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DISCRETE HEDGING IN THE MEAN/VARIANCE MODEL FOR EUROPEAN CALL OPTIONS

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We consider a portfolio with the call option and the relevant asset under the standard assumption that the market price is a random variable with a lognormal distribution. Minimizing the variance (hedging risk) of the portfolio on the maturity date of the option, we find the relative value of the asset per option unit. As a direct consequence, we obtain a statistically fair price of the call option explicitly. Unlike the well-known Black–Scholes theory, the portfolio cannot be risk-free, because no additional transactions within the contract are allowed, but the sequence of portfolios reduces the risk to zero asymptotically. This property is illustrated in the experimental section on the example of the daily stock prices of 18 leading Australian companies over a three year period.

1. Introduction

The typical asset S, as a process of geometrical Brownian motion [15–17], has the price defined by the equation

$$\frac{dS}{S} = \mu dt + \sigma dz,$$

where $\mu \in R$ and $\sigma \in R_+$ are the shift and volatility coefficients, and z is the standard Wiener process with $\mathrm{M}dz = 0$, $\mathrm{M}\,(dz)^2 = dt$. According to the Ito lemma,

$$d\log S(t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz. \tag{1}$$

Therefore,

$$\log S(t+T) \sim \mathrm{N}\left(\log S(t) + \left(\mu - \frac{\sigma^2}{2}\right) T,\ \sigma\sqrt{T}\right),$$

where $N(a,b)$ is a normal random variable with mean a and standard deviation b. Denote the corresponding density by

$$f_S(x) = \frac{1}{\sqrt{2\pi}\cdot x \cdot b(T)} \exp\left(-\frac{(\log x - a(T))^2}{2\cdot b^2(T)}\right),$$

where $a(T) = \log S(t) + \left(\mu - \frac{\sigma^2}{2}\right) T$, $b(T) = \sigma\sqrt{T}$.

Definition 1. The contract of the European call option allows its holder to purchase a unit of the underlying asset at a fixed price K (*strike price*) after the date $t+T$ in the future, or the holder of the call option may decide not to exercise the contract if the price of the underlying asset is less than the exercise price K. Accordingly, the price of the European call option with the maturity date $t+T$ is

$$C(t+T) = \psi(S(t+T) - K) = \max\{0, S(t+T) - K\}.$$

The fundamental problem of financial mathematics [8] is to find a reasonable hedge or an option price at the time t, prior to the expiration time.

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2. The expectation of the hedge (expected hedge — EH) and the Black–Scholes formula (BS)

In accordance with [14] and [19] we assume that

$$C_{\exp}(t) = e^{-r\cdot T} \cdot \mathrm{M}\psi(S(t+T) - K), \tag{2}$$

where r is a risk-free rate.

Assumption 1. Assume that the parameters t, T, r, and K are arbitrarily fixed. Then

$$e^{-r\cdot T} \cdot \mathrm{M}\psi\left(S(t+T) - K\right) = S(t) \cdot e^{(\mu - r)T} \cdot \Phi(\alpha) - K \cdot e^{-rT} \cdot \Phi(\beta), \tag{3}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal law, and

$$\alpha = \frac{\log\left(\frac{S(t)}{K}\right) + \left(\mu + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \ \beta = \alpha - \sigma\sqrt{T}.$$

The proof of formula (3) is given in [16], where it is also noted that (3) coincides with the Black–Scholes formula for the specific case $\mu = r$:

$$C_{BS}(t) = S(t) \cdot \Phi(\alpha_r) - K \cdot e^{-rT} \cdot \Phi(\beta_r), \tag{4}$$

where

$$\alpha_r = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \ \beta_r = \alpha_r - \sigma\sqrt{T}.$$

Remark 1. Similar results for the variance gamma processes are given in [11]. Let us also mention the papers [5, 6], where the formulas for call options were obtained with the methods based on the Fourier transform. However, assumption (2) neglects the fact that the seller may continue to actively trade in the stock market during the contract. The Black–Scholes formula (4) gives a unique price of the European call option [10] for the ideal and continuous trading process. With respect to these conditions the contract is self-financing and risk-free for both the seller and the buyer. On this financial market the problem of mean variance hedging in continuous time [7, 18] is to find the best approximation of the price through self-financing trading strategies, where the optimality criterion is the expected mean square error [4]. In a series of recent papers, this problem has been formulated and related to the task of linear-quadratic stochastic control [1, 2, 9].

According to [19], the original Black–Scholes formula is criticized on the grounds that it is based on completely unrealistic terms of risk-free operation and that the exact adherence to the contract may prevent maximization of the portfolio.

3. Hedging according to the mean/variance model

Consider the portfolio F, consisting of the call option C and h units of the underlying asset S. As a consequence, the price of the portfolio (the seller case) can be represented by the following formula:

$$F(t) = -C(t) + h \cdot S(t) \tag{5}$$

or

$$F(t) = \begin{cases} (h-1) \cdot S(t) + K, \text{ if } S(t) \geqslant K, \\ h \cdot S(t), \text{ if } S(t) < K. \end{cases} \tag{6}$$

According to the fundamental principles of the mean/variance model (MV) [12], we follow the rule that the investor considers the expected income as a desirable factor, and the variance as a non-desirable factor. This rule may be implemented in a different ways. For example, we can define the

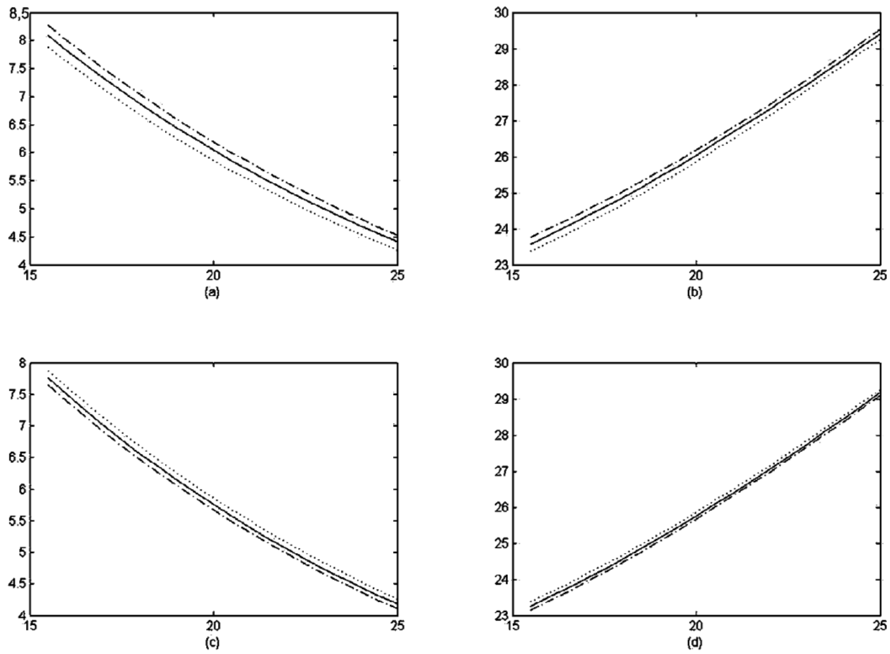

Fig. 1. The behavior of the call option cost C: (a), (c) — depending on the strike price K; (b) and (d) — the behavior of the sum of the call option and the strike price depending on K. We used the following parameters: (a) and (b) — $S(t) = 20,\ \mu = 0.1,\ r = 0.05,\ \sigma = 1,\ T = 180/365$; (c) and (d) — $S(t) = 20,\ \mu = 0.02,\ r = 0.05,\ \sigma = = 1,\ T = 180/365$; solid, chain, and dotted lines correspond to MV (10), EH (3), and BS (4) solutions respectively.

portfolio structure (or the investing strategy) by maximizing the ratio of mathematical expectation and the standard deviation of the portfolio, or we can minimize the variance of portfolio (5), assuming that its mathematical expectation is fixed. In our case we will minimize the variance of portfolio (5), assuming that the number of call options is fixed. In order to simplify the notations, we consider the portfolio with a single call option (Fig.1).

The following theorem is the main result of the paper and it determines the value of the parameter h, minimizing the variance of portfolio (6). The price of the call option hedging (or hedge) can be found by the formula

$$C_{\mathrm{MV}}(t) = h \cdot S(t) - e^{-rT}\mathrm{M}F(t+T). \tag{7}$$

Theorem 1. *Suppose that the cost of the portfolio F is defined by* (6). *Then the hedging problem*

$$\min_h Q_{\mathrm{var}}(F(t+T)),$$

where

$$Q_{\mathrm{var}}(F(t+T)) := \mathrm{M}\left[F(t+T) - \mathrm{M}F(t+T)\right]^2,$$

has a unique solution

$$h = \frac{A_4 - K \cdot A_2 + (A_2 + A_3)(K \cdot A_1 - A_2)}{A_4 + A_5 - (A_2 + A_3)^2}, \tag{8}$$

where the coefficients $A_i = A_i(K),\ i = \overline{1,5},$ are defined by the formulas

$$\begin{aligned}
A_1(K) &:= \int_K^\infty f_S(x)dx = \Phi\left(\frac{a(T)-\log K}{b(T)}\right),\\
A_2(K) &:= \int_K^\infty x f_S(x)dx = \exp\left(a(T)+\frac{b^2(T)}{2}\right)\Phi\left(b(T)+\frac{a(T)-\log K}{b(T)}\right),\\
A_3(K) &:= \int_0^K x f_S(x)dx = \exp\left(a(T)+\frac{b^2(T)}{2}\right) - A_2(K),\\
A_4(K) &:= \int_K^\infty x^2 f_S(x)dx = \exp(2(a(T)+b^2(T)))\Phi\left(2b(T)+\frac{a(T)-\log K}{b(T)}\right),\\
A_5(K) &:= \int_0^K x^2 f_S(x)dx = \exp(2(a(T)+b^2(T))) - A_4(K).
\end{aligned}$$

Proof. By the definition of the variance,

$$Q_{\text{var}}(F(t+T)) = \mathrm{M}F^2(t+T) - (\mathrm{M}F(t+T))^2, \tag{9}$$

where

$$\begin{aligned}
\mathrm{M}F^2(t+T) &= (h-1)^2A_4 + 2K(h-1)A_2 + K^2A_1 + h^2A_5,\\
\mathrm{M}F(t+T) &= h(A_2+A_3) - A_2 + K\cdot A_1.
\end{aligned}$$

Minimizing (9) as a function of h, we find the required solution (8).

Let us now find the hedge for the mean/variance model according to (7):

$$C_{\text{MV}}(t) = h\cdot S(t) - e^{-rT}\left[h(A_2+A_3) - A_2 + K\cdot A_1\right], \tag{10}$$

where the parameter h is defined in (8), and replace (2)–(3), using new notations:

$$C_{\exp}(t) = e^{-rT}(A_2 - K\cdot A_1). \tag{11}$$

The above call option is a portfolio with the risk-free asset

$$F(t) = -C(t) + h, \tag{12}$$

where h is a parameter. The corresponding standard deviation is invariant with respect to h and can be calculated by the following formula:

$$S_{\text{dev}}(F(t+T)) = \sqrt{A_1(1-A_1)K^2 + 2A_2K(A_1-1) + A_4 - A_2^2}.$$

Some calculation results with the use of this formula are presented in Fig. 2.

Remark 2. Note that the price (10) can be negative, unlike the price (11), which is always positive by definition. Using the relations

$$A_2 + A_3 = \exp\left(a+\frac{b^2}{2}\right), A_4 + A_5 = \exp(2(a+b^2)),$$

we can simplify (8):

$$h(K) = \frac{A_4 - K\cdot A_2 + \exp\left(a+\frac{b^2}{2}\right)(K\cdot A_1 - A_2)}{\exp(2a+b^2)(2\exp(b-1))}. \tag{13}$$

Let us consider some special properties of the coefficients $A_i(K),\ i = \overline{1,5}$:

$$A_1(K) \underset{K\to 0}{\longrightarrow} 1,\ A_3(K) \underset{K\to 0}{\longrightarrow} 0,\ A_5(K) \underset{K\to 0}{\longrightarrow} 0.$$

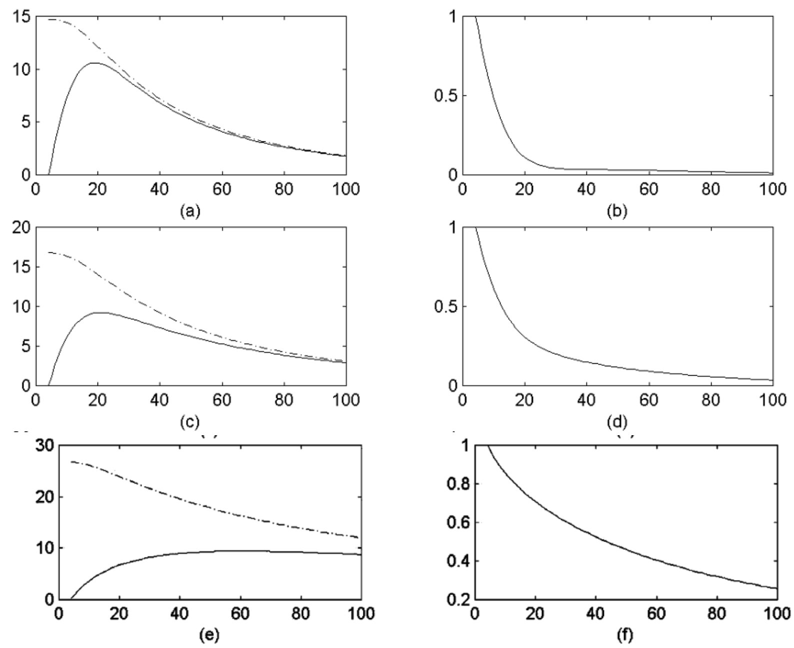

Fig. 2. (a), (c), (e) — standard deviation of the portfolio as a function of K, where the solid line corresponds to (10), and the chain line corresponds to the portfolio with the risk-free asset (2) (EH-method); (b), (d), (f) — the value of the parameter h as a function of K. We used the following parameters: (a) and (b) — $\mu = 0.1$, $\sigma = 0.9$; (c) and (d) — $\mu = 0.1$, $\sigma = 1.0$; (e) and (f) — $\mu = 0.1$, $\sigma = 1.4$. The other parameters are the same as in Fig. 1.

From what was said above, it follows that $h(K) \underset{K\to 0}{\longrightarrow} 1$ and $S_{\text{dev}}(K) \underset{K\to 0}{\longrightarrow} 0$ (see also Fig. 2).

Proposition 2. *Assume that $\sigma > 0$. Then $0 < h < 1$, where the asset split parameter h is defined in* (8).

The proof of Proposition 2 follows from the two lemmas.

Lemma 1. *For any $v \in R$ and $b \in R_+$ the following inequality holds*:

$$\frac{\Phi(v+b) - \Phi(v)}{\Phi(v) - \Phi(v-b)} < \exp\left(\frac{b^2}{2} - b \cdot v\right). \tag{14}$$

Proof. We have

$$\Phi(v) - \Phi(v-b) = \frac{1}{\sqrt{2\pi}} \int\limits_{v}^{v+b} \exp\left(-\frac{(t-b)^2}{2}\right) dt =$$

$$= \frac{\exp(-0.5b^2)}{\sqrt{2\pi}} \int\limits_{v}^{v+b} \exp(-0.5t^2 + bt)dt < \frac{\exp(-\frac{b^2}{2} + bv)}{\sqrt{2\pi}} \int\limits_{v}^{v+b} \exp\left(-\frac{t^2}{2}\right) dt =$$

$$= \frac{\exp(-\frac{b^2}{2} + bv)}{\sqrt{2\pi}} \left[\Phi(v+b) - \Phi(v)\right].$$

The proof is complete.

Lemma 2. *For any $v \in R$ and $b \in R_+$ the following inequality holds*:

$$\Phi(v) < \frac{e^{b^2}\Phi(v+b) + \exp(\frac{b^2}{2} - bv)\Phi(v-b)}{1 + \exp(\frac{b^2}{2} - bv)}. \tag{15}$$

Proof. We have

$$e^{b^2}\Phi(v+b)-\Phi(v) > e^{b^2}\left[\Phi(v+b)-\Phi(v)\right] = \frac{e^{b^2}}{\sqrt{2\pi}}\int\limits_{v}^{v+b}\exp\left(-\frac{t^2}{2}\right)dt =$$

$$= \frac{e^{b^2}}{\sqrt{2\pi}}\int\limits_{v-b}^{v}\exp\left(-\frac{(t+b)^2}{2}\right)dt =$$

$$= \frac{\exp(\frac{b^2}{2})}{\sqrt{2\pi}}\int\limits_{v-b}^{v}\exp\left(-\frac{t^2}{2}-bt\right)dt > \exp\left(\frac{b^2}{2}-bv\right)\left[\Phi(v)-\Phi(v-b)\right].$$

The proof is complete.

Using the definition of the coefficients $A_i(K)$, $i=1,5$, let us rewrite (13) in the following form:

$$h = \frac{e^{b^2}\Phi(v+b)-\Phi(v)-\exp\left({}^{b^2}\!/_2 - bv\right)\left[\Phi(v)-\Phi(v-b)\right]}{\exp(b^2)-1},$$

where

$$v = b + \frac{a-\log K}{b}.$$

Then the exact upper and lower bounds for h (as it is established in Proposition 2) are found from (14) and (15), if $\sigma > 0$.

Remark 3. Figure 1 (a), (c) illustrates the property of decrease of the call option price as a function of the strike price. It is interesting that the sum of the call option and the strike price is an increasing function of the strike price. This fact is quite understandable, since the second part of the deal (buying the assets) is not mandatory. According to Fig. 1, formulas (3) and (10) are more flexible compared to the Black–Scholes formula, which does not depend on the shift coefficient μ.

4. Experiments

The experimental part of the study was based on the daily stock prices of 18 leading Australian companies over a three-year period.

The estimate for the current volatility [13] was calculated on the basis of representation (1) by the formula

$$\hat{\sigma}_{i,t} = \sqrt{\frac{\sum_{j=1}^{n}\left(R_{i,t-j}-\bar{R}_{i,t}\right)^2}{n-1}}, \tag{16}$$

where

$$R_{i,t-j} = \log\frac{S_{i,t-j+1}}{S_{i,t-j}};$$

$S_{i,t}$ is a final price of the ith asset in a day $t > n$, and

$$\bar{R}_{i,t} = \frac{1}{n}\sum_{j=1}^{n}R_{i,t-j}.$$

The current shift (shift parameter) was estimated as follows:

$$\hat{\mu}_{i,t} = \bar{R}_{i,t} + \frac{1}{2}\hat{\sigma}^2_{i,t}. \tag{17}$$

The left column in Fig.3 illustrates the stock prices of Fairfax, Harvey Norman, Rio–Tinto, Tabcorp, and Westpac during the 1000 day period (where 2006 January 10 is the last day); the middle and the right columns illustrate the behavior of the mean and the standard deviation, which are calculated according to (17) and (16) with the use of the smoothing parameter $n = 120$.

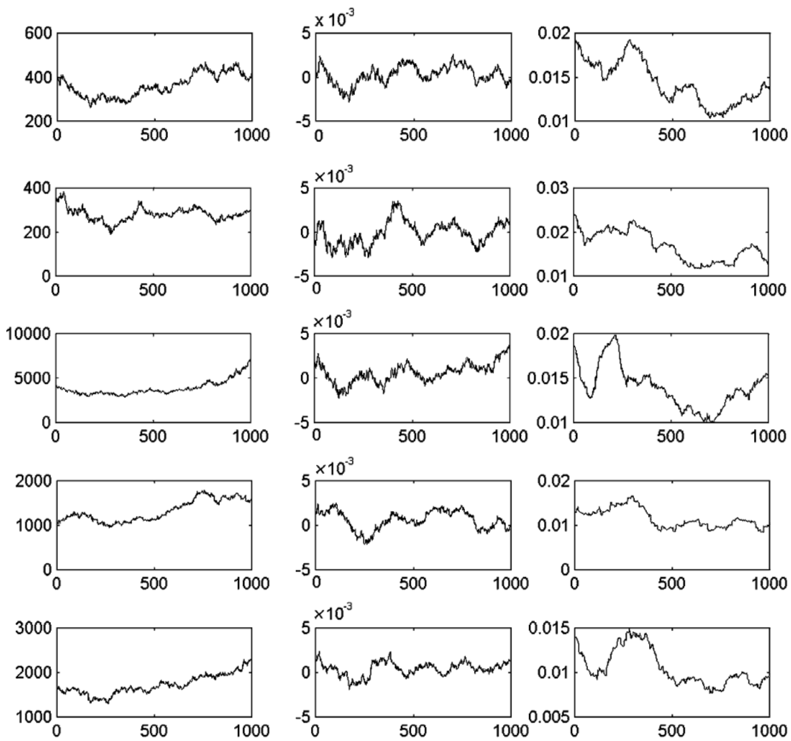

Fig. 3. Daily stock prices.

4.1. The mathematical expectation of the hedge

The call option price $C_{i,t}$ (the right column in Fig. 4) was calculated according to (11) under the following condition: $C_{i,t} \geqslant 0.03 \cdot S_{i,t}$ (administrative costs are no less than 3%). The strike price was calculated with the use of the estimates of the shift and volatility coefficients

$$K_{i,t} = S_{i,t} \cdot \exp\left(\frac{\mu_{i,t} \cdot T}{\beta + \gamma \cdot \sigma_{i,t}}\right), \ \beta = 1.1, \ \gamma = 20.$$

The left and the middle columns in Fig. 4 correspond to the income of the buyer $PB_{i,t}$ and the seller $PS_{i,t}$, which were calculated for 100 consecutive days (where 2006 January 10 is the last day). The calculations were made with the use of the following rules:

$$PB_{i,t+T+j+1} = PB_{i,t+T+j} + \\ + \begin{cases} S_{i,t+T+j+1} - K_{i,t+j+1} - C_{i,t+j+1}, \text{ if } S_{i,t+T+j+1} \geqslant K_{i,t+j+1}; \\ -C_{i,t+j+1}, \text{ if } S_{i,t+T+j+1} < K_{i,t+j+1}, \end{cases} \tag{18}$$

and

$$PS_{i,t+T+j+1} = PS_{i,t+T+j} + \\ + \begin{cases} K_{i,t+j+1} + C_{i,t+j+1} - S_{i,t+T+j+1}, \text{ if } S_{i,t+T+j+1} \geqslant K_{i,t+j+1}; \\ C_{i,t+j+1}, \text{ if } S_{i,t+T+j+1} K_{i,t+j+1}, \end{cases} \tag{19}$$

where the initial values of $PB_{i,t+T}$ and $PS_{i,t+T}$ are equal to zero.

In order to estimate the efficiency of this approach, we used m assets with the special weight coefficients: $w_i \propto (u_i)^{-1}$, $\sum_{i=1}^{m} w_i = 1$, where $\{u_i, i = 1, m,\}$ are the average stock prices for the considered period.

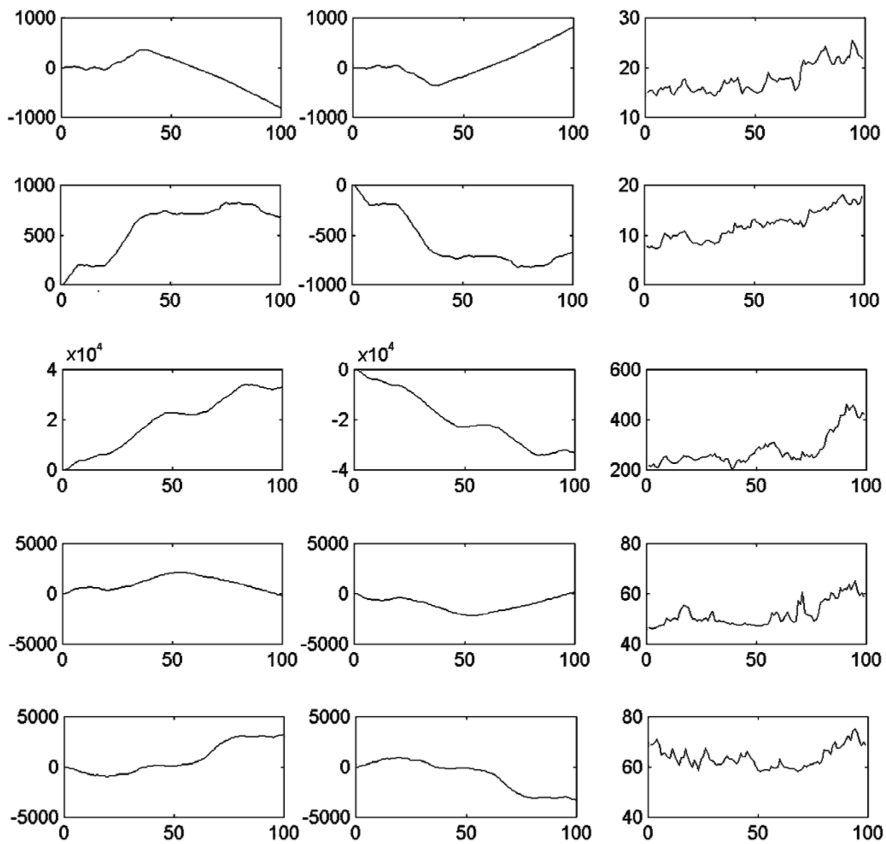

Fig. 4. The EH-method.

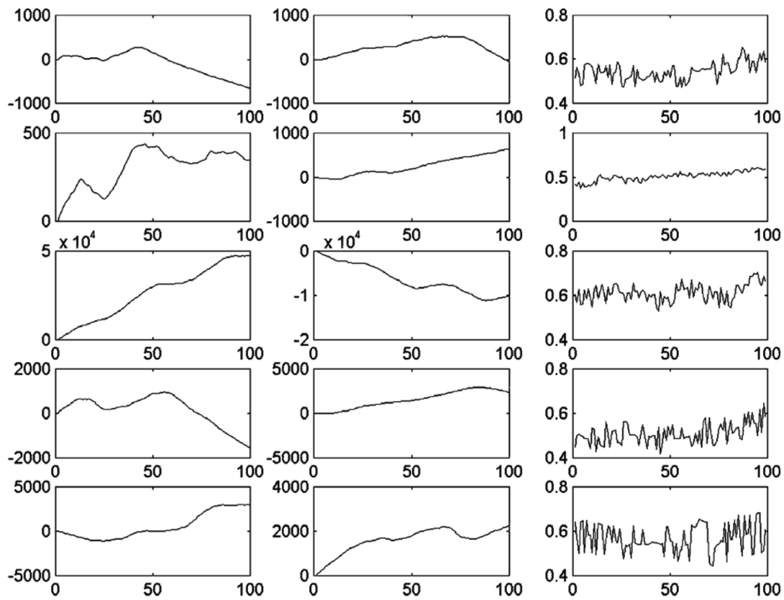

Fig. 5. MV method: the first two columns represent the change in the income of the buyer (18) and the seller (21) during the 100 day period; the third column represents the change of the parameter h (8).

Average incomes of the buyers AB_t and the sellers AS_t were calculated with the use of the formulas

$$AB_t = \sum_{i=1}^{m} w_i \cdot PB_{i,t}, \tag{20a}$$

$$AS_t = \sum_{i=1}^{m} w_i \cdot PS_{i,t}, \tag{20b}$$

$$AT_t = \sum_{i=1}^{m} w_i \cdot C_{i,t}. \tag{20c}$$

4.2. MV hedge

Here we need to modify (19) and (20a); all other formulas are the same as in the previous section:

$$PS_{i,t+T+j+1} = PS_{i,t+T+j} + C_{i,t+j+1} +$$
$$+ \begin{cases} (1-h_{i,j+1})(K_{i,t+j+1} - S_{i,t+T+j+1}), & \text{if } S_{i,t+T+j+1} \geqslant K_{i,t+j+1}; \\ h_{i,j+1}(S_{i,t+T+j+1} - S_{i,t+j+1}), & \text{otherwise}, \end{cases} . \tag{21}$$

and

$$AQ_t = \sum_{i=1}^{m} w_i \cdot (C_{i,t} + h_{i,t} S_{i,t}). \tag{22}$$

The seller's income in this case is higher than the buyer's income. The calculation results with the use of the EH and MV algorithms for $m = 18$ assets are presented in Fig. 6 and in Table 1.

The prices in the table are given in cents. The second column represents the average price during the 100 day period (see Figs. 4–6). Columns 3–6 represent the final profit of the buyer and the seller, obtained by applying the EH and MV methods.

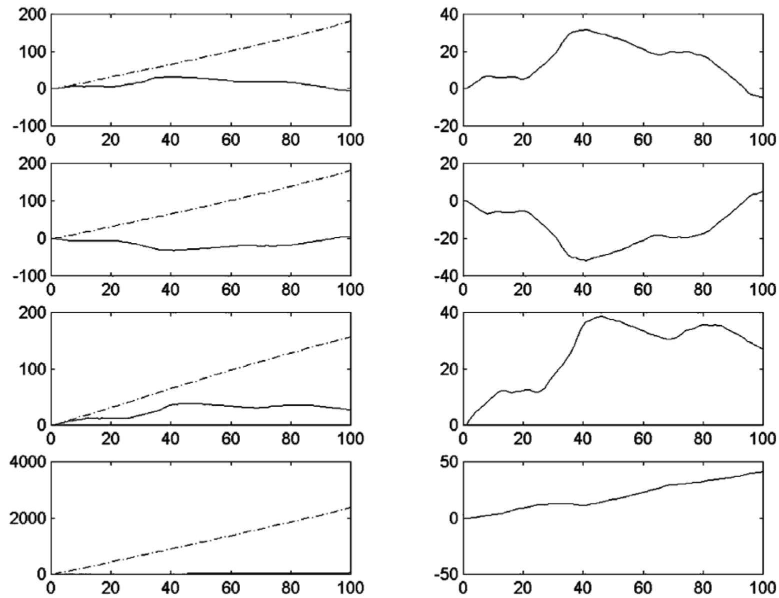

Fig. 6. The general case of $m = 18$ assets. The implementation of the portfolio based on the EH- (1st and 2nd rows) and MV (3d and 4th rows) algorithms, where the solid line represents the income of the buyer (1st and 3d rows) and the seller (2nd and 4th rows), and the chain line represents the average turnover of financial resources. All calculations were made in accordance with (20a–20c) and (22), where the parameter h is defined in (8). The right column illustrates the corresponding incomes.

Table 1

The name of the asset	Average price of the asset	**EH method**		**MV method**	
		Buying	Selling	Buying	Selling
ANZ Bank	2306.53	922.63	−922.63	1801.59	3125.66
CSS bank	3909.16	−1410.77	1410.77	−485.93	5702.04
Coles Myers	990.6	24.78	−24.78	461.42	1477.59
David Jones	236.11	1428.42	−1428.42	1844.33	−513.65
Fairfax	417.18	-816.73	816.73	−657.82	−58.73
Harvey Norman	277.67	671.77	−671.77	347.12	642.38
National Bank	3193.27	−4433.27	4433.27	−4172.65	6730.61
Publish.Brodcast.	1625.36	3416	−3416	2419.06	2119.44
Qantas	350.14	2234.44	−2234.44	1344.69	64.72
QBE Insurance	1813.72	1216.89	−1216.89	5023.97	1466.49
Rio–Tinto	5788.09	33156.66	−33156.66	47506.18	−10089.97
Santos	1141.14	−9401.15	9401.15	−2086.32	3910.58
Tabcorp	1606.01	−183.37	183.37	−1558.01	2335.71
TEN Network	342.78	−1192.97	1192.97	−1816.85	818.41
Telstra	412.26	−1372.16	1372.16	−1924.31	−1068.74
Westpac Bank	2104.37	3293.2	−3293.2	2949.07	2272.26
Woolworth	1633.53	−3596.63	3596.63	−2517.34	3440.02
Woodside Petroleum	3361.07	−21427.93	21427.93	3208.89	11465.86

5. Concluding notes

The classical equation (2) determines such hedging where the transactions are statistically profitable for the buyer if the price is lower, or for the seller if the price is higher. Any transaction is not risk-free, but a sequence of independent transactions may substantially reduce the risk (see Fig.6 and Table 1). In contrast, the risk-free formula (4) was obtained under the ideal assumption of absolute liquidity of the market. This means that every transaction is a continuous sequence of trades, which (as noted in many articles) cannot be implemented in real time.

Combining the call option with the corresponding asset gives an additional degree of flexibility. On the one hand, it helps to reduce the risk to the seller. On the other hand, the price of the call option will be reduced if the demand for the asset is historically high. In any case, the level of the MV hedge is located between the BS and *EH* prices. Therefore the MV hedge can be considered as a compromise between the two major decisions (see Fig. 1). Comparing the third and the fourth rows with the first two rows (see graphs in Fig. 6), which were built using the same governing parameters, we can see the benefits of the MV method in comparison with the EH method.

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