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OPERATOR ESTIMATES IN HOMOGENIZATION OF ELLIPTIC SYSTEMS OF EQUATIONS

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We study homogenization of nonselfadjoint second order elliptic systems with ε-periodic rapidly oscillating coefficients as $\varepsilon \to 0$. We obtain the L^2- and H^1-estimates for the homogenization error of order ε. The estimates admit the operator form and can be written in terms of the resolvents of the original and approximate systems in the operator norm $\|\cdot\|_{L^2\to L^2}$ or $\|\cdot\|_{L^2\to H^1}$. The shift method is used for obtaining such estimates. Bibliography: 20 titles.

1 Introduction: Operator Estimates in Homogenization and Shift Method

Error estimates have always been in the focus of homogenization theory (cf., for example, [1]–[3]). Such estimates are proved in different norms depending on the specific nature of the problem under consideration. For example, the H^1- and L^2-norms connected with energy estimates are natural for second order elliptic equations of divergence form. In the early results, the majorants in the estimates depend on the data of the problem in such a way that the estimates cannot be given an operator meaning. We explain by an example in what situations the error estimate in homogenization admits the operator interpretation. Let the original and homogenized equations be written in the resolvent form, and let their solutions can be represented as the action of the resolvents of the corresponding operators on the right-hand sides of the equations:

$$u_\varepsilon = (A^\varepsilon + 1)^{-1} f, \quad u_0 = (A^0 + 1)^{-1} f.$$

Assume that for small ε ($0 < \varepsilon \leqslant \varepsilon_0$) the following estimate holds:

$$\|u_\varepsilon - u_0\|_{L^2} \leqslant \varepsilon c \|f\|_{L^2}, \tag{1.1}$$

where the constant depends only on the dimension and ellipticity constant. Then, by the definition of the operator norm, the estimate (1.1) implies the following estimate in the operator norm for the difference of resolvents:

$$\|(A^\varepsilon + 1)^{-1} - (A^0 + 1)^{-1}\|_{L^2\to L^2} \leqslant \varepsilon c, \quad 0 < \varepsilon \leqslant \varepsilon_0.$$

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Thus, the error estimate (1.1) admits the operator interpretation.

Over last decade, there has been an increasing interest in operator error estimates in homogenization theory (cf. [4] and the references in [5]).

From the methodological point of view, this paper continues the study going back to the papers [6] and [7], where the *shift method* and its modification based on the Steklov average (or smoothing) were proposed.

We recall the main steps of the shift method and its modifications. First, an additional integration parameter is introduced, directly via shift or indirectly via Steklov averages. Second, a residual for the first approximation to the original equation is subject to a specific analysis. Note that the first approximation can involve a shift with respect fast variables or smoothing with respect to slow variables, which makes it possible to obtain operator estimates. In many cases, in particular, for scalar problems, it is possible to pass from the obtained H^1-estimates with shifted or smoothed first approximation to the standard first approximation, but this is not the case for a general vector problem because the classical first approximation not necessarily belongs to an H^1-space. Hence a question arises how to find a suitable form of the first approximation. This question was answered in [7] with the help of Steklov averages.

The universality of the shift method can be seen already in the early works [6]–[12] devoted to obtaining operator estimates in homogenization by the shift method. In particular, it was shown there that the method is applicable to equations of different type, elliptic and parabolic, scalar and vector, linear and nonlinear, not necessarily selfadjoint and, possibly, with various type degenerations (cf. also the references in [5]).

Another feature of the shift method is that it is applicable to problems with not only periodic but also locally periodic and many-scale coefficients (cf. [13] and the references in [5]). We recall that a function of the form $f(x, \frac{x}{\varepsilon})$ is said to be *locally periodic* if the function $f(x, y)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, is periodic only with respect to y, so that $f(x, \frac{x}{\varepsilon})$ slowly varies and simultaneously rapidly oscillates with respect to different groups of variables for small $\varepsilon > 0$. By a *multi-scale function* we mean a function that is periodic with respect to different groups of variables with different periods provided that the periods are infinitely small of different order as $\varepsilon \to 0$. For an example of a multi-scale function one can consider a function of the form $f(\frac{x}{\varepsilon}, \frac{x}{\delta})$, where $f(y, z)$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^d$, is 1-periodic with respect to all variables, $\delta \to 0$ and $\delta/\varepsilon \to 0$ as $\varepsilon \to 0$.

In [6, 7], as an example of a vector problem, the system of equations of elasticity theory with ε-periodic coefficients was intentionally discussed in order to clarify how the shift method works in the vector case and how technical difficulties of vector analysis could be reduced to similar ones in the scalar case,

For systems of elasticity theory the corresponding fourth order tensor possesses symmetry properties (cf., for example, [3, Chapter XII]). Respectively, a special language is required for problems of elasticity theory because of specific relations and inequalities. For example, if the problem is formulated in terms of symmetric gradients of vector-valued functions, then various Korn inequalities are considered as counterparts of the Friedrichs and Poincaré inequalities. However, the symmetry properties of a fourth order tensor are not essential in the shift method.

In this paper, we show that the tools used earlier for systems of elasticity theory can be also applied to a second order elliptic system of general form. For this purpose no new original concepts and ideas are required, but only the notation becomes much more cumbersome.

The main results of the paper are presented in Section 3 (Theorems 3.1 and 3.2).

Unlike problems of elasticity theory, the ellipticity condition for a tensor is formulated not as an algebraic relation, but in terms of the coercitivity inequality for the energy integral (cf. (3.4) below). Such an approach to the definition of ellipticity was used, for example, in the survey [14] on the G-convergence of elliptic operators. This ellipticity condition should be separately verified for each particular system. For the system of elasticity theory this condition follows from the algebraic positive definiteness condition for fourth order tensors and the Korn inequality for compactly supported vector-valued functions.

This paper was planned and discussed with Professor V. V. Zhikov. In fact, this paper could have appeared more than 10 years ago, after publication of the papers [6, 7]. However, other problems, more important and urgent, as it seemed to us then, distracted us from this topic.

After the untimely death of V. V. Zhikov, I consider it my duty to bring our plan to life.

2 Homogenization in the Scalar Case

For better understanding the results of the vector theory, we first discuss the scalar case. In this section, we consider the diffusion equation in the classical setting. In Section 3, where we proceed with a general elliptic system, we will see that there is a great similarity between the scalar and vector theories, but some differences are also observed. The differences are mainly caused by the absence of maximum principles for vector problems.

Considering the scalar case, we restrict ourselves to formulations and some remarks, whereas detailed proofs are given in the vector case.

2.1. L^2-estimate for homogenization. In homogenization theory, the following elliptic scalar equation in $\mathbb{R}^d$ ($d \geqslant 2$) is well studied (cf., for example, [1]–[3]):

$$\begin{aligned} &u_\varepsilon \in H^1(\mathbb{R}^d), \quad A^\varepsilon u_\varepsilon + u_\varepsilon = f, \quad f \in L^2(\mathbb{R}^d), \\ &A^\varepsilon = -\operatorname{div} a^\varepsilon(x)\nabla, \quad a^\varepsilon(x) = a(\varepsilon^{-1}x), \end{aligned} \tag{2.1}$$

where $a(x) = \{a_{jk}(x)\}_{j,k=1}^d$ is a measurable periodic matrix with real entries and the periodicity cell is the unit cube $\square = [-\frac{1}{2}, \frac{1}{2})^d$. While dealing with the scalar case, all the function spaces used in this section, for example, $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ are assumed (only for the sake of simplicity) to consist of real-valued functions.

Let the ellipticity and boundedness conditions hold:

$$\lambda|\xi|^2 \leqslant a\xi\cdot\xi, \quad a\xi\cdot\eta \leqslant \lambda^{-1}|\xi|\,|\eta| \quad \forall \xi, \eta \in \mathbb{R}^d \tag{2.2}$$

for some $\lambda > 0$. The operator A^ε depends on the parameter $\varepsilon \in (0, 1]$ and has rapidly oscillating ε-periodic coefficients for small ε.

With Equation (2.1) we associate the homogenized equation

$$u_0 \in H^1(\mathbb{R}^d), \quad A_0 u_0 + u_0 = -\operatorname{div} a^0 \nabla u_0 + u_0 = f, \tag{2.3}$$

where a^0 is a constant positive definite matrix. The matrix a^0 can be found by the known procedure in terms of solutions to auxiliary problems on the periodicity cell $\square$ (cf. (2.12) and (2.13)).

Solutions to Equations (2.1) and (2.3) are understood in the sense of the theory of distributions on $\mathbb{R}^d$, i.e., in the sense of integral identities on smooth compactly supported functions. For example, for Equation (2.1) we consider the integral identity

$$\int\limits_{\mathbb{R}^d} [a^\varepsilon(x)\nabla u_\varepsilon \nabla\cdot\varphi + u^\varepsilon\varphi]\, dx = \int\limits_{\mathbb{R}^d} f\varphi\, dx, \quad \varphi\in C_0^\infty(\mathbb{R}^d). \tag{2.4}$$

By closure, for a test function we can take any function $\varphi\in H^1(\mathbb{R}^d)$, in particular, the solution itself u_ε. Hence we have the energy equality and ε-uniform energy estimate

$$\lambda\|\nabla u_\varepsilon\|^2 + \|u_\varepsilon\|^2 \overset{(2.2)_1}{\leqslant} (a^\varepsilon\nabla u_\varepsilon, \nabla u_\varepsilon) + (u_\varepsilon, u_\varepsilon) = (f, u_\varepsilon) \leqslant \|f\|\|u_\varepsilon\| \leqslant \|f\|^2. \tag{2.5}$$

Therefore, the family u_ε is bounded in $H^1(\mathbb{R}^d)$ and, consequently, weakly compact in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. Hence we can talk about the limit function for the family u_ε. It turns out that the limit function is a solution to the homogenized equation. Various methods for proving this fact are known.

Throughout the paper, we use the simplified notation for the norm and inner product in $L^2(\mathbb{R}^d)$:

$$\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}, \quad (\,\cdot\,,\,\cdot\,) = (\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}. \tag{2.6}$$

We often use the same notation in the scalar and vector cases (for example, for the spaces $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)^d$).

The solvability of Equations (2.1) and (2.3) is proved by using the Lax–Milgram theorem. In the language of operators this means the following. By the condition (2.2), the differential operators A_ε and A_0 realize bounded mappings from $H^1(\mathbb{R}^d)$ to the dual $H^{-1}(\mathbb{R}^d) = (H^1(\mathbb{R}^d))^*$. Namely, with any $u\in H^1(\mathbb{R}^d)$ we can associate functionals $A_\varepsilon u$ and $A_0 u$ such that

$$\langle A_\varepsilon u, \varphi\rangle = (a^\varepsilon\nabla u, \nabla\varphi), \quad \varphi\in H^1(\mathbb{R}^d),$$
$$\langle A_0 u, \varphi\rangle = (a^0\nabla u, \nabla\varphi), \quad \varphi\in H^1(\mathbb{R}^d),$$

where $\langle f, \varphi\rangle$ denotes the value of a functional $f\in H^{-1}(\mathbb{R}^d)$ on a function $\varphi\in H^1(\mathbb{R}^d)$.

The first inequality in (2.5) and a similar inequality for the homogenized equation lead to the coercitivity property

$$\langle (A_\varepsilon+1)\varphi, \varphi\rangle \geqslant \lambda\|\varphi\|^2_{H^1(\mathbb{R}^d)},$$
$$\langle (A_0+1)\varphi, \varphi\rangle \geqslant \lambda\|\varphi\|^2_{H^1(\mathbb{R}^d)}$$

for any $\varphi\in H^1(\mathbb{R}^d)$. By the Lax–Milgram theorem, there exist resolvents

$$\begin{aligned}(A_\varepsilon{+}1)^{-1} &: H^{-1}(\mathbb{R}^d)\to H^1(\mathbb{R}^d),\\ (A_0{+}1)^{-1} &: H^{-1}(\mathbb{R}^d)\to H^1(\mathbb{R}^d).\end{aligned} \tag{2.7}$$

In particular, by the embeddings $H^1(\mathbb{R}^d)\subset L^2(\mathbb{R}^d)\subset H^{-1}(\mathbb{R}^d)$, these resolvents can be regarded as operators in $L^2(\mathbb{R}^d)$.

The result about the strong convergence of solutions $u_\varepsilon\to u_0$ in $L^2(\mathbb{R}^d)$ is well known. In the language of operators this means the strong resolvent convergence $(A_\varepsilon{+}1)^{-1}\to(A_0{+}1)^{-1}$

in $L^2(\mathbb{R}^d)$. The last convergence can be strengthened to the uniform resolvent convergence $\|(A_\varepsilon+1)^{-1}-(A_0+1)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}\to 0$ (cf., for example, [15]). For the uniform convergence rate the following exact estimate with respect to the parameter ε was proved in [4, 6, 7]:

$$\|(A_\varepsilon + 1)^{-1} - (A_0 + 1)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant c\varepsilon, \quad c = \mathrm{const}\,(d, \lambda), \tag{2.8}$$

which implies the following estimate for solutions to Equations (2.1) and (2.3):

$$\|u_\varepsilon - u_0\| \leqslant c\varepsilon\|f\| \tag{2.9}$$

with the same constant depending only on the dimension d and the ellipticity constant λ in (2.2).

To prove the estimate (2.8), a spectral approach was used in [4]. For the same purpose another method (possibly, simpler from the conceptual point of view) was proposed in [6] and [7]: the *shift method* and its modification based on the notion of a generalized shift. We note that a generalized shift is sometimes understood as the Steklov average (or smoothing) (cf. details in Subsection 2.4).

2.2. H^1-estimate for homogenization. According to (2.8), the resolvent $(A_0 + 1)^{-1}$ of the homogenized operator is taken for the zeroth approximation of the resolvent $(A_\varepsilon + 1)^{-1}$ of the original operator in the operator L^2-norm. If the resolvent $(A_\varepsilon + 1)^{-1}$ is regarded as an operator from $L^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$, then for its approximation we should take the sum of the constructed zeroth approximation and corrector, i.e., $(A_0 + 1)^{-1} + \mathscr{K}_\varepsilon$; moreover,

$$\|(A_\varepsilon + 1)^{-1} - (A_0 + 1)^{-1} - \mathscr{K}_\varepsilon\|_{L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant c\varepsilon, \quad c = \mathrm{const}\,(d, \lambda). \tag{2.10}$$

The operator $\mathscr{K}_\varepsilon : L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ is defined by

$$\mathscr{K}_\varepsilon f = \varepsilon N^\varepsilon \cdot \nabla(A_0 + 1)^{-1} f, \quad N^\varepsilon(x) = N(\varepsilon^{-1}x), \tag{2.11}$$

where $N(y) = \{N^j(y)\}_{j=1}^d$ is the periodic vector composed of the solutions to the problem on the cell

$$N^j \in H^1_{\mathrm{per}}(\square), \quad \mathrm{div}_y a(y)(e^j + \nabla_y N^j) = 0, \quad j = 1, \dots, d. \tag{2.12}$$

Here, $e^1, \dots, e^d$ is the canonical basis for the space $\mathbb{R}^d$ and $H^1_{\mathrm{per}}(\square)$ is the Sobolev space of 1-periodic functions with zero mean over the cell $\square$, equipped with the norm

$$\|\varphi\|_{H^1_{\mathrm{per}}(\square)} = (\nabla\varphi, \nabla\varphi)^{1/2}_{L^2(\square)}.$$

Using the solutions to the problem on the cell, we can introduce the homogenized matrix,

$$a^0 e^j = \langle a(e^j + \nabla_y N^j)\rangle, \quad j = 1, \dots, d, \tag{2.13}$$

where

$$\langle\cdot\rangle = \int_\square \cdot\, dy$$

denotes the mean value over the cell.

A solution to the problem (2.12) is understood in the sense of integral identity on smooth periodic functions

$$\langle a(e^j + \nabla_y N^j) \cdot \nabla\varphi\rangle = 0, \quad \varphi \in C^\infty_{\mathrm{per}}(\square) \tag{2.14}$$

which can be extended by closure to any test function in $H^1_{\mathrm{per}}(\square)$.

The solvability of the problem (2.12) is again established by the Lax–Milgram theorem applied to the operator $\mathscr{A} = \mathrm{div}_y(a(y)\nabla_y) : H^1_{\mathrm{per}}(\square) \to (H^1_{\mathrm{per}}(\square))^*$ acting from the space $H^1_{\mathrm{per}}(\square)$ to its dual by the identity (2.14). We preliminarily write the problem (2.12) as the operator equation

$$\mathscr{A} N^j = F^j, \quad F^j = -\mathrm{div}_y(a(y)e^j), \tag{2.15}$$

where $F^j \in (H^1_{\mathrm{per}}(\square))^*$ in view of the boundedness of the matrix a.

The operator (2.11) possess the property

$$\mathscr{K}_\varepsilon \text{ is a bounded operator from } L^2(\mathbb{R}^d) \text{ to } H^1(\mathbb{R}^d), \tag{2.16}$$

and the following estimate holds:

$$\|\mathscr{K}_\varepsilon\|_{L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant c, \quad c = \mathrm{const}\,(d,\lambda). \tag{2.17}$$

This fact is not obvious. Actually, it is a consequence of the L^∞-boundedness of solutions to the problem on the cell, which, in turn, is guaranteed by the generalized maximum principle valid for scalar, but not vector equations.

In the proof of the estimate (2.17), an important role is played by the elliptic estimate for solutions to the homogenized equation

$$\|u_0\|_{H^2(\mathbb{R}^d)} \leqslant c\|f\|, \quad c = \mathrm{const}\,(\lambda), \tag{2.18}$$

which can be easily proved by using the Fourier transform since the matrix $a^0 > 0$ is constant. The estimate (2.10) can be written in terms of the solutions to Equations (2.1), (2.3) and the corresponding corrector:

$$\|u_\varepsilon - u_0 - \varepsilon N^\varepsilon \cdot \nabla u_0\|_{H^1(\mathbb{R}^d)} \leqslant c\varepsilon\|f\|, \quad c = \mathrm{const}\,(d,\lambda). \tag{2.19}$$

The estimate (2.19) with majorant as above was first obtained in [7] and [9] with the help of the shift method or its modification based on Steklov averages.

Remarks. 1. In [7] and [9], the following estimate with a Steklov smoothed corrector was preliminarily proved:

$$\|u_\varepsilon - u_0 - \varepsilon N^\varepsilon \cdot \nabla S^\varepsilon u_0\|_{H^1(\mathbb{R}^d)} \leqslant c\varepsilon\|f\|, \quad c = \mathrm{const}\,(d,\lambda) \tag{2.20}$$

and only at the final stage it was proved that the smoothing operator S^ε can be omitted in this estimate. A counterpart of the estimate (2.20) in the vector case will be proved below. The fact that the smoothing operator S^ε is involved in the corrector makes it possible to overcome technical difficulties in estimating the residual in the equation for the first approximation. Note that such difficulties arise if the problem (2.1) is considered under minimal regularity assumptions.

2. Efficiency of the use of the smoothing operator S^ε in homogenization problems was first observed in [6]–[10].

3. In previous works, H^1-estimates for homogenization of Equation (2.1) have the form

$$\|u_\varepsilon - u_0 - \varepsilon N^\varepsilon \cdot \nabla u_0\|_{H^1(\mathbb{R}^d)} \leqslant C_0\varepsilon \tag{2.21}$$

under high regularity assumptions on the function f or the coefficient matrix a; moreover, the constant C_0 can depend on higher order Sobolev norms $\|f\|_{H^k}$ and the smoothness characteristics of the matrix a like the norms $\|a\|_{C^k}$ with rather large k. Under such a dependence of the constant C_0 on the data, the estimate (2.21) cannot be interpreted in the operator form.

3 Homogenization of Vector Equation

3.1. The original equation. We consider the following problem for vector-valued functions:

$$\begin{aligned} &u_\varepsilon \in \mathscr{H}, \quad A^\varepsilon u_\varepsilon + u_\varepsilon = f, \quad f \in \mathscr{H}^*, \\ &A^\varepsilon = D^* a^\varepsilon(x) D, \quad a^\varepsilon(x) = a(\varepsilon^{-1}x), \end{aligned} \tag{3.1}$$

where the ε-periodic coefficients are complex, $D = -i\nabla$ $(i^2 = -1)$, $\mathscr{H} = H^1(\mathbb{R}^d, \mathbb{C}^n) = H^1(\mathbb{R}^d)^n$, and $\mathscr{H}^*$ is the dual. We assume that $a(x) = \{a_{jk}(x)\}_{j,k=1}^d$ is a measurable bounded 1-periodic block $nd \times nd$-matrix with complex entries such that each block $a_{jk}(x)$ is an $n \times n$-matrix, $n \geqslant 1$. Any fourth order tensor acting as a linear operator in the space of complex $n \times d$-matrices can be written in this form. Denoting by $\mathbb{L}(E)$ the set of linear operators in the vector space E, we can write this assumption on $a(x)$ as

$$a \in L^\infty_{\text{per}}(\square, \mathbb{L}(\mathbb{C}^{n\times d})) = L^\infty_{\text{per}}(\square)^{n\times d\times n\times d}. \tag{3.2}$$

Equation (3.1) involves the gradient ∇u_ε of the vector-valued function u_ε which can be represented as an $(n\times d)$-matrix whose columns are n-dimensional vectors $\{\partial_j u_\varepsilon\}$, $\partial_j = \partial/\partial x_j$, $j = 1, \dots, d$. The action of the operator A^ε on the vector-valued function u_ε is defined by

$$A^\varepsilon u_\varepsilon = -\sum_{j,k=1}^d \frac{\partial}{\partial x_j}\Big(a_{jk}(\varepsilon^{-1}x)\frac{\partial u_\varepsilon}{\partial x_k}\Big).$$

Thus, we obtain a formula identical to that in the scalar case $n = 1$ (cf. (2.1)).

By definition, a solution to Equation (3.1) satisfies the integral identity

$$(a^\varepsilon D u_\varepsilon, D\varphi) + (u_\varepsilon, \varphi) = \langle f, \varphi\rangle, \quad \varphi \in \mathscr{H}. \tag{3.3}$$

Setting $\varphi = u_\varepsilon$ in (3.3), we obtain the energy equality

$$(a^\varepsilon D u_\varepsilon, D u_\varepsilon) + \|u_\varepsilon\|^2 = \langle f, u_\varepsilon\rangle,$$

where $\langle f, \varphi\rangle$ denotes the value of a functional $f \in \mathscr{H}^*$ on an element $\varphi \in \mathscr{H}$. Hereinafter, $(\, , \,)$ and $\|\cdot\|$ denote the inner product and norm in the space $L^2(\mathbb{R}^d)$. Unlike (2.6), this notation now deals with complex-valued functions most often of dimension n or nd.

We formulate a condition for a solution to the problem (3.1) to exist and be ε-uniformly bounded in the energy norm. By (3.2), the bounded operator $A^\varepsilon : \mathscr{H} \to \mathscr{H}^*$ sends a function $u \in \mathscr{H}$ to the functional $A^\varepsilon u$ defined by

$$\langle A^\varepsilon u, \varphi\rangle = (a^\varepsilon D u_\varepsilon, D\varphi), \quad \varphi \in \mathscr{H}.$$

We assume that there is $c_0 > 0$ such that

$$\operatorname{Re}(aDu, Du) \geqslant c_0\|Du\|^2 \quad \forall u \in C_0^\infty(\mathbb{R}^d)^n. \tag{3.4}$$

By closure, this inequality remains valid for any function in $\mathscr{H}$. A tensor (matrix) a satisfying the condition (3.4) is referred to as *elliptic*.

Applying homothety to the integrals in (3.4), we obtain a similar inequality with an ε-periodic matrix a^ε for all $\varepsilon \in (0,1]$ and the same constant, i.e.,

$$\operatorname{Re}(a^\varepsilon Du, Du) \geqslant c_0 \|Du\|^2 \quad \forall u \in \mathscr{H}, \tag{3.5}$$

which implies that the operator of the problem (3.1) is uniformly coercive, i.e.,

$$\operatorname{Re}\langle (A^\varepsilon + 1)u, u\rangle \geqslant c\|u\|^2_{\mathscr{H}},$$

where $c = \min(1, c_0)$. By the Lax–Milgram theorem, Equation (3.1) has a unique solution; moreover the following ε-uniform estimate holds:

$$\|u_\varepsilon\|_{\mathscr{H}} \leqslant \frac{1}{c}\|f\|_{\mathscr{H}^*}$$

or, in the operator form,

$$\|(A^\varepsilon + 1)^{-1}\|_{\mathscr{H}^* \to \mathscr{H}} \leqslant \frac{1}{c}. \tag{3.6}$$

Thus, the problem (3.1) with the bounded elliptic tensor $a(x)$ (cf. (3.2) and (3.4)) is well posed and a question about homogenization of this problem naturally arises.

The ellipticity condition guaranteeing the homogenization procedure was expressed in terms of the coercitivity estimate of type (3.4) already in the survey [14] devoted to the G-convergence of differential operators, where, in particular, homogenization of elliptic operators of an arbitrary even order with rapidly oscillating coefficients was discussed. This approach was further applied to higher order operators (cf., for example, [16] and [5]). For matrix-vector operators arising in the study of systems of equations this approach is also efficient.

3.2. Problem on the cell. In the set of smooth 1-periodic vector-valued functions of class $C^\infty_{\text{per}}(\square, \mathbb{C}^n) = C^\infty_{\text{per}}(\square)^n$ with zero mean over the cell $\square$, we introduce the norm

$$\left(\int_\square |Du|^2\,dx\right)^{1/2},$$

which is possible because of the Poincaré inequality. We denote by $\mathscr{H}_\square$ the completion of this set in this norm. The inequality (3.4) for a periodic tensor $a(x)$ and compactly supported test functions implies a similar inequality for periodic functions on the cell $\square$

$$\operatorname{Re}(aDv, Dv)_\square \geqslant c_0\|Dv\|^2_\square \quad \forall v \in C^\infty_{\text{per}}(\square)^n, \tag{3.7}$$

where $(\cdot,\cdot)_\square$ and $\|\cdot\|_\square$ denote the inner product and norm in the space $L^2(\square)$ of complex-valued functions most often of dimension n or nd. The inequality (3.7) will be proved below. By closure, the inequality (3.7) holds for all functions $u \in \mathscr{H}_\square$ and implies that $\mathscr{A} = D^*a(x)D$ is a coercive operator from $\mathscr{H}_\square$ to the dual $(\mathscr{H}_\square)^*$. The operator $\mathscr{A} = D^*a(x)D$ is bounded in view of (3.2).

For vector-valued functions N_ξ we consider the following problem on the cell:

$$N_\xi \in \mathscr{H}_\square, \quad D^*a(x)(DN_\xi + \xi) = 0, \tag{3.8}$$

where $\xi = \{\xi_{jk}\} \in \mathbb{C}^{n\times d}$ is a fixed matrix playing the role of a parameter. By definition, the solution satisfies the integral identity on periodic functions

$$(a(DN_\xi + \xi), D\varphi)_\square = 0, \quad \varphi \in \mathscr{H}_\square. \tag{3.9}$$

The problem (3.8) can be written as $\mathscr{A}N_\xi = F$ with $F = -D^*(a(x)\xi) \in (\mathscr{H}_\square)^*$. By the properties of the operator $\mathscr{A}$, this equation is uniquely solvable in view of the Lax–Milgram theorem and the solution satisfies the estimate

$$\|N_\xi\|_{\mathscr{H}_\square} \leqslant c\|\xi\|_{\mathbb{C}^{n\times d}}, \tag{3.10}$$

where the constant depends on the norm $\|a\|_{L^\infty(\square)}$ and the ellipticity constant c_0 in (3.4). One can observe a certain similarity in obtaining the solvability result for Equation (3.1) and the related problem (3.8) on the cell.

Since the problem (3.8) is linear, from the expansion $\xi = \sum_{j,k} e^{jk}\xi_{jk}$, of a constant $(n \times d)$-matrix $\xi = \{\xi_{jk}\}$ in the basis matrices e^{jk} in $\mathbb{C}^{n\times d}$ we obtain the representation

$$N_\xi(x) = \sum_{j,k} N^{jk}(x)\xi_{jk}, \tag{3.11}$$

where N^{jk} is the solution to the problem (3.8) with $\xi = e^{jk}$.

To conclude the subsection, we justify the key inequality (3.7) for the problem (3.8), We use arguments of [14], where a similar situation was considered.

Lemma 3.1. *If $a(x)$ is a bounded elliptic tensor, then* (3.4) *implies* (3.7).

Into the inequality (3.5) obtained from (3.4), we substitute a compactly supported function $\psi_\varepsilon(x) = \varepsilon v(\varepsilon^{-1}x)\varphi(x)$, $v \in C^\infty_{\text{per}}(\square)^n$, $\varphi \in C_0^\infty(\mathbb{R}^d)$, such that

$$D\psi_\varepsilon(x) = (Dv)(\varepsilon^{-1}x)\varphi(x) + \varepsilon v(\varepsilon^{-1}x) \otimes D\varphi(x),$$

where the tensor product of vectors is of order $O(\varepsilon)$. As a result, we have

$$\operatorname{Re}(a^\varepsilon D\psi_\varepsilon, D\psi_\varepsilon) \geqslant c_0\|D\psi_\varepsilon\|^2.$$

Letting $\varepsilon \to 0$, we find

$$\operatorname{Re}(aDv, Dv)_\square \int\limits_{\mathbb{R}^d} |\varphi(x)|^2\,dx \geqslant c_0\|Dv\|^2_\square \int\limits_{\mathbb{R}^d} |\varphi(x)|^2\,dx$$

in view of the mean value property of periodic functions. Reducing by a common factor, we obtain (3.7). We recall the *mean value property* of periodic functions (cf. [3, Chapter I, Section 1]): if $g \in L^p(\square)$, $p \geqslant 1$, then $g(\varepsilon^{-1}x) \rightharpoonup \langle g\rangle$ in $L^p_{\text{loc}}(\mathbb{R}^d)$ as $\varepsilon \to 0$, where

$$\langle g\rangle = \int\limits_\square g\,dx.$$

The symbol $\rightharpoonup$ means the weak convergence in the corresponding space. In particular, by the mean value property,

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^d} g(\varepsilon^{-1}x)\varphi(x)\,dx = \langle g\rangle \int\limits_{\mathbb{R}^d} \varphi(x)\,dx, \quad \varphi \in C_0^\infty(\mathbb{R}^d). \tag{3.12}$$

3.3. Homogenized equation. We introduce the homogenized tensor a^0 by

$$a^0\xi = \int_{\square} a(x)(DN_\xi(x) + \xi)\,dx = \langle a(\cdot)(DN_\xi(\cdot) + \xi)\rangle \quad \forall \xi \in \mathbb{C}^{n\times d} \tag{3.13}$$

where N_ξ is a solution to the problem (3.8).

Lemma 3.2. *The tensor a^0 is elliptic of class* (3.4).

Proof. We need to verify the inequality

$$\operatorname{Re}(a^0 Du, Du) \geqslant c_0\|Du\|^2 \quad \forall u \in C_0^\infty(\mathbb{R}^d)^n. \tag{3.14}$$

For this purpose we substitute the function

$$\psi_\varepsilon(x) = u(x) + \varepsilon \sum_{j,k} N^{jk}(\varepsilon^{-1}x) D_k u_j(x)$$

into (3.5), where N^{jk} is the same solution to the problem on the cell as in (3.11) and u_j, D_k are components of the vectors u, D respectively. Then we obtain the inequality

$$\operatorname{Re}(a^\varepsilon D\psi_\varepsilon, D\psi_\varepsilon) \geqslant c_0\|D\psi_\varepsilon\|^2 \tag{3.15}$$

and pass to the limit as $\varepsilon \to 0$. We note that

$$\begin{aligned} D\psi_\varepsilon &= Du + \sum_{j,k}(DN^{jk})(\varepsilon^{-1}x)D_k u_j + \varepsilon\sum_{j,k} N^{jk}(\varepsilon^{-1}x)\otimes D(D_k u_j) \\ &= \sum_{j,k}\Big(e^{jk} + (DN^{jk})(\varepsilon^{-1}x)\Big) D_k u_j + \varepsilon \sum_{j,k} N^{jk}(\varepsilon^{-1}x)\otimes D(D_k u_j). \end{aligned} \tag{3.16}$$

Therefore,

$$\begin{aligned} &\lim_{\varepsilon\to 0}(a^\varepsilon D\psi_\varepsilon, D\psi_\varepsilon) \\ &= \lim_{\varepsilon\to 0}\sum_{j,k,l,m}(a^\varepsilon(e^{jk} + (DN^{jk})(\varepsilon^{-1}x))D_k u_j, (e^{lm} + (DN^{lm})(\varepsilon^{-1}x))D_m u_l) \\ &= \sum_{j,k,l,m}(a(e^{jk} + DN^{jk}), (e^{lm} + DN^{lm}))_\square \int_{\mathbb{R}^d} D_k u_j \overline{D_m u_l}\,dx, \end{aligned} \tag{3.17}$$

where we used the mean value property (3.12) at each step.

We recall that N^{jk} is the solution to the problem (3.8) on the cell with $\xi = e^{jk}$

$$(a(e^{jk} + DN^{jk}), DN^{lm})_\square = 0$$

in view of the integral identity (3.9) with the test function $\varphi = N^{lm}$. Hence

$$\begin{aligned} (a(e^{jk} + DN^{jk}), (e^{lm} + DN^{lm}))_\square &= (a(e^{jk} + DN^{jk}), e^{lm})_\square \\ &= \langle a(e^{jk} + DN^{jk})\rangle \cdot e^{lm} = a^0 e^{jk}\cdot e^{lm}, \end{aligned} \tag{3.18}$$

where the definition (3.13) of the homogenized tensor is taken into account.

From (3.17) and (3.18) it follows that

$$\lim_{\varepsilon\to 0} \operatorname{Re}(a^\varepsilon D\psi_\varepsilon, D\psi_\varepsilon) = \sum_{j,k,l,m} \operatorname{Re} \int_{\mathbb{R}^d} a^0 e^{jk} D_k u_j \cdot e^{lm} \overline{D_m u_l}\, dx$$

$$= \operatorname{Re} \int_{\mathbb{R}^d} a^0 \Bigl(\sum_{j,k} e^{jk} D_k u_j \Bigr) \cdot \Bigl(\sum_{l,m} e^{lm} \overline{D_m u_l} \Bigr) dx = \operatorname{Re}(a^0 Du, Du). \tag{3.19}$$

Hence we obtain the limit of the left-hand side of (3.15).

Now, we pass to the limit on the right-hand side of (3.15). The structure of $D\psi_\varepsilon$ is such that (cf. (3.16)) $D\psi_\varepsilon \rightharpoonup Du$ in $L^2_{\text{loc}}(\mathbb{R}^d)$ in view of the mean value property (3.12). By the weak lower semicontinuity property,

$$\liminf_{\varepsilon\to 0} \|D\psi_\varepsilon\|^2 \geqslant \|Du\|^2. \tag{3.20}$$

Thus, (3.15), (3.19), and (3.20) imply (3.14). The lemma is proved. □

The homogenized problem for the problem (3.1) has the form

$$\begin{aligned} &u \in \mathscr{H}, \quad A^0 u + u = f, \quad f \in \mathscr{H}^*, \\ &A^0 = D^* a^0 D, \end{aligned} \tag{3.21}$$

where the tensor a^0 is defined by (3.13). Since the constant tensor a^0 is elliptic of class (3.1), this problem has a solution and an operator estimate of type (3.6) holds for the resolvent $(A^0+1)^{-1}$.

If $f \in L^2(\mathbb{R}^d)^n$, then the solution to the homogenized problem belongs to $H^2(\mathbb{R}^d)^n$ and

$$\|u\|_{H^2(\mathbb{R}^d)^n} \leqslant c_1 \|f\|_{L^2(\mathbb{R}^d)^n}, \quad c_1 = \text{const}\,(c_0, d), \tag{3.22}$$

which can be obtained by differentiating the equation in (3.21).

3.4. First approximation. For an approximation to the solution to the problem (3.1) in the energy norm we take

$$v_\varepsilon(x) = S^\varepsilon u(x) + \varepsilon \sum_{j,k} N^{jk}(\varepsilon^{-1}x) D_k S^\varepsilon u_j(x) = S^\varepsilon u(x) + K_\varepsilon(x), \tag{3.23}$$

where u is the solution to the homogenized problem, N^{jk} is the solution to the periodic problem (3.8) with $\xi = e^{jk}$, and

$$S^\varepsilon \varphi(x) = \int_{\square} \varphi(x - \varepsilon\omega)\, d\omega$$

is the Steklov average (or smoothing operator) defined for $\varphi \in L^1_{\text{loc}}(\mathbb{R}^d)$. It is natural to call a term $K_\varepsilon(x)$ in (3.23) the *corrector*.

The following property (first noted and proved in [7]) of the Steklov smoothing operator played a crucial role in the choice of the first approximation in the form (3.23).

Lemma 3.3. *If* $\varphi \in L^2(\mathbb{R}^d)$, $b \in L^2_{\text{per}}(\square)$, *and* $b^\varepsilon(x) = b(\varepsilon^{-1}x)$, *then* $b^\varepsilon S^\varepsilon \varphi \in L^2(\mathbb{R}^d)$ *and*

$$\|b^\varepsilon S^\varepsilon \varphi\|_{L^2(\mathbb{R}^d)} \leqslant \|b\|_{L^2(\square)} \|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{3.24}$$

Based on this property, it is easy to see that the most problem term $K_\varepsilon(x)$ in (3.23) belongs to the space $\mathscr{H}$ and the following estimates hold:

$$\begin{aligned} &\|K_\varepsilon\|_{L^2(\mathbb{R}^d)} \leqslant c\varepsilon\|u\|_{H^1(\mathbb{R}^d)}, \\ &\|DK_\varepsilon\|_{L^2(\mathbb{R}^d)} \leqslant c\|u\|_{H^2(\mathbb{R}^d)}, \end{aligned} \tag{3.25}$$

where the constant depends only on the norms $\|N^{jk}\|_{\mathscr{H}_\square}$. Without the smoothing operator in (3.23), it is impossible to guarantee that v_ε belongs to the energy space $\mathscr{H}$; moreover, both estimates in (3.25) become questionable.

Theorem 3.1. *Let u_ε be a solution to the problem* (3.1) *with a bounded elliptic tensor $a(x)$ and $f \in L^2(\mathbb{R}^d)^n$, and let v_ε be defined by* (3.23)*. Then*

$$\|u_\varepsilon - v_\varepsilon\|_{L^2(\mathbb{R}^d)^n} + \|D(u_\varepsilon - v_\varepsilon)\|_{L^2(\mathbb{R}^d)^{n\times d}} \leqslant \varepsilon C\|f\|_{L^2(\mathbb{R}^d)^n}, \tag{3.26}$$

where the constant C depends on the L^∞-norm of the tensor a (cf. (3.2)*), the ellipticity constant c_0 in* (3.4)*, and the dimension d.*

For the first approximation to u_ε we can take a slightly modified function v_ε with smoothing only in the corrector:

$$\widehat{v}_\varepsilon(x) = u(x) + \varepsilon\sum_{j,k} N^{jk}(\varepsilon^{-1}x)D_k S^\varepsilon u_j(x) = u(x) + K_\varepsilon(x). \tag{3.27}$$

Theorem 3.2. *Under the assumptions of Theorem* 3.1*,*

$$\|u_\varepsilon - u\|_{L^2(\mathbb{R}^d)^n} \leqslant \varepsilon C\|f\|_{L^2(\mathbb{R}^d)^n}, \tag{3.28}$$

$$\|u_\varepsilon - \widehat{v}_\varepsilon\|_{L^2(\mathbb{R}^d)^n} + \|D(u_\varepsilon - \widehat{v}_\varepsilon)\|_{L^2(\mathbb{R}^d)^{n\times d}} \leqslant \varepsilon C\|f\|_{L^2(\mathbb{R}^d)^n}, \tag{3.29}$$

where C is a constant of the same type as in (3.26)*.*

Theorems 3.1 and 3.2 are proved in the following section.

To obtain homogenization estimates, we use, in addition to (3.24), other properties of the Steklov smoothing operator:

$$\|S^\varepsilon\varphi\|_{L^2(\mathbb{R}^d)} \leqslant \|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{3.30}$$

$$\|S^\varepsilon\varphi - \varphi\|_{L^2(\mathbb{R}^d)} \leqslant (\sqrt{d}/2)\varepsilon\|\nabla\varphi\|_{L^2(\mathbb{R}^d)}, \tag{3.31}$$

$$\|S^\varepsilon\varphi - \varphi\|_{H^{-1}(\mathbb{R}^d)} \leqslant (\sqrt{d}/2)\varepsilon\|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{3.32}$$

The proof of (3.30)–(3.32) can be found, for example, in [7] or [5].

3.5. Remarks and examples. In a close setting, operator estimates for systems of elliptic equations were studied in [17, 18]. In [17], the coefficients are assumed to be sufficiently smooth, which makes it possible to apply the first approximation method in the classical form [1]–[3] without using any shift or smoothing. In [18], the spectral method in a version going back to [4] was used and, respectively, the smoothing operator naturally appears in the H^1-estimates, but of different kind than that in (3.29). In [17, 18], the matrix differential operator is allowed to involve lower order terms; moreover, in [18], the coefficients of lower order terms can be unbounded. Such a generalization can be also handled by our method: in [16, 5], the shift method was applied to the study of operators with lower order terms. In [17], locally periodic coefficients were admitted. The shift method is also adapted for operators with such coefficients (we refer to [13, 5] for homogenization estimates for operators with multi-scale coefficients).

In this paper, we do not pursue the goal of covering the most general operators, but we focus on operators of matrix-vector structure.

Finally, we give examples of operators satisfying the condition (3.4). First of all, this condition is satisfied by the selfadjoint operators studied in [4] and their nonselfadjoint generalizations.

The operator homogenization estimates in [4] are proved for second order factorized operators written as the products of mutually adjoint first order differential operators, for example,

$$A = \mathscr{X}^* \mathscr{X}, \quad \mathscr{X} = h\, b(D), \quad b(D) = \sum_{j=1}^{d} D_j b_j, \tag{3.33}$$

where b_j are $(m{\times}n)$-matrices with constant coefficients, $m \geqslant n$, rank $b(\xi) = n$, $0 \neq \xi \in \mathbb{R}^d$, and the 1-periodic $(m{\times}m)$-matrix $h = h(x)$ is bounded and boundedly invertible.

For many operators in mathematical physics a factorization of type (3.3) is already present from the very beginning or can be "imported" by some artificial tricks (cf. [4, 17]). Differential operators of the form (3.33) and similar ones were studied in [19, 20].

An operator of the form (3.33) can be written as

$$A = \sum_{j,k=1}^{d} D_j a_{jk} D_k, \quad a_{jk} = b_j^*\, g\, b_k, \tag{3.34}$$

where $g = h^* h$. One can take a nonfactorized operator of the form (3.34) with the same matrices b_j as above, but assuming only that $g \in L^\infty_{\text{per}}(\square, \mathbb{C}^{n\times n})$ and Re g is uniformly positive definite, instead of factorization of $g = g(x)$.

4 Proof of Estimates in Homogenization

We divide the proof of the error estimates in homogenization into several steps.

1° First of all, we note that the function $S^\varepsilon u$, in (3.23) is a solution to the homogenized equation with smoothed right-hand side

$$(A^0 + 1) S^\varepsilon u = S^\varepsilon f. \tag{4.1}$$

We find a residual of the approximation (3.23) in Equation (3.1):

$$\begin{aligned} A^\varepsilon (v_\varepsilon - u_\varepsilon) + (v_\varepsilon - u_\varepsilon) &= A^\varepsilon v_\varepsilon + v_\varepsilon - (A^\varepsilon u_\varepsilon + u_\varepsilon) \\ &= A^\varepsilon v_\varepsilon + v_\varepsilon - f = A^\varepsilon v_\varepsilon + v_\varepsilon - S^\varepsilon f + (S^\varepsilon f - f) \\ &\overset{(4.1)}{=} A^\varepsilon v_\varepsilon + v_\varepsilon - (A^0 + 1) S^\varepsilon u + (S^\varepsilon f - f) \\ &= A^\varepsilon v_\varepsilon - A^0 S^\varepsilon u + (v_\varepsilon - S^\varepsilon u) + (S^\varepsilon f - f). \end{aligned}$$

Recalling the structure of the function v_ε (cf. (3.23)) and the operators A^ε and A^0, we obtain the following equation for the difference $v_\varepsilon - u_\varepsilon$:

$$A^\varepsilon(v_\varepsilon - u_\varepsilon) + (v_\varepsilon - u_\varepsilon) = D^*(a^\varepsilon Dv_\varepsilon - a^0 DS^\varepsilon u) + K_\varepsilon + (S^\varepsilon f - f). \tag{4.2}$$

Since the resolvent $(A^\varepsilon + 1)^{-1}$ satisfies the operator estimate (3.6), we need to show that each component of the right-hand side of this equation is sufficiently small in the $\mathscr{H}^*$-norm. We note that the second and third terms satisfy the estimates

$$\begin{aligned} &\|K_\varepsilon\|_{\mathscr{H}^*} \leqslant C\varepsilon\|f\|_{L^2(\mathbb{R}^d)}, \\ &\|S^\varepsilon f - f\|_{\mathscr{H}^*} \leqslant C\varepsilon\|f\|_{L^2(\mathbb{R}^d)} \end{aligned} \tag{4.3}$$

in view of $(3.25)_1$, (3.10), the energy estimate for the solutions to the homogenized equation, and the properties of the Steklov average (3.32).

We study the first term on the right-hand side of (4.2). As in (3.16), we represent the gradient Dv_ε in the form

$$Dv_\varepsilon = \sum_{j,k}(e^{jk} + (DN^{jk})(\varepsilon^{-1}x))D_k S^\varepsilon u_j + \varepsilon\sum_{j,k} N^{jk}(\varepsilon^{-1}x) \otimes D(D_k S^\varepsilon u_j).$$

Then $a^\varepsilon Dv_\varepsilon - a^0 DS^\varepsilon u$ can be written as

$$a^\varepsilon Dv_\varepsilon - a^0 DS^\varepsilon u = \sum_{j,k} g^{jk}(\varepsilon^{-1}x)D_k S^\varepsilon u_j + \varepsilon a^\varepsilon \sum_{j,k} N^{jk}(\varepsilon^{-1}x) \otimes D(D_k S^\varepsilon u_j), \tag{4.4}$$

where

$$g^{jk}(y) = a(y)(e^{jk} + DN^{jk}(y)) - \langle a(e^{jk} + DN^{jk})\rangle, \tag{4.5}$$

provided that we take into account the definition of the averaged tensor (cf. (3.13)) and the representation of the averaged flux

$$a^0 Dw = a^0\sum_{j,k} e^{jk} D_k w_j = \sum_{j,k}(a^0 e^{jk})D_k w_j = \sum_{j,k}\langle a(e^{jk} + DN^{jk})\rangle D_k w_j, \quad w = S^\varepsilon u.$$

By (4.4), we have

$$D^*(a^\varepsilon Dv_\varepsilon - a^0 DS^\varepsilon u) = D^*\mathscr{S}_1 + D^*\mathscr{S}_2, \tag{4.6}$$

where

$$\mathscr{S}_1 = \sum_{j,k} g^{jk}(\varepsilon^{-1}x)D_k S^\varepsilon u_j, \quad \mathscr{S}_2 = \sum_{j,k} \varepsilon a^\varepsilon N^{jk}(\varepsilon^{-1}x) \otimes D(D_k S^\varepsilon u_j).$$

Our next goal is to prove the estimates

$$\begin{aligned} &\|D^*\mathscr{S}_1\|_{\mathscr{H}^*} \leqslant C\varepsilon\|f\|_{L^2(\mathbb{R}^d)}, \\ &\|D^*\mathscr{S}_2\|_{\mathscr{H}^*} \leqslant C\varepsilon\|f\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{4.7}$$

2° We consider the matrix (4.5) for any fixed pair j, k and the corresponding typical term $g^{jk}(\varepsilon^{-1}x)D_k S^\varepsilon u_j$ in $\mathscr{S}_1$. We simplify the notation as follows:

$$g^{jk}(y) = g(y), \quad g^{jk}(\varepsilon^{-1}x)D_k S^\varepsilon u_j = g(\varepsilon^{-1}x)z(x) = g^\varepsilon(x)z(x). \tag{4.8}$$

We need to estimate the norm $\|D^*(g^\varepsilon z)\|_{\mathscr{H}^*}$. By construction, the $(n\times d)$-matrix $g(y)=\{g_{hs}(y)\}$ possesses the properties $\langle g\rangle=0$ and $(g,D\varphi)_\square=0$ for all $\varphi\in C^\infty_{\mathrm{per}}(\square)^n$. Consequently, each row of the matrix g is a solenoidal vector $g_h=\{g_{hs}\}_{s=1}^d$ in $L^2_{\mathrm{per}}(\square)^d$ with zero mean over the cell, i.e., $\langle g_h\rangle=0$ and $(g,D\psi)_\square=0$ for all $\psi\in C^\infty_{\mathrm{per}}(\square)$. It is known that such a vector can be represented as the divergence of a skew-symmetric matrix (cf. [3, Chapter I, Section 1]). More exactly, there exists a matrix $G^h=\{G^h_{st}\}\in H^1_{\mathrm{per}}(\square,\mathbb{C}^{d\times d})$ such that

(i) $\|G^h\|_{H^1_{\mathrm{per}}(\square)^{d\times d}}\leqslant c\|g_h\|_{L^2_{\mathrm{per}}(\square)^d}$, $c=\mathrm{const}\,(d)$,

(ii) G^h is a skew-symmetric matrix, i.e., $G^h_{st}=-G^h_{ts}$ for all s,t,

(iii) $g_h=D^*G^h$, i.e., $g_{hs}=\sum_t D_tG^h_{st}$, $1\leqslant s\leqslant d$.

The solenoidal vector g_h related with the particular matrix (4.5) satisfies the estimate

$$\|g_h\|_{L^2_{\mathrm{per}}(\square)^d}\leqslant c,\quad c=\mathrm{const}\,(c_0,\|a\|_{L^\infty})$$

in view of (3.10). Therefore, by (i),

$$\|G^h\|_{H^1_{\mathrm{per}}(\square)^{d\times d}}\leqslant c,\quad c=\mathrm{const}\,(c_0,\|a\|_{L^\infty},d). \tag{4.9}$$

By (iii), we can represent each entry of the matrix $g(\varepsilon^{-1}x)z(x)$ as follows:

$$g_{hs}(\varepsilon^{-1}x)z(x)=\sum_t(D_tG^h_{st})(\varepsilon^{-1}x)z(x)=\sum_t D_t(\varepsilon G^h_{st}(\varepsilon^{-1}x)z(x))-\varepsilon\sum_t G^h_{st}(\varepsilon^{-1}x)D_tz(x). \tag{4.10}$$

Hence for any $\varphi\in C_0^\infty(\mathbb{R}^d)^n$

$$\begin{aligned}(g^\varepsilon z,D\varphi)&=\sum_{h,s}(g^\varepsilon_{hs}z,D_s\varphi_h)\\&=\sum_{h,s}\int_{\mathbb{R}^d}\Bigl(\sum_t D_t(\varepsilon G^h_{st}(\varepsilon^{-1}x)z(x))-\varepsilon\sum_t G^h_{st}(\varepsilon^{-1}x)D_tz(x)\Bigr)\overline{D_s\varphi_h(x)}\,dx\\&=\sum_h\int_{\mathbb{R}^d}\sum_{s,t}\varepsilon G^h_{st}(\varepsilon^{-1}x)z(x)\overline{D_tD_s\varphi_h(x)}\,dx-\varepsilon\sum_{h,s,t}\int_{\mathbb{R}^d}G^h_{st}(\varepsilon^{-1}x)D_tz(x)\overline{D_s\varphi_h(x)}\,dx,\end{aligned}$$

where

$$\int_{\mathbb{R}^d}\sum_{s,t}\varepsilon G^h_{st}(\varepsilon^{-1}x)z(x)\overline{D_tD_s\varphi_h(x)}\,dx=0,$$

since the matrix $G^h=\{G^h_{st}\}$ is skew-symmetric and the matrix of second order derivatives $DD\varphi_h=\{D_tD_s\varphi_h\}$ is symmetric. Thus,

$$(g^\varepsilon z,D\varphi)=-\varepsilon\sum_{h,s,t}\int_{\mathbb{R}^d}G^h_{st}(\varepsilon^{-1}x)D_tz(x)\overline{D_s\varphi_h(x)}\,dx,$$

which implies

$$\|D^*(g^\varepsilon z)\|_{\mathscr{H}^*}\leqslant\varepsilon\sum_h\|(G^h)^\varepsilon Dz\|_{L^2(\mathbb{R}^d)^d}. \tag{4.11}$$

Recalling that the smoothing $S^\varepsilon D_k u_j$ of components of the matrix-valued function Du (cf. (4.8)) is taken for z and $u \in H^2(\mathbb{R}^d)^n$, we see that Lemma 3.3 implies

$$\|(G^h)^\varepsilon Dz\|_{L^2(\mathbb{R}^d)^d} \leqslant \|G^h\|_{L^2_{\rm per}(\square)^{d\times d}} \|u\|_{H^2(\mathbb{R}^d)^n}. \tag{4.12}$$

From (4.11), (4.12), the elliptic estimate for u, and the estimate (4.9) for the matrix G^h it follows that

$$\|D^*(g^\varepsilon z)\|_{\mathscr{H}^*} \leqslant \varepsilon C \|f\|_{L^2(\mathbb{R}^d)^n}. \tag{4.13}$$

Thus, the first estimate in (4.7) is proved.

The terms of the sum $\mathscr{S}_2$ in (4.6) can be also estimated by using Lemma 3.3 since

$$\|N^{jk}(\varepsilon^{-1}x) \otimes D(D_k S^\varepsilon u_j)\|_{L^2(\mathbb{R}^d)^{n\times d}} \overset{(3.24)}{\leqslant} \sum_{j,k} \|N^{jk}\|_{L^2_{\rm per}(\square)^n} \|u\|_{H^2(\mathbb{R}^d)^n}$$

$$\overset{(3.22)}{\leqslant} \sum_{j,k} C \|N^{jk}\|_{L^2_{\rm per}(\square)^n} \|f\|_{L^2(\mathbb{R}^d)^n} \overset{(3.10)}{\leqslant} c\|f\|_{L^2(\mathbb{R}^d)^n}. \tag{4.14}$$

Thus, the second estimate in (4.7) is proved. From (4.2), (4.3), (4.6), and (4.7) we obtain (3.26).

3° In the estimate (3.26), we can replace v_ε with $\widehat{v}_\varepsilon$ since $v_\varepsilon - \widehat{v}_\varepsilon = S^\varepsilon u - u$ and

$$\|S^\varepsilon u - u\|_{L^2(\mathbb{R}^d)^n} + \|D(S^\varepsilon u - u)\|_{L^2(\mathbb{R}^d)^{n\times d}} \leqslant \varepsilon C \|f\|_{L^2(\mathbb{R}^d)^n}.$$

To obtain the last estimate, we use the property of the Steklov average (3.31) and the elliptic estimate (3.22). Thus, we proved the estimate (3.29).

4° To obtain (3.28), we ignore the term with gradient in (3.29) and note that $u_\varepsilon - u = (u_\varepsilon - \widehat{v}_\varepsilon) + K_\varepsilon(x)$, where $\|K_\varepsilon\|_{L^2(\mathbb{R}^d)^n} \leqslant \varepsilon C\|f\|_{L^2(\mathbb{R}^d)^n}$ by Lemma 3.3.

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