\{-\frac{\delta x(T)}{2}\right\}.$$

The analogous upper bound is true also for the term $\exp\{(e^{-r}-1+r)cT-r\delta T\}$. Hence, for sufficiently large T, we have

$$\mathbf{P}\left(\sup_{t\in[0,1]}|\eta_T(tT)-ct|>\delta\right)\le 2\exp\left\{-\frac{\delta x(T)}{2}\right\}.$$

This yields

$$\limsup_{T\to\infty}\frac{T}{x^2(T)}\ln\mathbf{P}\left(\sup_{t\in[0,1]}|\eta_T(t)-ct|>\delta\right)=-\infty.$$

Hence, condition 2) of Theorem 2.1 is satisfied.

Thus, all conditions of Theorem 2.1 are satisfied. Therefore,

$$w_T(\eta_T(t))=w_{T,\nu}(t)\overset{L.D.}{\sim}w_T(ct)$$

in m.s. $\mathbb{D}[0,1]$.

Since the family of random processes $w_T(t)$ satisfies $\left(I,\frac{x^2(T)}{T},\mathbb{C}[0,c]\right)$-l.d.p., Lemma 4.1 *(ii)* (see Section 4) implies that the family of processes $w_T(ct)$ satisfies $\left(\tilde{I},\frac{x^2(T)}{T},\mathbb{C}[0,1]\right)$-l.d.p., where

$$\tilde{I}(f)=\begin{cases}\frac{1}{2}\int_0^1(f'(t))^2dt, \text{ if } f\in\mathbb{AC}_0[0,1],\\ \infty, \text{ otherwise.}\end{cases}$$

Thus, Corollary 2.6 implies that the family of processes $w_{T,\nu}(t)$ satisfies $\mathbb{C}$–$\left(\tilde{I},\frac{x^2(T)}{T},\mathbb{D}[0,1]\right)$-l.d.p.

We note that the independence of the processes $w(t)$ and $\nu(t)$ was required nowhere.

3. The moderate deviations principle for trajectories of compound renewal processes

Let a sequence of independent identically distributed random vectors $\xi = (\tau, \zeta), \xi_2 = (\tau_2, \zeta_2), \xi_3 = (\tau_3, \zeta_3), \dots$, where $\tau > 0$, and a random vector $\xi_1 = (\tau_1, \zeta_1)$, $\tau_1 \geq 0$, which is independent of this sequence and has, generally speaking, a distribution different from that for $\xi = (\tau, \zeta)$, be given.

We set $T_0 = Z_0 = 0$ and denote

$$T_n := \sum_{j=1}^{n} \tau_j, \quad Z_n := \sum_{j=1}^{n} \zeta_j, \quad S_n := \sum_{j=1}^{n} \xi_j = (T_n, Z_n) \text{ for } n \geq 1.$$

Let, for $t > 0$,

$$\eta(t) := \min\{k \geq 0 : T_k \geq t\}, \quad \nu(t) := \max\{k \geq 0 : T_k < t\}. \tag{3.1}$$

It is clear that

$$\nu(t) = \eta(t) - 1. \tag{3.2}$$

Compound renewal process (c.r.p.) $Z(t);\ t \geq 0$, is defined by the equalities

$$Z(t) := Z_{\nu(t)} \quad \text{for} \quad t > 0, \quad Z(0) = 0. \tag{3.3}$$

In addition to c.r.p. $Z(t)$, we consider also the process

$$Y(t) := Z_{\eta(t)} = Z(t) + \zeta_{\eta(t)} \quad \text{for} \quad t > 0, \quad Y(0) = 0, \quad Y(+0) = \zeta_1.$$

We call it also c.r.p.

Agreement 1. Everywhere, unless otherwise stipulated, we assume that the Cramer condition holds in the form

[$\mathbf{C}_0$]. $\mathbf{E}e^{v|\xi|} < \infty$, $\mathbf{E}e^{v|\xi_1|} < \infty$ *for some* $v > 0$.

In addition, we assume that a random vector $\xi = (\tau, \zeta)$ is *nondegenerate*, i.e., for any $\lambda \in \mathbb{R}^2$, $|\lambda| \neq 0$, and $c \in \mathbb{R}$, the inequality $\mathbf{P}(\langle \lambda, \xi \rangle = c) < 1$ holds. In order to avoid repetitions, these two conditions will not be present in the statements of the main assertions.

If the distribution of a random vector ξ_1 coincides with the distribution of ξ, we call this case *homogeneous*. If the vectors ξ_1 and ξ have different distributions, we call this case *inhomogeneous*.

The standard commonly accepted model of c.r.p. assumes that the time of the appearance of the first jump τ_1 and its magnitude ζ_1 have a common distribution which is different, generally speaking, from the common distribution of (τ, ζ) (see, e.g., [18]). This case is realized, for example, for c.r.p. with stationary increments.

For $t \geq 0$, we denote

$$Z_1(t) := Z(t) - at, \quad Y_1(t) := Y(t) - at,$$

$$Z_2(t) := Z(t) - a_\zeta \nu(t), \quad Y_2(t) := Y(t) - a_\zeta \eta(t),$$

$$Z_3(t) := a_\zeta \left(\nu(t) - \frac{1}{a_\tau} t \right), \quad Y_3(t) := a_\zeta \left(\eta(t) - \frac{1}{a_\tau} t \right),$$

where $a := \frac{a_\zeta}{a_\tau}$, $a_\zeta := \mathbf{E}\zeta$, $a_\tau := \mathbf{E}\tau$.

It is easy to see that

$$Z_2(t) = Z_1(t) - Z_3(t), \quad Y_2(t) = Y_1(t) - Y_3(t), \quad t \geq 0. \tag{3.4}$$

Fix a function $x = x(T)$ such that

$$\lim_{T\to\infty} \frac{x(T)}{\sqrt{T}} = \infty, \quad \lim_{T\to\infty} \frac{x(T)}{T} = 0. \tag{3.5}$$

In what follows, we will drop the argument T of the function $x(T)$, if this is no hamper the statement.

The main object under study is two processes

$$\overline{z}_T = \overline{z}_T(t) = (z_{1,T}(t), z_{3,T}(t)) := \left(\frac{1}{x}Z_1(tT), \frac{1}{x}Z_3(tT)\right), \quad 0 \le t \le 1;$$

$$\overline{y}_T = \overline{y}_T(t) = (y_{1,T}(t), y_{3,T}(t)) := \left(\frac{1}{x}Y_1(tT), \frac{1}{x}Y_3(tT)\right), \quad 0 \le t \le 1.$$

These processes lie in the space $\mathbb{D}^2[0,1]$, where we will use the uniform metric $\rho = \rho(f, g)$.

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, we consider the function of deviations

$$\Lambda(\alpha) := \frac{1}{2}\alpha A \alpha^T = \frac{1}{2}\sum_{i,j=1}^{2} A_{ij}\alpha_i\alpha_j, \tag{3.6}$$

where $A = \|A_{ij}\|$ is the matrix reciprocal to the covariance matrix $B = \|\mathbf{E}\theta_i\theta_j\|$ of the random vector $\theta = (\theta_1, \theta_2) := (\zeta - a\tau, a_\zeta - a\tau)$.

For any function $f \in \mathbb{D}^2[0,1]$, we set

$$I(f) = \begin{cases} a_\tau \int_0^1 \Lambda(f'(t))dt, & \text{if} \quad f \in \mathbb{AC}_0^2[0,1], \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 3.1. ***(P.m.d. for c.r.p.)*** *Each of the processes*

$$\overline{z}_T = \overline{z}_T(t) = (z_{1,T}(t), z_{3,T}(t)), \quad \overline{y}_T = \overline{y}_T(t) = (y_{1,T}(t), y_{3,T}(t))$$

satisfies $\mathbb{C}$–$(I, \frac{x^2}{T}, \mathbb{D}^2[0,1])$*-l.d.p.*

Denote

$$z_{2,T}(t) := \frac{1}{x}Z_2(tT), \quad y_{2,T}(t) := \frac{1}{x}Y_2(tT).$$

We now define three functionals of deviations $I_1(f)$, $I_2(f)$, and $I_3(f)$, $f \in \mathbb{D}[0,1]$, by setting

$$I_i(f) := \frac{a_\tau}{2\sigma_i^2} I_0(f), \quad I_0(f) := \begin{cases} \int_0^1 (f'(t))^2 dt, & \text{if} \quad f \in \mathbb{AC}_0[0,1], \\ \infty, & \text{otherwise,} \end{cases}$$

where $\sigma_1^2 = \mathbf{D}(\zeta - a\tau)$, $\sigma_2^2 = \mathbf{D}\zeta$, and $\sigma_3^2 = a^2\mathbf{D}\tau$.

Since $z_{i,T}(t) = \beta_1^{(i)} z_{1,T}(t) + \beta_2^{(i)} z_{3,T}(t)$, where $\beta_1^{(1)} = 1$, $\beta_2^{(1)} = 0$; $\beta_1^{(2)} = 1$, $\beta_2^{(2)} = -1$; $\beta_1^{(3)} = 0$, and $\beta_2^{(3)} = 1$, we use Lemma 4.2 (see Section 4) and Theorem 3.1 and obtain the following proposition.

Corollary 3.2. *Each of the processes* $z_{i,T}(t)$, $y_{i,T}(t)$ *satisfies* $\mathbb{C}$–$(I_i, \frac{x^2}{T}, \mathbb{D}[0,1])$*-l.d.p.,* $i = 1, 2, 3$.

Remark 3.3. As was mentioned in Introduction, the starting point of the present work was the report by A. A. Borovkov, where he established, in particular, the principle of invariance for the processes $Z_1(t)$ and $Y_1(t)$ under the minimum condition $\mathbf{E}|\xi|^2 < \infty$. In other words, the weak convergence of the processes

$$\frac{Z_1(tT)}{\sigma_1\sqrt{T}}, \quad \frac{Y_1(tT)}{\sigma_1\sqrt{T}}; \ t \in [0,1]$$

to a Wiener process was proved. Under the condition $[\mathbf{C}_0]$, Corollary 3.2 extends, thus, the principle of invariance (in the logarithmic form) to the domain of moderate deviations defined by a function $x(T)$ of the form (3.5).

The proof of Theorem 3.1. Will be made for the first process

$$\overline{z}_T = \overline{z}_T(t) = (z_{1,T}(t), z_{3,T}(t)), \quad 0 \le t \le 1.$$

For the second process

$$\overline{y}_T = \overline{y}_T(t) = (y_{1,T}(t), y_{3,T}(t)), \quad 0 \le t \le 1,$$

the proof is analogous.

We need the following notations. By $\overline{\widetilde{Z}}(t)$, we denote a continuous random broken line (c.r.b.l.) constructed by the nodal points

$$(T_k, \ (Z_k - aT_k, \ a_\zeta k - aT_k)), \quad k = 0, 1, 2, \cdots .$$

By $\overline{\widetilde{S}}(t)$, we denote c.r.b.l. constructed by the nodal points

$$(k, \ (Z_k - aT_k, \ a_\zeta k - aT_k)), \quad k = 0, 1, 2, \cdots .$$

By $\widetilde{\nu}(t)$, we denote c.r.b.l. constructed by the nodal points

$$(T_k, \ k), \quad k = 0, 1, 2, \cdots .$$

Then it is easy to see that the following formula is valid:

$$\overline{\widetilde{Z}}(t) = \overline{\widetilde{S}}(\widetilde{\nu}(t)), \quad t \ge 0. \tag{3.7}$$

Denote

$$\overline{\widetilde{z}}_T(t) := \frac{1}{x}\overline{\widetilde{Z}}(tT), \quad 0 \le t \le 1;$$

$$\overline{\widetilde{s}}_T(t) := \frac{1}{x}\overline{\widetilde{S}}(tT), \quad 0 \le t < \infty;$$

$$\widetilde{\nu}_T(t) := \frac{1}{T}\widetilde{\nu}(tT), \quad 0 \le t < \infty.$$

Then, by virtue of (3.7), we have

$$\overline{\widetilde{z}}_T(t) = \overline{\widetilde{s}}_T(\widetilde{\nu}_T(t)), \quad 0 \le t \le 1. \tag{3.8}$$

In order to apply Theorem 2.1, we verify whether the conditions of this theorem are satisfied.

Lemma 3.4.
(i) For any $c > 0$, the family of random processes $\widetilde{\widetilde{s}}_T = \widetilde{\widetilde{s}}_T(t);\ 0 \le t \le c$, satisfies $(I_c, \frac{x^2}{T}, \mathbb{C}^2[0,c])$-l.d.p., where I_c is defined as

$$I_c := \begin{cases} \int_0^c \Lambda(f'(t))dt, & \text{if} \quad f \in \mathbb{AC}_0^2[0,c], \\ \infty, & \text{otherwise.} \end{cases}$$

(ii) For any $c > 0$, the family of random processes $\widetilde{\widetilde{s}}_T = \widetilde{\widetilde{s}}_T(t);\ 0 \le t \le c$ is e.t. in m.s. $\mathbb{C}^2[0,c]$ with n.f. $\frac{x^2}{T}$.
(iii) In the space $\mathbb{C}[0,1]$, the relation

$$\widetilde{\nu}_T \overset{L.D.}{\sim} \frac{1}{a_\tau} h,$$

where the function $h(t) = t$, is valid.
(iv) In the space $\mathbb{D}^2[0,1]$, the relation

$$\widetilde{\widetilde{z}}_T \overset{L.D.}{\sim} \widetilde{z}_T$$

holds.

Lemma 3.4 will be prove below. Now, we return to the proof of Theorem 3.1. By virtue of assertions $(i) - (iii)$ of Lemma 3.4, all conditions of Theorem 2.1 are satisfied. Therefore, this theorem, Lemma 4.1 (see Section 4), and Corollary 2.6 yield: the family of processes $\widetilde{\widetilde{z}}_T$ satisfies $\Big(I, \frac{x^2}{T}, \mathbb{C}^2[0,1]\Big)$-l.d.p. Applying assertion (iv) of Lemma 3.4 and using Definition 2.4, we get the assertion of Theorem 3.1. □

It remains to carry out

Proof of Lemma 3.1. $(i) - (ii)$. Consider firstly the case of homogeneity where the distributions of random vectors ξ_1 and ξ coincide. Assertions $(i) - (ii)$ of Lemma 3.1 follow from Theorems 5.2.1 and 5.2.2 [18], and Lemmas 4.1 and 4.3 (see Section 4).

Let the distributions of random vectors ξ_1 and ξ are different. Then we may consider that the independent random vectors

$$\xi_1^* = (\tau_1^*, \zeta_1^*), \xi_1 = (\tau_1, \zeta_1), \xi_2 = (\tau_2, \zeta_2), \cdots,$$

are defined on the probabilistic space. In this case, let the vectors $\xi_1, \xi_2, \cdots$ have the common distribution different from the distribution of the vector ξ_1^*. Then the sequence

$$\xi_1 = (\tau_1, \zeta_1), \xi_2 = (\tau_2, \zeta_2), \cdots$$

corresponds to the homogeneous case. By this sequence, we construct c.r.b.l.

$$\widetilde{\widetilde{s}}_T = \widetilde{\widetilde{s}}_T(t);\ 0 \le t \le c;$$

the sequence

$$\xi_1^* = (\tau_1^*, \zeta_1^*), \xi_2 = (\tau_2, \zeta_2), \cdots$$

corresponds to the inhomogeneous case. By this sequence, we construct c.r.b.l.

$$\widetilde{\widetilde{s}}_T^* = \widetilde{\widetilde{s}}_T^*(t);\ 0 \le t \le c.$$

It is easy to see that, in this case,

$$\rho(\widetilde{\widetilde{s}}_T, \widetilde{\widetilde{s}}_T^*) = \sup_{0 \le t \le c} |\widetilde{\widetilde{s}}_T(t) - \widetilde{\widetilde{s}}_T^*(t)| \le \frac{1}{x}(|\zeta_1 - \zeta_1^*| + |a|(|\tau_1 - \tau_1^*|)) \le \frac{1}{x}(|\zeta_1| + |\zeta_1^*| + |a|(\tau_1 + \tau_1^*)).$$

From whence, by virtue of condition $[\mathbf{C}_0]$, we obtain easily the relation $\widetilde{\widetilde{s}}_T \overset{L.D.}{\sim} \widetilde{\widetilde{s}}^*_T$. Thus, we have proved assertions (i) and (ii) in the general case.

(iii). We now prove assertion (iii) firstly in the homogeneous case. It is easy to see that $\{\nu(T) < N\} = \{T_N \geq T\}$. Therefore, $\{\nu(T) \geq N\} = \{T_N < T\}$. For $N = [cT]$ and $a_\tau N > T$ by virtue of the Chebyshev exponential inequality, we have

$$\mathbf{P}(\nu(T) \geq [cT]) = \mathbf{P}(T_N < T) \leq e^{-N\Lambda_\tau(\frac{T}{N})} = e^{-T\frac{N}{T}\Lambda_\tau(\frac{T}{N})}, \tag{3.9}$$

where

$$\Lambda_\tau(\alpha) := \sup_\lambda\{\lambda\alpha - \ln \mathbf{E}e^{\lambda\tau}\}$$

is a function of deviations of the random variable τ. Since the function of deviations $\Lambda_\tau(\alpha)$ decreases on the interval $(0, a_\tau)$, we have, for $\frac{T}{N} \leq \frac{a_\tau}{2}$,

$$\Lambda_\tau\left(\frac{T}{N}\right) \geq \Lambda_\tau\left(\frac{a_\tau}{2}\right) =: \delta,$$

$$\frac{N}{T}\Lambda_\tau\left(\frac{T}{N}\right) \geq \frac{2}{a_\tau}\Lambda_\tau\left(\frac{T}{N}\right) \geq \frac{2}{a_\tau}\delta =: \gamma_1 > 0.$$

Hence, for $c = \frac{3}{a_\tau}$ for all sufficiently large T by virtue of (3.9), we have

$$\mathbf{P}(\nu(T) \geq [cT]) \leq e^{-T\gamma_1}. \tag{3.10}$$

On the event $\{\nu(T) \leq [cT]\}$, we now estimate the quantity $\rho\left(\widetilde{\nu}_T, \frac{1}{a_\tau}h\right)$.

Since $\nu(T_k + 0) = k$, we have $\widetilde{\nu}(T_k) = k$ and, hence,

$$\rho\left(\widetilde{\nu}_T, \frac{1}{a_\tau}h\right) = \max_{0\leq t\leq 1}\left|\frac{1}{T}\widetilde{\nu}(tT) - \frac{1}{a_\tau}t\right| = \frac{1}{T}\max_{0\leq u\leq T}\left|\widetilde{\nu}(u) - \frac{1}{a_\tau}u\right| \leq \frac{1}{T}\max_{1\leq k\leq [cT]+1}\left|k - \frac{1}{a_\tau}T_k\right|.$$

Therefore,

$$\mathbf{P}\left(\rho\left(\widetilde{\nu}_T, \frac{1}{a_\tau}h\right) \geq \delta,\ \nu(T) \leq [cT]\right) \leq ([cT]+1)\max_{1\leq k\leq [cT]+1}\mathbf{P}\left(\left|k - \frac{1}{a_\tau}T_k\right| \geq T\delta\right).$$

For $1 \leq k \leq [cT] + 1$ with the help of the Chebyshev exponential inequality, we now estimate the probability

$$\mathbf{P}_k := \mathbf{P}\left(\left|k - \frac{1}{a_\tau}T_k\right| \geq T\delta\right) = \mathbf{P}(T_k \geq ka_\tau + T\delta a_\tau) + \mathbf{P}(T_k \leq ka_\tau - T\delta a_\tau).$$

We have

$$\mathbf{P}_k \leq e^{-k\Lambda_\tau(a_\tau + \frac{1}{k}T\delta a_\tau)} + e^{-k\Lambda_\tau(a_\tau - \frac{1}{k}T\delta a_\tau)}.$$

We note that, for $|\alpha| \geq \frac{\delta a_\tau}{c+1}$ by virtue of condition $[\mathbf{C}_0]$ for some $r = r_\delta > 0$, the relation $\Lambda_\tau(a_\tau + \alpha) \geq r|\alpha|$ holds. Therefore,

$$k\Lambda_\tau\left(a_\tau \pm \frac{1}{k}T\delta a_\tau\right) \geq kr\frac{1}{k}T\delta a_\tau = Tr\delta a_\tau.$$

We have got the estimate

$$\mathbf{P}_k \leq 2e^{-T\gamma_2}, \qquad \gamma_2 := r\delta a_\tau,$$

which is uniform in k in the limits $1 \leq k \leq [cT]+1$. From whence, we get

$$\mathbf{P}\left(\rho\left(\widetilde{\nu}_T, \frac{1}{a_\tau}h\right) \geq \delta, \quad \nu(T) \leq [cT]\right) \leq 2(cT+1)e^{-T\gamma_2}. \tag{3.11}$$

Estimates (3.10) and (3.11) yield assertion (iii).

The proof of (iii) in the inhomogeneous case is easily reduced to the proof in the homogeneous case. For this purpose, formula (3.9) should be replaced by

$$\mathbf{P}(\nu(T) \geq [cT]+1) = \mathbf{P}(T_{N+1} < T) \leq \mathbf{P}(\tau_2 + \ldots \tau_{n+1} + < T) \leq e^{-N\Lambda_\tau(\frac{T}{N})} = e^{-T\frac{N}{T}\Lambda_\tau(\frac{T}{N})},$$

and then we should repeat the above proof. In this case, we get inequality (3.10) in the inhomogeneous case as well.

We now prove (iv) at once in the general (inhomogeneous) case. We have

$$\rho_T := \rho(\widetilde{\overline{z}}_T, \overline{z}_T) = \sup_{0\leq t\leq 1} |\widetilde{\overline{z}}_T(t) - \overline{z}_T(t)| = \frac{1}{x} \sup_{0\leq u\leq T} \left|\widetilde{\overline{Z}}(u) - \overline{Z}(u)\right|.$$

Since the processes $\widetilde{\overline{Z}}(u)$ and $\overline{Z}(u)$ coincide for $u = T_k$, $k = 0, 1, \cdots$, we have, on the event $\{\nu(T) \leq [cT]\}$,

$$\rho_T \leq \frac{1}{x} \max_{1\leq k\leq [cT]+1} X_k,$$

where $X_k := \sqrt{(\zeta_k - a\tau_k)^2 + (a_\zeta - a\tau_k)^2}$. Therefore,

$$\mathbf{P}\left(\rho_T > \delta, \quad \nu(T) \leq [cT]\right) \leq ([cT]+1)\max\{\mathbf{P}(X_1 \geq x\delta), \mathbf{P}(X_2 \geq x\delta)\}.$$

The random variables X_1 and X_2 satisfy condition $[\mathbf{C}_0]$. Therefore, for some $M < \infty$ and $r_2 > 0$, the functions of deviations that correspond to X_1 and X_2 satisfy, for all $\alpha > 0$, the inequalities

$$\Lambda_{X_1}(\alpha) \geq -M + r_2\alpha, \quad \Lambda_{X_2}(\alpha) \geq -M + r_2\alpha.$$

Taking into account that, for all sufficiently large T, the relation $x \geq \sqrt{T}$ is satisfied, we get $(cT+1)e^{-x\gamma_3} = o(1)$ and

$$\mathbf{P}\left(\rho_T > \delta, \quad \nu(T) \leq [cT]\right) \leq e^{M - x\gamma_3}, \quad \gamma_3 := \frac{1}{2}r_2\delta. \tag{3.12}$$

Estimates (3.10) and (3.12) yield assertion (iv) of the lemma. □

4. Auxiliary results

We now prove several auxiliary lemmas.

Lemma 4.1. *(i) Let, for any function x satisfying condition* (3.5)*, the family of continuous processes $s_T(t) := \frac{1}{x}S(tT)$ satisfy $\left(I, \frac{x^2}{T}, \mathbb{C}^d[0,1]\right)$-l.d.p. Then, for any $c > 0$, the family satisfies also $\left(\tilde{I}, \frac{x^2}{T}, \mathbb{C}^d[0,c]\right)$-l.d.p., where $\tilde{I}(f) = I(g)$ for $g(t) = \frac{1}{\sqrt{c}}f(tc)$.*

(ii) Let, for any function x satisfying condition (3.5)*, the family of continuous random processes $s_T(t) := \frac{1}{x}S(tT)$ satisfy $\left(I, \frac{x^2}{T}, \mathbb{C}^d[0,c]\right)$-l.d.p. Then the family of processes $s_T(ct)$ satisfies $\left(\tilde{I}, \frac{x^2}{T}, \mathbb{C}^d[0,1]\right)$-l.d.p., where $\tilde{I}(f) = I(g)$ for $g(t) = f(t/c)$.*

Proof. We now prove assertion (i). Relation (3.5) implies that the family $s_r^c(t) := \frac{1}{\sqrt{cx(r/c)}} S(tr)$ satisfies $\left(I, \frac{x^2(r/c)}{r/c}, \mathbb{C}^d[0,1]\right)$-l.d.p. Then, changing the variable $T = \frac{r}{c}$, we get that the family of random processes $s_{Tc}^c(t) := \frac{1}{\sqrt{cx(T)}} S(tTc)$ satisfies $\left(I, \frac{x^2(T)}{T}, \mathbb{C}^d[0,1]\right)$-l.d.p.

Consider the continuous operator $\mathbf{F}$ that acts from $\mathbb{C}^d[0,1]$ into $\mathbb{C}^d[0,c]$ and maps a function $f(t)$ into $\sqrt{c}f(t/c)$. It is obvious that $\mathbf{F}s_{Tc}^c(\cdot) = s_T(\cdot)$. Therefore (see, e.g., Theorem 3.1 in [14]), we get that the family of processes $s_T(t)$ satisfies $\left(\tilde{I}, \frac{x^2(T)}{T}, \mathbb{C}^d[0,c]\right)$-l.d.p., where $\tilde{I}(f) = I(g)$, $g(t) = \frac{1}{\sqrt{c}} f(tc)$.

Assertion (ii) is proved analogously. □

Lemma 4.2. *The family of processes* $\beta_1 z_{1,T}(t) + \beta_2 z_{3,T}(t)$, *where* $|\beta_1| + |\beta_2| > 0$, *satisfies* $\mathbb{C}$–$\left(\hat{I}, \frac{x^2}{T}, \mathbb{D}[0,1]\right)$*-l.d.p., where*

$$\hat{I}(f) := \begin{cases} \frac{a_\tau}{2\mathbf{D}(\beta_1\theta_1+\beta_2\theta_2)} \int_0^1 (f'(t))^2 dt, & \text{if} \quad f \in \mathbb{AC}_0[0,1], \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. For the vector-function $g = (g_1, g_2)$, the operator $\mathbf{F}(g) := \beta_1 g_1 + \beta_2 g_2$ acting from $\mathbb{D}^2[0,1]$ into $\mathbb{D}[0,1]$ is continuous. Therefore, using the "contraction principle" (see, e.g., Theorem 3.1 in [14]), we get that the family of processes $\beta_1 z_{1,T}(t) + \beta_2 z_{3,T}(t)$ satisfies l.d.p. with the functional of deviations

$$\hat{I}(f) = \inf_{g:\ \beta_1 g_1 + \beta_2 g_2 = f} I(g).$$

We now show that the functional $\hat{I}(f)$ has the form claimed above.

If $f \notin \mathbb{AC}_0[0,1]$, then its preimage contains no functions from the set $\mathbb{AC}_0^2[0,1]$. Hence, $\inf_{g:\ \beta_1 g_1 + \beta_2 g_2 = f} I(g) = \infty$.

Let $f \in \mathbb{AC}_0[0,1]$ and $\beta_1 \neq 0$ (case $\beta_2 \neq 0$ is considered quite analogously). Then

$$\begin{aligned} \inf_{g:\ \beta_1 g_1 + \beta_2 g_2 = f} I(g) &= \inf_{g:\ \beta_1 g_1 + \beta_2 g_2 = f} \frac{a_\tau}{2\Delta_B} \int_0^1 (B_{22}(g_1'(t))^2 - 2B_{12} g_1'(t) g_2'(t) + B_{11}(g_2'(t))^2) dt \\ &= \inf_{g_2} \frac{a_\tau}{2\Delta_B} \int_0^1 \left(\frac{B_{22}}{\beta_1^2} (f'(t) - \beta_2 g_2'(t))^2 \right. \\ &- \left. \frac{2B_{12}}{\beta_1} (f'(t) - \beta_2 g_2'(t)) g_2'(t) + B_{11}(g_2'(t))^2 \right) dt \\ &= : \inf_{g_2} \frac{a_\tau}{2\Delta_B} \int_0^1 u(f'(t), g_2'(t)) dt. \end{aligned}$$

where Δ_B is the determinant of the covariance matrix $B = \|\mathbf{E}\theta_i\theta_j\|$, $B_{11} = \mathbf{D}\theta_1$, $B_{12} = \mathbf{E}\theta_1\theta_2$, $B_{22} = \mathbf{D}\theta_2$.

Separating the full square, we obtain

$$\begin{aligned} u(f'(t), g_2'(t)) = \frac{\mathbf{D}(\beta_1\theta_1 + \beta_2\theta_2)}{\beta_1^2} \left(g_2'(t) - f'(t) \frac{B_{12}\beta_1 + B_{22}\beta_2}{\mathbf{D}(\beta_1\theta_1 + \beta_2\theta_2)} \right)^2 \\ + (f'(t))^2 \frac{\Delta_B}{\mathbf{D}(\beta_1\theta_1 + \beta_2\theta_2)}. \end{aligned}$$

Hence, the infimum is attained on the function

$$g_2(t) = f(t) \frac{B_{12}\beta_1 + B_{22}\beta_2}{\mathbf{D}(\beta_1\theta_1 + \beta_2\theta_2)}.$$

Hence,

$$\inf_{g_2} \frac{a_\tau}{2\Delta_B} \int_0^1 u(f'(t), g_2'(t))dt = \frac{a_\tau}{2\mathbf{D}(\beta_1\theta_1 + \beta_2\theta_2)} \int_0^1 (f'(t))^2 dt.$$

□

In what follows, $[T]$ stands for the integer part of a number T.

Lemma 4.3. *Let, for any function x satisfying condition* (3.5)*, the sequence of continuous random processes $s_{[T]}(t) := \frac{1}{x([T])} S(t[T])$ satisfy $\left(I, \frac{x^2([T])}{[T]}, \mathbb{C}^d[0,1]\right)$-l.d.p. Then the family $s_T(t)$ satisfies $\left(I, \frac{x^2(T)}{T}, \mathbb{C}^d[0,1]\right)$-l.d.p.*

Proof. Lemma 4.1, (i), and the Pukhalskii theorem [19] imply that the sequence $s_{[T]}(t)$ is exponentially tight in m.s. $\mathbb{C}^d[0, 1+\Delta]$ with the normalizing function $\frac{x^2([T])}{[T]}$. Therefore, choosing

$$\eta_T(t) := \frac{tT}{[T]}, \ 0 \le t \le 1,$$

we fall into the conditions of Theorem 2.1. Hence, by virtue of the equality $\lim_{T\to\infty} \frac{T}{[T]} = 1$, Corollary 2.6, and Remark 2.5, we get: the family

$$\frac{1}{x([T])} S(tT), \ 0 \le t \le 1,$$

satisfies $\left(I, \frac{x^2([T])}{T}, \mathbb{C}^d[0,1]\right)$-l.d.p.

The subsequent proof will be performed by contradiction. Let the family s_T does not satisfy $\left(I, \frac{x^2(T)}{T}, \mathbb{C}^d[0,1]\right)$-l.d.p. Then there exist $\delta > 0$, $N < \infty$, a set $B \in \mathfrak{B}_{\mathbb{C}^d[0,1]}$, and a subsequence R_k, $\lim_{k\to\infty} R_k = \infty$ such that, for all k, at least one of three following conditions is satisfied:

$$\limsup_{k\to\infty} \frac{R_k}{x^2(R_k)} \ln \mathbf{P}(\, s_{R_k} \in B\,) \ge -I([B]) + \delta, \ I([B]) < \infty;$$

$$\limsup_{k\to\infty} \frac{R_k}{x^2(R_k)} \ln \mathbf{P}(\, s_{R_k} \in B\,) \ge -N, \ I([B]) = \infty;$$

$$\liminf_{k\to\infty} \frac{R_k}{x^2(R_k)} \ln \mathbf{P}(\, s_{R_k} \in B\,) \le -I((B)) - \delta.$$

For definiteness, let the first condition be satisfied (for the rest conditions, the proof is quite analogous).

Obviously, we may consider that $R_k - R_{k-1} > 1$.

Define $\tilde{x}(T)$ as follows:

$$\tilde{x}(T) = \begin{cases} x(R_k), & \text{if} \quad T \in [[R_k], [R_k]+1), \\ a_k T + b_k, & \text{if} \quad T \in [[R_k]+1, [R_{k+1}]), \end{cases}$$

where a_k and b_k are selected so that

$$a_k([R_k]+1) + b_k = x(R_k), \quad a_k[R_{k+1}] + b_k = x(R_{k+1}).$$

It is obvious that the function $\tilde{x}(T)$ satisfies condition (3.5).

Then, by virtue of the fact that the family of random processes $s_{T,\tilde{x}} := \frac{1}{\tilde{x}([T])} S(tT)$ satisfies $\Big(I, \frac{\tilde{x}^2([T])}{T}, \mathbb{C}^d[0,1]\Big)$-l.d.p., we have

$$-I([B]) \geq \limsup_{k\to\infty} \frac{R_k}{\tilde{x}^2([R_k])} \ln \mathbf{P}(\, s_{R_k,\tilde{x}} \in B\,) = \limsup_{k\to\infty} \frac{R_k}{x^2(R_k)} \ln \mathbf{P}(\, s_{R_k} \in B\,) \geq -I([B]) + \delta.$$

The obtained contradiction completes the proof. □

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