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EMPIRICAL BAYESIAN ESTIMATION IN THE MODEL OF COMPETING RISKS

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We study empirical semi-parametric Bayesian estimates of exponential functionals in the model of competing risks. For these estimates we establish the properties of the uniform strong consistency and iterated logarithm type laws.

1. Construction of estimates

Let $Z : (\Omega, \mathcal{A}) \longrightarrow (\mathcal{X}, \mathcal{B})$ be a random variable (r.v.) and $\{A^{(1)}, \dots, A^{(k)}\}$ be events, forming a partition of the space of elementary events Ω. With the use of continuous probability measure P defined on the measurable space $(\Omega, \mathcal{A})$, we define the continuous distributions $\{Q^{(i)}(B) = P(\omega \cap A^{(i)} : Z(\omega) \in \in B), B \in \mathcal{B}, i \in J\}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathcal{X}$ ($\mathcal{X} \subseteq R = (-\infty, \infty)$) and $J = \{1, \dots, k\}$. Note that for all $B \in \mathcal{B}$ the following relation holds:

$$Q^{(1)}(B) + \ldots + Q^{(k)}(B) = P(\omega : Z(\omega) \in B) = Q(B).$$

Let $\{\alpha^{(i)}(\cdot), i \in J\}$ be nonnegative continuous finitely additive measures on $(\mathcal{X}, \mathcal{B})$, and $\alpha(\cdot)$ be their sum: $\alpha(B) = \alpha^{(1)}(B) + \ldots + \alpha^{(k)}(B), B \in \mathcal{B}$. Denote by $\mathcal{D}(\alpha^{(1)}, \dots, \alpha^{(k)})$ the Dirichlet distribution with the parameter $(\alpha^{(1)}, \dots, \alpha^{(k)})$. Following [1, 2], we consider $(Q^{(1)}, \dots, Q^{(k)})$ as a random vector process on $(\mathcal{X}, \mathcal{B})$ with the prior Dirichlet distribution $\mathcal{D}(\alpha^{(1)}, \dots, \alpha^{(k)})$. Then the sub-distributions $H(t; i) = = P(Z \leqslant t, A^{(i)})$, $(t; i) \in R \times J$, are continuous r.v.s with the corresponding prior beta-distributions

$$\mathrm{Be}(\alpha^{(i)}(t); \alpha(R) - \alpha^{(i)}(t)),\ (t; i) \in R \times J,$$

where $\alpha^{(i)}(t) = \alpha^{(i)}((-\infty; t])$. In the considered generalized model of competing risks (MCR), from the Bayesian point of view the main interest is in joint properties of the random pairs $(Z; A^{(i)})$, $i \in J$, and the task is to estimate the functionals of the cumulative hazard functions (c.h.f.)

$$\Lambda(t; i) = \int\limits_{(-\infty;\ t]} \frac{dH(u; i)}{1 - H(u)},\ (t; i) \in R \times J, \tag{1}$$

from the independent random sample $S^{(n)} = \{(Z_j; \delta_j^{(1)}, \dots, \delta_j^{(k)}),\ j = \overline{1, n}\}$, where

$$H(t) = P(Z \leqslant t) = H(t; 1) + \ldots + H(t; k)$$

is the distribution function (d.f.) of the r.v. Z; $\delta_j^{(i)} = I(A_j^{(i)})$ is the indicator, and

$$\{(Z_j; A_j^{(1)}, \dots, A_j^{(k)}),\ j \geqslant 1\}$$

is a sequence of independent copies of the sample $(Z; A^{(1)}, \dots, A^{(k)})$. The exponential functionals

$$F(t; i) = 1 - \exp(-\Lambda(t; i)),\ (t; i) \in R \times J, \tag{2}$$

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are important functionals of c.h.f. (1), which possess the properties of sub-distributions [1]. From [1, 4, 5] it follows that Bayesian estimates of the distributions $H(t;i)$ and $H(t)$ with respect to the quadratic loss function are the statistics

$$H_n^{\alpha}(t;i) = q_n H_0(t;i) + (1-q_n)H_n(t;i) \quad \text{and} \quad H_n^{\alpha}(t) = q_n H_0(t) + (1-q_n)H_n(t),$$

where

$$q_n = \frac{\alpha(R)}{\alpha(R)+n}, \; H_0(t;i) = \frac{\alpha^{(i)}(t)}{\alpha(R)}, \quad H_0(t) = \sum_{i=1}^{k} H_0(t;i) = \frac{\alpha(t)}{\alpha(R)},$$

$$H_n(t;i) = \frac{1}{n}\sum_{k=1}^{n} I(Z_k \leqslant t; \delta_k^{(i)} = 1), \quad H_n(t) = \sum_{i=1}^{k} H_n(t;i).$$

In [1], with the use of these Bayesian estimates the author constructed and studied the asymptotic properties of the estimates for functionals (2) of the following three types:

$$F_{1n}^{\alpha}(t;i) = 1 - \exp(-\Lambda_n^{\alpha}(t;i)),$$

$$F_{2n}^{\alpha}(t;i) = 1 - \prod_{u \leqslant t}(1 - (\Lambda_n^{\alpha}(u;i) - \Lambda_n^{\alpha}(u;i))), \tag{3}$$

$$F_{3n}^{\alpha}(t;i) = 1 - (1 - H_n^{\alpha}(t;i))^{R_n^{\alpha}(t;i)},$$

where $R_n^{\alpha}(t;i) = \Lambda_n^{\alpha}(t;i)(\Lambda_n^{\alpha}(t))^{-1}, \; (t;i) \in R \times J$;

$$\Lambda_n^{\alpha}(t) = \int\limits_{(-\infty;\; t]} \frac{dH_n^{\alpha}(u)}{1 - H_n^{\alpha}(u)} = \sum_{i=1}^{n} \Lambda_n^{\alpha}(t;i), \quad \Lambda_n^{\alpha}(t;i) = \int\limits_{(-\infty;\; t]} \frac{dH_n^{\alpha}(u;i)}{1 - H_n^{\alpha}(u)},$$

and $\Lambda_n^{\alpha}(t)$ is the estimate for $\Lambda(t;i)$. For estimates (3) the author of [1] established the law of iterated logarithm (LIL) and the results of approximation by the same sequence of Gaussian processes. The main result of [2] is the theorem on the uniform closeness of order n^{-1} a.s. (almost surely) between estimates (3) and the Bayesian-type estimate proposed in [9] and studied by the authors of [6, 7, 10] in the simple model of right random censoring. In this paper we study analogs of estimates (3) in the case where the measures $\alpha^{(i)}(t) = \alpha^{(i)}(t, \theta^{(i)}), \; i = \overline{1,k}$, are defined up to some unknown parameter $\theta^{(i)} \in \Theta^{(i)} \subseteq R$, which is also subjected to estimation. Let $\widehat{\theta}^{(i)}$ be some estimate of $\theta^{(i)}$ from the sample $S^{(n)}$, and $\widehat{\alpha}^{(i)}(t) = \alpha^{(i)}(t, \widehat{\theta}^{(i)})$. Here $\alpha^{(i)}(R) = \beta_i, \; i = \overline{1,k}$, are unknown. Then $\widehat{\alpha}(t) = \alpha(t, \widehat{\theta}^{(i)})$, where $\widehat{\theta} = (\widehat{\theta}^{(1)}, \ldots, \widehat{\theta}^{(k)})$ is an estimate of the vector parameter $\theta = (\theta^{(1)}, \ldots, \theta^{(k)})$ and $\alpha(R) = \beta = \beta_1 + \ldots + \beta_k$ is a prior size of the sample. Let $\widehat{\theta}^{(i)}$ be $n^{1/2}$-consistent estimates, allowing the following representation for all $i \in J$ with probability 1:

$$(I(\theta_0)n)^{1/2}(\widehat{\theta}^{(i)} - \theta_0^{(i)}) = n^{-1/2}\sum_{j=1}^{n} l^{(i)}(Z_j; \delta_j^{(i)}) + \varepsilon_n^{(i)}, \tag{4}$$

where $\theta_0^{(i)}$ is the real value of $\theta^{(i)}, l^{(i)}$ is a measurable function (possibly depending on $\theta_0^{(i)}$) with $M_{\theta_0}^{*} l^{(i)}(Z_j; \delta_j^{(i)}) = 0$, $M_{\theta_0}^{*}[l^{(i)}(Z_j; \delta_j^{(i)})]^2 = I(\theta_0) \in (0;\infty)$, and $\varepsilon_n^{(i)} = O(n^{-\lambda})$, $\lambda > 0$. Here $M_{\theta_0}^{*}$ is the operator of mathematical expectation and is interpreted as $M_{\theta}^{*}[\cdot] = \mathcal{M}_{\theta}[M[\cdot]]$, where the first averaging M is over the joint distribution of the vector $(Z_j; \delta_j^{(1)}, \ldots, \delta_j^{(k)})$, and the second averaging $\mathcal{M}_{\theta}$ is over the Dirichlet distribution at the point θ. From [1] we have

$$\mathcal{M}_{\theta} H(t;i) = H_0(t;i) = \frac{\alpha^{(i)}(t;\theta)}{\alpha(R)}, \; i = \overline{1,k}.$$

The conditions necessary for the validity of (4) are given in Theorem 8.1 of [3]. In particular, from the representation (4) we obtain the following properties of the estimates $\widehat{\theta}^{(i)}$ for all $i \in J$:

$$n^{1/2}(\widehat{\theta}^{(i)} - \theta_0^{(i)}) \xrightarrow[n\to\infty]{d} N(0; I^{-1}(\theta_0)), \tag{5}$$

$$\overline{\lim_{n\to\infty}}(\widehat{\theta}^{(i)} - \theta_0^{(i)})(I(\theta_0))^{1/2} L_n = 1 \text{ a.s.} \tag{6}$$

for $L_n = \left(\frac{n}{2\log\log n}\right)^{1/2}$ and

$$\widehat{\theta}^{(i)} \xrightarrow[n\to\infty]{\text{a.s.}} \theta_0^{(i)}. \tag{7}$$

Let us replace in estimates (3) the measures $\alpha^{(i)}(t)$ with their parametric estimates $\widehat{\alpha}^{(i)}(t)$ and denote the resulting estimates of $F(t;i)$ by $\widehat{F}^{\alpha}_{mn}(t;i)$, $m = 1,2,3$, $(t;i) \in R \times J$. In this case the corresponding estimates of the Bayesian decision rules $H_0(t;i)$ and $H_0(t)$ are

$$\widehat{H}_0(t;i) = \frac{\widehat{\alpha}^{(i)}(t)}{\beta} \quad \text{and} \quad \widehat{H}_0(t) = \widehat{H}_0(t;1) + \ldots + \widehat{H}_0(t;k),$$

for which $\widehat{H}_0(\infty;i) = \frac{\beta_i}{\beta}$ are known and $\widehat{H}_0(\infty) = 1$. The estimates obtained by replacing $\alpha^{(i)}$ with $\widehat{\alpha}^{(i)}$ with good reason can be called empirical Bayesian estimates [4].

2. The choice of measure, reflecting the a priori idea of the studied model

Let us give an example of how the measures $\alpha^{(i)}(t;\theta^{(i)})$ can be chosen and the parameters $\theta^{(i)}$ can be estimated. Let

$$\alpha^{(i)}(t;\theta^{(i)}) = \beta_i\left(1 - \exp\left(-\frac{g(t)}{\theta^{(i)}}\right)\right),$$

where $t \geqslant 0$, $\theta^{(i)} > 0$, $\beta_i > 0$ (known), and $g(t)$ be strictly increasing, $\exists g'(t)$, $g(0) = 0$ and $g(\infty) = \infty$. Then for all $i = \overline{1,k}$: $\alpha^{(i)}(0;\theta^{(i)}) = 0$, $\alpha^{(i)}(\infty;\theta^{(i)}) = \beta_i$ and

$$\frac{d\alpha^{(i)}(t;\theta^{(i)})}{dt} = \beta_i \frac{g'(t)}{\theta^{(i)}} \exp\left(-\frac{g(t)}{\theta^{(i)}}\right) = \alpha^{(i)\prime}(t;\theta^{(i)}), \tag{8}$$

$$\frac{d\alpha^{(i)}(t;\theta^{(i)})}{d\theta^{(i)}} = \beta_i \frac{g(t)}{(\theta^{(i)})^2} \exp\left(-\frac{g(t)}{\theta^{(i)}}\right). \tag{9}$$

We construct the estimates of the parameters $\theta^{(i)}$ in two ways. Let $H(0) = 0$ and $\lim\limits_{x\to\infty} g(x)(1-H(x)) = 0$ a.s.

The method of maximum likelihood (MML). Let us construct the likelihood function from the sample $S^{(n)}$ with the use of the densities of prior representations

$$\frac{d}{dt}\Big[H_0(t;i)\Big] = \frac{1}{\beta}\alpha^{(i)\prime}(t;\theta^{(i)}) : L_n(\theta) = \prod_{j=1}^{n}\prod_{i=1}^{k}\left\{\frac{1}{\beta}\alpha^{(i)\prime}(Z_j;\theta^{(i)})\right\}^{\delta_j^{(l)}} =$$

$$= \frac{1}{\beta^n}\prod_{j=1}^{n} g'(Z_j) \prod_{l=1}^{k}\left(\frac{\beta_l}{\theta^{(l)}}\right)^{\sum\limits_{j=1}^{n}\delta_j^{(l)}} \cdot \exp\left(-\sum_{j=1}^{n}\sum_{l=1}^{k}\delta_j^{(l)}\frac{g(Z_j)}{\theta^{(i)}}\right).$$

Hence we obtain the system

$$\frac{d\log L_n(\theta)}{d\theta^{(i)}} = -\frac{1}{\theta^{(i)}}\sum_{j=1}^{n}\delta_j^{(i)} + \frac{1}{(\theta^{(i)})^2}\sum_{j=1}^{n}\delta_j^{(i)}g(Z_j) = 0,\ i=\overline{1,k},$$

and then the MML estimates (MLE) for $\theta^{(i)}$ in the form

$$\widehat{\theta}^{(i)} = \Bigl(\sum_{j=1}^{n}\delta_j^{(i)}\Bigr)^{-1}\cdot\sum_{j=1}^{n}\delta_j^{(i)}g(Z_j),\ i=\overline{1,k}. \tag{10}$$

The method of least squares (MLS). Before applying the MLS, let us calculate the following mathematical expectations:

$$Mg(Z_j) = \int_0^\infty g(t)dH(t) = \int_0^\infty (1-H(t))dg(t),$$

$$M_\theta^* g(Z_j) = \mathcal{M}_\theta[Mg(Z_j)] = \int_0^\infty \mathcal{M}_\theta(1-H(t))dg(t) = \int_0^\infty \Bigl[1-\frac{\alpha(t;\theta)}{\beta}\Bigr]dg(t) =$$

$$= \int_0^\infty \Bigl[1-\frac{1}{\beta}\sum_{i=1}^{k}\beta_i\Bigl(1-\exp\Bigl(-\frac{g(t)}{\theta^{(i)}}\Bigr)\Bigr)\Bigr]dg(t) = \frac{1}{\beta}\sum_{i=1}^{k}\beta_i\int_0^\infty \exp\Bigl(-\frac{g(t)}{\theta^{(i)}}\Bigr)dg(t) =$$

$$= \frac{1}{\beta}\sum_{i=1}^{k}\beta_i\theta^{(i)} = \sum_{i=1}^{k}\theta^{(i)}\mathcal{M}_\theta H(\infty;i) = \sum_{i=1}^{k}\theta^{(i)}\mathcal{M}_\theta[M\delta_j^{(i)}].$$

Thus,

$$M_\theta^* g(Z_j) - \sum_{i=1}^{k}\theta^{(i)}M_\theta^*\delta_j^{(i)} = 0. \tag{11}$$

With the use of (11) we estimate the parameters $\theta^{(i)}$ minimizing the empirical quadratic deviation

$$U_n^2(\theta) = \frac{1}{n}\sum_{j=1}^{n}[g(Z_j) - \sum_{l=1}^{k}\theta^{(i)}\delta_j^{(l)}]^2 \to \min,$$

which yields the MLS estimate (LSE) for $\theta^{(i)}$, $i=\overline{1,k}$. From the MLS system

$$\frac{dU_n^2(\theta)}{d\theta^{(i)}} = \frac{-2}{n}\sum_{j=1}^{n}[g(Z_j) - \sum_{l=1}^{k}\theta^{(l)}\delta_j^{(l)}]\delta_j^{(i)} = 0, \quad i=\overline{1,k},$$

we obtain estimates (10). Thus, the MLE and LSE of parameters $\theta^{(i)}$ coincide and are given by (10). Note that both methods are usual statistical methods and not Bayesian ones. Let us show that for estimates (10) representation (4) holds with

$$l^{(i)}(Z_j;\delta_j^{(i)}) = -\frac{1}{\theta^{(i)}}\delta_j^{(i)} + \frac{1}{(\theta^{(i)})^2}\delta_j^{(i)}g(Z_j), \quad i=\overline{1,k}.$$

Since for all $i=\overline{1,k}$

$$Ml^{(i)}(Z_j;\delta_j^{(i)}) = -\frac{H(\infty;i)}{\theta^{(i)}} + \frac{1}{(\theta^{(i)})^2}\int_0^\infty g(t)dH(t;i),$$

then

$$M_\theta^* l^{(i)}(Z_j;\delta_j^{(i)}) = -\frac{H_0(\infty;i)}{\theta^{(i)}} + \frac{1}{(\theta^{(i)})^2}\int_0^\infty g(t)dH_0(t;i) =$$

$$= -\frac{\beta_i}{\theta^{(i)}\beta} + \frac{\beta_i}{(\theta^{(i)})^2\beta}\int_0^\infty g(t)de^{-\frac{g(t)}{\theta^{(i)}}} = 0$$

and, similarly,

$$M_\theta^*[l^{(i)}(Z_j;\delta_j^{(i)})]^2 = M_\theta\Big\{\frac{H_0(\infty;i)}{(\theta^{(i)})^2} - \frac{2}{(\theta^{(i)})^3}\int_0^\infty g(t)dH(t;i) +$$

$$+\frac{1}{(\theta^{(i)})^4}\int_0^\infty g^2(t)dH(t;i)\Big\} = \frac{\beta_i}{(\theta^{(i)})^2\beta} + \frac{2\beta_i}{(\theta^{(i)})^3\beta}\int_0^\infty g(t)de^{-\frac{g(t)}{\theta^{(i)}}} -$$

$$-\frac{\beta_i}{(\theta^{(i)})^4\beta}\int_0^\infty g^2(t)de^{-\frac{g(t)}{\theta^{(i)}}} = \frac{\beta_i}{(\theta^{(i)})^2\beta} \in (0;\infty).$$

Thus, for estimates (10), representation (4) and, hence, properties (5)–(7) hold. The considered example of exponential measures $\alpha^{(i)}(t;\theta^{(i)})$ contains the corresponding measure from [8], which is obtained with $k = 1$ and $g(t) \equiv t$. However, we propose three types of estimates for $F(t;i)$, which differ from the multiplicative estimate from [8]. Since we consider more general measures than exponential, the estimates of the parameters of which satisfy representation (4), we can also consider a generalized analog of the estimate from [8] in the following form:

$$\widehat{F}_{4n}^\alpha(t;i) = 1 - (1 - \widehat{H}_n^\alpha(t))\prod_{j=1}^n\Big[\frac{1 - \widehat{H}_n^\alpha(Z_j) + \frac{1}{n+\beta}}{1 - \widehat{H}_n^\alpha(Z_j)}\Big]^{\gamma_j^{(i)}(t)}, \tag{12}$$

where $\gamma_j^{(i)}(t) = (1 - \delta_j^{(i)})I(Z_j \leqslant t)$. Note that the estimate of type (12) with the known parameters $\alpha^{(i)}$, which is Bayesian with respect to the quadratic risk in the MCR, was studied in [2], where the author established its uniform closeness of order $O(n^{-1})$ a.s. to estimates (3). In this sense, estimate (3) can be called asymptotically Bayesian, and $\{\widehat{F}_{mn}^\alpha(t;i), m = \overline{1,4}\}$ can be called empirically Bayesian. Since the difference between the estimates F_{mn}^α and $\widehat{F}_{mn}^\alpha$, $m = \overline{1,4}$, is that in the empirical estimates the prior sub-distributions $H_0(t;i)$ are estimated by $\widehat{H}_0(t;i)$, we provide only additional calculations in the proofs of the properties of empirical estimates. When studying the empirical estimates, the question arises concerning the choice of parameters β_i, $i = \overline{1,k}$. If, following the authors of [8], we proceed from the mean squared error of the estimates $H_n^\alpha(t;i)$:

$$M[H_n^\alpha(t;i) - H(t;i)]^2 = M[q_n(H_0(t;i) - H(t;i)) + (1 - q_n)(H_n(t;i) - H(t;i))]^2 =$$

$$= q_n^2(H_0(t;i) - H(t;i))^2 + (1 - q_n)^2\frac{H(t;i)(1 - H(t;i))}{n},$$

where the first term is the square of the bias introduced into the Bayesian estimate $H_n^\alpha(t;i)$ by the prior representation $H_0(t;i)$ of the sub-distribution $H(t;i)$, and the second one is the variance of the empirical sub-distribution $H_n(t;i)$, then it converges to zero for $\beta_i = O(n^{c_i})$, where $c_i < 1$, $i = \overline{1,k}$. If we require that both terms have the same order of smallness, then we must choose $\beta_i = O(n^{\frac{1}{2}})$, $i=\overline{1,k}$. For simplicity, we choose the prior sample size as $\alpha(R) = \beta = n^{\frac{1}{2}}$, and then

$$q_n = \frac{1}{1+n^{1/2}} < n^{-1/2}. \tag{13}$$

Let K_i be a compact set containing the point $\theta^{(i)}$, the derivatives $\dfrac{d\alpha^{(i)}(t;\theta^{(i)})}{d\theta^{(i)}}$ exist, and

$$\sup_{(t;\theta^{(i)})\in R\times K_i}\left|\frac{d\alpha^{(i)}(t;\theta^{(i)})}{d\theta^{(i)}}\right| = O(\beta_i), \quad i=\overline{1,k}. \tag{14}$$

According to (9), condition (14) holds for exponential measures. Then for each $i=\overline{1,k}$ the following inequality holds a.s.:

$$\begin{aligned}\sup_{(t;\theta^{(i)})\in R\times K_i}|\widehat{H}_n^\alpha(t;i) - H_n^\alpha(t;i)| &\leqslant q_n \cdot \sup_{(t;\theta^{(i)})\in R\times K_i}|\widehat{H}_0(t;i) - H_0(t;i)| \leqslant \\ &\leqslant \frac{q_n}{\beta}\cdot \sup_{(t;\theta^{(i)})\in R\times K_i}|\alpha^{(i)}(t;\widehat{\theta}^{(i)}) - \alpha^{(i)}(t;\theta^{(i)})| \leqslant \\ \leqslant \frac{q_n}{\beta}\sup_{(t;\theta^{(i)})\in R\times K_i}\left|\frac{d\alpha^{(i)}(t;\theta^{(i)})}{d\theta^{(i)}}\right|\cdot|\widehat{\theta}^{(i)} - \theta^{(i)}| &= O(q_n\cdot L_n^{-1}) = O\left(\frac{(\log\log n)^{1/2}}{n}\right).\end{aligned} \tag{15}$$

This inequality is derived with the use of the mean value theorem, LIL (6), and relations (13), (14), $\beta_{(i)} < \beta$, $i=\overline{1,k}$. Now with the use of (15) and the LIL for the empirical sub-distributions, we obtain

$$\begin{aligned}\sup_{(t;\theta^{(i)})\in R\times K_i}|\widehat{H}_n^\alpha(t;i) - H(t;i)| &\leqslant q_n \sup_{(t;\theta^{(i)})\in R\times K_i}|\widehat{H}_0(t;i) - H_0(t;i)| \leqslant \\ &\leqslant q_n \sup_{(t;\theta^{(i)})\in R\times K_i}|H_0(t;i) - H_n(t;i)| + \sup_{-\infty<t<\infty}|H_n(t;i) - H(t;i)| = \\ &= O(q_n\cdot L_n^{-1}) + O(q_n) + O(L_n^{-1}) = O(L_n^{-1}) = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \quad \text{a.s.}\end{aligned} \tag{16}$$

For the parametric Bayesian estimates $\widehat{\Lambda}_n^\alpha(t;i)$ of c.h.f. (1), the LIL holds. Let

$$T < T_H = \inf\{t\in R : H(t) = 1\}.$$

Theorem 1. *Assume that relations* (4) *and* (14) *hold. Then for all* $i\in J$

$$\sup_{-\infty<t\leqslant T}|\widehat{\Lambda}_n^\alpha(t;i) - \Lambda(t;i)| = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \quad \textit{a.s.}$$

Let the sequence $\{T_n,\, n\geqslant 1\}$ *be such that* $T_n < T_H$ *and* $T_n\to\infty$ *for* $n\to\infty$ *in a way that*

$$1 - H(T_n) \geqslant \left(\frac{\log\log n}{n}\right)^{1/2}.$$

Then for all $i\in J$

$$\sup_{-\infty<t\leqslant T_n}|(1-H(t))(\widehat{\Lambda}_n^\alpha(t;i) - \Lambda(t;i))| = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \quad \textit{a.s.}$$

The proof of Theorem 1 is the same as that of Theorem 3.4.1 from [1]. One just needs to take into consideration estimates (13), (15), and (16).

Let us now formulate the corresponding LIL-type results for all introduced estimates of the exponential functionals.

Theorem 2. *Under the conditions of Theorem* 1, *for* $m = \overline{1,4}$ *and* $i \in J$ *we have*

$$\sup_{-\infty<t<T_n} |\widehat{F}^{\alpha}_{mn}(t;i) - F(t;i)| = O\left(\left(\frac{\log\log n}{n}\right)^{1/2}\right) \text{ a.s.},$$

$$\sup_{-\infty<t<T_n} |(1-H(t))(F^{\alpha}_{n}(t;i) - F(t;i))| = O\left(\max\left\{\left(\frac{\log\log n}{n}\right)^{1/2}, \lambda^{(i)}(n)\right\}\right) \text{ a.s.},$$

where

$$\lambda^{(i)}(n) = (1 - H(T_n))(1 - F(T_n;i)).$$

The proof of Theorem 2 for $m = 1,2,3$ repeats that of Theorem 3.4.2 from [1], and for $m = 4$ repeats the proof of the main theorem of [2]. One just needs to use estimate (13) for q_n, from which for $m = 1,2,3$ and $i \in J$ it follows that

$$\sup_{-\infty<t<T} |\widehat{F}^{\alpha}_{mn}(t;i) - \widehat{F}^{\alpha}_{4n}(t;i)| = O\left(\frac{1}{n^{1/2}}\right) \text{ a.s.}$$

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