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NUMERICAL-ANALYTIC TECHNIQUE FOR THE SOLUTION OF NONSTATIONARY PROBLEMS OF HEAT CONDUCTION IN LOCALLY INHOMOGENEOUS MEDIA

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We propose a procedure of simultaneous application of the splitting method, boundary-element method, step-by-step time scheme, and iterative FD (Finite-Discrete) procedure for the construction of the integral representation of the solution of a nonstationary problem of heat conduction for a closed domain with Dirichlet condition given on its boundary containing a locally inhomogeneous subdomain whose physical characteristics depend on the coordinates. We perform a comprehensive numerical analysis of this approach with regard for the fact that the heat field is affected by the dependences of the heat-conduction coefficient and specific heat capacity of the material on the coordinates.

In various branches of economy and engineering and, in particular, in machine-building, instrument-making, and thermal power engineering, it is important to decrease the consumption of materials in inhomogeneous structural elements operating under the conditions of high thermal and mechanical loads and be able to evaluate their strength and reliability. For this purpose, it is necessary to know the heat fields in objects of any shape, i.e., to find the solutions of nonstationary problems of heat conduction. The linear mathematical models based on the assumption of piecewise constant dependences of the thermal characteristics of materials on the coordinates are not always capable to give adequate description of the actual processes [1, 2, 10, 11]. Models that take into account the dependences of the heat-conduction coefficient and specific heat capacity of the material of a body on the coordinates or temperature and lead to differential equations with variable coefficients or nonlinear equations prove to be more reliable [8, 9, 12–15]. As one of the approaches used to find the solutions of the obtained boundary-value problems of mathematical physics, we can mention the procedure of separation of the operator characterizing the influence of inhomogeneity and application of iterative methods with discretization of the local domain in which the analyzed physical characteristics depend on the coordinates to this operator [4–7].

Formulation of the Problem and the Choice of Numerical-Analytic Technique for Its Solution

Assume that the heat-conduction coefficient $\lambda(x)$ and specific heat capacity $c(x)$ of the material of the body occupying a domain $\Omega \subset \mathbf{R}^2$ depend on the coordinates in a certain part of this domain $\Omega_1 \subset \Omega$. Thus, we can write (see [4, 7])

$$\lambda(x) = \lambda_0 + \lambda_g(x)\chi(x) \quad \text{and} \quad c(x) = c_0 + c_g(x)\chi(x), \tag{1}$$

where $c_0 = \text{const}$, $\lambda_0 = \text{const}$, $c(x) \in C^1(\Omega)$, $\lambda(x) \in C^1(\Omega)$, $x = (x_1, x_2)$ are Cartesian coordinates, $\chi(x) = 0$,

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$x \in \Omega \backslash \Omega_1$, $\chi(x) = 1$, $x \in \Omega_1$, $\chi(x)$ is the characteristic function of the domain Ω_1, $\Gamma \cap \Gamma_1 = \varnothing$, Γ and Γ_1 are the boundaries of the domains Ω and Ω_1, respectively, and $\lambda_g(x)$ and $c_g(x)$ are functions equal to zero on the boundary Γ_1.

Separating the operator characterizing the influence of local inhomogeneity [9] in the differential equation with variable coefficients with an aim to find the unknown temperature $u(x,t)$ in the locally inhomogeneous body, we arrive at the equation [6]

$$a_0 \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = \frac{f(x,t)}{\lambda(x)} + P_x u(x,t) - P_t u(x,t), \tag{2}$$

with boundary conditions

$$u(x,t) = g(x,t), \qquad (x,t) \in \Gamma \times (0,T], \tag{3}$$

and initial conditions

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{4}$$

Here, $T = \{t : 0 < t < \infty\}$, t is time,

$$a_0 = \frac{c_0}{\lambda_0},$$

$$P_x u(x,t) = \frac{1}{\lambda(x)} \frac{\partial \lambda(x)}{\partial x_i} \frac{\partial u(x,t)}{\partial x_i}, \qquad P_t u(x,t) = \frac{c_g(x)\lambda_0 - c_0 \lambda_g(x)}{\lambda(x)\lambda_0} \frac{\partial u(x,t)}{\partial t},$$

$f(x,t)$ is the power of given internal sources in the domain Ω;

$$u = u(x,t) \in C^2(\Omega \times (0,T]) \cap C(\Omega \times (0,T]), \qquad g(x,t) \in C(\Gamma),$$

$u_0(x) \in C(\Omega)$, and $C^m(A)$ is the class of continuously differentiable functions with derivatives of order m in a domain A. Note that we use the Einstein rule of summation over the repeated indices.

To optimize the procedure of solution of the formulated problem, in view of the universality of the approach based on the direct solution of differential boundary-value problems and the advantages of the methods of integral equations for piecewise homogeneous media, we propose to combine these advantages in a single numerical-analytic technique. Its efficiency is explained by the transition to the integral representation of the solution (instead of the numerical differentiation in the method of finite differences and finite-element method) with discretization solely of the domain of local inhomogeneity and the boundary of the body.

Construction of Integral Representations of the Solution of the Problem by the Indirect Boundary-Element Method (IBEM)

In what follows, to simplify calculations, we set $f(x,t) \equiv 0$. We apply the IBEM [2] and introduce unknown fictitious heat sources. As a result, to find temperature, we get the following integral representation of the solution of problem (6), (3), (4):

$$u(x,t) = \int_{\Gamma}\int_{0}^{t} E(x,t,y,\tau)\varphi(y,\tau)\,d\tau\, d\Gamma(y) + \int_{\Omega} E(x,t,y,0)u_0(y)\,d\Omega(y)$$

$$+ \int_{\Omega_1}\int_{0}^{t} E(x,t,y,\tau)P_x u(y,\tau)\,d\tau\, d\Omega_1(y)$$

$$- \int_{\Omega_1}\int_{0}^{t} E(x,t,y,\tau)P_t u(y,\tau)\,d\tau\, d\Omega_1(y), \qquad (x,t)\in\Omega\times(0,T], \tag{5}$$

where

$$E(x,t,y,\tau) = \frac{e^{-\frac{r^2(x,y)}{4a_0(t-\tau)}}}{4\pi a_0(t-\tau)}$$

is the fundamental solution of Eq. (2),

$$r^2(x,y) = \sum_{i=1}^{2}(x_i - y_i)^2 .$$

Since the operators

$$P_x u(x,t) \quad \text{and} \quad P_t u(x,t)$$

contain the unknown derivatives of the required function with respect to the space coordinates and time, in view of (5), we get the following integral representations for these derivatives $((x,t)\in\Omega\times(0,T])$:

$$\frac{\partial u(x,t)}{\partial x_j} = \int_{\Gamma}\int_{0}^{t} \frac{\partial E(x,t,y,\tau)}{\partial x_j}\varphi(y,\tau)\,d\tau\, d\Gamma(y) + \int_{\Omega} \frac{\partial E(x,t,y,0)}{\partial x_j}u_0(y)\,d\Omega(y)$$

$$+ \int_{\Omega_1}\int_{0}^{t} \frac{\partial E(x,t,y,\tau)}{\partial x_j}P_x u(y,\tau)\,d\tau\, d\Omega_1(y)$$

$$- \int_{\Omega_1}\int_{0}^{t} \frac{\partial E(x,t,y,\tau)}{\partial x_j}P_t u(y,\tau)\,d\tau\, d\Omega_1(y), \tag{6}$$

$$\frac{\partial u(x,t)}{\partial t} = \int_{\Gamma}\int_{0}^{t} \frac{\partial E(x,t,y,\tau)}{\partial t}\varphi(y,\tau)\,d\tau\, d\Gamma(y) + \int_{\Omega} \frac{\partial E(x,t,y,0)}{\partial t}u_0(y)\,d\Omega(y)$$

$$+\int_{\Omega_1}\int_0^t \frac{\partial E(x,t,y,\tau)}{\partial t} P_x u(y,\tau)\,d\tau d\Omega_1(y)$$

$$-\int_{\Omega_1}\int_0^t \frac{\partial E(x,t,y,\tau)}{\partial t} P_t u(y,\tau)\,d\tau d\Omega_1(y)\,. \tag{7}$$

If we let the point x tend to the boundary of the body in relation (5), then we get the following boundary integral equation:

$$g(x,t) = -\frac{1}{2}\varphi(y,\tau) + \int_\Gamma\int_0^t E(x,t,y,\tau)\varphi(y,\tau)\,d\tau d\Gamma(y)$$

$$+\int_\Omega E(x,t,y,0)u_0(y)\,d\Omega(y)$$

$$+\int_{\Omega_1}\int_0^t E(x,t,y,\tau)P_x u(y,\tau)\,d\tau d\Omega_1(y)$$

$$-\int_{\Omega_1}\int_0^t E(x,t,y,\tau)P_t u(y,\tau)\,d\tau d\Omega_1(y), \qquad (x,t)\in\Gamma\times(0,T]. \tag{8}$$

By using the solution of Eq. (8), we can find the introduced unknown fictitious heat sources and determine the values of temperature at the inner points of the body and on its boundary according to relations (6) and (7).

Construction of a Discrete-Continual Model

For the analytic integration with respect to time, we split the interval $(0,T]$ into K equal subintervals

$$t_k = k\Delta t\,, \quad t_K = T\,,$$

and apply the scheme of sequential initial conditions (SSIC) [2] according to which, at the end of each time interval, we determine the values $u_k(x,t)$ and use them as initial values for the next step in relation (5). We split (discretize) the boundary of the body into curvilinear Hermitian elements of the second order [3]:

$$\Gamma = \bigcup_{i=1}^{N}\Gamma_i\,, \quad \Gamma_i\cap\Gamma_j = \varnothing, \quad i\neq j\,.$$

At the same time, the domain Ω_1 is split into eight-node Hermitian elements of the second order [3]:

$$\Omega = \bigcup_{q=1}^{N_{\text{inside}}}\Omega_q\,, \quad \Omega_q\cap\Omega_j = \varnothing, \quad q\neq j\,.$$

The unknown density of distribution in each time interval is approximated by beta-splines of order zero (constants):

$$\varphi_k(x) = \sum_{j=1}^{N} \chi_{kj}(x) d_{kj}, \qquad \chi_{ij} = \begin{cases} 1, & x \in \Gamma_j, \\ 0, & x \notin \Gamma_j, \end{cases} \qquad d_{kj} = \text{const},$$

i.e., in each time step, we determine the vector of unknown constants $d_k = (d_{k1}, \ldots, d_{kN})$.

In the last step of construction of the discrete-continual model, we apply the iterative FD procedure. As the initial approximation, in each time step, we take the solution of the homogeneous problem

$$u_k^0(x,t) = \sum_{i=1}^{N} d_{ki} \int_{\Gamma_i} E_\tau(x,t,y) d\Gamma_i(y) + \int_{\Omega} E(x,t,y,0) u_{k-1}(y,(k-1)\Delta t) d\Omega(y). \tag{9}$$

Then the integral representations (5)–(7) for the pth $(p = 1, \ldots, N_{\text{iter}})$ iteration of the kth time interval take the form $((x,t) \in \Omega \times (t_{k-1}, t_k])$:

$$\begin{aligned} u_k^p(x,t) = {} & \sum_{i=1}^{N} d_{ki}^p \int_{\Gamma_i} E_\tau(x,t,y) d\Gamma_i(y) \\ & + \int_{\Omega} E(x,t,y,0) u_{k-1}(y,(k-1)\Delta t) d\Omega(y) \\ & + \int_{\Omega_1} \int_0^t E(x,t,y,\tau) P_x u_k^{p-1}(y,\tau) d\tau d\Omega_1(y) \\ & - \int_{\Omega_1} \int_0^t E(x,t,y,\tau) P_t u_k^{p-1}(y,\tau) d\tau d\Omega_1(y), \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\partial u_k^p(x,t)}{\partial x_j} = {} & \sum_{i=1}^{N} d_{ki}^p \int_{\Gamma_i} Q_{j\tau}(x,t,y) d\Gamma_i(y) \\ & + \int_{\Omega} \frac{\partial E(x,t,y,0)}{\partial x_j} u_{k-1}(y,(k-1)\Delta t) d\Omega(y) \\ & + \int_{\Omega_1} \int_0^t \frac{\partial E(x,t,y,0)}{\partial x_j} P_x u_k^{p-1}(y,\tau) d\tau d\Omega_1(y) \\ & - \int_{\Omega_1} \int_0^t \frac{\partial E(x,t,y,0)}{\partial x_j} P_t u_k^{p-1}(y,\tau) d\tau d\Omega_1(y), \end{aligned} \tag{11}$$

$$\frac{\partial u_k^p(x,t)}{\partial t} = \sum_{i=1}^{N} d_{ki}^p \int_{\Gamma_i} Q_{t\tau}(x,t,y)d\Gamma_i(y)$$

$$+\int_{\Omega} \frac{\partial E(x,t,y,0)}{\partial t} u_{k-1}(y,(k-1)\Delta t)\,d\Omega(y)$$

$$+\int_{\Omega_1}\int_0^t \frac{\partial E(x,t,y,0)}{\partial t} P_x u_k^{p-1}(y,\tau)\,d\tau d\Omega_1(y)$$

$$-\int_{\Omega_1}\int_0^t \frac{\partial E(x,t,y,0)}{\partial t} P_t u_k^{p-1}(y,\tau)\,d\tau d\Omega_1(y), \qquad (12)$$

where

$$\Phi_\tau(x,t,y) = \int_0^t \Phi(x,t,y,\tau)d\tau, \qquad \Phi \in \{E, Q_j, Q_t\},$$

$d_k^p = \left(d_{k1}^p, \ldots, d_{kN}^p\right)$ is the intensity of unknown heat sources in the pth iteration for the kth time interval.

If the sequence of integral representations (10) is convergent, i.e.,

$$\exists \lim_{p\to\infty} u_k^p(x,t) = u_k(x,t),$$

then it is easy to see that (10) satisfies Eq. (2).

On the boundary of the body, we choose a set of collocation points $x = \left(x_1, x_2, \ldots, x_N\right)$ and satisfy the boundary integral equation (8) on this set. We get the following system of linear algebraic equations (SLAE) for the unknown heat sources introduced in boundary elements:

$$Ad_k^p = B_p, \qquad (13)$$

where

$$A = \{a_{ij}\}, \quad i = 1,\ldots,N, \quad j = 1,\ldots,N, \quad d_p^k = \{d_{kj}^p\}, \quad j = 1,\ldots,N, \quad B_p = \{b_i^p\}, \quad i = 1,\ldots,N,$$

are the coefficients of the matrix and the right-hand side for the pth step of iteration $(p = 1,\ldots,N_{\text{iter}})$. These coefficients are determined as follows:

$$a_{ij} = \int_{\Gamma_j}\int_0^{\Delta t} E(x_i, \Delta t, y, \tau)\,d\tau d\Gamma_j(y),$$

$$b_i^p = g(x_i, k\Delta t) - \int_\Omega E(x_i, \Delta t, y, 0) u_{k-1}(y) d\Omega(y)$$

$$- \int_{\Omega_1} \int_0^{\Delta t} E(x_i, \Delta t, y, \tau) P_x u_k^{p-1}(y, \tau) d\tau d\Omega_1(y)$$

$$+ \int_{\Omega_1} \int_0^{\Delta t} E(x_i, \Delta t, y, \tau) P_t u_k^{p-1}(y, \tau) d\tau d\Omega_1(y).$$

It is clear that, in finding the initial approximation, the corresponding SLAE takes the form

$$A d_k^0 = B_0,$$

i.e., is similar to (13) and, in view of (9), the parameters b_i^0 are given by the formulas

$$b_i = g(x_i, k\Delta t) - \int_\Omega E(x_i, \Delta t, y, 0) u_{k-1}(y) d\Omega(y).$$

After the completion of the iterative process, we find the values of the required function and its derivatives with respect to the coordinates and time in the set $\Omega \times (t_{k-1}, t_k]$ by using relations (10)–(12), respectively.

Note that, for $f(x,t) \neq 0$, we do not encounter any fundamental difficulties. In fact, the only difference is connected with the appearance of the terms

$$a_0 \int_\Omega \int_0^t E(x, t, y, \tau) f(y, \tau) d\tau d\Omega(y) \quad \text{and} \quad -a_0 \int_\Omega \int_0^{\Delta t} E(x_i, \Delta t, y, \tau) f(y, \tau) d\tau d\Omega(y)$$

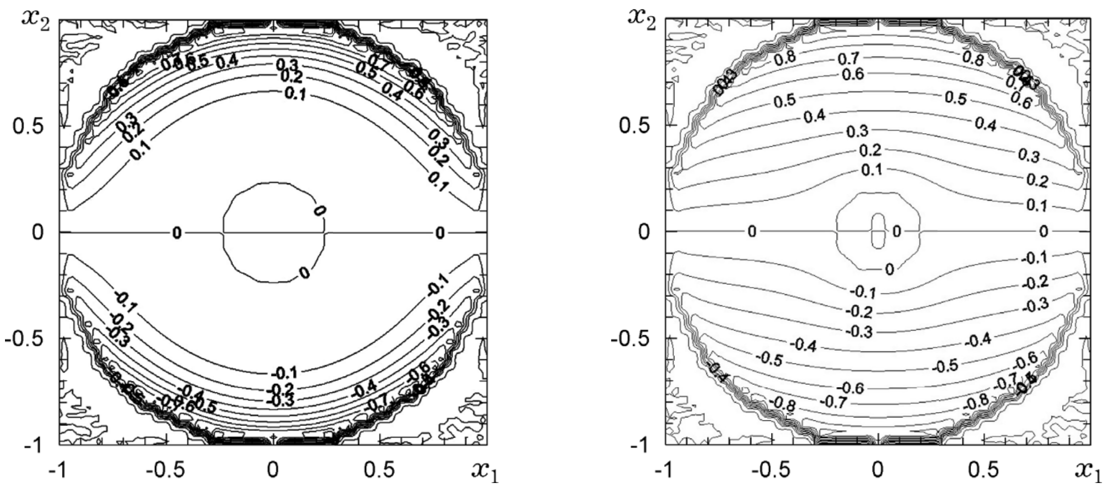
in the integral representation (5) and, hence, on the right-hand side of the SLAE (13).

Numerical Investigations

The proposed numerical-analytic approach was tested for a domain Ω chosen in the form of a circle of unit radius centered at $(0,0)$. The dependences of the heat-conduction coefficient and specific heat capacity on the coordinates in a rectangle Ω_1 with sizes $2k_1 \times 2k_2$ centered at $(0,0)$ were specified by the functions

$$\lambda(x) = \lambda_0 + k_{\lambda g}\left[1 + \cos\left(\frac{\pi x_1}{k_1}\right)\right]\left[1 + \cos\left(\frac{\pi x_2}{k_2}\right)\right],$$

$$c(x) = c_0 + k_{cg}\left[1 + \cos\left(\frac{\pi x_1}{k_1}\right)\right]\left[1 + \cos\left(\frac{\pi x_2}{k_2}\right)\right],$$



(a) **(b)**

Fig. 1. Propagation of the heat field in the inhomogeneous circle of unit radius ($\lambda_0 = 1$ W/(m·°C), $c_0 = 2$, $k_{\lambda g} = 2$, $k_{cg} = 0$) for different times: (a) 0.1 sec; (b) 0.2 sec.

where

$$\lambda_0 = 1 \text{ W/(m}\cdot°\text{C)}, \quad c_0 = 2 \text{ J/(kg}\cdot°\text{C)}, \quad k_1 = 0.25, \quad k_2 = 0.25, \quad \text{and} \quad k_{\lambda g} \quad \text{and} \quad k_{cg}$$

are constants.

The dimensions of all quantities are presented in the SI system of units, with the exception of temperature measured in degrees centigrade. We choose the step of splitting with respect to time $\Delta t = 0.05$, the number of boundary elements $N = 24$, the number of elements of discretization of the domain of local inhomogeneity $N_{\text{inside}} = 48$, the number of steps for the iterative procedure $N_{\text{iter}} = 3$, and the boundary (3) and initial (4) conditions in the form:

$$g(x,t) = x_2, \qquad u_0(x) = \begin{cases} 0, & r(x,0) < 0.95, \\ x_2\left(\dfrac{r(x,0) - 0.95}{0.05}\right), & r(x,0) \geq 0.95, \end{cases} \tag{14}$$

$$g(x,t) = x_2\left(1 + \frac{t}{4\Delta t}\right), \qquad u_0(x) = x_2. \tag{15}$$

Thus, in the first case, the initial condition is equal to zero almost everywhere inside the circle, except the inner near-boundary layer, where this condition linearly decreases from the value x_2 on the boundary to zero. In the second case, the boundary condition is chosen so that, at the end of the fourth time interval, the value of temperature on the boundary becomes twice higher.

In Fig. 1, we illustrate the process of propagation of heat in a locally inhomogeneous body with regard solely for the coordinates dependence of the heat-conduction coefficient in the second and fourth time steps under conditions (14).

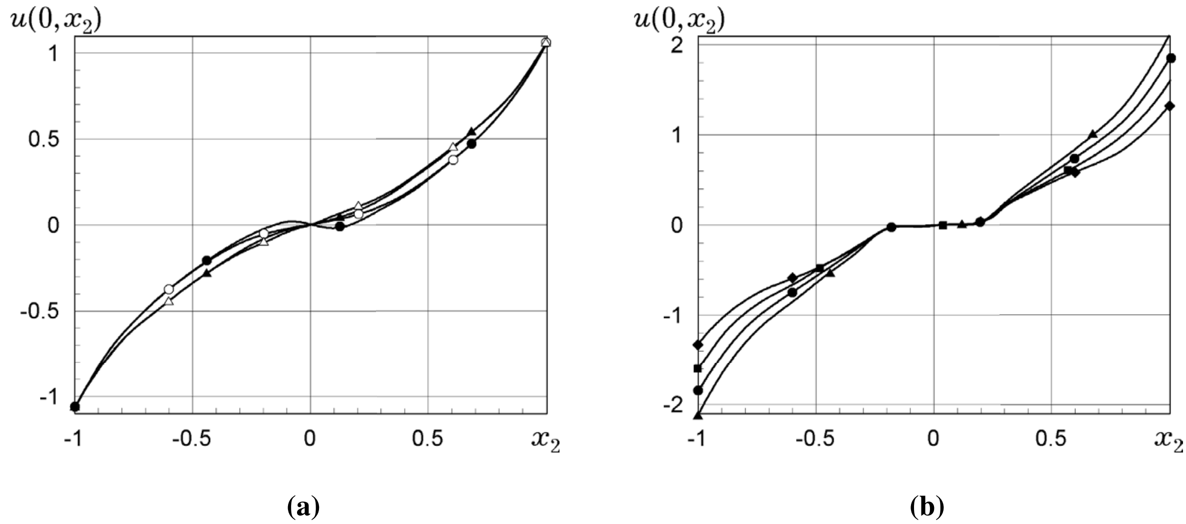


Fig. 2. Estimation of the influence of the coordinate dependence of heat-conduction coefficient on the distribution of temperature in the inhomogeneous circle of unit radius at different times for the boundary conditions (14) (a) and the initial conditions (15) (b).

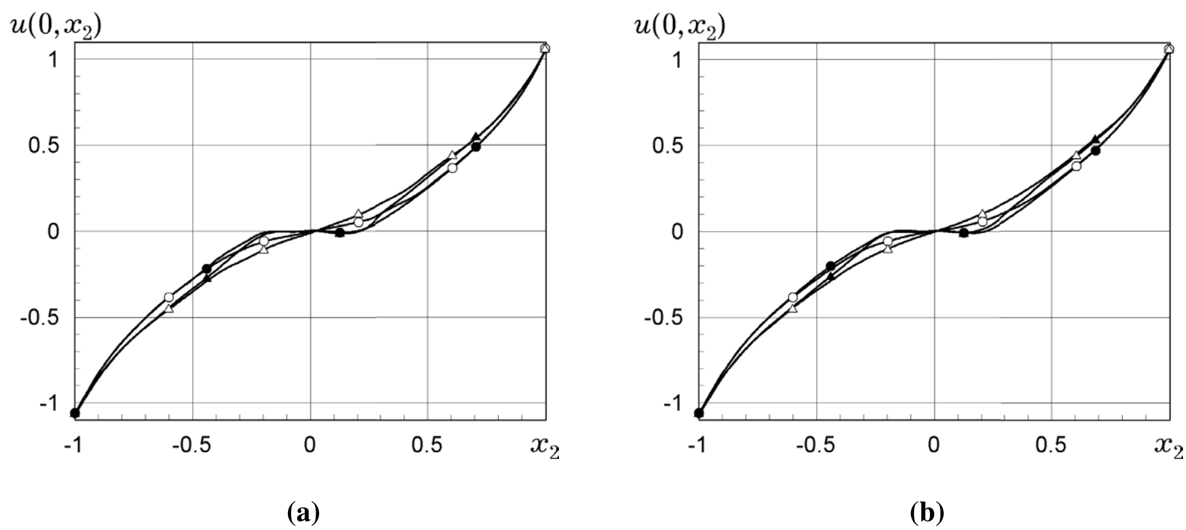


Fig. 3. Estimation of the influence of the coordinate dependence of the specific heat capacity (a) and the total influence of the coordinate dependences of both coefficients (b) on the distribution of temperature in the inhomogeneous circle of unit radius for different times.

It is easy to see that, at the end of the fourth time interval, the process becomes almost stationary and, in the rectangle Ω_1, where the heat-conduction coefficient is higher than λ_0, we get a more uniform distribution of temperature because the isotherms go around the rectangle. Note that, in the homogeneous circle ($k_{\lambda g} = 0$, $k_{cg} = 0$), the process also becomes stationary at the end of the fourth step.

We separately investigate the dependence of the heat-conduction coefficient on the coordinates ($k_{\lambda g} = 2$, $k_{cg} = 0$) in the same circle under conditions (14) (Fig. 2a) and (15) (Fig. 2b). A part of the obtained results is presented in Fig. 2 for different times: $t = 0.05$ (◆), 0.1 (■), 0.15 (●, ○), and 0.2 (▲, △). Note that, in Fig. 2a, the obtained results are compared with the homogeneous case (the plots with blank symbols). It is also worth noting that, in the locally inhomogeneous body (Fig. 2b), due to the high thermal conductivity inside the domain Ω_1, the temperature in this domain is identical almost everywhere (horizontal lines), which was not observed in the homogeneous body. Thus, the variations of temperature on the boundary lead to weaker changes inside the body due to its high heat capacity.

We also investigated the influence of the coordinate dependence of the specific heat capacity ($k_{\lambda g} = 0$, $k_{cg} = 5$) in the same circle and the total influence of the coordinate dependences of both coefficients under conditions (14). A part of the obtained results is presented in Fig. 3 for different times and compared with the homogeneous case.

In the initial period of time, prior to the penetration of heat into the domain of inhomogeneity, the solutions coincide. Later, the analyzed solutions become different and, finally, the process becomes stationary.

CONCLUSIONS

To find the solutions of nonstationary problems of heat conduction in locally inhomogeneous media, we propose a numerical-analytic technique that combines the indirect boundary-element method (with regard for its advantages for piecewise homogeneous media) with the procedure of separation of the operator characterizing the influence of the local domain of geometric inhomogeneity and subsequent discretization solely of the indicated domain. A more complex mathematical model that leads to a differential equation with variable coefficients and the iterative FD procedure enable us to combine the advantages of the indicated methods and optimize the numerical analyses of temperatures and heat fluxes in locally inhomogeneous objects of any shape. The developed numerical-analytic technique noticeably decreases the errors caused by the approximation of the boundary-value problem because the fundamental solution exactly satisfies the original equation (2) in the domain $\Omega \setminus \Omega_1$ and the initial conditions (4). The influence of the errors caused by the discretization and numerical integration is insignificant and is fairly well controlled due to the arbitrariness in the choice of the number of boundary elements, the number of elements of discretization of the domain of local inhomogeneity Ω_1, and the time step.

The numerical experiments reveal the necessity of taking into account the coordinate dependences of the heat-conduction coefficient and specific heat capacity of the material of the domain caused by the fact that the relative errors of the values of heat field obtained with and without regard for these dependences attain 8–10% and do not decrease with time.

The modular principle of the program realization of this approach makes it possible to unify the development of its components and leads to an increase in the universality and flexibility of the constructed mathematical model used for the solution of similar problems in piecewise homogeneous bodies with local domains of inhomogeneity. The accumulated results can be useful for the development of contemporary procedures of identification of local inhomogeneities in solid thermoelastic bodies.

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