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REALIZATION OF THE SPLINE-WAVELET DECOMPOSITION OF THE FIRST ORDER

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The aim of the paper is to present an orthogonal basis for the discrete wavelets in the general structure of the spline-wavelet decomposition. Decomposition of an original numerical flow without embedding in the standard functional spaces is discussed. It makes it possible to concentrate on simplification of the realization formulas: here, the simple formulas of decomposition and reconstruction are presented, an orthogonal wavelet basis is constructed, and an illustrative example is given. Finally, some estimates of the complexity of the method discussed for different software environments are provided. Bibliography: 6 *titles.*

Wavelet decompositions are widely used in treating numerical information flows; the volumes of flows constantly increase, which strongly stimulates further development of wavelet theory (see [1, 2]). The approach to the construction of wavelets used in the present paper is based on approximation relations (for example, see [3]), which ensures an efficient approximation automatically (most frequently, it is asymptotically optimal in the N-width of the standard compact sets).

Contrary to classical wavelet decompositions (see [1]), the approach proposed allows us to use irregular grids (both finite and infinite) without complicated investigations; this is very useful for saving computer resources in the case where singular variations of the flows considered occur. Besides, the well-known finite element approximations of R. Courant, M. Zlamal, and others can be applied in constructing wavelet decompositions in multivariate cases (and even on an arbitrary differentiable manifold, see [4]), which gives new opportunities of applying wavelets.

In the classical case, the construction of a wavelet basis in a certain functional space (frequently, in $L_2(\mathbb{R}^1)$) is (as a rule) quite complicated. The approach used in this paper does not require a preliminary construction of a wavelet basis (if desired, such a basis can be constructed after carrying out the main investigations). On the other hand, the knowledge of a wavelet basis allows one to save computer and network resources significantly.

In this connection, we note that we do not need the wavelet basis in a space of functions with a continual domain of definition. It is sufficient to have a suitable basis for the linear space of numerical wavelet flows. Hence it is only necessary to use functions with a discrete domain of definition (i.e., grid functions).

The aim of the present paper is to discuss the discrete spline-wavelet decomposition of the first order and to obtain some estimates of computer resources required for implementing the algorithms suggested.

Here, all the proofs are given for grid functions (i.e., for functions defined on a certain grid, which are frequently referred to in what follows as numerical information flows), and do not use mappings into function spaces with a continuous domain.

Now we will introduce some specific definitions, in particular, a finite dimensional space of original flows, a space of wavelet flows, and a space of main flows, which are associated with an original grid and with a coarsened grid, respectively. It has been proven that the discrete wavelet basis is the simplest set of unit coordinate vectors in the Euclidean space. As a result,

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simple formulas of decomposition and reconstruction are obtained. Computer resources needed for implementing the algorithms proposed for different software environments are evaluated. An illustrative example of the spline-wavelet decomposition is presented.

The paper is organized as follows. It consists of twelve sections. The first one presents the main idea of the general approach to the spline-wavelet decomposition on a one-dimensional model example. Here, for simplicity, the domain of definition of the functions considered is a continuous interval of the real axis. The second section introduces the original grid, which is the domain of definition for the grid function (specifically, for the original numerical flow).

In Sec. 3, we consider a coarsening of the original grid, the new grid being the domain of definition for the main flow. Section 4 is devoted to the calibration relations, connecting the discrete basis splines on the coarse grid with the discrete basis splines on the original grid.

Sections 5 and 6 discuss the discrete spline-wavelet decomposition. Here, the decomposition formulas in the general form are obtained. These formulas are specified for the situation considered in Secs. 7 and 8, where the coefficients of the above-mentioned formulas are computed based on representations of discrete splines. An illustrative example of a discrete wavelet decomposition is provided in Sec. 9.

The continuous image of the decomposition discussed is examined in Sec. 10. Section 11 is devoted to different variants of algorithms for an adaptive coarsening of the original grid. In Sec. 12, the time of computing the spline-wavelet decomposition with account for the software environment is evaluated.

1. The main idea of spline-wavelet decomposition (a nonclassical approach)

For the reader's convenience, the general idea of spline-wavelet decomposition (see [5]) is considered on a one-dimensional example. Here, for simplicity, the function domain is an interval of the real axis.

Let $\mathbb{L}$ be a linear function space defined on an interval $(\alpha,\beta)\subset\mathbb{R}^1$. Consider a grid

$$X : \ldots < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots, \tag{1.1}$$

$$\lim_{j\to-\infty} x_j = \alpha, \qquad \lim_{j\to-\infty} x_j = \beta, \tag{1.2}$$

and let $\varphi(t) \stackrel{\text{def}}{=} (\varphi_0(t),\varphi_1(t),\ldots,\varphi_m(t))$, $t\in(\alpha,\beta)$, be a vector function with components from the space $\mathbb{L}$: $\varphi_i\in\mathbb{L}$, $i\in\{0,1,\ldots,m\}$.

Consider a set G, $G \stackrel{\text{def}}{=} \{g^{(s)}\}_{s\in\mathbb{Z}}$, of linear functionals $g^{(s)}\in\mathbb{L}^*$ such that

$$\operatorname{supp} g^{(s)} \subset (x_s, x_{s+1}), \quad s\in\mathbb{Z}. \tag{1.3}$$

Let $\langle g^{(s)},u\rangle$ denote the result of the action of the functional $g^{(s)}$ on a function $u\in\mathbb{L}$. We set

$$\langle g^{(s)},\varphi\rangle \stackrel{\text{def}}{=} (\langle g^{(s)},\varphi_0\rangle,\langle g^{(s)},\varphi_1\rangle,\ldots,\langle g^{(s)},\varphi_m\rangle)^T.$$

Assume that

$$\det(\langle g^{(s)},\varphi\rangle,\langle g^{(s+1)},\varphi\rangle,\ldots,\langle g^{(s+m)},\varphi\rangle)\neq 0, \quad s\in\mathbb{Z}. \tag{1.4}$$

Let

$$\mathbf{a}_s \stackrel{\text{def}}{=} \langle g^{(s)},\varphi\rangle. \tag{1.5}$$

From (1.4) it follows that the vectors $\mathbf{a}_s,\mathbf{a}_{s+1},\ldots,\mathbf{a}_{s+m+1}$ are linearly independent.

Consider the approximation relations

$$\sum_{i=k-m}^{k} \mathbf{a}_i\omega_i(t) = \varphi(t), \quad t\in(x_k,x_{k+1}), \quad k\in\mathbb{Z}, \tag{1.6}$$

$$\operatorname{supp}\omega_s \subset [x_s, x_{s+m+1}], \qquad s\in\mathbb{Z}. \tag{1.7}$$

Relations (1.6)–(1.7) define the functions $\omega_i(t)$ uniquely.

Assume that $\omega_s \in \mathbb{L}$. By virtue of (1.3)–(1.7), we have

$$\langle g^{(j)}, \omega_s \rangle = \delta_{j,s}, \quad j, s \in \mathbb{Z}.$$

Consider the linear space $\mathbb{S} = \mathbb{S}(X, \varphi) \stackrel{\text{def}}{=\!=} \mathcal{L}\{\omega_s\}_{s\in\mathbb{Z}}$, where $\mathcal{L}$ is the linear hull of the functions in the braces.

If $\widetilde{X}$ is a subset of the grid X such that

$$\widetilde{X} : \ldots < \widetilde{x}_{-2} < \widetilde{x}_{-1} < \widetilde{x}_0 < \widetilde{x}_1 < \widetilde{x}_2 < \ldots,$$

$$\lim_{j\to-\infty} \widetilde{x}_j = \alpha, \qquad \lim_{j\to-\infty} \widetilde{x}_j = \beta, \qquad \widetilde{X} \subset X,$$

then we can define the functions $\widetilde{\omega}_i$, connected with the new grid $\widetilde{X}$, as above and consider the linear space $\widetilde{\mathbb{S}} = \mathbb{S}(\widetilde{X}, \varphi)$, which (under certain conditions) is a subspace of the space $\mathbb{S}(X, \varphi)$.

Consider the operator P, which projects the space $\mathbb{S}$ onto the space $\widetilde{\mathbb{S}}$:

$$Pu \stackrel{\text{def}}{=\!=} \sum_{s\in\mathbb{Z}} \langle \widetilde{g}^{(s)}, u \rangle \omega_s, \quad u \in \mathbb{S}(X, \varphi), \tag{1.8}$$

where $\{\widetilde{g}^{(s)}\}_{s\in\mathbb{Z}}$ is the fixed system of functionals biorthogonal to the system $\{\widetilde{\omega}_i\}_{i\in\mathbb{Z}}$.

Obviously, if a point $t \in (\widetilde{x}_k, \widetilde{x}_{k+1})$ is fixed, then the right-hand side of (1.8) involves at most $m + 1$ terms,

$$Pu(t) \stackrel{\text{def}}{=\!=} \sum_{s=k-m}^{k} \langle \widetilde{g}^{(s)}, u \rangle \omega_s(t), \quad t \in (\widetilde{x}_k, \widetilde{x}_{k+1}). \tag{1.9}$$

The projection operator P specifies the wavelet decomposition

$$\mathbb{S} = \widetilde{\mathbb{S}} \dot{+} \mathbb{W}. \tag{1.10}$$

Let $\mathbf{c} = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$ be an original numerical flow. Consider the function

$$u(t) \stackrel{\text{def}}{=\!=} \sum_j c_j \omega_j(t); \tag{1.11}$$

its projection $\widetilde{u} \stackrel{\text{def}}{=\!=} Pu$ onto the space $\widetilde{\mathbb{S}}$ can be written in the form

$$\widetilde{u} = \sum_i a_i \widetilde{\omega}_i. \tag{1.12}$$

Thus, we have the main numerical flow

$$\mathbf{a} \stackrel{\text{def}}{=\!=} (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots),$$

which corresponds to the coarsening $\widetilde{X}$ of the grid X, and also the wavelet flow

$$\mathbf{b} \stackrel{\text{def}}{=\!=} (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots),$$

which is defined by the expansion of the difference $w \stackrel{\text{def}}{=\!=} u - \widetilde{u}$ in the basis of the space $\mathbb{S}$: $w = \sum_s b_s \omega_s$ (see (1.9)–(1.12)).

The passage from the original flow $\mathbf{c}$ to the flows $\mathbf{a}$ and $\mathbf{b}$ is called *decomposition*, and the backward passage is referred to as the *reconstruction* of the original flow.

The decomposition and reconstruction formulas can be represented in matrix form as $\mathbf{a} = \mathfrak{Q}\mathbf{c}$, $\mathbf{b} = \mathbf{c} - \mathfrak{P}^T \mathfrak{Q}\mathbf{c}$, and $\mathbf{c} = \mathbf{b} + \mathfrak{P}^T \mathbf{a}$, where $\mathfrak{P}$ and $\mathfrak{Q}$ are the restriction and prolongation matrices, respectively.

In particular, for $m = 1$, $\varphi(t) = (1,t)^T$, and $X \setminus \widetilde{X} = \{x_{k+1}\}$, the decomposition formulas are as follows:

$$a_i = c_i \quad \text{for} \quad i \le k-1, \qquad a_i = c_{i+1} \quad \text{for} \quad i \ge k,$$

$$b_j = 0 \quad \text{for} \quad j \ne k, \qquad b_k = -\frac{x_{k+2} - x_{k+1}}{x_{k+2} - x_k} \cdot c_{k-1} + c_k - \frac{x_{k+1} - x_k}{x_{k+2} - x_k} \cdot c_{k+1},$$

whereas the reconstruction formulas can be written as

$$c_j = a_j + b_j \quad \text{for} \quad j \le k-1,$$

$$c_k = \frac{x_{k+2} - x_{k+1}}{x_{k+2} - x_k} \cdot a_{k-1} + \frac{x_{k+1} - x_k}{x_{k+2} - x_k} \cdot a_k + b_k,$$

$$c_j = a_{j-1} + b_j \quad \text{for} \quad j \ge k+1.$$

Note that if the closed interval $[a,b]$ is contained in the interval (α,β), then all the previous constructions hold for the restrictions of the functions considered to $[a,b]$. In this case, the grids, as well as the original, main, and wavelet flows, are finite.

This completes the description of the main idea of constructing spline-wavelet decompositions, and we pass to the realization of the idea, assuming that only grid functions are considered.

2. Notation

In contrast to the previous section, everywhere in what follows (except for Sec. 11) we consider grid functions with grid domains of the form (1.1)–(1.2). This is convenient in treating numerical flows, which are number sequences and can be regarded as functions defined on a grid (for example, on the set of integers).

On an interval (α,β), consider a grid

$$\Xi: \quad \ldots < \xi_{-2} < \xi_{-1} < \xi_0 < \xi_1 < \xi_2 \ldots, \tag{2.1}$$

$$\lim_{i \to -\infty} \xi_i = \alpha, \qquad \lim_{i \to +\infty} \xi_i = \beta. \tag{2.2}$$

Let $C(\Xi)$ be the set of functions $u(t)$ defined on the grid Ξ. Obviously, $C(\Xi)$ is a linear space. Below, we assume that

$$a, b \in \Xi, \qquad a^+ < b^-, \tag{2.3}$$

i.e., the equalities $a = \xi_i$ and $b = \xi_j$ hold for some $i, j \in \mathbb{Z}$, $i + 2 < j$. Set $[\![a,b]\!] \stackrel{\text{def}}{=} \{\xi_s \mid a \le \xi_s \le b,\ s \in \mathbb{Z}\}$, i.e., $[\![a,b]\!] = \{\xi_s \mid i \le s \le j,\ s \in \mathbb{Z}\}$. The set $[\![a,b]\!]$ is called a grid segment.

Let $C[\![a,b]\!]$ be the linear normed space of functions $u(t)$ defined on a grid segment $[\![a,b]\!]$, where the norm is defined by

$$\|u\|_{C[\![a,b]\!]} \stackrel{\text{def}}{=} \max_{t \in [\![a,b]\!]} |u(t)|. \tag{2.4}$$

Obviously, the space $C[\![a,b]\!]$ has finite dimension.

3. Grid coarsening

Let m be a positive integer. Consider the sets

$$J_m \stackrel{\text{def}}{=} \{0, 1, \ldots, m\}, \qquad J'_m \stackrel{\text{def}}{=} \{-1, 0, 1, \ldots, m\}.$$

Let $[\![a,b]\!]$, $a = \xi_0 < \xi_1 < \ldots < \xi_{M-1} < \xi_M = b$, be a grid segment, let $\{\omega_j(t)\}_{j \in J'_{M-1}}$ be the functions from the space $C[\![a,b]\!]$ (see (2.3)–(2.4)) defined by

$$\omega_j(\xi_s) = \delta_{s,j+1}, \qquad s \in J_M, \tag{3.1}$$

and let the linear functionals $g^{(i)}$, $i \in J'_{M-1}$, be defined by the relations

$$\langle g^{(i)}, u\rangle \stackrel{\text{def}}{=\!=} u(\xi_{i+1}), \qquad u \in C[\![a,b]\!]. \tag{3.2}$$

The system $\{\omega_j\}_{j\in J'_{M-1}}$ is a basis of the space $C[\![a,b]\!]$, which is called the *discrete basis.*

Lemma 1. *The functionals $\{g^{(i)}\}_{i\in J'_{M-1}}$ are biorthogonal to the discrete basis $\{\omega_j\}_{j\in J'_{M-1}}$, i.e.,*

$$\langle g^{(i)}, \omega_j\rangle = \delta_{i,j}, \qquad i,j \in J'_{M-1}. \tag{3.3}$$

Proof. Relations (3.3) readily follow from (3.1)–(3.2). □

In the sequel, we adopt the convention that if $c > d$, then the set $[\![c,d]\!]$ is empty.

Consider an integer K such that $5 \le K < M$. Let $\varkappa$ be an injective map of the set J_K into J_M such that

$$\varkappa(0) = 0, \quad \varkappa(i) < \varkappa(i+1), \quad \varkappa(K) = M. \tag{3.4}$$

Introduce the set $J^* \subset J_M$,

$$J^* \stackrel{\text{def}}{=\!=} \varkappa J_K. \tag{3.5}$$

Obviously, on this set the univalent inverse map $\varkappa^{-1}: r \longrightarrow s$, $r \in J^*$, $s \in J_K$, $J_K = \varkappa^{-1}J^*$ is defined.

Consider the new grid

$$\widehat{X}: \quad a = \widehat{x}_0 < \widehat{x}_1 < \ldots < \widehat{x}_K = b, \tag{3.6}$$

where $\widehat{x}_i \stackrel{\text{def}}{=\!=} \xi_{\varkappa(i)}$, $i \in J_K$.

Remark 1. In the sequel, we sometimes consider the *virtual* nodes ξ_{-1} and $\widehat{x}_{-1}$ of the grids $[\![a,b]\!]$ and $\widetilde{X}$ such that $\xi_{-1} = \widehat{x}_{-1} < a$; the above-mentioned nodes are virtual in the sense that they appear in formulas but do not affect the final results.

Let $\widehat{\omega}_j(t)$, $j \in J'_{K-1}$, be the functions defined in accordance with the formulas

$$\widehat{\omega}_{-1}(t) = (\widehat{x}_1 - t)(\widehat{x}_1 - \widehat{x}_0)^{-1}, \quad t \in [\![\widehat{x}_0, \widehat{x}_1^{-}]\!], \tag{3.7}$$

$$\widehat{\omega}_{-1}(t) = 0, \quad t \in [\![a,b]\!] \setminus [\![\widehat{x}_0, \widehat{x}_1^{-}]\!], \tag{3.8}$$

$$\widehat{\omega}_i(t) = (t - \widehat{x}_i)(\widehat{x}_{i+1} - \widehat{x}_i)^{-1}, \quad t \in [\![\widehat{x}_i^{+}, \widehat{x}_{i+1}^{-}]\!], \tag{3.9}$$

$$\widehat{\omega}_i(t) = (\widehat{x}_{i+2} - t)(\widehat{x}_{i+2} - \widehat{x}_{i+1})^{-1}, \quad t \in [\![\widehat{x}_{i+1}, \widehat{x}_{i+2}^{-}]\!], \tag{3.10}$$

$$\widehat{\omega}_i(t) = 0, \quad t \in [\![a,b]\!]\setminus[\![\widehat{x}_i^{+}, \widehat{x}_{i+2}^{-}]\!], \quad i \in J_{K-2}, \tag{3.11}$$

$$\widehat{\omega}_{K-1}(t) = (t - \widehat{x}_{K-1})(\widehat{x}_K - \widehat{x}_{K-1})^{-1}, \quad t \in [\![\widehat{x}_{K-1}^{+}, \widehat{x}_K]\!], \tag{3.12}$$

$$\widehat{\omega}_{K-1}(t) = 0, \quad t \in [\![a,b]\!]\setminus[\![\widehat{x}_{K-1}^{+}, \widehat{x}_K]\!]. \tag{3.13}$$

In terms of the nodes of the original grid, formulas (3.7)–(3.13) can be written as follows:

$$\widehat{\omega}_{-1}(t) = (\xi_{\varkappa(1)} - t)(\xi_{\varkappa(1)} - \xi_{\varkappa(0)})^{-1}, \quad t \in [\![\xi_{\varkappa(0)}, \xi_{\varkappa(1)}^{-}]\!], \tag{3.14}$$

$$\widehat{\omega}_{-1}(t) = 0, \quad t \in [\![a,b]\!]\setminus[\![\xi_{\varkappa(0)}, \xi_{\varkappa(1)}^{-}]\!], \tag{3.15}$$

$$\widehat{\omega}_i(t) = (t - \xi_{\varkappa(i)})(\xi_{\varkappa(i+1)} - \xi_{\varkappa(i)})^{-1}, \quad t \in [\![\xi_{\varkappa(i)}^{+}, \xi_{\varkappa(i+1)}]\!], \tag{3.16}$$

$$\widehat{\omega}_i(t) = (\xi_{\varkappa(i+2)} - t)(\xi_{\varkappa(i+2)} - \xi_{\varkappa(i+1)})^{-1}, \quad t \in [\![\xi_{\varkappa(i+1)}^{+}, \xi_{\varkappa(i+2)}^{-}]\!], \tag{3.17}$$

$$\widehat{\omega}_i(t) = 0, \quad t \in [\![a,b]\!]\setminus[\![\xi_{\varkappa(i)}^{+}, \xi_{\varkappa(i+2)}^{-}]\!], \quad i \in J_{K-2}, \tag{3.18}$$

$$\widehat{\omega}_{K-1}(t) = (t - \xi_{\varkappa(K-1)})(\xi_{\varkappa(K)} - \xi_{\varkappa(K-1)})^{-1}, \quad t \in [\![\xi_{\varkappa(K-1)}^{+}, \xi_{\varkappa(K)}]\!], \tag{3.19}$$

$$\widehat{\omega}_{K-1}(t) = 0, \quad t \in [\![a,b]\!]\setminus[\![\xi_{\varkappa(K-1)}^{+}, \xi_{\varkappa(K)}]\!]. \tag{3.20}$$

If $t \in [\![a, b]\!]$, then

$$\widehat{\omega}_i(t) = (t - \xi_{\varkappa(i)})(\xi_{\varkappa(i+1)} - \xi_{\varkappa(i)})^{-1}, \quad t \in [\![\xi^{\,+}_{\varkappa(i)}, \xi_{\varkappa(i+1)}]\!], \quad i \in J_{K-1}, \tag{3.21}$$

$$\widehat{\omega}_i(t) = (\xi_{\varkappa(i+2)} - t)(\xi_{\varkappa(i+2)} - \xi_{\varkappa(i+1)})^{-1}, \quad t \in [\![\xi_{\varkappa(i+1)}, \xi^{\,-}_{\varkappa(i+2)}]\!], \quad i \in J'_{K-2}; \tag{3.22}$$

$$\widehat{\omega}_i(t) = 0, \quad t \in [\![a, b]\!] \backslash [\![\xi^{\,+}_{\varkappa(i)}, \xi^{\,-}_{\varkappa(i+2)}]\!]. \tag{3.23}$$

Obviously, we have

$$\widehat{\omega}_i(\xi_{\varkappa(i+1)}) = 1, \quad i \in J'_{K-1}. \tag{3.24}$$

Remark 2. The set of nodes $t \in X$ at which a function $f \in C(X)$ is nonvanishing is called the support of the function f: $\operatorname{supp} f \overset{\text{def}}{=\!=} \{t \mid f(t) \neq 0\}$. The number of functions $\widehat{\omega}_i$ in the set $\{\widehat{\omega}_i \mid t \in \operatorname{supp}\widehat{\omega}_i\}$ is called the *multiplicity of the covering of the point* t by the supports of the functions $\widehat{\omega}_i$ and is denoted by $\kappa(t)$. By virtue of (3.21)–(3.24), we have $1 \leq \kappa(t) \leq 2$ and $\kappa(t) = 2$ for any $t \in X \setminus \widehat{X}$, i.e.,

$$\kappa(\xi_s) = 2, \qquad s \in J_M \setminus J^*. \tag{3.25}$$

In what follows, we use the notation

$$\operatorname{supp} \widehat{\omega}_i = [\![\widehat{x}^{+}_i, \widehat{x}^{\,-}_{i+2}]\!]. \tag{3.26}$$

4. Calibration relations

The splines $\widehat{\omega}_i$ can be represented as linear combinations of the splines ω_j,

$$\widehat{\omega}_i(t) = \sum_{j \in J'_{M-1}} \mathfrak{p}_{i,j} \omega_j(t), \quad t \in [\![a, b]\!], \quad i \in J'_{K-1}. \tag{4.1}$$

Relations (4.1) are called the *calibration relations*.

If $t \in X$ is fixed, then in the sum (4.1) there is an only one nonzero term. This sum can be considered an expansion of the vector $\widehat{\omega}_i$ in the vector system $\{\omega_j\}$, orthogonal in the Euclidean space $\mathbb{R}^{M+1}$.

Lemma 2. *The values* $\mathfrak{p}_{i,j}$ *satisfy the relations*

$$\mathfrak{p}_{-1,j} = \widehat{\omega}_{-1}(\xi_{j+1}), \qquad j \in \{\varkappa(0) - 1, \varkappa(0), \ldots, \varkappa(1) - 2\}, \tag{4.2}$$

$$\mathfrak{p}_{i,j} = \widehat{\omega}_i(\xi_{j+1}), \qquad j \in \{\varkappa(i), \varkappa(i) + 1, \ldots, \varkappa(i+2) - 2\}, \quad i \in J_{K-2}, \tag{4.3}$$

$$\mathfrak{p}_{K-1,j} = \widehat{\omega}_{K-1}(\xi_{j+1}), \qquad j \in \{\varkappa(K-1), \varkappa(K-1) + 1, \ldots, \varkappa(K) - 1\}; \tag{4.4}$$

the values $\mathfrak{p}_{r,s}$ $(r \in J'_{K-1},\ s \in J'_{M-1})$ *not occurring in* (4.2)–(4.4) *are equal to zero.*

Proof. By applying the functionals $g^{(j)}$ to (4.1) and taking into account relations (3.2)–(3.3), we obtain

$$\widehat{\omega}_{-1}(t) = \sum_{j=\varkappa(0)-1}^{\varkappa(1)-2} \widehat{\omega}_{-1}(\xi_{j+1})\omega_j(t), \quad t \in [\![\xi_{\varkappa(0)}, \xi^{\,+}_{\varkappa(1)}]\!], \tag{4.5}$$

$$\widehat{\omega}_{-1}(t) = 0, \quad t \in [\![a, b]\!] \backslash [\![\xi_{\varkappa(0)}, \xi^{\,+}_{\varkappa(1)}]\!], \tag{4.6}$$

$$\widehat{\omega}_i(t) = \sum_{j=\varkappa(i)}^{\varkappa(i+2)-2} \widehat{\omega}_i(\xi_{j+1})\omega_j(t), \quad t \in [\![\xi^{\,+}_{\varkappa(i)}, \xi^{\,-}_{\varkappa(i+2)}]\!], \tag{4.7}$$

$$\widehat{\omega}_i(t) = 0, \quad t \in [\![a, b]\!] \backslash [\![\xi^{\,+}_{\varkappa(i)}, \xi^{\,-}_{\varkappa(i+2)}]\!], \quad i \in J_{K-2}, \tag{4.8}$$

$$\widehat{\omega}_{K-1}(t)=\sum_{j=\varkappa(K-1)}^{\varkappa(K)-1}\widehat{\omega}_{K-1}(\xi_{j+1})\omega_j(t),\quad t\in[\![\xi^{\,+}_{\varkappa(K-1)},\xi_{\varkappa(K)}]\!],\tag{4.9}$$

$$\widehat{\omega}_{K-1}(t)=0,\quad t\in[\![a,b]\!]\backslash[\![\xi^{\,+}_{\varkappa(K-1)},\xi_{\varkappa(K)}]\!].\tag{4.10}$$

Relations (4.2)–(4.4) follow from (4.5)–(4.10). □

Lemma 3. *The values* $\mathfrak{p}_{i,j}$ *can be written in the following form:*

$$\begin{aligned}\mathfrak{p}_{-1,j}&=(\xi_{\varkappa(1)}-\xi_{\varkappa(0)})^{-1}(\xi_{\varkappa(1)}-\xi_{j+1}),\\ &\qquad j\in\{\varkappa(0)-1,\varkappa(0),\dots,\varkappa(1)-2\},\end{aligned}\tag{4.11}$$

$$\begin{aligned}\mathfrak{p}_{i,j}&=(\xi_{\varkappa(i+1)}-\xi_{\varkappa(i)})^{-1}(\xi_{j+1}-\xi_{\varkappa(i)}),\\ &\qquad j\in\{\varkappa(i),\varkappa(i)+1,\dots,\varkappa(i+1)-1\},\quad i\in J_{K-2},\end{aligned}\tag{4.12}$$

$$\begin{aligned}\mathfrak{p}_{i,j}&=(\xi_{\varkappa(i+2)}-\xi_{\varkappa(i+1)})^{-1}(\xi_{\varkappa(i+2)}-\xi_{j+1}),\\ &\qquad j\in\{\varkappa(i+1),\varkappa(i)+1,\dots,\varkappa(i+2)-2\},\ \ i\in J_{K-2},\end{aligned}\tag{4.13}$$

$$\begin{aligned}\mathfrak{p}_{K-1,j}&=(\xi_{\varkappa(K)}-\xi_{\varkappa(K-1)})^{-1}(\xi_{j+1}-\xi_{\varkappa(K-1)}),\\ &\qquad j\in\{\varkappa(K-1),\varkappa(K-1)+1,\dots,\varkappa(K)-1\};\end{aligned}\tag{4.14}$$

the values $\mathfrak{p}_{r,s}$, $r\in J'_{K-1}$, $s\in J'_{M-1}$, *which are not indicated in* (4.11)–(4.14), *equal zero.*

Proof. With account for relations (3.14)–(3.20), identities (4.5)–(4.10) can be written in the form

$$\widehat{\omega}_{-1}(t)=(\xi_{\varkappa(1)}-\xi_{\varkappa(0)})^{-1}\sum_{j=\varkappa(0)-1}^{\varkappa(1)-2}(\xi_{\varkappa(1)}-\xi_{j+1})\cdot\omega_j(t),\quad t\in[\![\xi_{\varkappa(0)},\xi^{\,+}_{\varkappa(1)}]\!],\tag{4.15}$$

$$\widehat{\omega}_{-1}(t)=0,\quad t\in[\![a,b]\!]\backslash[\![\xi_{\varkappa(0)},\xi^{\,+}_{\varkappa(1)}]\!],\tag{4.16}$$

$$\begin{aligned}\widehat{\omega}_i(t)&=(\xi_{\varkappa(i+1)}-\xi_{\varkappa(i)})^{-1}\sum_{j=\varkappa(i)}^{\varkappa(i+1)-1}(\xi_{j+1}-\xi_{\varkappa(i)})\cdot\omega_j(t),\\ &\qquad t\in[\![\xi^{\,+}_{\varkappa(i)},\xi_{\varkappa(i+1)}]\!],\end{aligned}\tag{4.17}$$

$$\begin{aligned}\widehat{\omega}_i(t)&=(\xi_{\varkappa(i+2)}-\xi_{\varkappa(i+1)})^{-1}\sum_{j=\varkappa(i+1)}^{\varkappa(i+2)-2}(\xi_{\varkappa(i+2)}-\xi_{j+1})\cdot\omega_j(t),\\ &\qquad t\in[\![\xi^{\,+}_{\varkappa(i+1)},\xi^{\,-}_{\varkappa(i+2)}]\!],\end{aligned}\tag{4.18}$$

$$\widehat{\omega}_i(t)=0,\quad t\in[\![a,b]\!]\setminus[\![\xi^{\,+}_{\varkappa(i)},\xi^{\,-}_{\varkappa(i+2)}]\!],\quad i\in J_{K-2},\tag{4.19}$$

$$\begin{aligned}\widehat{\omega}_{K-1}(t)&=(\xi_{\varkappa(K)}-\xi_{\varkappa(K-1)})^{-1}\sum_{j=\varkappa(K-1)}^{\varkappa(K)-1}(\xi_{j+1}-\xi_{\varkappa(K-1)})\cdot\omega_j(t),\\ &\qquad t\in[\![\xi^{\,+}_{\varkappa(K-1)},\xi_{\varkappa(K)}]\!],\end{aligned}\tag{4.20}$$

$$\widehat{\omega}_{K-1}(t)=0,\quad t\in[\![a,b]\!]\backslash[\![\xi^{\,+}_{\varkappa(K-1)},\xi_{\varkappa(K)}]\!].\tag{4.21}$$

Relations (4.11)–(4.14) follow from (4.15)–(4.21). □

Consider the functionals

$$\langle\widehat{g}^{(i)},u\rangle\stackrel{\text{def}}{=\!=}u(\widehat{x}_{i+1}),\quad u\in C[\![a,b]\!],\quad i\in J'_{K-1}.\tag{4.22}$$

Lemma 4. *The system of functionals* (4.22) *is biorthogonal to the spline system* $\{\widehat{\omega}_j\}_{j\in J'_{K-1}}$, *i.e.*,

$$\langle \widehat{g}^{(i)}, \widehat{\omega}_j\rangle = \delta_{i,j}, \quad i,j \in J'_{K-1}. \tag{4.23}$$

Proof. Relations (4.23) are readily obtained from (3.7)–(3.13) by applying the functionals (4.22). □

Lemma 5. *We have*

$$\widehat{g}^{(i)} = g^{(\varkappa(i+1)-1)}, \quad i \in J'_{K-1}. \tag{4.24}$$

Proof. In view of relations (3.2), (3.4)–(3.6), and (4.22), in the case where $u \in C[\![a,b]\!]$ and $i \in J'_{K-1}$ we have

$$\langle \widehat{g}^{(i)}, u\rangle = u(\widehat{x}_{i+1}) = u(\xi_{\varkappa(i+1)}) = \langle g^{\varkappa(i+1)-1}, u\rangle,$$

which amounts to equalities (4.24). □

Corollary 1. *We have*

$$\widehat{g}^{(\varkappa^{-1}(j+1)-1)} = g^{(j)}, \quad j+1 \in J^*. \tag{4.25}$$

Proof. Substitute $\varkappa(i+1)-1$ for j in (4.24). Since $i \in J'_{K-1}$, we have $i+1 \in J_K$, and, by the definition of J^* (see (3.5)), we have $\varkappa(i+1) \in J^*$. Thus, $j+1 \in J^*$ and $i = \varkappa^{-1}(j+1)-1$. Substituting the expression obtained into (4.24), we arrive at (4.25). □

5. The restriction matrix and its properties

The matrix $\mathfrak{P} \stackrel{\text{def}}{=} (\mathfrak{p}_{i,j})$, $i \in J'_{K-1}, j \in J'_{M-1}$, is called the restriction matrix; here, as above,

$$\mathfrak{p}_{i,j} = \langle g^{(j)}, \widehat{\omega}_i\rangle. \tag{5.1}$$

Introduce the ascendingly ordered subsets $J^0, J^1(r), J^2(r), J(r)$ of $\mathbb{Z}$ defined by the formulas

$$J^0 \stackrel{\text{def}}{=} \{-1,\dots,\varkappa(1)-2\}, \tag{5.2}$$

$$J^1(r) \stackrel{\text{def}}{=} \{\varkappa(r),\dots,\varkappa(r+1)-1\}, \quad r \in J_{K-1}, \tag{5.3}$$

$$J^2(r) \stackrel{\text{def}}{=} \{\varkappa(r+1),\dots,\varkappa(r+2)-2\}, \quad r \in J_{K-2}, \tag{5.4}$$

$$J(r) \stackrel{\text{def}}{=} J^1(r) \cup J^2(r), \quad r \in J_{K-2}, \qquad J(K-1) \stackrel{\text{def}}{=} J^1(K-1). \tag{5.5}$$

By convention, an ordered set is empty if the first number in the braces exceeds the last one.

Remark 3. In accordance with the above convention, the equality $J^2(r) = \varnothing$ is equivalent to $\varkappa(r+2)-\varkappa(r+1) < 2$. As $\varkappa(s)$ is an increasing integer function (whence $\varkappa(r+2)-\varkappa(r+1) \geq 1$), the set $J^2(r)$ can be empty only in the case where $\varkappa(r+2) = \varkappa(r+1)+1$. The set $J^1(r)$ cannot be empty because the inequality $\varkappa(r+1)-\varkappa(r) < 1$ is impossible for the function $\varkappa(s)$. As is readily seen, the sets J^0 and $J(K-1)$ neither can be empty.

Theorem 1. *The following calibration relations hold*:

$$\widehat{\omega}_r(t) = \sum_{q\in J'_{M-1}} \mathfrak{p}_{r,q}\omega_q(t), \quad t \in [\![a,b]\!], \quad r \in J'_{K-1}. \tag{5.6}$$

Here,

$$\mathfrak{p}_{-1,q} = \frac{\xi_{\varkappa(1)} - \xi_{q+1}}{\xi_{\varkappa(1)} - \xi_{\varkappa(0)}}, \qquad q \in J^0; \tag{5.7}$$

$$\mathfrak{p}_{r,q} = \frac{\xi_{q+1} - \xi_{\varkappa(r)}}{\xi_{\varkappa(r+1)} - \xi_{\varkappa(r)}}, \qquad q \in J^1(r), \quad r \in J_{K-1}; \tag{5.8}$$

$$\mathfrak{p}_{r,q} = \frac{\xi_{\varkappa(r+2)} - \xi_{q+1}}{\xi_{\varkappa(r+2)} - \xi_{\varkappa(r+1)}}, \qquad q \in J^2(r), \quad r \in J_{K-2}, \tag{5.9}$$

and the remaining entries of the matrix $\mathfrak{P}$, *not indicated in* (5.7)–(5.9), *are zero.*

Proof. We observe that relations (5.8) and (5.9) are not contradictory because for a fixed r, the sets $J^1(r)$ and $J^2(r)$ are disjoint. It is obvious that relations (5.6)–(5.9) are relations (4.15)–(4.21) written in a different form. □

Corollary 2. *For an arbitrary* $t \in [\![a, b]\!]$, *the calibration relations* (5.6)–(5.9) *can be represented in the form*

$$\widehat{\omega}_{-1}(t) = \sum_{j \in J^0} \mathfrak{p}_{-1,j}\omega_j(t); \tag{5.10}$$

$$\widehat{\omega}_i(t) = \sum_{j \in J(i)} \mathfrak{p}_{i,j}\omega_j(t), \qquad i \in J'_{K-2}; \tag{5.11}$$

$$\widehat{\omega}_{K-1}(t) = \sum_{j \in J^1(K-1)} \mathfrak{p}_{K-1,j}\omega_j(t), \quad t \in [\![a, b]\!], \tag{5.12}$$

and also in the form

$$\widehat{\omega}_{-1}(t) = \sum_{j \in J^0} \frac{\xi_{\varkappa(1)} - \xi_{j+1}}{\xi_{\varkappa(1)} - \xi_{\varkappa(0)}}\omega_j(t); \tag{5.13}$$

$$\widehat{\omega}_i(t) = \sum_{j \in J^1(i)} \frac{\xi_{j+1} - \xi_{\varkappa(i)}}{\xi_{\varkappa(i+1)} - \xi_{\varkappa(i)}}\omega_j(t) + \sum_{j \in J^2(i)} \frac{\xi_{\varkappa(i+2)} - \xi_{j+1}}{\xi_{\varkappa(i+2)} - \xi_{\varkappa(i+1)}}\omega_j(t), \quad i \in J_{K-2}; \tag{5.14}$$

$$\widehat{\omega}_{K-1}(t) = \sum_{j \in J^1(K-1)} \frac{\xi_{j+1} - \xi_{\varkappa(K-1)}}{\xi_{\varkappa(K)} - \xi_{\varkappa(K-1)}}\omega_j(t), \quad t \in [\![a, b]\!]. \tag{5.15}$$

Proof. Representations (5.10)–(5.15) follow from Theorem 1 (they are equivalent to relations (5.6)–(5.9)). □

Corollary 3. *The relations*

$$\mathfrak{p}_{i,\varkappa(i+1)-1} = 1, \qquad i \in J'_{K-1}, \tag{5.16}$$

are valid. If $i \in J_{K-2}$ *is fixed and*

$$\varkappa(i+1) = \varkappa(i) + 1, \quad \varkappa(i+2) = \varkappa(i+1) + 1, \tag{5.17}$$

then all the other entries of the ith row, except for the unit entry indicated in (5.16), *are zero. Therefore,*

$$\mathfrak{p}_{i,s} = \delta_{\varkappa(i+1)-1,\, s}, \qquad s \in J'_{M-1}. \tag{5.18}$$

If the condition $\varkappa(1) = 1$ *is fulfilled, then* (5.18) *holds for* $i = -1$, *and if* $\varkappa(K-1) = M-1$, *then* (5.18) *holds for* $i = K-1$.

Proof. If $i = -1$, then (5.16) is obtained from (5.7) by setting $q = -1$. If $i \in J_{K-1}$, then (5.16) follows from (5.8) with $q = \varkappa(i+1) - 1$.

If $i \in J_{K-2}$, then under condition (5.17) the index set $J^1(i)$ consists of a single element, $J^1(i) = \{\varkappa(i)\}$, and the set $J^2(i)$ is empty. Taking into account Theorem 1, we see that the entries $\mathfrak{p}_{i,q}$ of the matrix $\mathfrak{P}$ with $q \neq \varkappa(i+1) - 1$ are equal to zero, and therefore relation (5.18) is valid.

If $\varkappa(1) = 1$, then $J^0 = \{-1\}$, whence, by Theorem 1, (5.18) holds for $i = -1$, and if $\varkappa(K-1) = M-1$, then $J^1(K-1) = \{M-1\}$, and, by using Theorem 1 once again, we obtain (5.18) for $i = K-1$. □

Corollary 4. *If $j+1 \in J^*$, then the ith entry in the jth column of the matrix $\mathfrak{P}$, where $i = \varkappa^{-1}(j+1) - 1$, is unity; the remaining entries of the jth column are zero; thus,*

$$\mathfrak{p}_{r,j} = \delta_{r,\varkappa^{-1}(j+1)-1}, \qquad r \in J'_{K-1}. \tag{5.19}$$

Proof. We use (4.23), (4.25), and (5.1). For an arbitrary $r \in J'_{K-1}$, we have

$$p_{r,j} = \langle g^{(j)}, \widehat{\omega}_r \rangle = \langle \widehat{g}^{(\varkappa^{-1}(j+1)-1)}, \widehat{\omega}_r \rangle, \qquad r \in J'_{K-1},$$

and relation (5.19) follows. □

6. Discrete spline-wavelet decomposition

By $\mathbb{S}(\widehat{X})$ we denote the linear hull of the functions $\widehat{\omega}_j$,

$$\mathbb{S}(\widehat{X}) \stackrel{\text{def}}{=\!=} \mathcal{L}\{\widehat{\omega}_i(t) \mid t \in [\![a,b]\!], \quad i \in J'_{K-1}\}.$$

The linear space $\mathbb{S}(\widehat{X})$ is called *the space of discrete splines of the first degree on the grid $\widehat{X}$*.

In this connection, we note that the constructions suggested earlier in the continuous case (cf. [3]) are also applicable in the case of discrete numerical flows. Here, we present these constructions for the reader's convenience.

Since $\mathbb{S}(\widehat{X}) \subset C[\![a,b]\!]$, we can consider the operator P projecting the space $C[\![a,b]\!]$ onto the space $\mathbb{S}(\widehat{X})$,

$$Pu \stackrel{\text{def}}{=\!=} \sum_{i \in J'_{K-1}} \langle \widehat{g}^{(i)}, u \rangle \, \widehat{\omega}_i, \qquad u \in C[\![a,b]\!]. \tag{6.1}$$

We set $Q = I - P$, where I is the identity operator in $C[\![a,b]\!]$.

Thus, in accordance with (6.1), we obtain the direct decomposition

$$C[\![a,b]\!] = \mathbb{S}(\widehat{X}) \dot{+} \mathbb{W}, \tag{6.2}$$

where $\mathbb{W} \stackrel{\text{def}}{=\!=} QC[\![a,b]\!]$. The space $\mathbb{S}(\widehat{X})$ is called the *main space*, and $\mathbb{W}$ is called the *wavelet space* in the *wavelet decomposition* (6.2) of the space $C[\![a,b]\!]$.

If $u \in C[\![a,b]\!]$, then, by using relation (6.2), we obtain two representations of the element u:

$$u = \sum_{s \in J'_{M-1}} c_s \omega_s \tag{6.3}$$

and also

$$u = \widehat{u} + w, \tag{6.4}$$

where

$$\widehat{u} \stackrel{\text{def}}{=\!=} \sum_{i \in J'_{K-1}} a_i \widehat{\omega}_i, \qquad w \stackrel{\text{def}}{=\!=} \sum_{j \in J'_{M-1}} b_j \omega_j,$$

$$a_i \stackrel{\text{def}}{=\!=} \langle \widehat{g}^{(i)}, u \rangle, \quad i \in J'_{K-1}; \quad b_j,\ c_s \in \mathbb{R}^1, \quad j, s \in J'_{M-1}. \tag{6.5}$$

It is obvious that relation (6.4) is the wavelet decomposition of the element $u \in C[\![a,b]\!]$, where $\widehat{u} \in \mathbb{S}(\widehat{X})$, $w \in \mathbb{W}$.

By (6.3)–(6.4), we have

$$\sum_{j \in J'_{M-1}} c_j \omega_j = \sum_{i \in J'_{K-1}} a_i \sum_{j \in J'_{M-1}} \mathfrak{p}_{i,j} \omega_j + \sum_{j \in J'_{M-1}} b_j \omega_j,$$

and, by taking into account the linear independence of the system $\{\omega_j\}_{j \in J'_{M-1}}$, we obtain the *formulas of reconstruction*

$$c_j = \sum_{i \in J'_{K-1}} a_i \mathfrak{p}_{i,j} + b_j, \qquad j \in J'_{M-1}. \tag{6.6}$$

Using representation (6.5), we can write (6.6) in the form

$$c_j = \sum_{i \in J'_{K-1}} \langle \widehat{g}^{(i)}, u \rangle \mathfrak{p}_{i,j} + b_j, \qquad j \in J'_{M-1}.$$

Substituting the expression for u from relation (6.3), we have

$$c_j = \sum_{i \in J'_{K-1}} \sum_{s \in J'_{M-1}} c_s \langle \widehat{g}^{(i)}, \omega_s \rangle \mathfrak{p}_{i,j} + b_j, \qquad j \in J'_{M-1},$$

implying that

$$b_j = c_j - \sum_{i \in J'_{K-1}} \sum_{s \in J'_{M-1}} c_s \langle \widehat{g}^{(i)}, \omega_s \rangle \mathfrak{p}_{i,j}, \qquad j \in J'_{M-1}. \tag{6.7}$$

On substituting (6.3) into (6.5), we obtain

$$a_i = \langle \widehat{g}^{(i)}, \sum_{s \in J'_{M-1}} c_s \omega_s \rangle, \qquad i \in J'_{K-1},$$

whence

$$a_i = \sum_{s \in J'_{M-1}} c_s \langle \widehat{g}^{(i)}, \omega_s \rangle, \qquad i \in J'_{K-1}. \tag{6.8}$$

Relations (6.7) and (6.8) are called the *formulas of decomposition.*

7. Prolongation matrix

Set

$$\mathfrak{q}_{s,j} \overset{\text{def}}{=\!=} \langle \widehat{g}^{(s)}, \omega_j \rangle \tag{7.1}$$

and consider the matrix $\mathfrak{Q} \overset{\text{def}}{=\!=} (\mathfrak{q}_{s,j})_{s \in J'_{K-1}, j \in J'_{M-1}}$; this matrix is called *the prolongation matrix.*

Lemma 6. *The following relations hold*:

$$\mathfrak{q}_{s,j} = \delta_{\varkappa(s+1)-1,\, j}, \quad s \in J'_{K-1}, \quad j \in J'_{M-1}. \tag{7.2}$$

Proof. By (3.3), (4.22), (4.24), (6.8), and (7.1), we have

$$\mathfrak{q}_{s,j} = \langle \widehat{g}^{(s)}, \omega_j \rangle = \langle g^{\varkappa(s+1)-1}, \omega_j \rangle = \delta_{\varkappa(s+1)-1,\, j},$$

which proves (7.2). □

Corollary 5. *The following assertions are valid*:

1) *the columns* $\mathfrak{q}^{(j)} \stackrel{\text{def}}{=\!=} (\mathfrak{q}_{sj})_{s\in J'_{K-1}}$ *of the matrix* $\mathfrak{Q}$ *whose numbers* j *satisfy* $j+1 \notin J^*$ *are zero*;

2) *the remaining columns have unit entry in position* s_0*, where* $\varkappa(s_0+1) = j+1 \in J^*$*, and zero entries elsewhere.*

Proof. By (7.2), we have $\mathfrak{q}_{s,j} = \delta_{\varkappa(s+1),\, j+1}$. Since $\varkappa(s+1) \in J^*$ and $j+1 \notin J^*$, we conclude that $\varkappa(s+1) \neq j+1$. Therefore, $\mathfrak{q}_{s,j} = 0$, which proves the first assertion.

If $j+1 \in J^*$, then there exists a unique s_0 such that $s_0+1 \in J_K$ and $j+1 = \varkappa(s_0+1)$. From (7.2) we have $\mathfrak{q}_{s_0,j} = 1$ and $\mathfrak{q}_{s,j} = 0$ if $s \neq s_0$. This proves the second assertion. □

Corollary 6. *The matrix* $\mathfrak{Q}$ *is the left inverse to the matrix* $\mathfrak{P}^T$*, i.e.,*

$$\mathfrak{Q}\mathfrak{P}^T = I, \tag{7.3}$$

where I *is the identity matrix of order* $K+1$*.*

Proof. Consider an entry $[\mathfrak{Q}\mathfrak{P}^T]_{i,j}$ of the matrix $\mathfrak{Q}\mathfrak{P}^T$. We have

$$[\mathfrak{Q}\mathfrak{P}^T]_{i,j} = \sum_{s\in J'_{M-1}} \mathfrak{q}_{is}\mathfrak{p}_{js}, \quad i,j \in J'_{K-1}.$$

By (7.2), we obtain

$$[\mathfrak{Q}\mathfrak{P}^T]_{i,j} = \sum_{s\in J'_{M-1}} \delta_{\varkappa(i+1)-1,s}\mathfrak{p}_{js} = \mathfrak{p}_{j,\varkappa(i+1)-1}. \tag{7.4}$$

Using representation (4.22) and formula (4.23), we derive

$$\mathfrak{p}_{j,\varkappa(i+1)-1} = \langle g^{\varkappa(i+1)-1}, \widehat{\omega}_j\rangle = \langle \widehat{g}^{(i)}, \widehat{\omega}_j\rangle = \delta_{i,j}, \quad i,j \in J'_{K-1}. \tag{7.5}$$

Relations (7.4) and (7.5) prove (7.3). □

8. Flows. Formulas of decomposition

Introduce into consideration the three vectors

$$\mathbf{a} \stackrel{\text{def}}{=\!=} (a_{-1}, a_0, a_1, a_2, \dots, a_{K-1}),$$

$$\mathbf{b} \stackrel{\text{def}}{=\!=} (b_{-1}, b_0, b_1, b_2, \dots, b_{M-1}),$$

$$\mathbf{c} \stackrel{\text{def}}{=\!=} (c_{-1}, c_0, c_1, c_2, \dots, c_{M-1}).$$

These vectors are called the main, wavelet, and original numerical flows, respectively. By $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ the linear spaces of these flows are denoted.

The relations

$$\widetilde{u} = \sum_{i\in J'_{K-1}} a_i\widetilde{\omega}_i, \quad w = \sum_{i\in J'_{N-1}} b_i\omega_i, \quad u = \sum_{i\in J'_{N-1}} c_i\omega_i \tag{8.1}$$

establish linear isomorphisms of the spaces just introduced and the spaces $\mathbb{S}(\widehat{X})$, $\mathbb{W}$, $C[\![a,b]\!]$:

$$\mathcal{A} \sim \mathbb{S}(\widehat{X}), \qquad \mathcal{B} \sim \mathbb{W}, \qquad \mathcal{C} \sim C[\![a,b]\!]. \tag{8.2}$$

In view of these isomorphisms, we have

$$\mathcal{C} = \mathcal{A} \dotplus \mathcal{B}.$$

Using the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and the matrices $\mathfrak{P}$, $\mathfrak{Q}$, we write formulas (6.6)–(6.8) in the form

$$\mathbf{c} = \mathbf{b} + \mathfrak{P}^T \mathbf{a}, \tag{8.3}$$

$$\mathbf{a} = \mathfrak{Q}\mathbf{c}, \qquad \mathbf{b} = \mathbf{c} - \mathfrak{P}^T \mathfrak{Q}\mathbf{c}. \tag{8.4}$$

Lemma 7. *The entries* $[\mathfrak{P}^T \mathfrak{Q}]_{i,j}$, $i, j \in J'_{M-1}$, *of the matrix product* $\mathfrak{P}^T \mathfrak{Q}$ *are computed by the formulas*

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j} = 0, \quad i \in J'_{M-1}, \quad j+1 \in J_M \backslash J^*; \tag{8.5}$$

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j}, = \mathfrak{p}_{\varkappa^{-1}(j+1)-1,i}, \qquad i \in J'_{M-1}, \quad j+1 \in J^*. \tag{8.6}$$

Proof. If $j+1 \in J^*$, then, by (7.2), we have

$$\mathfrak{q}_{s,j} = \delta_{\varkappa(s+1)-1,j} = \delta_{\varkappa(s+1),j+1} = \delta_{s+1,\varkappa^{-1}(j+1)} = \delta_{s,\varkappa^{-1}(j+1)-1}, \tag{8.7}$$

$$s \in J'_{K-1}, \quad j+1 \in J^*. \tag{8.8}$$

If $j+1 \in J_M \setminus J^*$, then we obtain

$$\mathfrak{q}_{s,j} = \delta_{\varkappa(s+1),j+1} = 0, \qquad s \in J'_{K-1}, \quad j+1 \in J_M \setminus J^*. \tag{8.9}$$

If $j+1 \in J^*$, then from (8.7)–(8.8) it follows that

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j} = \sum_{s \in J'_{K-1}} \mathfrak{p}_{s,i} \mathfrak{q}_{s,j} = \mathfrak{p}_{\varkappa^{-1}(j+1)-1,i}, \qquad i \in J'_{M-1}.$$

If $j+1 \in J_M \setminus J^*$, then, using (8.9), we find

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j} = 0, \qquad i \in J'_{M-1}, \quad j+1 \in J_M \setminus J^*.$$

This completes the proof of (8.5)–(8.6). □

Corollary 7. *If* $i+1,\ j+1 \in J^*$, *then*

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j} = \delta_{i,j}. \tag{8.10}$$

Proof. Taking into account that $j+1 \in J^*$, we set

$$r = \varkappa^{-1}(j+1) - 1. \tag{8.11}$$

Obviously, $r \in J'_{K-1}$ and $j = \varkappa(r+1) - 1$. In accordance with (8.6), we have

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j} = \mathfrak{p}_{\varkappa^{-1}(j+1)-1,i} = \mathfrak{p}_{r,i}, \qquad i \in J'_{M-1}, \quad j+1 \in J^*. \tag{8.12}$$

Since $i+1 \in J^*$, the assumptions of Corollary 4 (with $j = i$) are satisfied. Therefore, by (5.19), we obtain

$$\mathfrak{p}_{r,i} = \delta_{r,\varkappa^{-1}(i+1)-1}. \tag{8.13}$$

Taking into account (8.13), from (8.12) we derive

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j} = \delta_{r,\varkappa^{-1}(i+1)-1}. \tag{8.14}$$

Taking r from (8.11) and substituting it into relation (8.14), we deduce

$$[\mathfrak{P}^T \mathfrak{Q}]_{i,j} = \delta_{\varkappa^{-1}(j+1)-1,\varkappa^{-1}(i+1)-1} = \delta_{i,j}.$$

This proves (8.10). □

Theorem 2. *The formulas of decomposition can be written in the form*

$$a_i = c_{\varkappa(i+1)-1}, \qquad i \in {J'}_{K-1}, \tag{8.15}$$

$$b_q = 0, \qquad q+1 \in J^*, \tag{8.16}$$

$$b_q = c_q - \sum_{j \in {J'}_{K-1}} \langle g^{(q)}, \widehat{\omega}_j \rangle c_{\varkappa(j+1)-1}, \quad q+1 \in J_M \setminus J^*. \tag{8.17}$$

Proof. By (6.8) and (7.1)–(7.2), for $i \in {J'}_{K-1}$ we have

$$a_i = \sum_{j \in {J'}_{M-1}} \mathfrak{q}_{ij} c_j = \sum_{j \in {J'}_{M-1}} \delta_{\varkappa(i+1)-1} c_j = c_{\varkappa(i+1)-1},$$

which proves (8.15).

By the second relation in (8.4), we have

$$b_q = c_q - [\mathfrak{P}^T \mathfrak{Q} \mathbf{c}]_q, \qquad q \in {J'}_{M-1},$$

whence

$$b_q = c_q - \sum_{s \in {J'}_{M-1}} [\mathfrak{P}^T \mathfrak{Q}]_{q,s}\, c_s.$$

By taking into account that, by (8.5),

$$[\mathfrak{P}^T \mathfrak{Q}]_{q,s} = 0, \qquad s+1 \notin J^*, \quad q \in {J'}_{M-1},$$

we obtain

$$b_q = c_q - \sum_{s+1 \in J^*} [\mathfrak{P}^T \mathfrak{Q}]_{q,s}\, c_s.$$

From (8.6) it follows that

$$b_q = c_q - \sum_{s+1 \in J^*} \mathfrak{p}_{\varkappa^{-1}(s+1)-1,q}\, c_s. \tag{8.18}$$

From (5.19) with $q+1 \in J^*$ we have

$$\mathfrak{p}_{\varkappa^{-1}(s+1)-1,q} = \delta_{s,q},$$

implying that

$$b_q = c_q - \sum_{s+1 \in J^*} \delta_{s,q}\, c_s = 0, \quad q+1 \in J^*.$$

This completes the proof of (8.16).

Now consider the case $q+1 \in J_M \setminus J^*$.

Using the bijective map $\varkappa : J_K \longrightarrow J^*$ (see (3.4)–(3.5)), we change the index of summation in (8.18):

$$j = \varkappa^{-1}(s+1) - 1 \iff s = \varkappa(j+1) - 1.$$

It is clear that $j \in {J'}_{K-1}$. As a result, we obtain

$$b_q = c_q - \sum_{j \in {J'}_{K-1}} \mathfrak{p}_{j,\,q}\, c_{\varkappa(j+1)-1}.$$

Applying (5.1), we deduce relation (8.17). □

Theorem 3. *If $q+1 \in J_M \setminus J^*$, then the wavelet flow satisfies the relation*

$$b_q = c_q - (\widehat{x}_{s+1} - \widehat{x}_s)^{-1} \Big[(\widehat{x}_{s+1} - \xi_{q+1}) c_{\varkappa(s)-1} + (\xi_{q+1} - \widehat{x}_s) c_{\varkappa(s+1)-1} \Big], \tag{8.19}$$

where

$$\widehat{x}_s < \xi_{q+1} < \widehat{x}_{s+1}. \tag{8.20}$$

Proof. First we note that the nonzero terms in relation (8.17) necessarily satisfy the condition $\langle g^{(q)}, \widehat{\omega}_j\rangle \neq 0$. By virtue of definition (3.2), the latter inequality is equivalent to the relation $\widehat{\omega}_j(\xi_{q+1}) \neq 0$.

Since $q+1 \in J_M \setminus J^*$, there exists a unique $s \in J_K$ such that inequality (8.20) holds. Taking into account that $\operatorname{supp}\widehat{\omega}_j = [\![\widehat{x}_j^+, \widehat{x}_{j+2}^-]\!]$ (see (3.26)), we see that the sum in (8.17) involves at most two terms, which correspond to $j = s-1$ and $j = s$. Thus,

$$b_q = c_q - \langle g^{(q)}, \widehat{\omega}_{s-1}\rangle c_{\varkappa(s)-1} - \langle g^{(q)}, \widehat{\omega}_s\rangle c_{\varkappa(s+1)-1}, \tag{8.21}$$

and, in order to complete the proof, it only remains to use (3.7)–(3.13) and (8.21). □

Corollary 8. *Formula* (8.17) *can be written in the form*

$$b_q = c_q - \mathfrak{p}_{s-1,q} c_{\varkappa(s)-1} - \mathfrak{p}_{s,q} c_{\varkappa(s+1)-1}, \tag{8.22}$$

where s satisfies relation (8.20).

Corollary 9. *The space $\mathcal{B}$ of wavelet flows can be represented as*

$$\mathcal{B} = \{\mathbf{b} \mid \mathbf{b} = (b_{-1}, b_0, \ldots, b_{M-1}),\ b_{j-1} \in \mathbb{R}^1,\ \ j \in J_M \setminus J^*;\ b_{i-1} = 0,\ i \in J^*\}, \tag{8.23}$$

and it is the linear hull of the orthonormal[1] *system*

$$\mathbf{e}_j \stackrel{\text{def}}{=\!=} ([\mathbf{e}_j]_{-1}, [\mathbf{e}_j]_0, \ldots, [\mathbf{e}_j]_{M-1}),$$

where $[\mathbf{e}_j]_i = \delta_{i,j}$, $i, j \in J'_{M-1}$.

Proof. It is known (see [5]) that the space of wavelet flows is the kernel of the operator $\mathfrak{Q}$. Since

$$\mathfrak{Q}\mathbf{b} = 0 \quad \iff \quad \sum_{j \in J'_{M-1}} \mathfrak{q}_{ij}\, b_j = 0, \quad i \in J'_{K-1},$$

by virtue of relation (7.2), we have

$$\sum_{j \in J'_{M-1}} \delta_{\varkappa(i+1)-1,j}\, q_{ij}\, b_j = 0, \quad i \in J'_{K-1} \quad \iff \quad b_{\varkappa(i+1)-1} = 0, \quad i \in J'_{K-1}.$$

Therefore (also see (8.16)), we conclude that

$$b_{j-1} = 0, \quad j \in J^*.$$

The remaining components of the wavelet flow are arbitrary; they define the space $\mathcal{B}$ of wavelet flows in representation (8.2). Thus, (8.23) holds. □

9. An illustrative example of the spline-wavelet decomposition

In this section, we give an illustrative example.

Set $M = 10$ and, on the grid segment $[\![a, b]\!]$, consider the grid

$$a = \xi_0 < \xi_1 < \ldots < \xi_9 < \xi_{10} = b.$$

Choose the discrete basis (3.1) of the space $C[\![a, b]\!]$,

$$\omega_j(\xi_s) = \delta_{s,j+1}, \qquad s \in J_{10}. \tag{9.1}$$

Consider the linear functionals $g^{(i)}$, $i \in J'_9$, defined by the formula

$$\langle g^{(i)}, u\rangle \stackrel{\text{def}}{=\!=} u(\xi_{i+1}), \qquad u \in C[\![a, b]\!], \tag{9.2}$$

where

$$J_{10} = \{0, 1, 2, \ldots, 9, 10\}, \qquad J'_9 = \{-1, 0, 1, 2, \ldots, 9\}.$$

[1]In the Euclidean space $\mathbb{R}^{M+1}$ with the standard inner product.

Consider the new grid

$$\widehat{X}: \quad a = \widehat{x}_0 < \widehat{x}_1 < \ldots < \widehat{x}_8 = b, \tag{9.3}$$

where $\widehat{x}_i \stackrel{\text{def}}{=} \xi_{\varkappa(i)}$, $i \in J_8$. The map $\varkappa(i)$ is defined by the relations

$$\varkappa(0) = 0,\ \varkappa(1) = 1,\ \ldots, \varkappa(5) = 5,\ \varkappa(6) = 8,\ \varkappa(7) = 9,\ \varkappa(8) = 10.$$

Thus, in the case considered, we have $K = 8$ and

$$J^* = \{0, 1, 2, 3, 4, 5, 8, 9, 10\},$$

whence $J_{10} \setminus J^* = \{6, 7\}$.

From (5.2) we obtain $J^0 = \{-1\}$, and from (5.3) we have $J^1(s) = \{s\}$ for $s \in \{0, 1, 2, 3, 4\}$, $J^1(5) = \{5, 6, 7\}$, $J^1(6) = \{8\}$.

Applying (5.4), we find that $J^2(s) = \varnothing$ for $s \in \{0, 1, 2, 3, 5, 6\}$, $J^2(4) = \{5, 6\}$; the second formula in (5.5) yields $J^1(7) = \{9\}$.

Finally, in accordance with the first formula in (5.5), we have $J(s) = \{s\}$ for $s \in \{0, 1, 2, 3\}$, $J(4) = \{4, 5, 6\}$, $J(5) = \{5, 6, 7\}$, and $J(s) = \{s + 2\}$ for $s \in \{6, 7\}$.

Let $\widehat{\omega}_j(t)$, $j \in J'_7$, be the functions defined by (3.7)–(3.13) and consider the vectors

$$\widehat{\omega}(t) \stackrel{\text{def}}{=} (\widehat{\omega}_{-1}(t), \widehat{\omega}_0(t), \ldots, \widehat{\omega}_7(t))^T,$$

$$\omega(t) \stackrel{\text{def}}{=} (\omega_{-1}(t), \omega_0(t), \ldots, \omega_9(t))^T.$$

From Theorem 1 (see (5.7)–(5.9)) it follows that the calibration relations $\widehat{\omega}(t) = \mathfrak{P}\omega(t)$ are defined by the matrix

$$\mathfrak{P} \stackrel{\text{def}}{=} \begin{array}{c} \\ -\widehat{1} \\ \widehat{0} \\ \widehat{1} \\ \widehat{2} \\ \widehat{3} \\ \widehat{4} \\ \widehat{5} \\ \widehat{6} \\ \widehat{7} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{\xi_8-\xi_6}{\xi_8-\xi_5} & \frac{\xi_8-\xi_7}{\xi_8-\xi_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\xi_6-\xi_5}{\xi_8-\xi_5} & \frac{\xi_7-\xi_5}{\xi_8-\xi_5} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Thus,

$$\widehat{\omega}_i(t) = \omega_i(t) \quad \text{for } i \in \{-1, 0, 1, 2, 3\}, \tag{9.4}$$

$$\widehat{\omega}_4(t) = \omega_4(t) + \frac{\xi_8 - \xi_6}{\xi_8 - \xi_5} \cdot \omega_5(t) + \frac{\xi_7 - \xi_6}{\xi_8 - \xi_5} \cdot \omega_6(t), \tag{9.5}$$

$$\widehat{\omega}_5(t) = \frac{\xi_6 - \xi_5}{\xi_8 - \xi_5} \cdot \omega_5(t) + \frac{\xi_7 - \xi_5}{\xi_8 - \xi_5} \cdot \omega_6(t) + \omega_7(t), \tag{9.6}$$

$$\widehat{\omega}_i(t) = \omega_{i+2}(t) \quad \text{for } i \in \{6, 7\}. \tag{9.7}$$

Formulas (9.4)–(9.7) are an illustration of (4.15)–(4.21).

Now we illustrate Corollary 3. It is clear that conditions (5.17) are fulfilled for $i \in \{0, 1, 2, 3, 6\}$, and relations (5.18) hold for $s \in J'_9$. Since $\varkappa(1) = 1$, (5.18) holds for $i = -1$, whence $p_{-1,s} = \delta_{-1,s}$ for $s \in J'_9$. As $K = 8$, we have $\varkappa(K - 1) = \varkappa(7) = 9$, and the condition $\varkappa(\,-1) = M - 1$ is fulfilled. Therefore, (5.18) is valid for $i = K - 1 = 7$. Thus, $p_{7,s} = \delta_{9,s}$ for $s \in J'_9$.

It is even easier to illustrate Corollary 4. In the case considered, the condition $j+1 \in J^*$ amounts to the condition

$$j \in \{-1, 0, 1, 2, 3, 4, 7, 8, 9\}.$$

In what follows, we will need the transposed matrix

$$\mathfrak{P}^T \stackrel{\text{def}}{=\!=} \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} \widehat{-1} & \widehat{0} & \widehat{1} & \widehat{2} & \widehat{3} & \widehat{4} & \widehat{5} & \widehat{6} & \widehat{7} \end{array} \\ \left(\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\xi_8-\xi_6}{\xi_8-\xi_5} & \frac{\xi_6-\xi_5}{\xi_8-\xi_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\xi_8-\xi_7}{\xi_8-\xi_5} & \frac{\xi_7-\xi_5}{\xi_8-\xi_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}.$$

The projection of the space $C[\![a,b]\!]$ of the original flows $\mathbb{W}$ onto the space $\mathbb{S}(\widehat{X})$ of the main flows is defined by the method of prolongation of the system of functionals $\{g^{(j)}_{j\in J'_7}\}$ biorthogonal to the system of coordinate splines $\{\widehat{\omega}_j\}_{j\in J'_7}$ onto the space $C[\![a,b]\!]$ (see (6.1), (6.8), and (8.2)). Ultimately, the projection is equivalent to applying the matrix of prolongation $\mathfrak{Q}$ to the original flow $\mathbf{c} \in \mathcal{C}$: $\mathbf{a} = \mathfrak{Q}\mathbf{c}$. In accordance with (7.2), the matrix $\mathfrak{Q}$ can be written as

$$\mathfrak{Q} = \begin{array}{c} \\ \widehat{-1} \\ \widehat{0} \\ \widehat{1} \\ \widehat{2} \\ \widehat{3} \\ \widehat{4} \\ \widehat{5} \\ \widehat{6} \\ \widehat{7} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array},$$

whence

$$\mathbf{a}_i = \mathbf{c}_i \qquad \text{for} \quad i \in \{-1, 0, 1, 2, 3, 4\} \tag{9.8}$$

and

$$\mathbf{a}_i = \mathbf{c}_{i+2} \qquad \text{for} \quad i \in \{7, 8\}. \tag{9.9}$$

It is nondifficult to see that $\mathfrak{Q}\mathfrak{P}^T = I$, where I is the identity matrix of order 9. From (8.5)–(8.6) it follows that $\mathfrak{P}^T\mathfrak{Q}$ is a square matrix of order 11 of the form

$$\mathfrak{P}^T\mathfrak{Q} = \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left(\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\xi_8-\xi_6}{\xi_8-\xi_5} & 0 & 0 & \frac{\xi_6-\xi_5}{\xi_8-\xi_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\xi_8-\xi_7}{\xi_8-\xi_5} & 0 & 0 & \frac{\xi_7-\xi_5}{\xi_8-\xi_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array}.$$

Certainly, the components of the wavelet flow $\mathbf{b} = \mathbf{c} - \mathfrak{P}^T\mathfrak{Q}\mathbf{c}$ with numbers j such that $j+1 \in J^*$ are zero (see (8.19)). In the case considered, we obtain

$$\mathbf{b}_j = 0, \qquad j \in \{-1,0,1,2,3,4,7,8,9\}, \tag{9.10}$$

$$\mathbf{b}_5 = -\frac{\xi_8-\xi_6}{\xi_8-\xi_5}\mathbf{c}_4 + \mathbf{c}_5 - \frac{\xi_6-\xi_5}{\xi_8-\xi_5}\mathbf{c}_7, \tag{9.11}$$

$$\mathbf{b}_6 = -\frac{\xi_8-\xi_7}{\xi_8-\xi_5}\mathbf{c}_4 + \mathbf{c}_6 - \frac{\xi_7-\xi_5}{\xi_8-\xi_5}\mathbf{c}_7. \tag{9.12}$$

Thus, (9.8)–(9.12) are the formulas of decomposition.

Using (8.3), we obtain the following reconstruction formulas for the original flow $\mathbf{c}$:

$$c_j = a_j + b_j, \qquad j \in \{-1,0,1,2,3,4\}, \tag{9.13}$$

$$c_5 = \frac{\xi_8-\xi_6}{\xi_8-\xi_5}\cdot a_4 + \frac{\xi_6-\xi_5}{\xi_8-\xi_5}\cdot a_5 + b_5, \tag{9.14}$$

$$c_6 = \frac{\xi_8-\xi_7}{\xi_8-\xi_5}\cdot a_4 + \frac{\xi_7-\xi_5}{\xi_8-\xi_5}\cdot a_5 + b_6, \tag{9.15}$$

$$c_j = a_{j-2} + b_j, \qquad j \in \{7,8,9\}. \tag{9.16}$$

Formulas (9.13)–(9.16) illustrate relations (8.3).

10. Continual image of the discrete wavelet decomposition

In this paper, as was mentioned in the introduction, in contrast with most approaches to wavelet decompositions (see [1–2]), the standard function spaces (such as $C[a,b]$, L_2, etc.) are not used. Here, we only consider finite-dimensional flow spaces. The approach used permits one to support the algorithmic structure at all steps of the tracing. However, in some situations it is necessary to apply the infinite-dimensional function spaces mentioned above (for example, in the case where the original flow is obtained by discretization and digitization of a continuous signal arriving from a device of the analog type). In this case, the notions and results discussed above remain valid for an adjusted domain of definition of relevant functions. This situation will be referred to as the *continual case*.

Consider the continual case in more detail. The domain of variation of the argument t becomes either the entire real axis $\mathbb{R}^1$ or an interval $[a,b] \subset \mathbb{R}^1$, and the set $[\![a,b]\!]$ is a grid on the interval $[a,b]$. Some notions and symbols are redefined in such a way that the functions introduced can be regarded as the results of extending the domain of the mappings considered above, and the designations of these mappings are transferred to the functions mentioned above.

The discrete basis (3.1) is replaced with the following system of piecewise-linear functions:

$$\omega_{-1}(t) = (\xi_1 - t)(\xi_1 - \xi_0)^{-1}, \quad t \in (\xi_0, \xi_1), \tag{10.1}$$

$$\omega_{-1}(t) = 0, \quad t \in [a, b] \setminus (\xi_0, \xi_1), \tag{10.2}$$

$$\omega_i(t) = (t - \xi_i)(\xi_{i+1} - \xi_i)^{-1}, \quad t \in (\xi_i, \xi_{i+1}), \tag{10.3}$$

$$\omega_i(t) = (\xi_{i+2} - t)(\xi_{i+2} - \xi_{i+1})^{-1}, \quad t \in (\xi_{i+1}, \xi_{i+2}), \tag{10.4}$$

$$\omega_i(t) = 0, \quad t \in [a, b] \setminus (\xi_i, \xi_{i+2}), \quad i \in J_{M-2}, \tag{10.5}$$

$$\omega_{M-1}(t) = (t - \xi_{M-1})(\xi_M - \xi_{M-1})^{-1}, \quad t \in (\xi_{M-1}, \xi_M), \tag{10.6}$$

$$\omega_{M-1}(t) = 0, \quad t \in [a, b] \setminus (\xi_{M-1}, \xi_M). \tag{10.7}$$

The functions $\widehat{\omega}_j(t)$ defined by (3.7)–(3.13) as functions of the discrete argument $t \in [\![a, b]\!]$ are redefined by extending their domains to the interval $[a, b]$ as follows :

$$\widehat{\omega}_{-1}(t) = (\widehat{x}_1 - t)(\widehat{x}_1 - \widehat{x}_0)^{-1}, \quad t \in (\widehat{x}_0, \widehat{x}_1), \tag{10.8}$$

$$\widehat{\omega}_{-1}(t) = 0, \quad t \in [a, b] \setminus (\widehat{x}_0, \widehat{x}_1), \tag{10.9}$$

$$\widehat{\omega}_i(t) = (t - \widehat{x}_i)(\widehat{x}_{i+1} - \widehat{x}_i)^{-1}, \quad t \in (\widehat{x}_i, \widehat{x}_{i+1}), \tag{10.10}$$

$$\widehat{\omega}_i(t) = (\widehat{x}_{i+2} - t)(\widehat{x}_{i+2} - \widehat{x}_{i+1})^{-1}, \quad t \in (\widehat{x}_{i+1}, \widehat{x}_{i+2}), \tag{10.11}$$

$$\widehat{\omega}_i(t) = 0, \quad t \in [a, b] \setminus (\widehat{x}_i, \widehat{x}_{i+2}), \quad i \in J_{K-2}, \tag{10.12}$$

$$\widehat{\omega}_{K-1}(t) = (t - \widehat{x}_{K-1})(\widehat{x}_K - \widehat{x}_{K-1})^{-1}, \quad t \in (\widehat{x}_{K-1}, \widehat{x}_K), \tag{10.13}$$

$$\widehat{\omega}_{K-1}(t) = 0, \quad t \in [a, b] \setminus (\widehat{x}_{K-1}, \widehat{x}_K). \tag{10.14}$$

The passage to the enlarged domain of definition does not affect the statements of the lemmas and theorems presented in Secs. 1–8.

Considering the situation from the above-mentioned viewpoint, we recall that

$$\mathbb{S}(\widehat{X}) = \mathcal{L}\{\widehat{\omega}_i(t) \mid t \in [\![a, b]\!], \quad i \in J'_{K-1}\}$$

and introduce the following notation:

$$X \stackrel{\text{def}}{=} [\![a, b]\!], \qquad \mathbb{S}(X) \stackrel{\text{def}}{=} C[\![a, b]\!],$$

$$\widetilde{\mathbb{S}}(X) \stackrel{\text{def}}{=} \mathcal{L}\{\omega_j(t) \mid t \in [a, b],\ j \in J'_{M-1}\},$$

$$\widetilde{\mathbb{S}}(\widehat{X}) \stackrel{\text{def}}{=} \mathcal{L}\{\widehat{\omega}_i(t) \mid t \in [a, b],\ i \in J'_{K-1}\},$$

where the functions $\omega_j(t)$ and $\widehat{\omega}_i(t)$ are defined by relations (10.1)–(10.7) and (10.8)–(10.14), respectively.

Obviously, we have the isomorphisms

$$\mathcal{C} \sim \widetilde{\mathbb{S}}(X) \sim \mathbb{S}(X), \qquad \mathcal{A} \sim \widetilde{\mathbb{S}}(\widehat{X}) \sim \mathbb{S}(\widehat{X})$$

and also the embedding

$$\widetilde{\mathbb{S}}(\widehat{X}) \subset \widetilde{\mathbb{S}}(X).$$

In the continual case, by taking into account (6.2)–(6.4) and the isomorphisms indicated above, we obtain the wavelet decomposition

$$\widetilde{\mathbb{S}}(X) = \widetilde{\mathbb{S}}(\widehat{X}) \dotplus \widetilde{\mathbb{W}},$$

where

$$\widetilde{\mathbb{W}} \stackrel{\text{def}}{=} \mathcal{L}\{\omega_{j-1}(t) \mid t \in [a, b],\ j \in J_M \backslash J^*\}.$$

It is clear that the spaces $\mathcal{B}$, $\mathbb{W}$, and $\widetilde{\mathbb{W}}$ are linearly isomorphic: $\mathcal{B} \sim \mathbb{W} \sim \widetilde{\mathbb{W}}$.

11.1. Preliminary remarks. In this paper, the coarsening of a grid is considered as given a priory and defined by a map $\varkappa$. In [6], some variants of adaptive grid coarsening are proposed. They provide for an a priori estimate of the deviation of the main flow from the original one and for an evaluation of volumes for the data used in the case of different irregularity properties of the original flow, and also limit characterizations of the latter volumes in the case where the original flow is generated by a differentiable function. For completeness, we recall the statements of the main results established in [6].

One of the central results of [6] is as follows. If the original flow is generated by a function $u \in C^1[a,b]$, then (under the condition of equal deviations) the limit of the ratio of the data volume for the main flow associated with an irregular adaptive grid to the data volume associated with the analogous flow corresponding to an equidistant grid equals $\frac{1}{b-a}\int_a^b |u'(t)|dt/\|u'\|_{C[a,b]}$. If $u \in C^2[a,b]$, then the corresponding limit is equal to $\frac{1}{b-a}\int_a^b \sqrt{|u''(t)|}dt/\|\sqrt{u''}\|_{C[a,b]}$.

11.2. A grid of the adaptive type. Consider a function $f \in C(\Xi)$ that satisfies the inequality

$$f(t) \geq c, \quad t \in \Xi, \tag{11.1}$$

where c is a positive constant.

Assume that

$$\varepsilon \in (\varepsilon^*, \varepsilon^{**}), \tag{11.2}$$

where

$$\varepsilon^* \stackrel{\text{def}}{=} \max_{\xi \in [\![a,\, b^-]\!]} \max_{t \in \{\xi, \xi^+\}} f(t)(\xi^+ - \xi), \quad \varepsilon^{**} \stackrel{\text{def}}{=} (b-a)\|f\|_{C[\![a,\, b]\!]}. \tag{11.3}$$

Lemma 8. *If conditions* (11.1)–(11.3) *are fulfilled, then there exist a unique positive integer* $K = K(f, \varepsilon, \Xi)$ *and a unique grid*

$$\widetilde{X} = \widetilde{X}(f, \varepsilon, \Xi): \quad a = \widetilde{x}_0 < \widetilde{x}_1 < \ldots < \widetilde{x}_K \leq \widetilde{x}_{K+1} = b \tag{11.4}$$

such that

$$\max_{t \in [\![\widetilde{x}_s,\, \widetilde{x}_{s+1}]\!]} f(t)(\widetilde{x}_{s+1} - \widetilde{x}_s) \leq \varepsilon < \max_{t \in [\![\widetilde{x}_s,\, \widetilde{x}^+_{s+1}]\!]} f(t)(\widetilde{x}^+_{s+1} - \widetilde{x}_s), \quad s \in \{0, 1, \ldots, K-1\}, \tag{11.5}$$

and

$$\max_{t \in [\![\widetilde{x}_K,\, b]\!]} f(t)(b - \widetilde{x}_K) \leq \varepsilon, \quad \widetilde{X} \subset \Xi. \tag{11.6}$$

Lemma 8 is proved by induction passage from the nodes $\widetilde{x}_1, \ldots, \widetilde{x}_s$ to the nodes $\widetilde{x}_1, \ldots, \widetilde{x}_{s+1}$.

The grid (11.4) with the properties (11.5)–(11.6) is referred to as a *grid of the adaptive type* for the discrete function f with the property (1.1).

The integer-valued function $K(f, \varepsilon, \Xi)$ is monotonic, i.e., if $\varepsilon' \leq \varepsilon''$, then $K(f, \varepsilon', \Xi) \geq K(f, \varepsilon'', \Xi)$.

11.3. A pseudo-equidistant grid. Assume that

$$\varepsilon \in (\overline{\varepsilon}^*, \varepsilon^{**}), \tag{11.7}$$

where

$$\overline{\varepsilon}^* \stackrel{\text{def}}{=} \max_{\xi \in [\![a,\, b^-]\!]} (\xi^+ - \xi)\|f\|_{C[\![a,\, b]\!]}, \quad \varepsilon^{**} \stackrel{\text{def}}{=} (b-a)\|f\|_{C[\![a,\, b]\!]}. \tag{11.8}$$

For this ε, we find the values

$$N = N(f, \varepsilon, \Xi) \stackrel{\text{def}}{=} \lceil \varepsilon^{**}/\varepsilon \rceil \tag{11.9}$$

and

$$h = h(f, \varepsilon, \Xi) \stackrel{\text{def}}{=\!=} \frac{b-a}{N+1}; \tag{11.10}$$

here, if r is a real, then $\lceil r \rceil$ is the integer defined by the conditions $0 \le \lceil r \rceil - r < 1$.

On the grid section $[\![a, b]\!]$, consider a grid

$$\overline{X} = \overline{X}(f, \varepsilon, \Xi): \quad a = \overline{x}_0 < \overline{x}_1 < \ldots < \overline{x}_N = b, \quad \overline{X} \subset \Xi, \tag{11.11}$$

where

$$\overline{x}_{s+1} - \overline{x}_s \le h < \overline{x}^+_{s+1} - \overline{x}_s, \quad s \in \{0, 1, \ldots, N-1\}, \tag{11.12}$$

$$\overline{x}_{N+1} - \overline{x}_N \le h. \tag{11.13}$$

The grid (11.11) possessing the properties (11.12)–(11.13) is called a *pseudo-equidistant grid with step size* h. By (11.9), we have

$$(b-a)\|f\|_{C[\![a,\,b]\!]} - \varepsilon < N\varepsilon \le (b-a)\|f\|_{C[\![a,\,b]\!]}, \tag{11.14}$$

$$\max_{t \in [\![\overline{x}_s,\, \overline{x}_{s+1}]\!]} f(t)\,(\overline{x}_{s+1} - \overline{x}_s) \le \varepsilon, \quad s \in \{0, 1, \ldots, N\}. \tag{11.15}$$

Lemma 9. *If* ε *satisfies conditions* (11.7)–(11.8)*, then the grid* (11.11) *with the properties* (11.12)–(11.13) *is defined uniquely, and relations* (11.14)–(11.15) *are satisfied.*

The existence and uniqueness of the grid (11.11) are proved by induction, the proof being similar to that of Lemma 8.

11.4. Limit relations. Assume that a function $f(t)$ is continuous on $[a, b]$ and satisfies the condition

$$f(t) \ge c > 0, \quad t \in [a, b]. \tag{11.16}$$

Consider a sequence of grids $\Xi(\lambda)$ of the form (2.1)–(2.2),

$$\Xi(\lambda): \quad \ldots < \xi_{-2}(\lambda) < \xi_{-1}(\lambda) < \xi_0(\lambda) < \xi_1(\lambda) < \xi_2(\lambda) \ldots, \tag{11.17}$$

dependent on a parameter $\lambda > 0$ and such that $a, b \in \Xi(\lambda)$.

Introduce the notation

$$[\![a, b]\!]_\lambda \stackrel{\text{def}}{=\!=} \Xi(\lambda) \cap [a, b], \quad h_\lambda \stackrel{\text{def}}{=\!=} \max_{\xi \in [\![a,\, b^-\ ]\!]_\lambda} (\xi^+ - \xi).$$

Theorem 4. *If a function* $f(t) \in C[a, b]$ *satisfies condition* (11.16) *and the grid sequence* (11.17) *is such that* $\lim_{\lambda \to +0} h_\lambda = 0$*, then*

$$\lim_{\varepsilon \to +0} \lim_{\lambda \to +0} \frac{K}{N} = \frac{\frac{1}{b-a}\int_a^b f(t)dt}{\|f\|_{C[a,b]}}. \tag{11.18}$$

11.5. Variants of approximation for the discrete flow. Let $\widehat{X}$ be a subset of the grid Ξ,

$$\widehat{X}: \quad a = \widehat{x}_0 < \widehat{x}_1 < \widehat{x}_2 < \ldots < \widehat{x}_{\widehat{K}} < \widehat{x}_{\widehat{K}+1} = b, \quad \widehat{X} \subset \Xi, \tag{11.19}$$

and let $u(t)$ be a discrete function defined on $[\![a, b]\!]$. Consider the piecewise-linear interpolation of the function u,

$$\widetilde{u}(t) \stackrel{\text{def}}{=\!=} u(\widehat{x}_j) + \frac{u(\widehat{x}_{j+1}) - u(\widehat{x}_j)}{\widehat{x}_{j+1} - \widehat{x}_j}(t - \widehat{x}_j), \quad t \in [\widehat{x}_j, \widehat{x}_{j+1}), \quad j \in \{0, 1, \ldots, \widehat{K}\}. \tag{11.20}$$

For $u \in C[\![a, b]\!]$ and $\xi \in [\![a^+, b^-]\!]$, consider the difference ratios

$$D_\Xi u(\xi) \overset{\text{def}}{=\!=} \frac{u(\xi^+) - u(\xi)}{\xi^+ - \xi},$$
$$D_\Xi^2 u(\xi) \overset{\text{def}}{=\!=} \frac{D_\Xi u(\xi) - D_\Xi u(\xi^-)}{\xi - \xi^-}.$$

Theorem 5. *For the piecewise-linear interpolation* (11.19)–(11.20) *and* $t \in [\![\widehat{x}_j, \widehat{x}_{j+1}]\!]$ *the following inequalities hold*:

$$|u(t) - \widetilde{u}(t)| \le (\widehat{x}_{j+1} - \widehat{x}_j) \max_{\xi \in [\![\widehat{x}_j, \widehat{x}_{j+1}^-]\!]} |D_\Xi u(\xi)|, \tag{11.21}$$
$$|u(t) - \widetilde{u}(t)| \le (\widehat{x}_{j+1} - \widehat{x}_j)^2 \max_{\xi \in [\![\widehat{x}_j^+, \widehat{x}_{j+1}^-]\!]} |D_\Xi^2 u(\xi)|. \tag{11.22}$$

If $u \in C^1[a, b]$, then, by (11.21), we have

$$|u(t) - \widetilde{u}(t)| \le \max_{\xi \in [\widehat{x}_j, \widehat{x}_{j+1}]} |u'(\xi)|(\widehat{x}_{j+1} - \widehat{x}_j),$$

and if $u \in C^2[a, b]$, then, by (11.22), we obtain

$$|u(t) - \widetilde{u}(t)| \le \max_{\zeta \in [\widehat{x}_j, \widehat{x}_{j+1}]} |u''(\zeta)|(\widehat{x}_{j+1} - \widehat{x}_j)^2, \quad t \in (\widehat{x}_j, \widehat{x}_{j+1}).$$

11.6. On the number of nodes for a grid of the adaptive type

Theorem 6. *Assume that*

$$|D_\Xi u(t)| \ge c > 0, \qquad t \in [\![y, z]\!]. \tag{11.23}$$

If $\eta > 0$ *and the grid* $\widehat{X}$ *coincides with the grid* $\widetilde{X}(|D_\Xi u(t)|, \eta, \Xi)$, *then the following assertions are valid:*

1) *the number of nodes* $K'_{u,\Xi}(\eta) \overset{\text{def}}{=\!=} K(|D_\Xi u(t)|, \eta, \Xi)$ *of the grid satisfies the relations*

$$\begin{aligned}\sum_{s=0}^{K-1} \max_{t \in [\![\widetilde{x}_s,\, \widetilde{x}_{s+1}]\!]} |D_\Xi u(t)|(\widetilde{x}_{s+1} - \widetilde{x}_s)/\eta \le K'_{u,\Xi}(\eta) \\ < \sum_{s=0}^{K-1} \max_{t \in [\![\widetilde{x}_s,\, \widetilde{x}_{s+1}^+]\!]} |D_\Xi u(t)|(\widetilde{x}_{s+1}^+ - \widetilde{x}_s)/\eta;\end{aligned} \tag{11.24}$$

2)

$$|u(t) - \widetilde{u}(t)| \le \eta, \qquad t \in [a, b];$$

3) *if* $u \in C^1[a, b]$, $|u'(t)| \ge c > 0$ *for all* $t \in [a, b]$, *and a grid sequence of the form* (11.17) *with the property* $\lim_{\lambda \to +0} h_\lambda = 0$ *is considered, then*

$$\lim_{\eta' \to +0} \lim_{\lambda \to +0} K'_{u,\Xi(\lambda)}(\eta')\eta' = \int_a^b |u'(t)|\, dt.$$

Theorem 7. *Assume that*

$$|D_\Xi^2 u(t)| \ge c > 0, \qquad t \in [\![y, z]\!].$$

If $\eta > 0$ *and the grids* $\widehat{X}$ *and* $\widetilde{X}(\sqrt{|D_\Xi^2 u(t)|}, \eta, \Xi)$ *coincide, then the following assertions hold:*

1) *the number of nodes* $K''_{u,\Xi}(\eta) \stackrel{\text{def}}{=} K(\sqrt{|D^2_\Xi u(t)|}, \eta, \Xi)$ *of the grid satisfies the relations*

$$\sum_{s=0}^{K-1} \max_{t\in[\![\widetilde{x}_s,\, \widetilde{x}_{s+1}]\!]} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}_{s+1}-\widetilde{x}_s)/\eta \le K''_{u,\Xi}(\eta) < \sum_{s=0}^{K-1} \max_{t\in[\![\widetilde{x}_s,\, \widetilde{x}^+_{s+1}]\!]} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}^+_{s+1} - \widetilde{x}_s)/\eta;$$

2)

$$|u(t) - \widetilde{u}(t)| \le \eta^2, \quad t \in [a, b];$$

3) *if* $u \in C^2[a, b]$, $|u''(t)| \ge c > 0$ *for all* $t \in [a, b]$, *and a sequence of grids of the form* (11.17) *with the property* $\lim_{\lambda\to+0} h_\lambda = 0$ *is considered, then*

$$\lim_{\eta'\to+0} \lim_{\lambda\to+0} K''_{u,\Xi(\lambda)}(\eta')\eta' = \int_a^b \sqrt{|u''(t)|}\, dt.$$

The proof follows from (11.24).

11.7. On the number of nodes for a pseudo-equidistant grid

Theorem 8. *If the grids* $\widehat{X}$ *and* $\overline{X}(|D_\Xi u|, \eta, \Xi)$ *coincide, then*

1) *the number* $N'_{u,\Xi}(\eta) \stackrel{\text{def}}{=} N(|D_\Xi u|, \eta, \Xi)$ *of interior nodes of the grid satisfies the relations*

$$(b - a)\|D_\Xi u\|_{C[\![a,\, b]\!]}/\eta - 1 < N'_{u,\Xi}(\eta) \le (b - a)\|D_\Xi u\|_{C[\![a,\, b]\!]}/\eta, \tag{11.25}$$

and

2)

$$|u(t) - \widetilde{u}(t)| \le \eta, \quad t \in [\![a, b]\!]. \tag{11.26}$$

Proof. Setting $\widehat{X} \stackrel{\text{def}}{=} \overline{X}(|D_\Xi u|, \eta, \Xi)$ and applying (11.14), we come to relation (11.25). Inequality (11.26) is obtained by using relation (11.15) with $f = |D_\Xi u|$ and $\varepsilon = \eta$. □

Theorem 9. *If the grid* $\widehat{X}$ *coincides with the grid* $\overline{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$, *then the following assertions hold*:

1) *the number* $N''_{u,\Xi}(\eta) \stackrel{\text{def}}{=} N(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$ *of interior nodes of the grid satisfies the relations*

$$(b - a)\|\,|D^2_\Xi u|^{1/2}\|_{C[\![a,\, b]\!]}/\eta - 1 < N''_{u,\Xi}(\eta) \le (b - a)\|\,|D^2_\Xi u|^{1/2}\|_{C[\![a,\, b]\!]}/\eta, \tag{11.27}$$

and

2)

$$|u(t) - \widetilde{u}(t)| \le \eta^2, \qquad t \in [\![a, b]\!]. \tag{11.28}$$

Proof. By using (11.14) for $\widehat{X} \stackrel{\text{def}}{=} \overline{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$, we obtain relation (11.27). Inequality (11.28) is obtained by applying (11.25) with $f = \sqrt{|D^2_\Xi u|}$ and $\varepsilon = \eta$. □

11.8. Comparison of the numbers of nodes for the same approximation. A comparison of the numbers of nodes for an adaptive irregular grid and for a pseudo-equidistant grid reduces to applying Theorems 7–9 to one and the same function f for suitable approximations on different grids.

Theorem 10. *Assume that the assumptions of Theorem 7 are satisfied. Consider a discrete function $u(t)$, $t \in [\![a, b]\!]$, approximated by $\widetilde{u}(t)$ for two choices of the grid, namely,*

$$\widehat{X} \stackrel{\text{def}}{=} \overline{X}(|D_\Xi u|, \eta, \Xi) \quad \textit{and} \quad \widehat{X} \stackrel{\text{def}}{=} \widetilde{X}(|D_\Xi u|, \eta, \Xi).$$

Then the following inequalities are valid:

$$\frac{(b-a)\|D_\Xi u\|_{C[\![a,\,b]\!]} - \eta}{\sum\limits_{s=0}^{K-1} \max_{t\in[\![\widetilde{x}_s,\, \widetilde{x}^+_{s+1}]\!]} |D_\Xi u(t)|(\widetilde{x}^+_{s+1} - \widetilde{x}_s)} < \frac{N'_{u,\Xi}(\eta)}{K'_{u,\Xi}(\eta)}$$
$$\leq \frac{(b-a)\|D_\Xi u\|_{C[\![a,\,b]\!]}}{\sum\limits_{s=0}^{K-1} \max_{t\in[\![\widetilde{x}_s,\, \widetilde{x}_{s+1}]\!]} |D_\Xi u(t)|(\widetilde{x}_{s+1} - \widetilde{x}_s)}. \tag{11.29}$$

Theorem 11. *Let $u \in C^1[a, b]$,*

$$\|u'\|_{C[a,b]} \neq 0, \tag{11.30}$$

and let a system of grids of the form (11.17) *satisfy the condition* $\lim_{\lambda\to+0} h_\lambda = 0$. *Then*

$$\lim_{\eta\to+0} \lim_{\lambda\to+0} \frac{N'_{u,\Xi}(\eta)}{K'_{u,\Xi}(\eta)} = \frac{\frac{1}{b-a}\int_a^b |u'(t)|dt}{\|u'\|_{C[a,b]}}. \tag{11.31}$$

Proof. In order to prove (11.31), consider relation (11.29) and pass to the limit as $\lambda \to +0$ and then $\eta \to +0$. By (10.30), there exists a constant $c > 0$ such that (11.23) holds. This completes the proof of (11.31). □

Theorem 12. *Let a discrete function $u(t)$ be approximated by $\widetilde{u}(t)$, $|\widetilde{u}(t) - u(t)| \leq \eta^2$, using two choices of the grid, namely, the pseudo-equidistant grid of the form $\widehat{X} \stackrel{\text{def}}{=} \overline{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$ and the grid $\widehat{X} \stackrel{\text{def}}{=} \widetilde{X}(\sqrt{|D^2_\Xi u|}, \eta, \Xi)$. Then*

$$\frac{(b-a)\|\,|D^2_\Xi u|^{1/2}\|_{C[\![a,\,b]\!]} - \eta}{\sum\limits_{s=0}^{K-1} \max_{t\in[\![\widetilde{x}_s,\, \widetilde{x}^+_{s+1}]\!]} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}^+_{s+1} - \widetilde{x}_s)} < \frac{N''_{u,\Xi}(\eta)}{K''_{u,\Xi}(\eta)}$$
$$\leq \frac{(b-a)\|\,|D^2_\Xi u|^{1/2}\|_{C[\![a,\,b]\!]}}{\sum\limits_{s=0}^{K-1} \max_{t\in[\![\widetilde{x}_s,\, \widetilde{x}_{s+1}]\!]} \sqrt{|D^2_\Xi u(t)|}(\widetilde{x}_{s+1} - \widetilde{x}_s)}. \tag{11.32}$$

Theorem 13. *Consider a grid family of the form* (11.17) *such that* $\lim_{\lambda\to+\infty} h_\lambda = 0$ *and a function $u \in C^2[a, b]$ with the property*

$$\|u''\|_{C[a,b]} \neq 0.$$

Then

$$\lim_{\eta\to+0} \lim_{\lambda\to+0} \frac{N''_{u,\Xi}(\eta)}{K''_{u,\Xi}(\eta)} = \frac{\frac{1}{b-a}\int_a^b \sqrt{|u''(t)|}dt}{\|\sqrt{|u''|}\|_{C[a,b]}}. \tag{11.33}$$

Proof. Relation (11.33) is obtained from inequalities (11.32) by passing to the limit as in the proof of (11.31). □

Problem statement. Computation of the spline-wavelet decomposition consists of two steps. The first step realizes the decomposition formulas, and the second one realizes the reconstruction formulas.

The computations related to the decomposition formulas involve the two problems of finding the main and wavelet flows. As a rule, the first problem is more important than the second one because, in most cases, the original flow is characterized by the main flow.

In computing the spline-wavelet decomposition, one needs to construct an embedded grid $\widehat{X}$ given on an original grid X. Usually, the structure of $\widehat{X}$ is defined by the original flow $\mathbf{c}$ (see Sec. 11) and can require considerable computer resources. Note that the algorithm used here is optimal (in a certain sense). A drawback of this algorithm is that it is an essentially sequential process, whence the use of a single-processor computer system is more efficient. On the other hand, it is possible to split the original flow into a few parts, which can be processed concurrently. But this leads to a certain loss of efficiency. It is preferable to use a successive algorithm (or a parallel algorithm of small width) in the case where the flow is processed in the real time frame, in which case the flow has not been received completely. If the entire flow has already been received, then its processing on a parallel computer system can be very efficient.

In what follows, we assume that the computer system considered is a successive or a parallel computer of discrete operation with synchronous cycles. Instead of the usual time, the discrete time, measured in computer cycles, is considered. Thus, the time unit is the cycle, and the run time for an operation is a positive integer (see [7]).

In implementing the above algorithms, one needs to extract elements from arrays (i.e., to assign their values to auxiliary variables), to immerse elements into arrays (i.e., to assign the value of an auxiliary variable to an element of an array), and also to perform arithmetic and logical operations.

Introduce an additional notation. Let $\mathbf{A}$ be an array with elements A_i of a type $f_{\mathbf{A}}$ (for simplicity, we may assume that $f_{\mathbf{A}} = f$, where f is the type `real`; specific representations of the types $f_{\mathbf{A}}$ and f are not discussed here).

Let $\breve{T}_{\mathbf{A}}$ be the time (i.e., the length of the time interval[2]) for extracting an element A_i of the array $\mathbf{A}$ into a simple (auxiliary) variable. By $\widehat{T}_{\mathbf{A}}$ we denote the time needed to immerse the value of a simple variable into an element A_i of the array $\mathbf{A}$[3].

For brevity, below we use the same symbols both for the objects considered and for the arrays in which they are stored in the computer memory (this concerns the number flows $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and their elements, and also X, $\widehat{X}$, J_s, J^*, etc.). All the arrays considered are assumed to be dynamic (i.e., the length of an array is not fixed beforehand and can be changed).

First consider the formulas of decomposition (8.15)–(8.16):

$$a_i = c_{\varkappa(i+1)-1}, \qquad i \in J'_{K-1}, \tag{12.1}$$

$$b_q = 0, \qquad q+1 \in J^*. \tag{12.2}$$

If $q+1 \in J_M \setminus J^*$, then relations (8.19)–(8.20) can be written in the form

$$b_q = c_q - (\xi_{\varkappa(i+1)} - \xi_{\varkappa(i)})^{-1}\Big[(\xi_{\varkappa(i+1)} - \xi_{q+1})c_{\varkappa(i)-1} + (\xi_{q+1} - \xi_{\varkappa(i)})c_{\varkappa(i+1)-1}\Big], \tag{12.3}$$

where

$$\varkappa(i) + 1 \le q+1 \le \varkappa(i+1) - 1. \tag{12.4}$$

[2]Recall that we discuss the discrete time, and the unit of time is equal to the cycle of the computer system.

[3]We assume that $\widehat{T}_{\mathbf{A}}$ is independent of the number i of the element A_i, but it may depend on the type of elements and the length of the array.

Such formulas can be computed in a number of ways. We discuss one of them, assuming that

$$a = 0, \quad b = M, \quad \xi_i = i. \tag{12.5}$$

Thus,

$$X \overset{\text{def}}{=\!=} \{a = 0, 1, 2, \ldots, M-1, M = b\}.$$

In this case, $\widehat{x}_i = \varkappa(i)$, $i \in \{0, 1, \ldots, K\}$,

$$\widehat{X} = \{a = \varkappa(0), \varkappa(1), \varkappa(2), \ldots, \varkappa(K-1), \varkappa(K) = b\}.$$

Obviously, $X = J_M$ and $\widehat{X} = J^*$.

Under condition (12.5), relations (12.3)–(12.4) can be written as

$$b_q = c_q - (\varkappa(i+1) - \varkappa(i))^{-1}\Big[(\varkappa(i+1) - q - 1)c_{\varkappa(i)-1} + ((q + 1 - \varkappa(i))c_{\varkappa(i+1)-1}\Big], \tag{12.6}$$

where

$$\varkappa(i) + 1 \le q + 1 \le \varkappa(i+1) - 1. \tag{12.7}$$

Assume that the realization of the algorithm for searching for $j = \varkappa(i)$ requires τ_i cycles, an additive operation requires t_a cycles, and a multiplicative operation requires t_m cycles.

As a rule, we do not indicate the names of intermediate variables, though their presence is implied (for instance, instead of the assignment $j := \varkappa(i+1)$, where j is a simple variable used later, we simply "compute" $\varkappa(i+1)$).

The realization of the decomposition process is represented as a sequence of steps. Consider the $(i+1)$st step of this process. We use the subscript i to denote the state of an array at the ith step of the process.

Before the beginning of the $(i+1)$st step, the state of arrays is described by the following characteristics:

a/ the value of $\varkappa(i)$ has been calculated (and has been saved in a simple auxiliary variable);

b/ the array X_i has been represented in the form

$$X_i = \{0, 1, 2, \ldots, i-1, i\};$$

c/ the array $J_i^* = \widehat{X}_i$ has been written in the form

$$\widehat{X}_i = \{a = \varkappa(0), \varkappa(1), \varkappa(2), \ldots, \varkappa(i)\};$$

d/ the array $\mathbf{a}$ has been filled up to the element $a_{\varkappa(i-1)}$, i.e.,

$$\mathbf{a}_i = \{a_{\varkappa(0)}, a_{\varkappa(1)}, \ldots, a_{\varkappa(i-1)}\};$$

e/ the array $\mathbf{b}$ has been filled up to the element $b_{\varkappa(i-1)-1}$, i.e.,

$$\mathbf{b}_i = \{b_{\varkappa(i_0+1)}, \ldots, b_{\varkappa(i_1)-1}\},$$

where $i_0 = \min\{i \mid i \in X \setminus \widehat{X}\}$, $i_1 = \max\{i \mid i \in X \setminus \widehat{X}\}$;

f/ the value $c_{\varkappa(i)-1}$ has been computed.

The $(i+1)$st step consists of the following operations:

1. First, the element $\varkappa(i+1)$ is computed, which requires $\tau_i + t_a$ cycles.

2. The array $\widehat{X}_i$ is extended with the subsequent element $\varkappa(i+1)$, which requires $\widehat{T}_{\widehat{X}}$ cycles; then the array takes the form

$$\widehat{X}_{i+1} = \{a = \varkappa(0), \varkappa(1), \varkappa(2), \ldots, \varkappa(i), \varkappa(i+1)\}.$$

3.The value $\varkappa(i+1) - 1$ is computed, and then the element $c_{\varkappa(i+1)-1}$ is computed and extracted from the array $\mathbf{c}$, which requires $t_a + \breve{T}_{\mathbf{c}}$ cycles.

4. The element $a_i = c_{\varkappa(i+1)-1}$ is added to the array $\mathbf{a}$, which requires $\widehat{T}_{\mathbf{a}}$ cycles (recall that the value $\varkappa(i+1)-1$ has already been computed and assigned to a simple variable, the name of which is not indicated in accordance with the above agreement).

5. For every

$$q \in \{\varkappa(i), \varkappa(i)+1, \ldots, \varkappa(i+1)-2\}, \tag{12.8}$$

the element b_q is computed by formulas (12.6)–(12.7) (note that the set (12.8) is nonempty because $q+1 \in J_M \setminus J^*$). Since the elements $\varkappa(i)$, $\varkappa(i+1)$, and $c_{\varkappa(i)-1}$ have already been computed, it is only necessary to extract the element $c_{\varkappa(i+1)-1}$ from the array $\mathbf{c}$ and to compute the difference $\varkappa(i+1)-\varkappa(i)$; this requires $\breve{T}_{\mathbf{c}} + t_a$ cycles.

In order to compute b_q, we put $q = \varkappa(i)+j$, so that

$$b_{\varkappa(i)+j} = c_{\varkappa(i)+j} - (\varkappa(i+1)-\varkappa(i))^{-1}\Big[(\varkappa(i+1)-\varkappa(i)+j-1)c_{\varkappa(i)-1} + (j+1)c_{\varkappa(i+1)-1}\Big]; \tag{12.9}$$

then we organize a loop over $j \in \{0, 1, \ldots, \varkappa(i+1)-\varkappa(i)-2\}$.

At the jth iteration of the loop one needs

5.1) to extract $c_{\varkappa(i)+j}$ from the array $\mathbf{c}$, which requires $t_a + \breve{T}_{\mathbf{c}}$ cycles;

5.2) to perform the four additive and two multiplicative operations in the square brackets of relation (12.9) (here, the operations performed earlier are not taken into account); this requires $4t_a + 2t_m$ cycles;

5.3) to perform one multiplicative and one additive operations outside of the square brackets; this requires $t_a + t_m$ cycles;

5.4) to immerse the value $b_q = b_{\varkappa(i)+j}$ into the array $\mathbf{b}$, which requires $\widehat{T}_{\mathbf{b}}$ cycles.

Thus, the realization of one loop iteration requires $6t_a + 3t_m + \breve{T}_{\mathbf{c}}$ cycles. The number of such iterations is equal to $\varkappa(i+1)-\varkappa(i)-1$ (obviously, the iterations are absent in the case where $\varkappa(i+1) = \varkappa(i)+1$). With account for the preparatory operations, the computation of all the necessary values b_q at the $(i+1)$st step requires $\breve{T}_{\mathbf{c}} + t_a + (6t_a + 3t_m + \breve{T}_{\mathbf{c}} + \widehat{T}_{\mathbf{b}})(\varkappa(i+1)-\varkappa(i)-1)$ time units.

Now it is clear that the realization of the entire $(i+1)$st step requires

$$\tau_i + 3t_a + \widehat{T}_{\widehat{X}} + \breve{T}_{\mathbf{c}} + \widehat{T}_{\mathbf{a}} + \breve{T}_{\mathbf{c}} + (6t_a + 3t_m + \breve{T}_{\mathbf{c}} + \widehat{T}_{\mathbf{b}})(\varkappa(i+1)-\varkappa(i)-1) \tag{12.10}$$

cycles.

Theorem 14. *The discrete time T^* required to realize the decomposition algorithm is given by the formula*

$$T^* = \sum_{i=0}^{K-1} \tau_i + K(3t_a + \widehat{T}_{\widehat{X}} + 2\breve{T}_{\mathbf{c}} + \widehat{T}_{\mathbf{a}}) + (6t_a + 3t_m + \breve{T}_{\mathbf{c}} + \widehat{T}_{\mathbf{b}})(M-K). \tag{12.11}$$

Proof. The number of steps is equal to the number K of segments $[\![\varkappa(i), \varkappa(i+1)]\!]$. Taking into account (12.1)–(12.2) and using (12.10), we have

$$T^* \stackrel{\text{def}}{=} \sum_{i=0}^{K-1} \Big(\tau_i + 3t_a + \widehat{T}_{\widehat{X}} + 2\breve{T}_{\mathbf{c}} + \widehat{T}_{\mathbf{a}} + (6t_a + 3t_m + \breve{T}_{\mathbf{c}} + \widehat{T}_{\mathbf{b}})(\varkappa(i+1)-\varkappa(i)-1)\Big).$$

It is clear that

$$\sum_{i=0}^{K-1} (\varkappa(i+1)-\varkappa(i)-1) = \varkappa(K) - \varkappa(0) - K = M - K,$$

and (12.11) follows. □

The time for the realization of the reconstruction algorithm can be evaluated similarly. This will be a done in a forthcoming publication.

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