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GALERKIN METHOD FOR SEMILINEAR PARABOLIC EQUATION WITH VARYING TIME DIRECTION

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We establish the unique solvability of the first boundary value problem for a semilinear second order parabolic equation with varying time direction by using a modified Galerkin method and the regularization method. We obtain a priori error estimates for approximate solutions. Bibliography: 2 titles.

Let Ω be a bounded domain in R^n with smooth boundary S. We denote $\Omega_t = \Omega \times \{t\}$ for $0 \leqslant t \leqslant T$ and $S_T = S \times (0, T)$. In the cylindrical domain $Q = \Omega \times (0, T)$, we consider the semilinear parabolic equation

$$Lu \equiv k(x,t)u_t - \Delta u + c(x,t)u + |u|^\rho u = f(x,t), \tag{1}$$

where the coefficients are assumed to be sufficiently smooth in $\overline{Q}$. We note that the sign of $k(x,t)$ can be arbitrarily changed inside the domain. We introduce the sets

$$S_0^{\pm} = \{(x,0) : k(x,0) \gtrless 0, x \in \Omega\},$$
$$S_T^{\pm} = \{(x,T) : k(x,T) \gtrless 0, x \in \Omega\}$$

and put $p = \rho + 2$, $-1 < \rho < 2/(n-1)$.

Boundary value problem. Find a solution to Equation (1) in Q such that

$$u\,|_{S_T} = 0, \quad u\,|_{\overline{S_0^+}} = 0, \quad u\,|_{\overline{S_T^-}} = 0. \tag{2}$$

Let $L_p(Q)$, $1 < p < \infty$, be the Banach space of measurable p-integrable functions in Q equipped with the norm

$$\|u\|_{L_p(Q)} = \left(\int_Q |u(x,t)|^p dxdt \right)^{1/p}.$$

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The space $L_2(Q)$ is a Hilbert space relative to the inner product

$$(u,v)=\int\limits_Q uv dQ=\int\limits_0^T (u,v)_0 dt, \quad ||u||^2=(u,u).$$

Denote by $W_2^{m,s}(Q)$ the anisotropic Sobolev space equipped with the norm

$$\|u\|_{m,s}^2=\int\limits_Q\Big[\sum_{|\alpha|\leqslant m}(D^\alpha u)^2+(D_t^s u)^2\Big]dQ$$

and by C_L the class of functions in $W_2^{2,1}(Q)$ satisfying the boundary conditions (2).

Lemma 1 (cf. [1])**.** *Assume that* $c-\frac{1}{2}k_t\geqslant 0$, $(x,t)\in\overline{Q}$. *Then for any* $u\in C_L$

$$\int\limits_Q |u|^p dQ+\|u\|_{1,0}^2\leqslant C_1(Lu,u), \quad C_1>0. \tag{3}$$

We consider vector-valued functions $y(t)=(y_1(t),\dots,y_m(t))^*$, $f(t)=(f_1(t),\dots,f_m(t))^*$, $g(y)=(g_1(y),\dots,g_m(y))^*$ in R^m and continuous symmetric matrices $K(t)=(k_{ij}(t))_{m\times m}$, $B(t)=(b_{ij}(t))_{m\times m}$, $0\leqslant t\leqslant T$, such that the following assumptions hold:

1) $g_k(y)$ is continuous with respect to y and $f_k\in L_2(0,T)$ for $k=\overline{1,m}$,

2) $\sum\limits_{k=1}^m g_k(y)y_k\geqslant 0$ for continuous vector-valued functions y, $t\in[0,T]$,

3) $B(t)-\frac{1}{2}K_t\geqslant \delta E$, $\delta>0$, $t\in[0,T]$.

We look for a solution to the nonlinear system of second order ordinary differential equations

$$Ly=-\varepsilon y''+K(t)y'+B(t)y+g(y)=f(t), \quad \varepsilon>0, \quad 0<t<T, \tag{4}$$

$$y(0)=0, \quad y(T)=0 \tag{5}$$

with the boundary conditions

$$y(0)=0, \quad y'(T)=0, \qquad K(T)\geqslant 0, \tag{5'}$$

$$y'(0)=0, \quad y(T)=0, \qquad K(0)\leqslant 0, \tag{5''}$$

$$y'(0)=0, \quad y'(T)=0, \qquad K(0)\leqslant 0,\ K(T)\geqslant 0. \tag{5'''}$$

We denote by $\widetilde{W}_2^1(0,T)$ the closure of the set of smooth functions $u(t)$ satisfying (5) in the $W_2^1(0,T)$-norm.

Definition 1. A vector-valued function $y(t)$, ($y_k\in\widetilde{W}_2^1(0,T)$) is a *weak solution* to the boundary value problem (4), (5) if

$$a_i(y,\eta)=\int\limits_0^T\Bigg[\varepsilon y'\eta'+\sum_{j=1}^m k_{ij}y_j'\eta+\sum_{j=1}^m b_{ij}y_j\eta+g_i(y)\eta\Bigg]dt=\int\limits_0^T f_i\eta dt \tag{6}$$

for all $\eta\in\widetilde{W}_2^1(0,T)$, $i=\overline{1,m}$.

Let functions $\psi_k(t)$, $k = 1, \ldots$, be orthonormal in $L_2(0,T)$ and satisfy the equation

$$-u'' = \gamma u, \quad 0 < t < T,$$

and the boundary conditions (5).

We look for approximate solutions $y^N = (y_1^N, \ldots, y_m^N)^*$ to the problem (4), (5) in the form

$$y_k^N(t) = \sum_{k=1}^{N} c_{kl}^N \psi_l(t), \quad k = \overline{1,m},$$

where the constants c_{kl}^N are determined by the system of nonlinear algebraic equations

$$a_i(y^N, \psi_l) = \int_0^T f_i(t)\psi_l dt, \quad i = \overline{1,m},\ l = \overline{1,N}. \tag{7}$$

Lemma 2. *Let Assumptions* 1)–3) *hold. Then the system of algebraic equations* (7) *has at least one solution.*

Proof. We set

$$A_{il}(c) \equiv a_i(y^N, \psi_l), \quad h_{il} = \int_0^T f_i \psi_l dt.$$

Then the system (7) takes the form

$$A_{il}(c) \equiv h_{il}, \quad i = \overline{1,m},\ l = \overline{1,N}. \tag{8}$$

By the properties of ψ_k and Assumptions 1)–3),

$$\sum_{i=1}^{m}\sum_{l=1}^{N} A_{il}(c)c_{il}^N \geqslant \delta|c|^2 + \frac{1}{2}\sum_{i,j=1}^{m} k_{ij}(t) y_i^N y_j^N \Big|_{t=0}^{t=T} + \int_0^T \sum_{k=1}^{m} g_k(y^N) y_k^N dt \geqslant \delta|c|^2,$$

where

$$|c|^2 = \sum_{i=1}^{m}\sum_{l=1}^{N} (c_{kl}^N)^2.$$

Then the solvability of the system (8) follows according to [2]. □

Theorem 1. *Let Assumptions* 1)–3) *hold. Then the boundary value problem* (4), (5) *has a unique weak solution* $y(t), y_k(t) \in \widetilde{W}_2^1(0,T)$, $k = \overline{1,m}$.

Proof. By Lemma 2, the boundary value problem (4), (5) has at least one approximate solution y^N. By (7), we obtain the a priori estimate

$$\int_0^T \sum_{i=1}^{m} \Big[\varepsilon\Big(\frac{dy_i^N}{dt}\Big)^2 + (y_i^N)^2\Big] dt \leqslant C_2 \sum_{i=1}^{m} \|f_i\|_{L_2(0,T)}^2, \quad C_2 > 0, \tag{9}$$

which implies the existence of a function $y_i \in \widetilde{W}_2^1(0,T)$, $i = \overline{1,m}$, such that $y_i^{N_k} \to y_i$ weakly in $W_2^1(0,T)$. Since $W_2^1(0,T)$ is compactly embedded itno $C[0,T]$, we have $y_i^{N_k} \to y_i$ strongly in

$C[0,T]$ for $i=\overline{1,m}$. Then the sequence $g_i(y^{N_k})$ strongly converges to $g_i(y)$ in $C[0,T]$ for $y=(y_1,\dots,y_m)$. Passing to the limit in (7) as $N=N_k\to\infty$, we obtain (6) for any $\eta(t)\in \widetilde{W}_2^1(0,T)$, i.e., $y(t)$ is a weak solution to the boundary value problem (4), (5).

It is easy to show that any weak solution $y(t)$ to the boundary value problem (4), (5) satisfies the estimate (9) which implies the uniqueness of the weak solution. □

Remark 1. According to the theory of ordinary differential equations, from Assumption 1) it follows that $\dfrac{d^2y}{dt^2}\in L_2(0,T)$, $k=\overline{1,m}$, where $y=(y_1,\dots,y_m)$ is a weak solution to the boundary value problem (4), (5).

For $\varepsilon>0$ we set $L_\varepsilon u\equiv -\varepsilon u_{tt}+Lu$. For basis functions we take the solutions $\varphi_k(x)$ to the problem

$$-\Delta\varphi=\lambda\varphi,\quad \varphi|_S=0.$$

Moreover, $\varphi_k(x)$ form an orthonormal basis for the space $L_2(\Omega)$ and the corresponding eigenvalues λ_k are such that $0<\lambda_1\leqslant\lambda_2\leqslant\cdots$ and $\lambda_k\to+\infty$ as $k\to\infty$.

We look for approximate solutions to the boundary value problem (1), (2) in the form

$$u^{N,\varepsilon}(x,t)=\sum_{k=1}^{N}c_k^{N,\varepsilon}(t)\varphi_k(x).$$

We consider the boundary value problem for the system of nonlinear second order ordinary differential equations

$$(L_\varepsilon u^{N,\varepsilon},\varphi_l)_0=(f,\varphi_l)_0 \tag{10}$$

with the boundary conditions

$$c_l^{N,\varepsilon}|_{t=0}=0,\quad c_l^{N,\varepsilon}|_{t=T}=0,\quad l=\overline{1,N}. \tag{11}$$

or

$$c_l^{N,\varepsilon}(0)=0,\quad D_tc_l^{N,\varepsilon}|_{t=T}=0,\quad l=\overline{1,N},\qquad k(x,T)\geqslant 0, \tag{11$'$}$$

$$D_tc_l^{N,\varepsilon}|_{t=0}=0,\quad c_l^{N,\varepsilon}(0)=0,\quad l=\overline{1,N},\qquad k(x,0)\leqslant 0, \tag{11$''$}$$

$$D_tc_l^{N,\varepsilon}|_{t=0}=0,\quad D_tc_l^{N,\varepsilon}|_{t=T}=0,\quad l=\overline{1,N},\qquad k(x,0)\leqslant 0,\ k(x,T)\geqslant 0. \tag{11$'''$}$$

Lemma 3. *Assume that $c-\frac12k_t\geqslant\delta>0$, $f\in L_2(Q)$, $-1<\rho\leqslant 2/(n-1)$. Then there exists a unique solution $c_k^{N,\varepsilon}$ in $W_2^2(0,T)$ to the boundary value problem* (10), (11).

Proof. We show that the boundary value problem (10), (11) satisfies the assumptions of Lemma 2. We set $m=N$,

$$y=\begin{pmatrix}c_1^{N,\varepsilon}(t)\\ \cdot\\ \cdot\\ c_N^{N,\varepsilon}(t)\end{pmatrix},$$

$k_{ij}=(k(x,t)\varphi_i,\varphi_j)_0$, $b_{ij}=\lambda_i\delta_{ij}+(c\varphi_i,\varphi_j)_0$, and $g_i(y)=(|u^{N,\varepsilon}|^\rho u^{N,\varepsilon},\varphi_l)_0$. Then we have

$$\sum_{i,j=1}^{N}\Big(b_{ij}-\frac12k_{ijt}\Big)\xi_i\xi_j=\sum_{i=1}^{N}\lambda_i\xi_i^2+\int_\Omega\Big(c-\frac12k_t\Big)\Big(\sum_{i=1}^{N}\xi_i\varphi_i\Big)^2dx\geqslant\delta|\xi|^2\quad\forall\xi\in R^N.$$

On the other hand,

$$\sum_{k=1}^{N} g_k(y) y_k = \int\limits_{\Omega} \left|u^{N,\varepsilon}\right|^{\rho+2} dx \geqslant 0.$$

The existence of a solution to the boundary value problem (10), (11) follows from Theorem 1 and Remark 1. The solution is unique since the function $|t|^\rho t$ is monotone. □

Theorem 2. *Assume that* $c - \frac{1}{2}k_t \geqslant \delta > 0$, $c + \frac{1}{2}k_t \geqslant \delta > 0$, *and* $f \in W_2^{0,1}(Q)$. *Let one of the following cases holds*:

1) $k(x,0) > 0,\ k(x,T) < 0,\ f(x,0) = 0,\ f(x,T) = 0,$

2) $k(x,0) > 0,\ k(x,T) \geqslant 0,\ f(x,0) = 0,$

3) $k(x,0) \leqslant 0,\ k(x,T) < 0,\ f(x,T) = 0,$

4) $k(x,0) \leqslant 0,\ k(x,T) \geqslant 0.$

Then the boundary value problem (1), (2) *has a unique solution in* $W_2^{2,1}(Q)$ *and the following estimate holds*:

$$\int\limits_{Q} \Big[(u_t)^2 + \sum_{i=1}^{n} (u_{tx_i})^2 + (\Delta u)^2\Big] dQ + (\rho+1) \int\limits_{Q} \Big(|u|^\rho (u_t)^2 + |u|^\rho \sum_{i=1}^{n} (u_{x_i})^2\Big) dQ$$

$$\leqslant C_3(\|f\|^2 + \|f_t\|^2), \quad C_3 > 0.$$

Proof. In the case of semilinear parabolic equations with varying time direction, this theorem was proved in [1] by the stationary Galerkin method. Here, we use the modified (nonstationary) Galerkin method and the method of elliptic regularization.

We begin with the case $k(x,0) > 0$, $k(x,T) < 0$, $f(x,0) = 0$, $f(x,T) = 0$. We look for approximate solutions $u^{N,\varepsilon}(x,t)$ to the boundary value problem (1), (2) in the form

$$u^{N,\varepsilon}(x,t) \equiv v(x,t) = \sum_{k=1}^{N} c_k^{N,\varepsilon}(t) \varphi_k(x),$$

where $c_k^{N,\varepsilon} \in W_2^2(0,T)$ are solutions to the boundary value problem (10), (11). We multiply (10) by $c_l^{N,\varepsilon}$ and take the sum with respect to l from 1 to N. Integrating with respect to t, we find

$$(f, v) = \varepsilon \|v_t\|^2 + (Lv, v).$$

By the estimate (3), we obtain the a priori estimate

$$\varepsilon \|v_t\|^2 + \int\limits_{Q} |v|^p dQ + \|v\|_{1,0}^2 \leqslant C_4 \|f\|^2, \quad C_4 > 0. \tag{12}$$

Integrating by parts, from (10) and (11) we find

$$-(f, v_{tt}) = \varepsilon \|v_{tt}\|^2 + \int\limits_{Q} \Big[\Big(c + \frac{1}{2}k_t\Big) v_t^2 + \sum_{i=1}^{n} v_{tx_i}^2 + c_t v_t v + (\rho+1)|v|^\rho v_t^2\Big] dQ$$

$$+ \frac{1}{2} \int\limits_{\Omega_0} k v_t^2 dx - \frac{1}{2} \int\limits_{\Omega_T} k v_t^2 dx - \int\limits_{\Omega} \Big[\sum_{i=1}^{n} v_{x_i} v_{tx_i} + c v v_t + |v|^\rho v v_t\Big] dx \Big|_{t=0}^{t=T}.$$

Taking into account the boundary conditions (11) and using the estimate (12) and the Cauchy inequality $|ab| \leqslant \gamma a^2 + \frac{1}{4\gamma}b^2$, $\gamma > 0$, we arrive at the estimate

$$\varepsilon\|v_{tt}\|^2 + \|v_t\|_{1,0}^2 + \int\limits_Q |v|^\rho v_t^2 dQ \leqslant C_5(\|f\|^2 + \|f_t\|^2), \quad C_5 > 0. \tag{13}$$

Integrating by parts, from (10) and (11) we get

$$-(f, \Delta v) = \int\limits_Q \Big[(\Delta v)^2 - k v_t \Delta v - c v \Delta v + (\rho+1)\|v\|^\rho \sum_{i=1}^n v_{x_i}^2\Big] dQ$$

$$+ \varepsilon \int\limits_Q \sum_{i=1}^n v_{tx_i}^2 dQ + \varepsilon \int\limits_\Omega v_t \Delta v dx\Big|_{t=0}^{t=T}. \tag{14}$$

Using the estimate (13) and the Cauchy inequality, from (14) we obtain the a priori estimate

$$\varepsilon \int\limits_Q \sum_{i=1}^n v_{tx_i}^2 dQ + \int\limits_Q (\Delta v)^2 dQ + (\rho+1) \int\limits_Q |v|^\rho \sum_{i=1}^n v_{x_i}^2 dQ \leqslant C_6(\|f\|^2 + \|f_t\|^2), \quad C_6 > 0. \tag{15}$$

By (12), (13), and (15), the approximate solutions $u^{N,\varepsilon}(x,t)$ satisfy the a priori estimate

$$\int\limits_Q \Big[(u_t^{N,\varepsilon})^2 + \sum_{i=1}^n (u_{tx_i}^{N,\varepsilon})^2 + (\Delta u^{N,\varepsilon})^2\Big] dQ + (\rho+1) \int\limits_Q \Big(|u^{N,\varepsilon}|^\rho (u_t^{N,\varepsilon})^2 + |u^{N,\varepsilon}|^\rho \sum_{i=1}^n (u_{x_i}^{N,\varepsilon})^2\Big) dQ$$

$$\leqslant C_7(\|f\|^2 + \|f_t\|^2), \quad C_7 > 0, \tag{16}$$

which allows us to complete the proof by standard arguments since $|u^{N,\varepsilon}|^\rho u^{N,\varepsilon}$ is bounded in $L_2(Q)$ in view of the embedding theorem. The remaining cases are considered in a similar way.

The uniqueness of a solution to the boundary value problem (1), (2) in the class $W_2^{2,1}(Q)$ follows from the inequality $(|u|^\rho u - |v|^\rho v)(u - v) \geqslant 0$. □

Theorem 3. *Let all the assumptions Theorem* 2 *be satisfied. Then*

$$\big\|u - u^{N,\varepsilon}\big\|_{1,0} \leqslant C_8\big(\|f\| + \|f_t\|\big)\big(\varepsilon^{1/2} + \lambda_{N+1}^{-1/2}\big), \quad C_8 > 0, \tag{17}$$

where $u(x,t)$ is an exact solution to the boundary value problem (1), (2).

Proof. We again consider the case $k(x,0) > 0$, $k(x,T) < 0$, $f(x,0) = 0$, $f(x,T) = 0$. Let $u(x,t)$ be an exact solution to the boundary value problem (1), (2) which exists in view of Theorem 2. We represent Lu in the form $Lu = L_0 u + |u|^\rho u$. We note that the function $u(x,t)$ can be represented as the Fourier series

$$u(x,t) = \sum_{k=1}^\infty c_k(t)\varphi_k(x), \quad c_k(t) = (f, \varphi_k)_0.$$

Moreover,

$$-\Delta u = \sum_{k=1}^\infty c_k(t)\lambda_k\varphi_k(x).$$

Hence

$$\sum_{k=1}^{\infty} \lambda_k^2 \int\limits_0^T c_k^2(t)dt = \|\Delta u\|^2 \leqslant C_9(\|f\|^2 + \|f_t\|^2), \quad C_9 > 0. \tag{18}$$

On the other hand, for $u(x,t)$ we have

$$(Lu, \varphi_l) = (f, \varphi_l), \quad l = 1, 2, \ldots \tag{19}$$

In the space $L_2(Q)$, we introduce the linear manifold

$$H_N = \left\{ \eta(x,t) = \sum_{l=1}^{N} d_l(t)\varphi_l(x): \; d_l \in W_2^1(0,T), \; d_l(0) = d_l(T) = 0 \right\}.$$

By (10) and (19),

$$(L_\varepsilon u^{N,\varepsilon}, \eta) = (f, \eta), \quad (Lu, \eta) = (f, \eta) \quad \forall \eta \in H_N.$$

Hence

$$(L_0(u - u^{N,\varepsilon}) + |u|^\rho u - |u^{N,\varepsilon}|^\rho u^{N,\varepsilon}, \eta) = -\varepsilon(u_{tt}^{N,\varepsilon}, \eta) \quad \forall \eta \in H_N.$$

Setting $\eta = w - u^{N,\varepsilon} = (u - u^{N,\varepsilon}) + (w - u)$, $w \in H_N$, from the last identity we have

$$\begin{aligned} &(L_0(u - u^{N,\varepsilon}), u - u^{N,\varepsilon}) + (|u|^\rho u - |u^{N,\varepsilon}|^\rho u^{N,\varepsilon}, u - u^{N,\varepsilon}) \\ &= \varepsilon(u_t^{N,\varepsilon}, w_t - u_t^{N,\varepsilon}) + (f - Lu^{N,\varepsilon}, u - w) - \varepsilon\left(\int\limits_\Omega u_t^{N,\varepsilon}(w - u)dx \right)\Bigg|_{t=0}^{t=T}. \end{aligned} \tag{20}$$

We note that $\{L_0 u^{N,\varepsilon}\}$ is bounded in $L_2(Q)$ and $|u^{N,\varepsilon}|^\rho u^{N,\varepsilon}$ is bounded in $L_2(Q)$ by the embedding theorem for $-1 < \rho \leqslant 2/(n-1)$. Since $|\tau|^\rho \tau$ is monotone, from (20) we get

$$\|u - u^{N,\varepsilon}\|_{1,0}^2 \leqslant C_{10}[\varepsilon \|u_t^{N,\varepsilon}\| \, \|w_t - u_t^{N,\varepsilon}\| + \|f - Lu^{N,\varepsilon}\| \, \|u - w\|], \quad C_{10} > 0. \tag{21}$$

For $w = \sum\limits_{k=1}^{N} c_k(t)\varphi_k(x) \in H_N$ we have

$$\|u - w\|^2 = \sum_{k=N+1}^{\infty} \int\limits_0^T c_k^2(t)dt \leqslant \lambda_{N+1}^{-2} \sum_{k=N+1}^{\infty} \lambda_k^2 \int\limits_0^T c_k^2(t)dt$$

which implies

$$\|u - w\| \leqslant C_{11}\lambda_{N+1}^{-1}(\|f\| + \|f_t\|), \quad C_{11} > 0. \tag{22}$$

Taking into account (3), (16), (18), and (22), from (21) we derive the error estimate for the modified Galerkin method (17). We can consider the remaining cases in a similar way with the linear manifold

$$H_N = \begin{cases} \left\{ \eta(x,t) = \sum\limits_{l=1}^{N} d_l(t)\varphi_l(x), d_l \in W_2^1(0,T) : d_l(0) = 0 \right\}, & k(x,0) > 0, \; k(x,T) \geqslant 0, \\ \left\{ \eta(x,t) = \sum\limits_{l=1}^{N} d_l(t)\varphi_l(x), \; d_l \in W_2^1(0,T) : d_l(T) = 0 \right\}, & k(x,0) \leqslant 0, \; k(x,T) < 0, \\ \left\{ \eta(x,t) = \sum\limits_{l=1}^{N} d_l(t)\varphi_l(x), \; d_l \in W_2^1(0,T) \right\}, & k(x,0) \leqslant 0, \; k(x,T) \geqslant 0. \end{cases}$$

Theorem 3 is proved. □

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