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TIGHTNESS OF SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED PSEUDO-POISSON PROCESSES IN THE SKOROKHOD SPACE

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We consider a pseudo-Poisson process of the following simple type. This process is a Poissonian subordinator for a sequence of i.i.d. random variables with finite variance. Further we consider sums of i.i.d. copies of a pseudo-Poisson process. For a family of distributions of these random sums, we prove the tightness (relative compactness) in the Skorokhod space. Under the conditions of the Central Limit Theorem for vectors, we establish the weak convergence in the functional Skorokhod space of the examined sums to the Ornstein–Uhlenbeck process. Bibliography: 3 *titles.*

Let us define a process of Poisson Stochastic Index $\psi(s)$, $s \geq 0$, by the following Poisson random change of "mathematical" time for a sequence of random variables $(\xi) = \xi_0, \xi_1, \ldots$:

$$\psi(s) \stackrel{\triangle}{=} \xi_{\Pi(s)}, \tag{$\star$}$$

where we assume that the leading Poisson process $\Pi(s)$ with intensity $\lambda > 0$ is independent of the sequence (ξ). The sequence (ξ) is said to be "forming" or "subordinated." We use the abbreviation PSI for the "Poisson Stochastic Index." Let us note that processes PSI are Poisson subordinators for sequences.

Trajectories of processes PSI are right-continuous, with left limits (RCLL), piecewise constant, having jumps only at moments of jumps of the leading Poisson process, but not necessarily at all these moments.

Processes PSI are studied in the classical monograph [2, Chap. X] (they are called "Pseudo-Poisson processes" there) in the case where the forming sequence is a Markov chain. In this case, PSI processes are called pseudo-Poisson processes. Remark that under the Markov property for the forming sequence, the corresponding pseudo-Poisson process is Markov and has continuous time. Thus, the Poisson subordination for sequences allows one to embed these sequences into continuous time preserving the Markov property.

In this paper, we consider sums of independent identically distributed (i.i.d.) pseudo-Poisson processes for which every forming sequence consists of i.i.d. random variables. In this case, even a sum of two independent PSI processes loses the Markov property; nevertheless, the limit of infinite sums of terms of i.i.d. pseudo-Poisson processes (under a proper normalization) gets the Markov property again since the limit of these sums is the Ornstein–Uhlenbeck process, which we canonically define as a stationary Gaussian Markov process. Here the convergence of finite-dimensional distributions is a direct corollary of the Central Limit Theorem for vectors. A difficultly in obtaining the corresponding functional limit theorem is the tightness (relative compactness) of a family of pre-limit distributions. Thus, the main goal of the paper is a proof of this tightness.

Let us fix a large enough $N \in \mathbf{N}$ and consider a sum of i.i.d. processes PSI (ψ_i) (independent copies of the process $\psi(s)$) given on an interval $[0, \Theta]$,

$$\Psi_N(s) = \sum_{i=1}^{N} \psi_i(s), \quad s \in [0, \Theta]. \tag{1}$$

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We denote the corresponding forming sequences of the processes (ψ_i) by $(\xi)(i) = \xi_0(i), \xi_1(i), \dots,$ $i = 1, \dots, N$. All the random variables $(\xi_n(i))$ are assumed to be totally independent over all combinations of indices $n \geq 0$, $i = 1, \dots, N$, and identically distributed with zero mean and variance $1/N$. All the leading Poisson processes are totally independent and they are independent of the forming sequences.

Theorem 1. *Let us consider random broken lines constructed by values of the process* $\Psi_N(s)$ *as elements of the Skorokhod space* $D_{[0,\Theta]}$, $\Theta < \infty$. *The family of these broken lines is a relatively compact set in* $D_{[0,\Theta]}$.

Proof. Let us fix points $0 \leq v < u < s \leq \Theta$ of the time axis. To prove the tightness of the family of distributions generated by these broken lines, we use the following criterion from P. Billingsley's monograph [1, p. 128]:

$$\mathbf{P}\big\{|\Psi_N(u) - \Psi_N(v)| \geq \varepsilon, \quad |\Psi_N(s) - \Psi_N(u)| \geq \varepsilon\big\} \leq \frac{1}{\varepsilon^{2\gamma}} \, |F(s) - F(v)|^{2\alpha} \tag{2}$$

for some continuous function F that is nondecreasing on $[0, \Theta]$ and for parameters $\gamma \geq 0$ and $\alpha > 1/2$ for every positive $\varepsilon < 1$.

The main idea is to split the set of summands which forms the random function Ψ into special subsets on which an independence appears. Then we use this independence in our proof (in one or another way).

We split the set of summation indices $\{1, \dots, N\}$ in (1) into four random subsets taking into account whether there are any jumps of the leading Poisson processes on the time intervals $[v, u)$ and $[u, s]$ or not. As we show below, the following random set plays the main role here:

$$\mathcal{A} = \{i = i(\omega) : \ \Pi_i(u-)(\omega) > \Pi_i(v)(\omega), \, \Pi_i(s)(\omega) > \Pi_i(u)(\omega)\} \, . \tag{3}$$

In other words, the set $\mathcal{A}$ is the set of indices for processes (ψ_i) that have replacements (with their independent copies) of terms of the forming sequences both on the interval $[v, u)$ and on the interval $[u, s]$. The set $\mathcal{A}$ is measurable with respect to the direct product of σ-algebras generated by trajectories of N independent Poisson processes up to moment s (including this moment), $\bigotimes_{i=1}^{N} \sigma\{\Pi_i(\leq s)\}$.

In a similar way, we define the following random subsets. $\mathcal{B}$ is the set of indices of processes (ψ_i) that have replacements of terms of the forming sequences on the interval $[v, u)$ and have no replacements on the interval $[u, s]$; $\mathcal{C}$ is the set of indices of processes (ψ_i) that have no replacements of terms of the forming sequences on the interval $[v, u)$ and have replacements on the interval $[u, s]$; $\mathcal{D}$ is the set of indices of processes (ψ_i) that have no replacements of terms of the forming sequences both on the interval $[v, u)$ and on the interval $[u, s]$.

Formally,

$$\mathcal{B} = \{i : \ \Pi_i(u-) > \Pi_i(v), \, \Pi_i(s) = \Pi_i(u)\} \, ,$$
$$\mathcal{C} = \{i : \ \Pi_i(u-) = \Pi_i(v), \, \Pi_i(s) > \Pi_i(u)\} \, ,$$

and

$$\mathcal{D} = \{i : \ \Pi_i(u-) = \Pi_i(v), \, \Pi_i(s) = \Pi_i(u)\} \, .$$

Let us note that though the sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ are random, the Law of Large Numbers for the Poisson indicators implies that the cardinalities of these sets are asymptotically degenerate, i.e., the following equivalences are valid as $N \to \infty$:

$$\frac{\#\mathcal{A}}{N} \sim \left(1 - e^{-\lambda(u-v)}\right)\left(1 - e^{-\lambda(s-u)}\right) \stackrel{\triangle}{=} e_{\mathcal{A}},$$
$$\frac{\#\mathcal{B}}{N} \sim \left(1 - e^{-\lambda(u-v)}\right) e^{-\lambda(s-u)} \stackrel{\triangle}{=} e_{\mathcal{B}},$$
$$\frac{\#\mathcal{C}}{N} \sim e^{-\lambda(u-v)}\left(1 - e^{-\lambda(s-u)}\right) \stackrel{\triangle}{=} e_{\mathcal{C}},$$

and

$$\frac{\#\mathcal{D}}{N} \sim e^{-\lambda(u-v)}\, e^{-\lambda(s-u)} \stackrel{\triangle}{=} e_{\mathcal{D}}.$$

It is obvious that $\#\mathcal{A} + \#\mathcal{B} + \#\mathcal{C} + \#\mathcal{D} = N$ (where the sign $\#$ denotes the cardinality of a set).

These equivalences obviously imply the following equalities for the mathematical expectations of the indicators of the events that the index of the corresponding process PSI belongs to one (and only one) random set $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, or $\mathcal{D}$:

$$\mathbf{E}\,\mathbb{1}_{\mathcal{A}}(i) = \mathbf{E}\,\mathbb{1}(i \in \mathcal{A}) = e_{\mathcal{A}}, \quad \mathbf{E}\,\mathbb{1}_{\mathcal{B}}(i) = \mathbf{E}\,\mathbb{1}(i \in \mathcal{B}) = e_{\mathcal{B}},$$

$$\mathbf{E}\,\mathbb{1}_{\mathcal{C}}(i) = \mathbf{E}\,\mathbb{1}(i \in \mathcal{C}) = e_{\mathcal{C}}, \quad \mathbf{E}\,\mathbb{1}_{\mathcal{D}}(i) = \mathbf{E}\,\mathbb{1}(i \in \mathcal{D}) = e_{\mathcal{D}},$$

for all $i = 1, \ldots, N$ and for each of the introduced four types of indicators.

Additionally, we need in estimations the second moments for the cardinalities of the random sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. We easily calculate these characteristics applying the calculated above mathematical expectations for i.i.d. indicators of type $\mathbb{1}_{\mathcal{A}}(i)$, $i = 1, \ldots, N$, and for the sets $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, respectively.

Let us consider the increments $\bigl(\Psi_N(u) - \Psi_N(v)\bigr)$ and $\bigl(\Psi_N(s) - \Psi_N(u)\bigr)$ from the criterion of relative compactness. Let us also stipulate that we exclude from our considerations the case where any leading Poisson process has a jump at level u since the corresponding probability equals zero. Note that the given increments are dependent because they may have identical terms (from the forming sequences) at level u. Note that on the sets $\mathcal{B}$ and $\mathcal{C}$, one of the considered increments is equal to zero, and on $\mathcal{D}$, both given increments are equal to zero. We also note that both considered increments are independent on all pairwise disjoint sets of summation of the processes ψ.

Let us denote

$$X_{\mathcal{F}} \stackrel{\triangle}{=} \sum_{i \in \mathcal{F}} \bigl(\psi_i(u) - \psi_i(v)\bigr), \qquad \mathcal{F} = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}.$$

By the construction of the sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$, the increments $X_{\mathcal{C}}$ and $X_{\mathcal{D}}$ take zero values; hence, $\Psi_N(u) - \Psi_N(v) = X_{\mathcal{A}} + X_{\mathcal{B}}$. Analogously we introduce the notation for the interval $[\,u, s\,]$:

$$Z_{\mathcal{F}} \stackrel{\triangle}{=} \sum_{i \in \mathcal{F}} \bigl(\psi_i(s) - \psi_i(u)\bigr), \qquad \mathcal{F} = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}.$$

The increments $Z_{\mathcal{B}}$ and $Z_{\mathcal{D}}$ take zero values; thus, the equality $\Psi_N(s) - \Psi_N(u) = Z_{\mathcal{A}} + Z_{\mathcal{C}}$ is valid. Hence, the probability from the left-hand side of (2) satisfies the following chain of estimates:

$$\begin{aligned}
\mathbf{P}\big\{|\Psi_N(u)-\Psi_N(v)|\geq\varepsilon,\ |\Psi_N(s)-\Psi_N(u)|\geq\varepsilon\big\}&\\
=\mathbf{P}\big\{|X_{\mathcal{A}}+X_{\mathcal{B}}|\geq\varepsilon,\ |Z_{\mathcal{A}}+Z_{\mathcal{C}}|\geq\varepsilon\big\}&\\
\leq\mathbf{P}\big\{|X_{\mathcal{A}}|+|X_{\mathcal{B}}|\geq\varepsilon,\ |Z_{\mathcal{A}}|+|Z_{\mathcal{C}}|\geq\varepsilon\big\}&\qquad(4)\\
\leq\ \mathbf{P}\big\{\{|X_{\mathcal{A}}|\geq\varepsilon/2\}\cup\{|X_{\mathcal{B}}|\geq\varepsilon/2\},\ \{|Z_{\mathcal{A}}|\geq\varepsilon/2\}\cup\{|Z_{\mathcal{C}}|\geq\varepsilon/2\}\big\}&\\
\leq\mathbf{P}\{|X_{\mathcal{A}}|\geq\varepsilon/2\big\}+\mathbf{P}\{|Z_{\mathcal{A}}|\geq\varepsilon/2\big\}+\mathbf{P}\{|X_{\mathcal{B}}|\geq\varepsilon/2,|Z_{\mathcal{C}}|\geq\varepsilon/2\}.&
\end{aligned}$$

In this chain, the latter probability contains two dependent events. However, these dependent events are conditionally independent under the condition of the σ-algebra which is generated by random sets of indices $\mathcal{B}$ and $\mathcal{C}$. Moreover, these two events contain pairs of conditionally independent summands if the cardinalities $\#\mathcal{B}$ and $\#\mathcal{C}$ are fixed.

Let us first make an agreement that every sum over the empty set of indices is equal to 0 and every product over the empty set of indices is equal to 1. First we estimate the latter probability at the end of expression (4). We use the formula of total probability (assuming that k and m are integer and nonnegative) and next the Chebyshev inequality:

$$\begin{aligned}
&\mathbf{P}\Big\{|X_{\mathcal{B}}|\geq\varepsilon/2,|Z_{\mathcal{C}}|\geq\varepsilon/2\Big\}\\
&=\sum_{(k,m):k+m\leq N}\mathbf{P}\Big\{|X_{\mathcal{B}}|\geq\varepsilon/2,|Z_{\mathcal{C}}|\geq\varepsilon/2;\{\#\mathcal{B}=k,\#\mathcal{C}=m\}\Big\}\\
&=\sum_{(k,m):k+m\leq N}\mathbf{P}\Big\{\Big|\sum_{i=1}^{k}\big(\xi_1(i)-\xi_0(i)\big)\Big|\geq\varepsilon/2,\Big|\sum_{j=k+1}^{k+m}\big(\xi_1(j)-\xi_0(j)\big)\Big|\geq\varepsilon/2;\{\#\mathcal{B}=k,\#\mathcal{C}=m\}\Big\}\\
&=\sum_{(k,m):k+m\leq N}\mathbf{P}\Big\{\Big|\sum_{i=1}^{k}\big(\xi_1(i)-\xi_0(i)\big)\Big|\geq\varepsilon/2\Big\}\qquad(5)\\
&\qquad\times\mathbf{P}\Big\{\Big|\sum_{j=k+1}^{k+m}\big(\xi_1(j)-\xi_0(j)\big)\Big|\geq\varepsilon/2;\Big\}\mathbf{P}\Big\{\#\mathcal{B}=k,\#\mathcal{C}=m\Big\}\\
&\leq\sum_{(k,m):k+m\leq N}\frac{4k\mathbf{D}(\xi_1(1)-\xi_0(1))}{\varepsilon^2}\,\frac{4m\mathbf{D}(\xi_1(1)-\xi_0(1))}{\varepsilon^2}\times\mathbf{P}\Big\{\#\mathcal{B}=k,\#\mathcal{C}=m\Big\}\\
&=\frac{64}{N^2\varepsilon^4}\sum_{(k,m):k+m\leq N}km\mathbf{P}\Big\{\#\mathcal{B}=k,\#\mathcal{C}=m\Big\}=\frac{64}{N^2\,\varepsilon^4}\mathbf{E}\{\#\mathcal{B}\#\mathcal{C}\}.
\end{aligned}$$

Thus, we need to estimate the mathematical expectation of the product of the cardinalities of the random sets $\mathcal{B}$ and $\mathcal{C}$. This is not difficult if we calculate $\mathbf{E}\,(\#\mathcal{B})^2$, $\mathbf{E}\,(\#\mathcal{C})^2$, and $\mathbf{E}\,(\#\{\mathcal{B}\cup\mathcal{C}\})^2$.

We calculate $\mathbf{E}\,(\#\mathcal{B})^2$ using the fact that the leading Poisson processes are independent and the event $\mathcal{B}$ is determined by realizations of each of the N Poisson processes and taking into account the notation $e_{\mathcal{B}}$:

$$\begin{aligned}
\mathbf{E}\,(\#\mathcal{B})^2&=\mathbf{E}\Big(\sum_{i=1}^{N}\mathbb{I}_{\mathcal{B}}(i)\Big)^2=\sum_{i=1}^{N}\mathbf{E}\,\mathbb{I}_{\mathcal{B}}(i)+\sum_{i\neq k}^{N}\mathbf{E}\,\{\mathbb{I}_{\mathcal{B}}(i)\,\mathbb{I}_{\mathcal{B}}(k)\}\\
&=\sum_{i=1}^{N}\mathbf{E}\,\mathbb{I}_{\mathcal{B}}(i)+\sum_{i\neq k}^{N}\mathbf{E}\,\{\mathbb{I}_{\mathcal{B}}(i)\}\,\mathbf{E}\,\{\mathbb{I}_{\mathcal{B}}(k)\}=N\,e_{\mathcal{B}}+N\,(N-1)\,e_{\mathcal{B}}^2.
\end{aligned}$$

Analogously, we calculate

$$\mathbf{E}\,(\#\mathcal{C})^2 = N\,e_{\mathcal{C}} + N\,(N-1)\,e_{\mathcal{C}}^2.$$

Note that the set $\{\mathcal{B} \cup \mathcal{C}\}$ is the set of Poisson processes that have jumps precisely on one of the two intervals $[v,u)$ and $[u,s]$; moreover, the sets $\mathcal{B}$ and $\mathcal{C}$ are disjoint. Analogously, we calculate

$$\mathbf{E}\,(\#\{\mathcal{B} \cup \mathcal{C}\})^2 = \mathbf{E}\,(\#\mathcal{B} + \#\mathcal{C})^2 = N\,e_{\mathcal{B}\cup\mathcal{C}} + N\,(N-1)\,e_{\mathcal{B}\cup\mathcal{C}}^2,$$

where

$$e_{\mathcal{B}\cup\mathcal{C}} = \bigl(1 - e^{-\lambda(u-v)}\bigr)\,e^{-\lambda(s-u)} + e^{-\lambda(u-v)}\,\bigl(1 - e^{-\lambda(s-u)}\bigr).$$

After that, it is easy to calculate the claimed mathematical expectation of the product of the cardinalities of the random sets $\mathcal{B}$ and $\mathcal{C}$:

$$2\,\mathbf{E}\,\{\#\mathcal{B}\,\#\mathcal{C}\} = \mathbf{E}\,(\#\mathcal{B} + \#\mathcal{C})^2 - \mathbf{E}\,(\#\mathcal{B})^2 - \mathbf{E}\,(\#\mathcal{C})^2$$
$$= N\,C(v;u;s) + N\,(N-1)\,2\,\bigl(1 - e^{-\lambda(u-v)}\bigr)\,e^{-\lambda(s-v)}\,\bigl(1 - e^{-\lambda(s-u)}\bigr),$$

where the constant $C(v;u;s)$ depends only on the time cuts $(v;u;s)$.

Substitute the derived estimate for the mathematical expectation of the cardinalities of the random sets $\mathcal{B}$ and $\mathcal{C}$ into (5) to obtain the following estimate of the latter probability at the end of expression (4):

$$\mathbf{P}\bigl\{|X_{\mathcal{B}}| \geq \varepsilon/2,\ |Z_{\mathcal{C}}| \geq \varepsilon/2\bigr\} \leq \frac{64}{\varepsilon^4}\,\bigl(1 - e^{-\lambda(s-u)}\bigr)\,e^{-\lambda(s-v)}\,\bigl(1 - e^{-\lambda(u-v)}\bigr) + \frac{C}{\varepsilon^4\,N}, \tag{6}$$

where the constant C does not depend on ε and N. Estimate from above the value $\exp\{-\lambda\,(s-v)\}$ by 1 and choose a sufficiently large N (with the aim to get rid of the term $C/(\varepsilon^4\,N)$ in the right-hand side of (6); for instance, with the help of estimation of $\exp\{-\lambda\,(s-v)\}$ from above by 1). Thus, we derive the following estimate:

$$\mathbf{P}\bigl\{|X_{\mathcal{B}}| \geq \varepsilon/2,\ |Z_{\mathcal{C}}| \geq \varepsilon/2\bigr\} \leq \frac{64}{\varepsilon^4}\,\bigl(1 - e^{-\lambda(s-u)}\bigr)\bigl(1 - e^{-\lambda(u-v)}\bigr). \tag{7}$$

Finally, we estimate the probabilities $\mathbf{P}\bigl\{|X_{\mathcal{A}}| \geq \varepsilon/2\bigr\}$ and $\mathbf{P}\bigl\{|Z_{\mathcal{A}}| \geq \varepsilon/2\bigr\}$ at the end of inequality (4). As we see below, their estimates are similar to (7).

Let us consider the following events: $A_k = \{\omega : \#\mathcal{A}(\omega) = k\}$ for $k = 0,1,\ldots,N$. It is obvious that these events (A_k), $k = 0,1,\ldots,N$, are pairwise disjoint and the only possible. Thus, we apply the total probability formula (for convenience, with $\epsilon = \varepsilon/2$) and then use the conditional independence of $X_{\mathcal{A}}$ to show that

$$\mathbf{P}\bigl\{|X_{\mathcal{A}}| \geq \epsilon\bigr\} = \sum_{k\leq N} \mathbf{P}\Bigl\{\Bigl|\sum_{i=1}^{k}(\zeta_i - \eta_i)\Bigr| \geq \epsilon\Bigr\}\mathbf{P}\{A_k\},$$

where the sequences (η_i) and (ζ_i), $i \in \mathbf{N}$, are independent and each sequence is i.i.d. with distribution similar to that of $\xi_0(1)$. Then we apply the Chebyshev inequality taking into account that $\mathbf{E}\,(\xi_0(1))^2 = 1/N$:

$$\sum_{k\leq N} \mathbf{P}\Bigl\{\Bigl|\sum_{i=1}^{k}(\zeta_i - \eta_i)\Bigr| \geq \epsilon\Bigr\}\,\mathbf{P}\{A_k\} \leq \frac{2}{\epsilon^2}\sum_{k\leq N} k\,\mathbf{P}\{A_k\} = \frac{2}{\epsilon^2}\,\mathbf{E}\,\{\#\mathcal{A}\}. \tag{8}$$

It is obvious that the same estimate is valid for $\mathbf{P}\{|Z_{\mathcal{A}}| \geq \epsilon\}$. It remains to take into account that $\mathbf{E}\,\{\#\mathcal{A}\} = N e_{\mathcal{A}}$ since, as was noted above, $\mathbf{E}\,\mathbb{1}_{\mathcal{A}}(i) = e_{\mathcal{A}}$, $i = 1,\ldots,N$.

As a result, we derive the following estimate with the term in the right-hand side of the type of $e_{\mathcal{A}}$:

$$\mathbf{P}\bigl\{|X_{\mathcal{A}}| \geq \varepsilon/2\bigr\} + \mathbf{P}\bigl\{|Z_{\mathcal{A}}| \geq \varepsilon/2\bigr\} \leq \frac{4}{\varepsilon^2}\,\bigl(1 - e^{-\lambda(u-v)}\bigr)\,\bigl(1 - e^{-\lambda(s-u)}\bigr). \tag{9}$$

Combining (9) with (7), we obtain the following estimate for $0 < \varepsilon < 1$:

$$\mathbf{P}\bigl\{|\Psi_N(u) - \Psi_N(v)| \ge \varepsilon,\ |\Psi_N(s) - \Psi_N(u)| \ge \varepsilon\bigr\} \le \frac{68}{\varepsilon^4}\bigl(1 - e^{-\lambda(u-v)}\bigr)\bigl(1 - e^{-\lambda(s-u)}\bigr). \quad (10)$$

Let us reduce the right-hand side to the form needed in criterion (2),

$$\bigl(1-e^{-\lambda(u-v)}\bigr)\bigl(1-e^{-\lambda(s-u)}\bigr) \le \lambda\,(u-v)\,\lambda\,(s-u) \le \lambda^2\Bigl(\frac{(u-v)+(s-u)}{2}\Bigr)^2 = \frac{\lambda^2}{4}\,(s-v)^2. \quad (11)$$

Here we first use the inequality $1-e^{-x} \le x$ for $x \ge 0$, and then the classical inequality between the arithmetic mean and geometric mean.

Collecting estimates (10) and (11), we get for $0 < \varepsilon < 1$ the necessary inequality (2):

$$\mathbf{P}\bigl\{|\Psi_N(u) - \Psi_N(v)| \ge \varepsilon,\ |\Psi_N(s) - \Psi_N(u)| \ge \varepsilon\bigr\} \le \frac{17\,\lambda^2}{\varepsilon^4}(s-v)^2 \quad (12)$$

with the function $F(s) = \sqrt{17}\,\lambda\,s,\ s \in [0,\Theta]$, and parameters $\gamma = 2$ and $\alpha = 1$. It is the main case in criterion (2) (see the proof of this criterion in [1, p.128]).

Note that if $\varepsilon \ge 1$, inequality (12) remains the same, with the only replacement of ε^4 by ε^2. We also note that if one requires the existence of the fourth moment for terms of the forming sequences, then if it possible to derive an analog of inequality (12) for all positive ε. □

Corollary 1. *Let us consider random broken lines constructed by values of the process $\Psi_N(s)$ as elements of the Skorokhod space $D_{[0,\Theta]}$, $\Theta < \infty$. The family of these broken lines is a relatively compact (tight) set in the Skorokhod space $D_{[0,\infty)}$.*

The proof of Corollary 1 directly follows from the construction of the space $D_{[0,\infty)}$, which is constructed for every finite positive Θ on the base of the spaces $D_{[0,\Theta]}$.

Proposition 1 (Autocovariance for sums of PSI processes). *Consider, according to definition ($\star$), the PSI process $\xi'_{\Pi(s)}$, $s \ge 0$, for a forming sequence $(\xi'_0, \xi'_1, \ldots)$ consisting of i.i.d. random variables with zero mean and unit variance.*

For a natural N, consider the sum of i.i.d. processes PSI such that their leading Poisson processes are i.i.d. and their i.i.d. forming sequences consist, in turn, of i.i.d. random variables $\Psi_N(s) = \sum_{i=1}^{N} \psi_i(s)$, where $\psi_i(s) = \xi_{\Pi_i(s)}(i)$ is the ith independent copy for $\psi(s) = \xi_{\Pi(s)}$, and the following notation under the normalization by $\sqrt{N}$ is adopted: $\xi_0 = (1/\sqrt{N})\xi'_0$, $\xi_1 = (1/\sqrt{N})\xi'_1, \ldots$.

For any natural N and for every nonnegative s and r, the following equality for covariances is fulfilled:

$$\operatorname{cov}\bigl(\Psi_N(r), \Psi_N(r+s)\bigr) = \exp\{-\lambda\,s\}. \quad (13)$$

Note that the right-hand side of equality (13) does not depend of r, which proves the stationarity in the wide sense for the process $\Psi_N(s)$ for an arbitrary natural N.

Proof. First we calculate the covariance for one term in $\Psi(s)$. We refer to the independence between the random variables (ξ_j), $j \ge 0$, and the Poisson process $\Pi(s)$, to the equality $\mathbf{E}\,\xi_j^2 = 1/N$, and to the property of independence and homogeneity for increments of a Poisson process.

The following representation in the form of an infinite sum of random terms weighted with random indicators is obvious for a process PSI:

$$\xi_{\Pi(s)} \equiv \xi_\Pi(s) = \sum_{j=0}^{\infty} \xi_j\,\mathbb{I}\{\Pi(s) = j\}. \quad (14)$$

Applying representation (14), we obtain the following chain of equalities:

$$\begin{aligned}\operatorname{cov}\bigl(\xi_{\Pi(r)},\, \xi_{\Pi(r+s)}\bigr) &= \mathbf{E}\Bigl\{\sum_{j=0}^{\infty} \xi_j\, \mathbb{I}\{\Pi(r)=j\} \sum_{i=0}^{\infty} \xi_i\, \mathbb{I}\{\Pi(r+s)=i\}\Bigr\} \\ &= \mathbf{E}\Bigl\{\sum_{j=0}^{\infty} \xi_j^2\, \mathbb{I}\{\Pi(r)=\Pi(r+s)=j\}\Bigr\} = \sum_{j=0}^{\infty} \mathbf{E}\,\{\xi_j^2\}\, \mathbf{P}\{\Pi(r)=\Pi(r+s)=j\} \\ &= \frac{1}{N} \sum_{j=0}^{\infty} \mathbf{P}\{\Pi(r)=\Pi(r+s)=j\} = \frac{1}{N}\,\mathbf{P}\{\Pi(r)=\Pi(r+s)\} \\ &= \frac{1}{N}\,\mathbf{P}\{\Pi(0)=\Pi(s)\} = \frac{1}{N}\,\mathbf{P}\{\Pi(s)=0\} = \frac{1}{N}\,\exp\{-\lambda\, s\}.\end{aligned}$$

Since the random elements $\xi_{\Pi_i(s)}(i)$ and $\xi_{\Pi_j(s)}(j)$ are identically distributed and independent for $i \neq j$, the final calculations are very simple, and we conclude that

$$\operatorname{cov}\bigl(\Psi_N(r), \Psi_N(r+s)\bigr) = \exp\{-\lambda\, s\}. \qquad \square$$

Let us consider $(\Psi_N(s))$, $N \in \mathbf{N}$, according to Proposition 1. The assumptions of Proposition 1 allow us to apply the Central Limit Theorem for vectors; thus, the convergence of finite-dimensional distributions of $(\Psi_N(s))$, $s \geq 0$, takes place as $N \to \infty$. The tightness is proved in Theorem 1. Thus, the following functional limit theorem is valid.

Theorem 2. *The sequence of piecewise random functions $(\Psi_N(s))$ defined in Proposition 1 tends, as $N \to \infty$, to the Ornstein-Uhlenbeck process in the Skorokhod space $D_{[0,\Theta]}$ for every finite positive Θ as well as in the Skorokhod space $D_{[0,\infty)}$.*

The result obtained can be applied in the stochastic financial mathematics, first of all, in stochastic models for interest rates, because since the pioneering paper [3] of O. Vasićek, processes of Ornstein–Uhlenbeck type are the basic ones for description of dynamics of financial instruments of that kind.

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