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PERTURBATION BOUNDS FOR MARKOV CHAINS WITH GENERAL STATE SPACE

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The aim of this paper is to investigate the stability of Markov chains with general state space. We present new conditions for the strong stability of Markov chains after a small perturbation of their transition kernels. Also, we obtain perturbation bounds with respect to different quantities.

Introduction

This paper considers the stability problem of Markov chains where the term "stability" is used to designate robustness, continuity, insensitivity to perturbations, etc. The problem of stability consists of two essential questions: first, the qualitative question, i.e., whether the system is stable or not, namely, whether a small perturbation of its parameters (inputs) generates only a small deviation in its characteristics (outputs). If we can answer positively to the first question, then we consider the second, quantitative, question, i.e., to provide a bound to the deviation of outputs due to the perturbation in the inputs. This bound is, in general, an upper bound of the exact deviation, and we refer to it as perturbation bound, approximation error, or simply, error.

For finite Markov chain, the problem of stability has been considered by many authors since Schweitzer's paper [22]. Meyer singly and with co-authors used the group inverse notion [4] to derive several bounds (see, e.g. [17]). The coefficient of ergodicity has been used by Seneta [23, 24]. Those authors used linear algebra and matrix analysis arguments, and their results are given in terms of the norm $\|\cdot\|_1$. Also, several results have been obtained on the absolute and relative deviation of the individual stationary probabilities (see for example [5] where several bounds are collected and compared). In [19] it is shown that a finite irreducible Markov chain is strongly stable (with respect to the norm $\|\cdot\|$) and that most perturbation bounds are also valid for an infinite irreducible Markov chain whenever it is strongly stable.

For a general state space, the problem of stability of Markov chains was first studied by Rossberg [21] and the first stability method was elaborated by Kalashnikov [8]. Kalashnikov's inspiration for this method came from Laypunov's direct method for differential equations and he called it "test functions method." Stoyan [25, 26] proposed a different approach based on the weak convergence theory and attempted a comparison with the test functions method. The method elaborated by Zolotarev [27] uses "probability metrics" arguments (a chapter is devoted to this method in [20]) while the method of Borovkov [3] uses renewal theory arguments. The strong stability method [1] makes use of the operator theory and provides results for Markov chains in a general state space with respect to a general class of norms. Kartashov has given many qualitative and quantitative results in [10, 11] based on the latter method.

In this paper, we study the stability of Markov chains with a general state space and with respect to a general class of norms. We obtain conditions for the strong stability and derive perturbation bounds.

The paper is organized as follows: in Section 1 we introduce notations. In Section 2 we recall basic definitions and results concerning the strong stability criteria. The main results of this paper are presented in Section 3. Conditions for the stability of Markov chains and quantitative estimates are obtained.

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1. Preliminaries and notations

Let $X = (X_t)_{t\geqslant 0}$ be a homogeneous Markov chain with values in a measurable space $(E, \mathcal{E})$, where the σ-algebra $\mathcal{E}$ is countably generated. Assume that the chain X is given by a regular transition kernel $P(x, A)$, $x \in E$, $A \in \mathcal{E}$ and admits a unique stationary measure π.

Denote by $m\mathcal{E}$ ($m\mathcal{E}^+$) the space of finite (nonnegative) measures over $\mathcal{E}$, and $f\mathcal{E}$ ($f\mathcal{E}^+$) the space of bounded (nonnegative) measurable functions over E.

We associate to every transition kernel $P(x, A)$ the linear mappings $\mathcal{L}_P : m\mathcal{E} \to m\mathcal{E}$ and $\mathcal{L}^*_P : f\mathcal{E} \to \to f\mathcal{E}$ whose values for $\mu \in m\mathcal{E}$ and $f \in f\mathcal{E}$ are respectively defined as

$$\mu P(A) = \mathcal{L}_P(\mu)(A) = \int_E \mu(dx)P(x, A), \quad \forall A \in \mathcal{E},$$

$$Pf(x) = \mathcal{L}^*_P(f)(x) = \int_E P(x, dy)f(y), \quad \forall x \in E,$$

and to every function $f \in f\mathcal{E}$ we associate the linear mapping $f : m\mathcal{E} \to \mathbb{R}$ such that

$$\mu f = \int_E \mu(dx)f(x),$$

provided that these integrals are well defined. The product of two transition kernels P and Q is the kernel defined as

$$PQ(x, A) = \int_E P(x, dy)Q(y, A) \text{ for } x \in E,\ A \in \mathcal{E}.$$

For $\mu \in m\mathcal{E}$ and $f \in f\mathcal{E}$ the symbol $f \circ \mu$ stands for the direct product:

$$(f \circ \mu)(x, A) = f(x)\mu(A),\ x \in E,\ A \in \mathcal{E}.$$

Consider in $m\mathcal{E}$ the Banach space $\mathcal{M} = \{\mu \in (m\mathcal{E}) : \|\mu\| < \infty\}$ with a norm $\|\cdot\|$ compatible with the structural order in $m\mathcal{E}$, i.e.,

$$\|\mu_1\| \leqslant \|\mu_1 + \mu_2\| \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}^+, \tag{1}$$

$$\|\mu_1\| \leqslant \|\mu_1 - \mu_2\| \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}^+ \text{ and } \mu_1 \perp \mu_2, \tag{2}$$

$$|\mu|(E) \leqslant k\|\mu\| \quad \text{for} \quad \mu \in \mathcal{M}, \tag{3}$$

where $|\mu|$ is the variation of the measure μ, k is a positive finite constant, and $\mathcal{M}^+ = \mathcal{M} \cap (m\mathcal{E}^+)$.

The induced norm on the space $f\mathcal{E}$ is defined as

$$\|f\| = \sup\left(|\mu f|, \|\mu\| \leqslant 1\right).$$

The induced norm on the space $\mathcal{B}$ of bounded linear operators is

$$\|P\| = \sup\left(\|\mu P\|, \|\mu\| \leqslant 1\right).$$

In what follows we assume that the operator $P : \mathcal{M} \to \mathcal{M}$ corresponding to the transition kernel of the chain X is bounded, i.e.,

$$\mathcal{M}P \subset \mathcal{M},\ \|P\| < \infty, \tag{4}$$

where $\mathcal{M}P = \{\mu_1 P : \mu_1 \in \mathcal{M}\}$. Denote by $\Pi = \mathbf{1} \circ \pi$ the stationary projector of the kernel P, where $\mathbf{1}$ is a function identically equal to 1, and by I the identity operator.

2. Strong stability

The strong stability method introduced in early 1980s [1] is applicable to all operations research models governed by Markov chains. It has been applied to queueing systems (see, e.g., [2]) and inventory systems [18]. For basic definitions and results on this method we refer to [12].

Definition 1. A Markov chain X with transition kernel P and stationary measure π is said to be strongly stable with respect to the norm $\|\cdot\|$, if every stochastic kernel Q in the neighborhood ($\|Q-P\| \leqslant \varepsilon$ for a certain $\varepsilon > 0$) admits a unique stationary measure ν and

$$\|\nu - \pi\| \longrightarrow 0 \ \text{ as } \ \|Q-P\| \longrightarrow 0.$$

An interesting result is the fact that strong stability is equivalent to uniform ergodicity.

Theorem 1 [1, 12]. *A Markov chain X is strongly stable for a norm if and only if it is uniformly ergodic in this norm.*

So it is now possible to use some results concerning uniform ergodicity in the context of the strong stability.

Theorem 2. *A Markov chain X having the transition kernel P and the stationary measure π is uniformly ergodic in the norm $\|\cdot\|$, if and only if the operator $(I-P+\Pi)$ has a bounded inverse*:

$$\|(I-P-\Pi)^{-1}\| < \infty. \tag{5}$$

Moreover, it follows from (5) *that $\|\Pi\| < \infty$.*

For the proof of this theorem see [12].

3. Criteria for the strong stability and perturbation bounds

3.1. Fundamental operator

For a Markov chain, the existence and the boundedness of the operator $R = (I-P+\Pi)^{-1}$ determines its uniform ergodicity and its strong stability. Moreover, many quantitative estimates are expressed by means of R. The role that R plays in this context is remarkable. In the case of discrete (finite and denumerable) Markov chains the quantity $(I-P+\Pi)$ is called the fundamental matrix (see, [14] for the finite case and [15] for the denumerable case). Its properties are well known and its role in perturbation theory is well understood.

In what follows we refer to $R = (I-P+\Pi)^{-1}$ as the fundamental operator of the chain X. This operator has the following properties:

- $R\Pi = \Pi R = \Pi$;
- $R(I-P) = (I-P)R = I - \Pi$.

Furthermore, when the operator $Z = \sum_{i=0}^{\infty}(P-\Pi)^i$ exists, it is equal to R.

The first stability inequalities were derived with respect to the fundamental operator R (see, e.g., [9, 12]). For instance, it has been shown that for a strongly stable Markov chain X, every stochastic kernel Q in the neighborhood of P satisfies

$$\nu = \nu\Delta R + \pi \tag{6}$$

and

$$\|\pi - \nu\| \leqslant \|\nu\|\|R\|\|\Delta\|,$$

where ν is the unique stationary measure corresponding to the kernel Q and $\Delta = Q - P$.

From a practical point of view, the last inequality may be useless except for few situations where $\|\nu\|$ can be estimated (e.g., for the norm $\|\cdot\|_1$). In general, ν is unknown and we need further development to obtain a useful bound such as the following (see, e.g., [9, 12]):

$$\nu(I - P + \Pi) = \nu(Q - P + \Pi) = \nu(Q - P) + \nu(\mathbf{1}o\pi) = \nu\Delta + \pi,$$

$$\nu = \nu\Delta R + \pi R = \nu\Delta R + \pi,$$

$$\nu - \pi = \nu\Delta R,$$

$$\|\pi - \nu\| \leqslant \|\nu\|\|R\|\|\Delta\|.$$

Theorem 3. *Suppose that the chain X is strongly stable with respect to the norm $\|\cdot\|$. Then every stochastic kernel Q in the neighborhood of P satisfying $\|\Delta\| < \|R\|^{-1}$ has a unique stationary measure ν :*

$$\nu = \pi(I - \Delta R)^{-1} = \pi\sum_{i=0}^{\infty}(\Delta R)^i$$

and

$$\|\nu - \pi\| \leqslant \frac{\|\pi\|\|\Delta\|\|R\|}{1 - \|\Delta\|\|R\|}.$$

3.2. Drazin inverse-group inverse

In 1958, Drazin [6] introduced a generalized inverse of an element in a semi-group or an associative ring. Let S be an algebraic semi-group (or an associative ring). An element $a \in S$ admits a Drazin inverse if there exists $x \in S$ such that for some k

$$a^{k+1}x = a^k, \ xax = x, \text{ and } ax = xa. \tag{7}$$

If the Drazin inverse of a exists, then the smallest nonnegative integer k satisfying (7) is called the index of a and is denoted $\operatorname{ind}(a)$. For all a in S there exists at most one x satisfying (7). If so, we denote this element by a^D.

In particular, when $\operatorname{ind}(a) = 1$, the element x satisfying (7) is called the group inverse of a and is denoted by $a^{\#}$.

Drazin inverses have many applications. For the theory of generalized inverses of operators and matrices and their applications to Markov chains, see [4, 7].

We call the group inverse of an operator A the unique operator $A^{\#}$ (whenever it exists) satisfying

$$AA^{\#}A = A, \ A^{\#}AA^{\#} = A^{\#}, \text{ and } AA^{\#} = A^{\#}A.$$

In the remaining part of this paper, A designates the operator $A = I - P$. It is relatively easy to show that there exists a tight relationship between R and $A^{\#}$. In particular, if the operator R exists and is bounded, then $A^{\#}$ exists and is bounded and vice versa. Moreover,

$$R = \Pi + A^{\#},$$

and consequently

$$\|R\| - \|\Pi\| \leqslant \|A^{\#}\| \leqslant \|R\| + \|\Pi\|.$$

The operator $A^{\#}$ has also the following properties:

- $\Pi A^{\#} = A^{\#}\Pi = 0,$
- $AA^{\#} = A^{\#}A = I - \Pi,$

and when the operator $W = \sum_{i=0}^{\infty}(P^i - \Pi)$ exists, it is a unique group inverse of A.

It is now clear that $A^{\#}$ can play the same important role that R plays, and this is the context of the strong stability theory. In particular, conditions for strong stability and perturbation bounds are derived in the same manner when replacing R by $A^{\#}$ and we obtain

Theorem 4. *A Markov chain X is strongly stable with respect to the norm $\|\cdot\|$, if and only if the group inverse $A^{\#}$ of the operator $A = (I - P)$ exists and is bounded, i.e.,*

$$\|(I - P)^{\#}\| < \infty.$$

The group inverse $A^{\#}$ of the operator $A = (I - P)$ may be used in the same manner as R. Indeed, in addition to the necessary and sufficient condition given by the last theorem, we can obtain quantitative estimates using $A^{\#}$.

Theorem 5. *Assume that the chain X is strongly stable with respect to the norm $\|\cdot\|$. Then for every stochastic kernel Q neighbor of P we have*

$$\nu - \pi = \nu\Delta A^{\#}$$

and

$$\|\nu - \pi\| \leqslant \|\nu\| \; \|A^{\#}\| \; \|\Delta\|.$$

Theorem 6. *If a chain X is strongly stable with respect to the norm $\|\cdot\|$, then every stochastic kernel Q in the neighborhood of P satisfying $\|\Delta\| < \|A^{\#}\|^{-1}$ has a unique stationary measure ν,*

$$\nu = \pi(I - \Delta A^{\#})^{-1} = \pi\sum_{i=0}^{\infty}(\Delta A^{\#})^i \tag{8}$$

and

$$\|\nu - \pi\| \leqslant \frac{\|\pi\| \; \|A^{\#}\| \; \|\Delta\|}{1 - \|\Delta\| \; \|A^{\#}\|}. \tag{9}$$

Inequality (9) appears in [11, Theorem 3]. In his paper, Kartashov uses the operator $R_0 = (I - P + \Pi)^{-1} - \Pi$ (i.e., $A^{\#}$) and refers to it as the generalized potential of the chain X.

3.3. Ergodicity coefficient

For a bounded operator B satisfying $B\mathbf{1} = a\mathbf{1}$ for some constant a, we define its ergodicity coefficient:

$$\Lambda_t(B) = \sup_{\|\bar{\mu}\|\leqslant 1,\ \bar{\mu}(E)=0} \|\bar{\mu}B^t\|.$$

Observe that this quantity is related to the norm $\|\cdot\|$. The operator B can have different ergodicity coefficients for different norms. Furthermore, if we consider the subspace

$$\mathcal{M}_0 = \{\mu \in \mathcal{M}, \mu(E) = 0\}$$

then B^t is an operator over $\mathcal{M}_0$ since $\mu B^t\mathbf{1} = 0$ for $\mu \in \mathcal{M}_0$. Therefore, $\Lambda_t(B)$ is an ordinary operator norm of B in the space $\mathcal{M}_0$.

Lemma 1. *For the bounded linear operator B satisfying $B\mathbf{1} = a\mathbf{1}$ for some constant a we have*

$$\Lambda_t(B) \leqslant \|B^t\|,$$

$$\Lambda_t(B) \leqslant (\Lambda_1(B))^t,$$

$$\Lambda_1(B^t) = \Lambda_t(B).$$

Proof. Observe that

$$\{\|\bar{\mu}\| \leqslant 1,\ \bar{\mu}(E) = 0\} \subset \{\|\bar{\mu}\| \leqslant 1\}.$$

Then

$$\Lambda_t(B) = \sup\{\|\bar{\mu}B^t\| : \ \|\bar{\mu}\| \leqslant 1,\ \bar{\mu}(E) = 0\} \leqslant \sup\{\|\bar{\mu}B^t\| : \ \|\bar{\mu}\| \leqslant 1\} = \|B^t\|.$$

Now denote by $\bar{\mu}_i = \|\bar{\mu}B^i\|^{-1}\bar{\mu}B^i$ the measures satisfying $\|\bar{\mu}_i\| \leqslant 1$ and $\bar{\mu}_i(E) = 0$. Then

$$\|\bar{\mu}B^t\| = \|\bar{\mu}B^{t-1}B\| = \|\bar{\mu}B^{t-1}\|\|\bar{\mu}_{t-1}B\| = \ldots = \prod_{i=0}^{t-1}\|\bar{\mu}_i B\|.$$

Thus,

$$\sup_{\|\bar{\mu}\|\leqslant 1,\ \bar{\mu}(E)=0}\|\bar{\mu}B^t\| \leqslant \prod_{i=0}^{t-1}\sup_{\|\bar{\mu}\|\leqslant 1,\ \bar{\mu}(E)=0}\|\bar{\mu}B\|,$$

$$\Lambda_t(B) \leqslant (\Lambda_1(B))^t.$$

Finally,

$$\Lambda_1(B^t) = \sup_{\|\bar{\mu}\|\leqslant 1,\ \bar{\mu}(E)=0}\|\bar{\mu}(B^t)\| = \sup_{\|\bar{\mu}\|\leqslant 1,\ \bar{\mu}(E)=0}\|\bar{\mu}B^t\| = \Lambda_t(B).$$

Lemma 2. *Consider the operator $D = \|\Delta\|^{-1}\Delta$. Then for a bounded operator B we have*

$$\|DB^t\| \leqslant \Lambda_t(B).$$

Proof. We see that

$$\|DB^t\| = \sup\{\|\mu DB^t\| : \ \|\mu\| \leqslant 1\}.$$

The measure $\bar{\mu} = (\mu D)$ satisfies

$$\|\bar{\mu}\| = \|(\mu D)\| \leqslant \|\mu\|\|D\| \leqslant 1 \text{ and } \bar{\mu}(E) = (\mu D)(E) = 0.$$

Thus, we obtain

$$\|DB^t\| \leqslant \sup\{\|\bar{\mu}B^t\| : \ \|\bar{\mu}\| \leqslant 1,\ \bar{\mu}(E) = 0\} = \Lambda_t(B).$$

Lemma 3. *For a Markov chain X the folowing relation holds*:

$$\Lambda_t(R) = \Lambda_t(A^{\#}).$$

Proof. For a measure $\bar{\mu}$ satisfying $\|\bar{\mu}\| \leqslant 1,\ \bar{\mu}(E) = 0$,

$$\|\bar{\mu}R^t\| = \|\bar{\mu}(\Pi + A^{\#})^t\| = \|\bar{\mu}(A^{\#})^t\|.$$

since $\bar{\mu}\Pi = \bar{\mu}(\mathbf{1} \circ \pi) = (\bar{\mu}\mathbf{1})\pi = 0$.

Lemma 4. *The folowing relation holds*:

$$\|\Delta P^t\| \leqslant \|\Delta\|(\Lambda_1(P))^t.$$

Proof. Consider the family of operators

$$D_t = \|\Delta P^t\|^{-1}\Delta P^t.$$

We aim at showing by recurrence that

$$\|\Delta P^n\| = \|\Delta\|\prod_{i=0}^{n-1}\|D_i P\|. \tag{10}$$

For $n = 1$ we have

$$\|\Delta P\| = \|\Delta\| \|(\|\Delta\|^{-1}\Delta)P\| = \|\Delta\| \|D_0 P\|.$$

Assume that

$$\|\Delta P^{n-1}\| = \|\Delta\| \prod_{i=0}^{n-2} \|D_i P\|.$$

So, we show that

$$\|\Delta P^n\| = \|\Delta P^{n-1}\| \|(\|\Delta P^{n-1}\|^{-1}\Delta P^{n-1})P\| = \left(\|\Delta\| \prod_{i=0}^{n-2} \|D_i P\|\right) \|D_{n-1}P\|,$$

which gives us

$$\|\Delta P^n\| = \|\Delta\| \prod_{i=0}^{n-1} \|D_i P\|.$$

Now we show that

$$\|D_t P\| \leqslant \Lambda_1(P) \ \forall t.$$

We have

$$\|D_t P\| = \sup\{\|\mu D_t P\| : \ \|\mu\| \leqslant 1\}.$$

But the measure $\bar{\mu} = (\mu D_t)$ satisfies

$$\|\bar{\mu}\| = \|(\mu D_t)\| \leqslant \|\mu\| \|D_t\| \leqslant 1 \text{ and } \bar{\mu}(E) = (\mu D_t)(E) = 0.$$

Thus, we get

$$\|D_t P\| \leqslant \sup\{\|\bar{\mu}P\| : \ \|\bar{\mu}\| \leqslant 1, \ \bar{\mu}(E) = 0, \ \bar{\mu} \in \mathcal{M}P\} = \Lambda_t(P).$$

Substituting this in (10), we obtain the desired result.

Now we obtain the following bound with respect to the ergodicity coefficient.

Theorem 7. *Let a chain X be strongly stable with respect to the norm $\|\cdot\|$. Then for every stochastic kernel Q in the neighborhood of P we have*

$$\|\nu - \pi\| \leqslant \|\nu\| \|\Delta\| \Lambda_1(R) = \|\nu\| \|\Delta\| \Lambda_1(A^{\#}). \tag{11}$$

Proof. According to Theorem 5 we have

$$\nu - \pi = \nu \Delta A^{\#} \Rightarrow \|\nu - \pi\| = \|\nu\| \|\Delta A^{\#}\| \Rightarrow \|\nu - \pi\| = \|\nu\| \|\Delta\| \|D A^{\#}\|.$$

Then Lemma 2 gives the result.

From this theorem we can get the following condition of strong stability.

Theorem 8. *A Markov chain X with transition kernel P and stationary measure π is strongly stable with respect to the norm $\|\cdot\|$, if and only if R exists ($A^{\#}$ exists) and the coefficient of ergodicity $\Lambda_1(R)$ ($\Lambda_1(A^{\#})$) of the fundamental operator R (the group inverse $A^{\#}$ of $A = I - P$) in this norm is finite, i.e.,*

$$\Lambda_1(R) = \Lambda_1(A^{\#}) < \infty.$$

Proof. First we prove that the condition is necessary. The strong stability of the chain X implies that R (and $A^{\#}$) exists and is bounded. Then by Lemmas 1 and 3 we have

$$\Lambda_1(A^{\#}) = \Lambda_1(R) \leqslant \|R\| < \infty.$$

To prove the sufficiency, consider inequality (11). Setting $C = \|\nu\|\Lambda_1(R) = \|\nu\|\Lambda_1(A^{\#}) < \infty$, we obtain

$$\|\nu - \pi\| \leqslant C\|\Delta\|.$$

Then $\|\nu - \pi\| \to 0$ as $\|\Delta\| \to 0$, which implies the strong stability of the chain X.

Theorem 9. *Let a chain X be strongly stable with respect to the norm $\|\cdot\|$. Then every stochastic kernel Q in the neighborhood of P satisfying $\|\Delta\| < (\Lambda_1(A^{\#}))^{-1}$ has a unique stationary measure ν and*

$$\|\nu - \pi\| \leqslant \frac{\|\pi\|\|\Delta\|\Lambda_1(A^{\#})}{1 - \|\Delta\|\Lambda_1(A^{\#})}.$$

Proof. Suppose that $\|\Delta\| < (\Lambda_1(A^{\#}))^{-1}$. Then

$$\|\Delta A^{\#}\| \leqslant \|\Delta\|\|A^{\#}\| \leqslant \|\Delta\|\Lambda_1(A^{\#}) < 1.$$

Therefore the operator $(I - \Delta A^{\#})$ is invertible (then, it is bijective) and

$$\nu = \pi(I - \Delta A^{\#})^{-1} = \pi \sum_{i=0}^{\infty} (\Delta A^{\#})^i,$$

which implies the uniqueness of ν. Furthermore,

$$\nu - \pi = \pi \sum_{i=1}^{\infty} (\Delta A^{\#})^i,$$

$$\|\nu - \pi\| \leqslant \|\pi\| \sum_{i=1}^{\infty} \|\Delta A^{\#}\|^i,$$

$$\|\nu - \pi\| \leqslant \|\pi\| \sum_{i=1}^{\infty} (\|\Delta\|\Lambda_1(A^{\#}))^i,$$

if $\|\Delta\| \leqslant (\Lambda_1(A^{\#}))^{-1}$ then

$$\|\nu - \pi\| \leqslant \frac{\|\pi\|\|\Delta\|\Lambda_1(A^{\#})}{1 - \|\Delta\|\Lambda_1(A^{\#})},$$

completing the proof.

Theorem 10. *Let X be a Markov chain with transition kernel P and stationary distribution π. Suppose that there exists $n \geqslant 0$ such that $\Lambda_n(P) < 1$. Then the following statements are true*:

(*a*) *The fundamental operator R exists and is bounded. Moreover,*

$$R = \sum_{i=0}^{\infty} (P - \Pi)^i = \Pi + \sum_{i=0}^{\infty} (P^i - \Pi).$$

(*b*) *The group inverse $A^{\#}$ of $A = I - P$ exists and is bounded. Moreover,*

$$A^{\#} = \sum_{i=0}^{\infty} (P^i - \Pi) \text{ and } \|A^{\#}\| \leqslant n\tau p_n (1 - \Lambda_n(P))^{-1},$$

where $\tau = \max_{0 \leqslant i < n} \|P^i\|$ *and* $p_n = \max_{0 \leqslant i < n} \|P^i - \Pi\|$.

(*c*) *The chain X is strongly stable in the norm $\|\cdot\|$.*

Proof. Let us prove (a) and (b) simultaneously. Since $P^t\Pi = \Pi$, $\forall t \geqslant 0$, it is easy to prove that

$$\sum_{i=0}^{\infty}(P-\Pi)^t = \Pi + \sum_{i=0}^{\infty}(P^i - \Pi)$$

and

$$\|\sum_{i=0}^{\infty}(P-\Pi)^t\| \leqslant \|\Pi\| + \|\sum_{i=0}^{\infty}(P^i - \Pi)\|.$$

It suffices to prove that the series $W = \sum_{i=0}^{\infty}(P^i - \Pi)$ converges in $\mathcal{B}$. Assume that there exists $n \geqslant 0$ such that $\Lambda_n(P) < 1$. Then

$$\left\|\sum_{t=0}^{\infty}(P^t - \Pi)\right\| \leqslant \sum_{t=0}^{\infty}\|P^t - \Pi\| \leqslant \sum_{s=0}^{\infty}\sum_{r=0}^{n-1}\|P^{sn+r} - \Pi\| \leqslant$$

$$\leqslant \sum_{s=0}^{\infty}\sum_{r=0}^{n-1}\|P^r\|\|P^{sn} - \Pi\| \leqslant \sum_{s=0}^{\infty}\left(\|P^{sn} - \Pi\|\sum_{r=0}^{n-1}\|P^r\|\right) \leqslant$$

$$\leqslant n\tau\sum_{s=0}^{\infty}\|P^{sn} - \Pi\| = n\tau\sum_{s=0}^{\infty}\|(I-\Pi)P^{sn}\| =$$

$$= n\tau\sum_{s=0}^{\infty}\sup_{\|\mu\|\leqslant 1}\|\mu(I-\Pi)P^{sn}\|,$$

where the measure $\bar{\mu} = \mu(I-\Pi)\|I-\Pi\|^{-1}$ satisfies the condition

$$\|\bar{\mu}\| = \|\mu(I-\Pi)\|I-\Pi\|^{-1}\| \leqslant \|\mu\|\|(I-\Pi)\|\|I-\Pi\|^{-1} \leqslant 1$$

and

$$\bar{\mu}(E) = (\mu(I-\Pi)\|I-\Pi\|^{-1})(E) = (\mu(I-\Pi))(E)\|I-\Pi\|^{-1} = 0.$$

Then

$$\sup_{\|\mu\|\leqslant 1}\|\mu(I-\Pi)P^{sn}\| \leqslant \sup_{\|\bar{\mu}\|\leqslant 1, \bar{\mu}(E)=0}\|I-\Pi\|\|\bar{\mu}P^{sn}\| \leqslant$$

$$\leqslant \|I-\Pi\|\sup_{\|\bar{\mu}\|\leqslant 1, \bar{\mu}(E)=0}\|\bar{\mu}(P^n)^s\| \leqslant \|I-\Pi\|\Lambda_s(P^n) \leqslant \|I-\Pi\|(\Lambda_n(P))^s$$

by Lemma 1. Finally,

$$\|\sum_{t=0}^{\infty}(P^t - \Pi)\| \leqslant n\tau\sum_{s=0}^{\infty}\|I-\Pi\|(\Lambda_n(P))^s \leqslant n\tau p_n\frac{1}{1-\Lambda_n(P)}.$$

So, the operator $W = \sum_{i=0}^{\infty}(P^t - \Pi)$ exists (the series is absolutely convergent) and is bounded. This also implies the existence of the operator $\sum_{i=0}^{\infty}(P-\Pi)^t$.

We obtain the strong stability of the chain X from (b) (or (c) and Theorem 4).

To avoid difficulties in computing certain constants, we may use some simplifications like

$$p_n = \max_{0\leqslant i<n}\|P^i - \Pi\| \leqslant \max_{0\leqslant i<n}\|P^i(I-\Pi)\| \leqslant \max_{0\leqslant i<n}\|P^i\|\|I-\Pi\| = \tau\|I-\Pi\|.$$

Also,

$$\|I-\Pi\| \leqslant \|I\| + \|\Pi\| \leqslant 1 + \|\mathbf{1}\|\|\pi\|.$$

Theorem 11. *Under the condition of Theorem* 10, *for every stochastic kernel Q in the neighborhood of P we have*

$$\|\nu - \pi\| \leqslant \frac{\|\nu\| \, \|\Delta\| n\tau p_n}{1 - \Lambda_n(P)}, \tag{12}$$

$$\|\nu - \pi\| \leqslant \|\nu\|\|\Delta\| n\tau(1 - \Lambda_n(P))^{-1}. \tag{13}$$

If, in addition, $\|\Delta\| \leqslant np_n\tau(1 - \Lambda_n(P))^{-1}$, *then the stationary measure ν of Q is unique and*

$$\|\nu - \pi\| \leqslant \frac{\|\pi\| \, \|\Delta\| np_n\tau}{1 - \Lambda_n(P) - np_n\tau\|\Delta\|}. \tag{14}$$

Proof. Inequality (12) (resp. inequality (14)) follows from Theorem 5 (resp. from Theorem 6) by replacing $\|A^{\#}\|$ by its upper bound given in Theorem 10. Let us prove inequality (13). For the chain X we have

$$A^{\#} = \sum_{i=0}^{\infty}(P^i - \Pi) \text{ and } \nu - \pi = \nu\Delta A^{\#}.$$

Thus,

$$\nu - \pi = \nu\Delta\sum_{i=0}^{\infty}(P^i - \Pi) \Rightarrow \nu - \pi = \nu\sum_{i=0}^{\infty}\Delta P^i,$$

since $\Delta\Pi = 0$. Furthermore,

$$\nu - \pi = \nu\sum_{s=0}^{\infty}\sum_{r=0}^{n-1}\Delta P^{sn+r},$$

$$\nu - \pi = \nu\sum_{s=0}^{\infty}\sum_{r=0}^{n-1}\Delta P^{sn}P^r = \nu\sum_{s=0}^{\infty}\left(\Delta P^{sn}\sum_{r=0}^{n-1}P^r\right),$$

$$\nu - \pi = \nu\sum_{s=0}^{\infty}\Delta P^{sn}\sum_{r=0}^{n-1}P^r,$$

$$\|\nu - \pi\| = \|\nu\sum_{s=0}^{\infty}\Delta P^{sn}\|$$

So,

$$\|\nu - \pi\| \leqslant \|\nu\|\sum_{s=0}^{\infty}\left(\|\Delta P^{sn}\|\sum_{r=0}^{n-1}\|P^r\|\right),$$

$$\|\nu - \pi\| \leqslant \|\nu\|\sum_{s=0}^{\infty}\left(\|\Delta\|\|D_sP^n\|\sum_{r=0}^{n-1}\|P^r\|\right),$$

$$\|\nu - \pi\| \leqslant \|\nu\|\|\Delta\|\left(\sum_{r=0}^{n-1}\|P^r\|\right)\sum_{s=0}^{\infty}\|D_sP^n\|.$$

Using Lemma 2 we obtain

$$\|\nu - \pi\| \leqslant \|\nu\|\|\Delta\| n\tau\sum_{s=0}^{\infty}(\Lambda_n(P))^s,$$

where $\Lambda_n(P) < 1$. Then

$$\|\nu - \pi\| \leqslant \|\nu\|\|\Delta\| n\tau(1 - \Lambda_n(P))^{-1}.$$

Remark 1. From inequality (13) we observe that the condition $\Lambda_n(P) < 1$ for some n is sufficient for the strong stability of the chain X. Kartashov [10] proved that a Harris recurrent aperiodic Markov chain is uniformly ergodic (and strongly stable) if and only if $\Lambda_t(P) < 1$ for some $t \geqslant 1$.

Lemma 5. *If the Markov chain X satisfies the conditions of Theorem* 10, *then*

$$\Lambda_1(A^{\#}) = \Lambda_1(R) \leqslant n\tau(1 - \Lambda_n(P))^{-1}.$$

Proof. Assume that the chain X satisfies the required condition. Then we have

$$\Lambda_1(R) = \Lambda_1(A^{\#}) = \sup\{\|\bar{\mu}A^{\#}\| : \ \|\bar{\mu}\| \leqslant 1, \ \bar{\mu}(E) = 0, \ \bar{\mu} \in \mathcal{M}B\},$$

where $A^{\#} = \sum\limits_{t=0}^{\infty}(P^t - \Pi)$ and $\bar{\mu}\Pi = 0$. Then

$$\|\bar{\mu}A^{\#}\| = \|\bar{\mu}\sum_{t=0}^{\infty} P^t\| \leqslant \sum_{t=0}^{\infty} \|\bar{\mu}P^t\| = \sum_{s=0}^{\infty}\sum_{r=0}^{n-1} \|\bar{\mu}P^{sn+r}\| \leqslant \sum_{s=0}^{\infty} \|\bar{\mu}(P^n)^s\| \sum_{r=0}^{n-1} \|P^r\| \leqslant$$

$$\leqslant n\tau \sum_{s=0}^{\infty} \Lambda_s(P^n) \leqslant n\tau \sum_{s=0}^{\infty} (\Lambda_n(P))^s \leqslant n\tau(1 - \Lambda_n(P))^{-1}.$$

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