

NONLINEAR NORMAL MODES OF STRONGLY NONLINEAR PERIODICALLY EXCITED PIECEWISE LINEAR SYSTEMS

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A method for the numerical analysis of nonlinear normal modes of forced vibrations in strongly nonlinear systems with piecewise linear elastic characteristics is proposed. The approach is based on the combination of the Shaw–Pierre method of nonlinear normal modes with the Rauscher technique. As a result of application of this approach, the nonautonomous piecewise linear system is transformed into an autonomous system. For this system, we determine the Shaw–Pierre nonlinear normal modes. We also study the nonlinear torsional vibrations of the power transmission in a three-cylinder transport engine.

Introduction

The systems with finitely many degrees of freedom and piecewise linear elastic characteristics describe a broad class of engineering objects [1, pp. 59–74]. Numerous approaches were developed for the investigation of systems of this kind. Thus, in [21, 9], it was shown that the approximation of piecewise linear characteristics by polynomial curves may lead to incorrect results in the study of stability of solutions and to significant errors in the numerical analysis of periodic motions of systems with strong nonlinearities. The analytic investigation of free oscillations in systems with piecewise linear elastic characteristic was carried out in [10, 18, 20]. The procedure of finding Shaw–Pierre nonlinear normal modes of free oscillations in piecewise linear systems is presented in [13]. The response of piecewise linear systems to the action of harmonic excitations was analyzed in [2, 12, 15, 22]. The chaotic oscillations in piecewise linear systems with two and three degrees of freedom were studied in [6, 7]. For the analysis of piecewise linear systems, it was proposed to use amplitude surfaces [4]. The nonlinear normal modes of free oscillations of piecewise linear systems were considered in [11, 13].

There are only several works devoted to the numerical analysis of nonlinear normal modes under the conditions of forced vibrations. It seems likely that the Shaw–Pierre nonlinear normal modes under the conditions of forced vibrations were first studied in [14]. The possibility of combination of the method of nonlinear normal modes with the Rauscher technique for the analysis of the Kauderer–Rosenberg nonlinear normal modes under the conditions of forced vibrations was studied in the book [19, pp. 230–238]. The forced and parametric vibrations were computed with the help of the method of nonlinear normal modes and the Rauscher technique in [3, 5]. A detailed survey of works dealing with the theory of nonlinear normal modes can be found in [8, 16].

In the present work, forced vibrations in strongly nonlinear piecewise linear systems with any number of degrees of freedom are studied by the Shaw–Pierre method of nonlinear normal modes and the Rauscher technique. As a result of application of this approach, the nonautonomous dynamical systems are reduced to equivalent autonomous systems, which are used to compute the Shaw–Pierre nonlinear normal modes.

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Statement of the Problem

We study a system with N degrees of freedom subjected to the action of external harmonic excitations. The motion of this system is analyzed in a system of generalized coordinates x_i , $i = 1, \dots, N$:

$$\begin{aligned} \ddot{x}_i + f_i(x_1, \dots, x_N) &= 0, \quad i = 1, \dots, k-1, k+1, \dots, N, \\ \ddot{x}_k + f_k(x_1, \dots, x_N) &= A \cos \omega t. \end{aligned} \quad (1)$$

The nonlinear functions f_i , $i = 1, \dots, N$, are trilinear, depend on the vector of generalized coordinates $x = [x_1, \dots, x_N]^T$, and take the form

$$f_i(x) = \begin{cases} k_{1i}x, & \Delta_{13} < h^T x < \Delta_{12}, \\ k_{2i}x + b_{2i}, & h^T x \geq \Delta_{12}, \\ k_{3i}x + b_{3i}, & h^T x \leq \Delta_{13}, \end{cases} \quad (2)$$

where k_{1i} , k_{2i} , and k_{3i} are the row vectors of coefficients, b_{2i} and b_{3i} are free terms guaranteeing the continuity of the elastic characteristic at the points of kinks, and h is the column vector.

The aim of the present work is to determine the nonlinear normal modes under the conditions of forced vibrations in the strongly nonlinear system (1). The detailed description of the definitions and properties of nonlinear normal modes can be found in the book [1] and in the works [17, 21, 22].

Numerical Analysis of Nonlinear Normal Modes for Nonautonomous Systems

We propose a procedure of finding forced vibrations in strongly nonlinear piecewise linear systems based on the Rauscher technique of reduction of nonautonomous dynamical systems to equivalent autonomous systems. In the obtained autonomous dynamical system, we seek the Shaw–Pierre nonlinear normal modes. In this way, we determine the nonlinear normal modes under the conditions of forced vibrations.

The dynamical system (1) without external action can be rewritten in the normal coordinates of one of its linear parts. As this part, we choose the first row of function (2). Then the free oscillations in this part are described by the dynamical system

$$\ddot{x} + K_1 x = 0, \quad (3)$$

where the matrix K_1 has the block structure:

$$K_1 = [k_{11} \ k_{12} \ \dots \ k_{1N}]^T.$$

The blocks of this matrix are presented in Eq. (2). The dynamical system (3) has the vector of modal coordinates $\eta = [\eta_1, \dots, \eta_N]$ connected with the vector x by the following nonsingular linear transformation of

coordinates:

$$x = Q\eta,$$

where Q is the matrix of eigenvectors of the matrix K_1 . In the modal coordinates, the dynamical system (3) takes the form

$$\ddot{\eta}_i + \omega_i^2 \eta_i = 0, \quad i = 1, \dots, N,$$

where ω_i^2 are the eigenvalues of the matrix K_1 .

The dynamical system (1) can be represented in the modal coordinates η as follows:

$$\ddot{\eta} + \Lambda_1 \eta = \tilde{f}(\eta) + \frac{Q_{*,k}}{M_k} A \cos \omega t, \quad (4)$$

where M_k is the mass of the k th object in the system, Λ_1 is a diagonal matrix formed by the eigenvalues of the matrix K_1 , $Q_{*,k} = (Q_{1,k}, \dots, Q_{N,k})$ is the k th column of the matrix Q , k is the number of mass subjected to the action of external periodic forces in system (1). The vector function $\tilde{f}(\eta)$ can be represented in the form

$$\tilde{f}(\eta) = \begin{cases} 0, & \Delta_{13} < h^T Q \eta < \Delta_{12}, \\ Z_2 \eta + \xi_2, & \eta \geq \Delta_{12}, \\ Z_3 \eta + \xi_3, & \eta \leq \Delta_{13}, \end{cases}$$

where

$$Z_2 = \Lambda_1 - Q^{-1} K_2 Q, \quad Z_3 = \Lambda_1 - Q^{-1} K_3 Q,$$

$$\xi_2 = Q^{-1} b_2, \quad \xi_3 = Q^{-1} b_3,$$

$$K_2 = \begin{bmatrix} k_{21} \\ k_{22} \\ \dots \\ k_{2N} \end{bmatrix}, \quad K_3 = \begin{bmatrix} k_{31} \\ k_{32} \\ \dots \\ k_{3N} \end{bmatrix},$$

$$b_2 = \begin{bmatrix} b_{21} \\ b_{22} \\ \dots \\ b_{2N} \end{bmatrix}, \quad b_3 = \begin{bmatrix} b_{31} \\ b_{32} \\ \dots \\ b_{3N} \end{bmatrix}.$$

First, we consider the nonlinear normal modes in the autonomous dynamical system that follows from (4) if the external periodic forces are neglected. According to the procedure of finding the Shaw–Pierre nonlinear normal modes [1, 3], we choose one of the modal coordinates η and its velocity as driving. We denote these coordinates by $(\eta_1, \dot{\eta}_1)$. All remaining coordinates of the vector η are regarded as driven. In this case, the driven coordinates of the nonlinear normal modes can be expressed via the driving coordinates in the following way:

$$\eta_k = \Theta_k(\eta_1, \dot{\eta}_1), \quad \dot{\eta}_k = \Xi_k(\eta_1, \dot{\eta}_1), \quad k = 2, \dots, N. \quad (5)$$

We now perform a change of variables. As the driving coordinates, we choose coordinates a and φ connected with η_1 and $\dot{\eta}_1$ by the formulas

$$\eta_1 = a \cos \varphi, \quad \dot{\eta}_1 = -a\omega_1 \sin \varphi. \quad (6)$$

We introduce representation (6) in the first relation in (5). The function obtained as a result can be represented in the form of truncated Fourier series as follows [1]:

$$\eta_i = P_i(a, \varphi) = \sum_{\ell=1}^{N_a} \sum_{m=1}^{N_\varphi} C_i^{\ell, m} A_\ell(a) \cos(m-1)\varphi, \quad (7)$$

where $A_\ell(a)$ are basis functions, N_a is the number of basis functions A_ℓ , and N_φ is the number of elements of the expansion in φ .

We now study forced vibrations in the essentially nonlinear system (4). It is known [1] that the forced vibrations in this system are well described in the harmonic approximation:

$$\eta_i = \gamma_i \cos \omega t. \quad (8)$$

Comparing relations (8) and (6), we get

$$A \cos(\omega t) = A \cos \varphi = A \frac{\eta_1}{\sqrt{\eta_1^2 + \dot{\eta}_1^2 \frac{1}{\omega_1^2}}}. \quad (9)$$

Thus, the external force in the dynamical system (4) is expressed in terms of the generalized coordinate η_1 and velocity $\dot{\eta}_1$. Hence, the nonautonomous dynamical system (4) can be replaced by the so-called equivalent pseudoautonomous dynamical system. For this purpose, we insert relations (9) in (4) and get the following pseudoautonomous dynamical system:

$$\ddot{\eta} + \Lambda_1 \eta = g(\eta), \quad (10)$$

where the elements of the matrix $g = [g_1, \dots, g_N]$ are given by the formula

$$g_i(\eta) = f_i(\eta) + \frac{AQ_{i,k}\eta_1}{M_k \sqrt{\eta_1^2 + \dot{\eta}_1^2 \frac{1}{\omega_1^2}}}.$$

According to [16], the modal coordinates (a, φ) satisfy the system of ordinary differential equations:

$$\dot{a} = -\frac{g_1(a, \varphi)}{\omega_1} \cos \varphi, \quad \dot{\varphi} = \omega_1 - \frac{g_1(a, \varphi)}{a\omega_1} \sin \varphi. \quad (11)$$

The determination of the invariant manifolds of system (1) is reduced to finding the coefficients

$$C_i^{\ell, m}, \quad i = 2, \dots, N, \quad \ell = 1, \dots, N_a, \quad m = 1, \dots, N_\varphi,$$

of expansion (7). For this purpose, we represent the first and second derivatives in Eqs. (10) in the form

$$\begin{aligned} \dot{\eta}_i &= \sum_{\ell=1}^{N_a} \sum_{m=1}^{N_\varphi} C_i^{\ell, m} \left(A'_\ell \cos(m-1)\varphi \cdot \dot{a} - A_\ell(m-1) \sin(m-1)\varphi \cdot \dot{\varphi} \right) \\ &= \sum_{\ell=1}^{N_a} \sum_{m=1}^{N_\varphi} C_i^{\ell, m} \left(\frac{g_1(a, \varphi)}{\omega_1} (-A'_\ell \cos \varphi \cos(m-1)\varphi \right. \\ &\quad \left. + (m-1) \frac{A_\ell}{a} \sin \varphi \sin(m-1)\varphi \right) - \omega_1(m-1) A_\ell \sin(m-1)\varphi, \\ \ddot{\eta}_i &= \sum_{\ell=1}^{N_a} \sum_{m=1}^{N_\varphi} C_i^{\ell, m} \left(A''_\ell \cos(m-1)\varphi \cdot \dot{a}^2 \right. \\ &\quad \left. + A'_\ell \cos(m-1)\varphi \cdot \ddot{a} - 2(m-1) A'_\ell \sin(m-1)\varphi \cdot \dot{a} \dot{\varphi} \right. \\ &\quad \left. - A_\ell(m-1)^2 \dot{\varphi}^2 \cos(m-1)\varphi - A_\ell \sin(m-1)\varphi \cdot \ddot{\varphi} \right) \\ &= \sum_{\ell=1}^{N_a} \sum_{m=1}^{N_\varphi} C_i^{\ell, m} \left(\cos(m-1)\varphi \left(A''_\ell \dot{a}^2 + A'_\ell \ddot{a} \right) \right. \\ &\quad \left. - \sin(m-1)\varphi \left(2(m-1) A'_\ell \cdot \dot{a} \dot{\varphi} + A_\ell \ddot{\varphi} \right) \right), \end{aligned} \quad (12)$$

where

$$A'_\ell = \frac{dA_\ell}{da} \quad \text{and} \quad A''_\ell = \frac{d^2 A_\ell}{da^2}.$$

We now represent the second derivatives of modal coordinates in Eqs. (12) in the following form:

$$\ddot{a} = \frac{d}{dt} \left(-\frac{g_1}{\omega_1} \sin \varphi \right) = -\frac{1}{\omega_1} \left(\left(\frac{\partial g_1}{\partial a} \dot{a} + \frac{\partial g_1}{\partial \varphi} \dot{\varphi} \right) \sin \varphi + g_1 \cos \varphi \right), \quad (13)$$

$$\begin{aligned}\ddot{\varphi} &= \frac{d}{dt} \left(\omega_1 - \frac{g_1}{a\omega_1} \cos \varphi \right) \\ &= -\frac{1}{\omega_1} \left(\left(\frac{\partial g_1}{\partial a} \dot{a} + \frac{\partial g_1}{\partial \varphi} \dot{\varphi} - \frac{g_1}{a} \dot{a} \right) \frac{1}{a} \cos \varphi - \frac{g_1}{a} \dot{\varphi} \sin \varphi \right),\end{aligned}\quad (14)$$

where

$$\frac{\partial g_1}{\partial a} = \begin{cases} 0, & \Delta_2 < h^T Q \eta < \Delta_3, \\ \sum_{i=1}^N Z_{2_{1i}} \frac{\partial \eta_i}{\partial a}, & h^T Q \eta \leq \Delta_2, \\ \sum_{i=1}^N Z_{3_{1i}} \frac{\partial \eta_i}{\partial a}, & h^T Q \eta \geq \Delta_3, \end{cases}$$

$$\frac{\partial g_1}{\partial \varphi} = \begin{cases} -\frac{A Q_{i,k}}{M_k} \sin \varphi, & \Delta_2 \leq h^T Q \eta \leq \Delta_3, \\ \sum_{i=1}^N Z_{2_{1i}} \frac{\partial \eta_i}{\partial \varphi} - \frac{A Q_{i,k}}{M_k} \sin \varphi, & h^T Q \eta < \Delta_2, \\ \sum_{i=1}^N Z_{3_{1i}} \frac{\partial \eta_i}{\partial \varphi} - \frac{A Q_{i,k}}{M_k} \sin \varphi, & h^T Q \eta > \Delta_3, \end{cases}\quad (15)$$

$$\frac{\partial \eta_i}{\partial a} = \begin{cases} \cos \varphi, & i = 1, \\ \sum_{\ell=1}^{N_a} \sum_{m=1}^{N_\varphi} C_i^{\ell,m} A'_\ell \cos(m-1)\varphi & i \neq 1, \end{cases}$$

$$\frac{\partial \eta_i}{\partial \varphi} = \begin{cases} -a \sin \varphi, & i = 1, \\ -\sum_{\ell=1}^{N_a} \sum_{m=1}^{N_\varphi} C_i^{\ell,m} (m-1) A_\ell \sin(m-1)\varphi, & i \neq 1. \end{cases}\quad (16)$$

We now substitute relations (15) and (16) in (13) and (14) and then insert the results in Eq. (10). Finally, we get the discrepancy vector for the solutions of the system of equations (10):

$$Z(a, \varphi, C) = \ddot{\eta}(a, \varphi, C) + \Lambda_1 \eta(a, \varphi, C) - g(a, \varphi, C),\quad (17)$$

where C is a vector formed by the elements $C_i^{\ell,m}$, $i = 2, \dots, N$, $\ell = 1, \dots, N_a$, $m = 1, \dots, N_\varphi$. By virtue of relations (6) and (7), the first element of the discrepancy vector (17) is equal to zero. The remaining elements

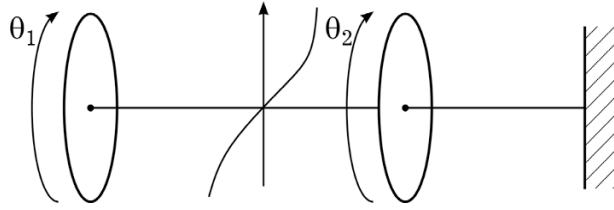


Fig. 1. Mechanical system.

are nonzero. In order to find the nonlinear normal modes, we use the Galerkin method and write the conditions of orthogonality of the discrepancy vector $Z(a, \varphi, C)$ to the basis functions $A_\ell \cos(m-1)\varphi$:

$$\int_0^{2\pi} \int_{a_0}^{a_1} (\ddot{\eta}(a, \varphi, C) + \Lambda_1 \eta(a, \varphi, C) - g(a, \varphi, C)) A_\ell \cos[(m-1)\varphi] da d\varphi = 0. \quad (18)$$

Thus, relations (18) form a system of $(N-1)N_a N_\varphi$ nonlinear algebraic equations for the elements of the vector C . They can be solved numerically by the Newton–Raphson method.

To study the motions on the invariant manifolds, we choose the initial values of the modal variables $a(0)$ and $\varphi(0)$. The motion on the form is determined by the numerical integration of the equations of modal dynamics (11). To analyze the stability of the obtained periodic motions, we numerically solve the equations in variations. We construct the corresponding fundamental matrix and determine its eigenvalues (multipliers) [1].

Numerical Analysis of Nonlinear Torsional Vibrations

We consider a model of torsional vibrations of the power transmissions in engines of internal combustion. The computational scheme of a system of this kind is presented in Fig. 1. We analyze the power transmission of a three-cylinder transport engine with oppositely moving pistons. The concentrated disks of the model describe the torsional vibrations of two crankshafts of the engine. The nonlinear elastic characteristic describes the final transmission between the crankshafts. It consists of five gears one of which contains an elastic clutch. The forced torsional vibrations of the power transmission are described by the following system of ordinary differential equations:

$$I_1 \ddot{\theta}_1 + f(\theta_1 - \theta_2) = A \cos(\omega t), \quad I_2 \ddot{\theta}_2 - f(\theta_1 - \theta_2) + c_1 \theta_2 = 0, \quad (19)$$

where

$$f(\theta_1 - \theta_2) = \begin{cases} c_1(\theta_1 - \theta_2), & -\Delta < \theta_1 - \theta_2 < \Delta, \\ c_2(\theta_1 - \theta_2) + \Delta(c_2 - c_1), & \theta_1 - \theta_2 \geq \Delta, \\ c_2(\theta_1 - \theta_2) - \Delta(c_2 - c_1), & \theta_1 - \theta_2 \leq -\Delta, \end{cases}$$

θ_1 and θ_2 are generalized coordinates used to describe the vibrations of two crankshafts of the engine, and c_1 is the stiffness of the gear transmission between the crankshafts.

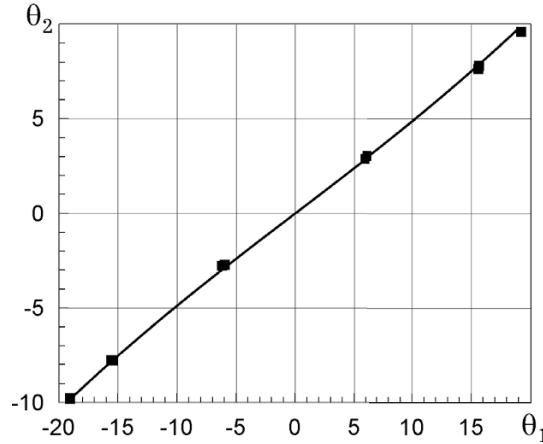


Fig. 2. Nonlinear normal mode in the configuration space for $\omega = 17.186$ rad/sec.

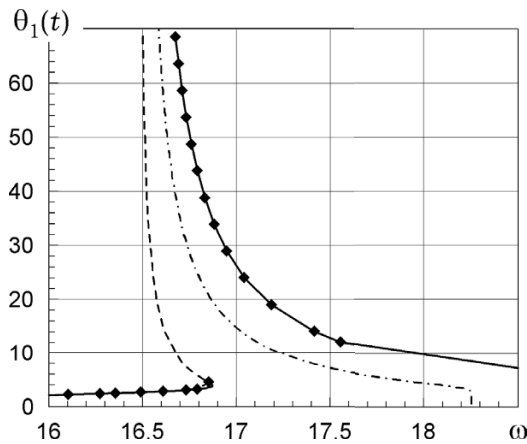


Fig. 3. Amplitude-frequency characteristic of the response.

We performed the numerical analysis for the following dimensionless values of the parameters of the system:

$$c_1 = 300, \quad c_2 = 150, \quad \Delta = 1.04, \quad I_1 = 0.3, \quad I_2 = 0.45, \quad A = 77.7.$$

The eigenfrequencies of linear low-amplitude vibrations of the system are as follows:

$$\omega_1 = 18.25 \quad \text{and} \quad \omega_2 = 44.72 .$$

The nonlinear normal modes under the conditions of forced vibrations were studied by using the approach described above. The numerically obtained nonlinear normal modes were rearranged in the configuration space (θ_1, θ_2) . The results of numerical modeling of the nonlinear normal modes were studied for the amplitude of vibration $a(0) = 20 \Delta$. The frequency of perturbing force was set equal to $\omega = 17.186$ rad/sec. The results of numerical analysis of this nonlinear normal mode performed by using the proposed algorithm are illustrated by the continuous line in Fig. 2. To confirm the obtained nonlinear normal mode, we carried out the

direct numerical integration of the system of differential equations (19) for 100 periods of action of the perturbing force. The results of calculations are presented in the form of squares in Fig. 2. It is easy to see that the obtained trajectories are close to each other, which confirms the validity of our calculations of the nonlinear normal mode.

We also computed the nonlinear normal modes in the region of the first main resonance of the system. The results are shown in Fig. 3, where the amplitude of vibrations $\theta_1(t)$ is plotted on the ordinate and the frequency of perturbing force ω is plotted on the abscissa. The dash-dotted line reflects the skeleton curve of free vibrations, the solid line shows the branches of stable forced vibrations of the system, and the dashed line corresponds to the regions of unstable forced vibrations of the system. In order to verify the accumulated results, we performed the direct numerical integration of the dynamical system (19) with initial conditions corresponding to the nonlinear normal mode of forced vibrations. The results of this direct integration are shown by symbols in the amplitude-frequency characteristic. It is easy to see that the data obtained by using both methods are quite close.

CONCLUSIONS

We propose a new approach to the determination of the Shaw–Pierre nonlinear normal modes of forced vibrations in strongly nonlinear systems with piecewise linear elastic characteristic. It is based on the combination of the method of Shaw–Pierre nonlinear normal modes for piecewise linear systems in the process of free vibration with the Rauscher technique.

We studied forced vibrations in a strongly nonlinear system with two degrees of freedom, which are nonlinear normal modes. They are presented in the amplitude-frequency characteristic, which has the ordinary form. Thus, the forced vibrations in strongly nonlinear piecewise linear systems can be represented in the form of nonlinear normal modes.

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