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SPHERICAL TRANSFORMATION OF GENERALIZED POISSON SHIFT AND PROPERTIES OF WEIGHTED LEBESGUE CLASSES OF FUNCTIONS

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We obtain a formula for the spherical transformation of generalized shift of a function depending on multiple-axial spherical symmetry. This formula shows that the generalized shift order depends on the dimension of the spherically symmetric part of the Euclidean space. The formula can be used for reducing some problems in weighted function spaces to the case of function spaces without weight. For an example we prove the global continuity with respect to shift and show that functions of class $C^\infty_{ev,0}$ are dense in the weighted Lebesgue classes. Bibliography: 6 *titles.*

1 The Main Notions and Notation

Introduce the notation $x' \in \mathbb{R}_n$, $x'' \in \mathbb{R}_{N-n}$, $\mathbb{R}_N = \mathbb{R}_n \times \mathbb{R}_{N-n}$, $\mathbb{R}^+_N$ is the Euclidean n-dimensional half-space of $\mathbb{R}_N$ defined by $x_1 > 0, \dots, x_n > 0$, γ_k are fixed positive numbers, $k = \overline{1,n}$, $n \leqslant N$, and $S^+_r(N) = \{x : |x| = r\}^+ = \{x : |x| = r\} \cap \mathbb{R}^+_N$. The *area of the weighted n-dimensional half-sphere* $S^+_r(N)$ is defined by formula (1.2.5) in [1]

$$|S^+_r(N)|_\gamma = \int\limits_{S^+_r(N)} (x')^\gamma dS = r^{n+|\gamma|-1} \frac{\pi^{\frac{N-n}{2}} \prod\limits_{k=1}^n \Gamma\left(\frac{\gamma_k+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{N+|\gamma|}{2}\right)} \tag{1.1}$$

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where $(x')^\gamma = \prod\limits_{k=1}^{n} x_k^{\gamma_k}$. We consider the singular Bessel differential operator

$$B_{\gamma_k} = \frac{\partial^2}{\partial x_k^2} + \frac{\gamma_k}{x_k}\frac{\partial}{\partial x_k},$$

$k = \overline{1,n}$. Denote $B_{x'}^{\alpha'} = B_{\gamma_1}^{\alpha_1} \dots B_{\gamma_n}^{\alpha_n}$ and $D_{x''}^{\alpha''} = D_{x_{n+1}}^{\alpha_{n+1}} \dots D_{x_N}^{\alpha_N}$, $D_{x_j} = \frac{\partial}{\partial x_j}$. The *mixed BD-derivative* of order $|\alpha| = 2|\beta'| + |\beta''|$ of a function f is defined by $B_{x'}^{\beta'} D_{x''}^{\beta''} f(x', x'')$. We say that a function is *x'-even* if it is even with respect to each component of the vector x'. The set $C_{\rm ev}^\infty$ of infinitely differentiable x'-even functions is invariant under taking the BD-derivative. The set of infinitely differentiable x'-even compactly supported functions is denoted by $C_{{\rm ev},0}^\infty$. We introduce the *multi-dimensional mixed generalized shift of order* $\gamma = (\gamma_1, \dots \gamma_n)$ by

$$({}^\gamma T^y f)(x) = ({}^\gamma T_{x'}^{y'} f)(x'\ x'' - y'') = \prod_{k=1}^{n} ({}^{\gamma_k} T_{x_k}^{y_k} f)(x'\ x'' - y'')$$

where the generalized Poisson shift acts on each weight variable x_k $(k = \overline{1,n})$

$$({}^{\gamma_k} T_{x_k}^{y_k} f)(x_k, x^k) = \frac{\Gamma\left(\frac{\gamma_k+1}{2}\right)}{\Gamma(1/2)\Gamma(\gamma_k/2)} \int\limits_0^\pi f(\sqrt{x_k^2 + y_k^2 - 2x_k y_k \cos\theta_k}\ x^k) \sin^{\gamma_k - 1}\theta_k d\theta_k$$

and $x^k = (x_1, \dots x_{k-1}, x_{k+1}, \dots x_N)$, $k = \overline{1,n}$ (cf. [2, 3]). In particular, the shift commutes with the singular Bessel differential operator: $B_{\gamma_k}({}^{\gamma_k} T_{x_k}^{y_k} f)(x_k, x^k) = ({}^{\gamma_k} T_{x_k}^{y_k} B_\gamma f)(x_k, x^k)$. Convolutions of the form

$$(u * v)_\gamma = \int\limits_{\mathbb{R}_N^+} ({}^\gamma T^y u)(x) v(y) (y')^\gamma \, dy$$

are called *mixed generalized convolutions* generated by the mixed generalized shift of order γ.

2 Spherical Transformation of Generalized Shift

Denote by $({}^\gamma T^x f(y) \star g(y))$ one or several arithmetical operations for functions $({}^\gamma T^x f)(y)$ and $g(y)$. We begin with the case $n = 1$ (γ is a fixed positive number), i.e., the generalized shift acts only on one variable

$$({}^\gamma T^x f)(y) = ({}^\gamma T_{y_1}^{x_1} f)(y_1, y' - x') = \frac{\Gamma(\gamma + 1/2)}{\Gamma(\frac{\gamma}{2})\Gamma(\frac{1}{2})} \int\limits_0^\pi f(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\alpha}, x' - y') \sin\alpha d\alpha,$$

where $x, y \in \overline{\mathbb{R}_m^+}$, $\mathbb{R}_m^+ = \{x = (x_1, x') = (x_1, x_2, \dots x_m), x_1 > 0\}$.

Theorem 2.1. *Assume that* $\gamma{>}0$, $\gamma = [\gamma] + \{\gamma\}$, *where* $[\gamma]$ *and* $\{\gamma\}$ *are the integer and fractional parts of* γ, $[\gamma] = m - 1$, $m \geqslant 1$, *and* $\{\gamma\} = \mu$. *Let* $f = f(|x|)$ *and* $g = g(|x|)$ *be radial functions in* $\mathbb{R}_m^+ = \{x : x_1 > 0\}$, *and let the weighted Lebesgue integral of the operation* $(\star)$

$$[f \star g]_\mu = \int\limits_{\mathbb{R}_m^+} ({}^\mu T_{y_1}^{x_1} f(\sqrt{y_1^2 + |x' - y'|^2}) \star g(|y|))\, y_1^\mu \, dy < \infty.$$

Then

$$[f\,\star\,g]_\mu=|S_1^+(m)|_\mu\;[f(r)\,\star\,g(r)]_\gamma=|S_1^+(m)|_\mu\int\limits_0^\infty(\,^\gamma T^\rho f(r)\,\star\,g(r))\,r^\gamma\,dr, \tag{2.1}$$

where $\rho=|x|$, $S_1^+(m)=\{|y|=1\}^+=\{y:|y|=1,y_1>0\}$,

$$|S_1^+(m)|_\mu=\int\limits_{S_1^+(m)}y_1^\mu\,dS=\frac{\pi^{m-1/2}\,\Gamma\,(\mu+1/2)}{\Gamma\,(m+\mu/2)}\,. \tag{2.2}$$

Proof. For $[\gamma]=0$ (i.e., $m=1$ and $\mu=\{\gamma\}\neq0$) the equality (2.1) becomes identity since it connects two one-dimensional integrals and $|S_1^+(1)|_\mu=1$ (cf. (2.2)). Therefore, we consider the case $[\gamma]=m-1$, $m\geqslant2$. In this case, the generalized shift acts on the first variable. Introduce the normalizing constant

$$C(\mu)=\frac{\Gamma(\mu+1/2)}{\Gamma(\mu/2)\,\Gamma(1/2)}\,.$$

We have

$$[f\star g]_\mu=\int\limits_{\mathbb{R}_m^+}(\,^\mu T_{y_1}^{x_1}f(\sqrt{y_1^2+|x'-y'|^2})\,\star\,g(|y|))\,y_1^\mu\,dy$$

$$=C(\mu)\int\limits_{\mathbb{R}_m^+}\left(\int\limits_0^\pi f(\sqrt{y_1^2+x_1^2-2x_1y_1\,\cos\alpha+|x'-y'|^2}\,)\sin^{\mu-1}\alpha d\alpha\,\star\,g(|y|)\right)y_1^\mu\,dy$$

$$=C(\mu)\int\limits_{\mathbb{R}_m^+}\int\limits_0^\pi(f(\sqrt{(y_1\,\cos\alpha-x_1)^2+y_1^2\sin^2\alpha+|x'-y'|^2}\,)\;\sin^{\mu-1}\alpha\;d\alpha\,\star\,g(|y|))\,y_1^\mu\,dy.$$

We recall that $y\in\mathbb{R}_m^+$ and, consequently, $y_1>0$. Therefore, the pair (y_1,α) can be regarded as the polar coordinates of the point (z_1,z_2) on the half-plane $z_2>0$:

$$z_1=y_1\cos\alpha,\;\;z_2=y_1\sin\alpha\quad(0\leqslant\alpha\leqslant\pi). \tag{2.3}$$

Taking into account that $\sin^{\mu-1}\alpha\,d\alpha\;y_1^\mu\,dy=z_2^{\mu-1}\,dz_1dz_2$, we write

$$[f\,\star\,g]_\mu=C(\mu)\int\limits_{\mathbb{R}_{m+1}^+}\left(f(\sqrt{(z_1-x_1)^2+z_2^2+|x'-y'|^2}\,)\,\star\,g(|z|)\right)z_2^{\mu-1}\,dz\,,$$

where $z=(z_1,z_2,y')\in\mathbb{R}_{m+1}^+=\{z:z_2>0\}$. We simplify the obtained relation by rotation about the Oz_2-axis in such a way that the direction of the Oz_1–axis coincides with the direction of the radius-vector of the point $\widetilde{x}=(x_1,0,x')$:

$$[f\,\star\,g]_\mu=C(\mu)\int\limits_{\mathbb{R}_{m+1}^+}(f(\sqrt{(z_1-|x|)^2+z_2^2+|y'|^2}\,)\,\star\,g(|z|))\,z_2^{\mu-1}\,dz\,.$$

We introduce the spherical coordinates $z = r\theta$, $|\theta| = 1$ in $\mathbb{R}^+_{m+1}$:

$$\left\{\begin{array}{l} z_1 = r\cos\varphi_1, \\ y_2 = r\sin\varphi_1\ \cos\varphi_2 \\ \dots\dots\dots\dots \\ y_m = r\sin\varphi_1\ \sin\varphi_2\dots\ \sin\varphi_{m-1}\cos\varphi_m, \\ z_2 = r\sin\varphi_1\ \sin\varphi_2\dots\ \sin\varphi_{m-1}\sin\varphi_m, \end{array}\right. \quad \Bigg| \quad 0 \leqslant \varphi_i \leqslant \pi, \quad i = \overline{1,m}\,.$$

It is clear that $z_2 > 0$ under this change of variables. A formula for an element of the unit sphere in an Euclidean space of an arbitrary dimension is well known. In $\mathbb{R}_{m+1}$, we have

$$dS = \sin^{m-1}\varphi_1 d\varphi_1 \sin^{m-2}\varphi_2\, d\varphi_2\ \dots \sin\varphi_{m-1}\, d\varphi_{m-1}\ d\varphi_m\,.$$

Hence

$$|S_1^+(m+1)|_{\mu-1} = \int\limits_{S_1^+(m+1)} z_2^{\mu-1} dS$$

$$= \int\limits_0^\pi \sin^{m+\mu-2}\varphi_1 d\varphi_1 \int\limits_0^\pi \sin^{m-2}\varphi_2\, d\varphi_2 \dots \int\limits_0^\pi \sin\varphi_{m-1}\, d\varphi_{m-1} \int\limits_0^\pi d\varphi_m\,.$$

Therefore,

$$(f\,\star\,g)_\mu = \frac{C(\mu)\ |S_1^+|_\mu}{\int\limits_0^\pi (\sin\varphi_1)^{m+\mu-2}\, d\varphi_1} 0 \int\limits_0^\infty \int\limits_0^\pi (f(\sqrt{r^2\theta_1^2+\rho^2-2r\rho\cos\varphi+r^2(\theta\,')^2+r^2\theta_m^2}\,)$$

$$\times \sin^{m+\mu-2}\varphi\ d\varphi \star g(r))\, r^{m+\mu-1}dr = \frac{C(\mu)\ |S_1^+|_\mu}{\int\limits_0^\pi (\sin\varphi_1)^{m+\mu-2}\, d\varphi_1}$$

$$\times \int\limits_0^\infty \left(\int\limits_0^\pi f(\sqrt{r^2\theta_1^2+\rho^2-2r\rho\cos\varphi+r^2(\theta\,')^2+r^2\theta_m^2}\,) \sin^{m+\mu-2}\varphi\ d\varphi \star g(r) \right) r^{m+\mu-1}\, dr\,.$$

We note that $r^2\theta_1^2+\rho^2-2r\rho\cos\varphi+r^2(\theta\,')^2+r^2\theta_m^2 = r^2 + \rho^2 - 2r\rho\ \cos\alpha$. By the formula for the Euler β-function, we have

$$\int\limits_0^\pi (\sin\varphi_1)^{m+\mu-2}\, d\varphi_1 = 2\int\limits_0^{\pi/2} (\sin\varphi)^{2m+\mu-1/2-1}\, d\varphi = \frac{\Gamma(m+\mu-1/2)\Gamma(1/2)}{\Gamma(m+\mu/2)}\,.$$

The obtained coefficient is the normalizing constant of the generalized shift of order $m-1+\mu = \gamma$. Consequently,

$$(f\,\star\,g)_\mu = C(\mu)|S_1^+(m+1)|_{\mu-1} \int\limits_0^\infty {}^\gamma T^\rho f(r)\,\star\,g(r) r^\gamma\, dr\,.$$

It remains to compute the coefficient. According to formula (2.2) (or formula (1.1) applied to one weight variable), we have

$$|S_1^+(m+1)|_{\mu-1} = \pi^{m/2}\frac{\Gamma(\mu/2)}{\Gamma(m+\mu/2)}\,.$$

Hence

$$\begin{aligned} C(\mu)|S_1^+(m+1)|_{\mu-1} &= \frac{\Gamma(\mu+1/2)}{\Gamma(\mu/2)\,\Gamma(1/2)}\,.\pi^{m/2}\frac{\Gamma(\mu/2)}{\Gamma(m+\mu/2)} \\ &= \pi^{m-1/2}\frac{\Gamma(\mu+1/2)}{\Gamma(m+\mu/2)} = |S_1(m)|_\mu, \end{aligned}$$

where the last equality follows from (2.2). □

Remark 2.1. In the case $\mu = \{\gamma\} = 0$, the relation (2.1) should be understood in the following sense: a usual shift acts on the left-hand side, whereas the operation $[\star]_\gamma$ on the right-hand side is satisfied by one-dimensional functions with the generalized Poisson shift of integer order $[\gamma] = m-1$. The corresponding equality for generalized convolutions is known (cf., for example, [3, formula (2.1)]).

We need the following assertion about extension of Euclidean spaces, based on spherical symmetry. Let $\gamma = (\gamma_1, \dots \gamma_n)$ be a multiindex consisting of positive numbers. We represent $\gamma = [\gamma] + \{\gamma\}$, where $[\gamma]$ and $\{\gamma\}$ are multi-indices of integer and fractional parts of numbers γ_i (some of them can vanish, but they cannot vanish simultaneously). Let $f(x', x'')$ be defined in the n-dimensional half-space $\mathbb{R}_N^+$. We consider the function $\widetilde{f}(\xi, x'') = f(|\xi^{(1)}|, \dots |\xi^{(n)}|, x'')$, where $x_i = |\xi^{(k)}|$, $\xi^{(k)} = (\xi_1^{(k)}, \dots \xi_{m_k}^{(k)}) \in \mathbb{R}_{m_k}^+$, $(\xi, x'') \in \mathbb{R}_m^+ \times \mathbb{R}_{N-n} = \mathbb{R}_{N-n+m}^+$, $m = \sum\limits_{k=1}^{n} m_k$, $m_k = [\gamma_k] + 1$. Since this function is even with respect to $\xi_i^{(k)}$, we can assume that all the components of ξ are positive. We set $\mathbb{R}_{N-n+m}^+ = \{(\xi, x'') : \xi_1^{(k)} > 0, k = \overline{1,n}\}$.

Theorem 2.2. *Let ${}^\gamma T^y$ be the mixed generalized shift corresponding to the multiindex γ, and let f and g be summable functions in $\mathbb{R}_N^+$ for which there exists a weighted integral of the operation $(\star)$ generated by this shift. Then*

$$\begin{aligned} [f \star g]_\gamma = \int\limits_{\mathbb{R}_N^+} (({}^\gamma T^x f)(y) \star g(y))(y')^\gamma\, dy = \left(\prod_{k=1}^{n} |S_1^+(m_k)|_{\{\gamma_k\}}\right)^{-1} \\ \times \int\limits_{\mathbb{R}_{N-n+m}^+} (({}^{\{\gamma\}}T_\eta^\xi \widetilde{f})(\eta, x''-y'') \star \widetilde{g}(\eta, y''))\eta^{\{\gamma\}}\, d\eta\, dy''\,, \end{aligned} \tag{2.4}$$

where ${}^{\{\gamma\}}T_\eta^\xi$ is the mixed generalized shift (acting only on the first coordinate $\eta_1^{(k)}$ of the vector $\eta^{(k)} \in \mathbb{R}_{m_k}$, whereas usual shifts act on the remaining variables)

$$|S_1^+(m_k)|_{\{\gamma\}} = \int\limits_{S_1^+(m_k)} (\xi_1^{(k)})^{\{\gamma_k\}}\, dS = \frac{\pi^{m_k-1/2}\,\Gamma(\{\gamma_k\}+1/2)}{\Gamma(m_k+\{\gamma_k\}/2)}\,.$$

Proof. We have

$$\int\limits_{\mathbb{R}^+_{N-n+m}} (({}^{\{\gamma\}}T^\xi_\eta \widetilde{f})(\eta, x''-y'') \star \widetilde{g}(\eta, y''))\eta^{\{\gamma\}}\, d\eta\, dy''$$

$$= \int\limits_{\mathbb{R}_{N-n}} dy'' \int\limits_{\mathbb{R}^+_{m_1}} \dots \int\limits_{\mathbb{R}^+_{m_n}} ({}^{\{\gamma\}}T^\xi_\eta \widetilde{f}(\eta, x''-y'') \star \widetilde{g}(\eta, y'')) \prod_{k=1}^n (\eta_1^{(k)})^{\{\gamma_k\}}\, d\eta^{(1)} \dots d\eta^{(n)}\,.$$

Applying formula (2.1) to each integral over $\mathbb{R}^+_{m_k}$, we obtain (2.4). □

3 Properties of Weighted Lebesgue Classes of Functions Connected with Generalized Shift

We consider functions defined in $\mathbb{R}^+_N$ and assume that n is fixed, $1 \leqslant n \leqslant N$ (the case $n = 0$ corresponds to the classical theory). We denote by Ω^+ a bounded domain adjacent to the coordinate hyperplanes $x_1 = 0, \dots, x_n = 0$. The boundary of Ω^+ consists of two parts: Γ^+ in $\mathbb{R}^+_n$ and Γ_0 in the hyperplane $x_1 = 0, \dots, x_n = 0$. Since we will consider x'-even functions, Γ_0 is the symmetry surface. Therefore, Ω^+ is understood as a partially closed domain $\Omega^+ = \Omega \cup \Gamma_0$. An interior subdomain of Ω^+ adjacent to the symmetry surface is called s-interior. We denote by Ω^+_δ an s-interior subdomain of Ω^+ at distance at least δ from Γ^+. We denote by $L^\gamma_p(\Omega^+)$ $(p \geqslant 1)$ the closure of the set of measurable x'-even functions in the norm

$$\|f\|_{L^\gamma_p(\Omega^+)} = \left[\int\limits_{\Omega^+} |f(x)|^p\, (x')^\gamma\, dx\right]^{1/p}. \tag{3.1}$$

Definition 3.1. A function $f \in L^\gamma_p(\Omega^+)$ is *globally continuous* in $L^\gamma_p(\Omega^+)$ for a mixed generalized shift of order γ if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\|\,{}^\gamma T^h f(x) - f(x)\|_{L^\gamma_p(\Omega^+_\delta)} < \varepsilon \quad \forall\, |h| < \delta\,. \tag{3.2}$$

By (3.1), the function $|f|^p$ can have singularity on Γ_0 that is controlled by the weight $(x')^\gamma$. We use the approach due to Kipriyanov (cf., for example, [4]). For $x \in \overline{\mathbb{R}^+_N}$ we introduce the spherical coordinates $x = (r, \theta)$, $|\theta| = 1$ and set $f(x) = 0$, $x \notin \overline{\Omega^+}$. Then (3.1) takes the form

$$\|f\|_{L^\gamma_p(\Omega^+)} = \left[\int\limits_0^\infty \int\limits_{S^+_1(N)} |f(r\theta)|^p\, r^{N-1+|\gamma|}\, (\theta')^\gamma\, dS dr\right]^{1/p}.$$

Hence such a function can have singularity at $r = 0$ controlled by the weight connected with the dimension of the Euclidean space corresponding to the domain Ω. Further we use the fact that the integrability of functions with singularity is improved with increasing the dimension of the integration domain, which allows us to use Theorems 2.1 and 2.2.

Let us consider (3.2). It is clear that if the singularity is caused by the weight x_1^γ, then it is located on the hyperplane $x_1 = 0$. The generalized shift sends such a singularity to the hyperplane $x_1 = y_1$, but this singularity is not controlled by the weight since this weight is left "unshifted." The same phenomenon is observed in the classical theory. In th case under consideration, it is expected that the singularity as well as the weight degree can have an arbitrarily large order. Nevertheless, the following assertion is valid.

Theorem 3.1. *Any function $f \in L_p^\gamma(\Omega_N^+)$ is globally continuous for a mixed generalized shift of order γ.*

Proof. For functions of class L_p, $p \geqslant 1$, this theorem is well known. Therefore, we consider only the case of the power weight $x_i^{\gamma_i}$. Assume that the function under consideration is continuous outside Γ_0. We first assume that the multiindex γ consists of natural numbers. We use Theorem 2.2, where for the operation $(\star)$ we take the difference

$$({}^\gamma T_x^h f \star g) = ({}^\gamma T_x^h f)(x) - f(x) = ({}^\gamma T_h^x f)(h) - f(x) = ({}^\gamma T_x^h (f(h) - f(x)) \quad p \geqslant 1 .$$

By the obvious symmetry of the generalized shift, $T_x^h f(x) = T_h^x f(h).$, we have

$$\|({}^\gamma T_x^h f)(x) - f(x)\|_{L_p^\gamma} = \|{}^\gamma T_h^x (f(h) - f(x))\|_{L_p^\gamma}.$$

As was proved in [5], $\|{}^\gamma T^h f(x)\|_{L_p^\gamma} \leqslant \|f(x)\|_{L_p^\gamma}$. Moreover,

$$\|{}^\gamma T^h \varphi(x)\|_{L_p^\gamma} \leqslant C(\gamma) \left[\int_0^\pi \sin^{\gamma-1}\alpha \, d\alpha \int_{\mathbb{R}_m^+} \left| \varphi(\sqrt{x_1^2 + h_1^2 - 2x_1h_1\cos\alpha}\, x' - h') \right|^p x_1^\gamma \, dx \right]^{1/p} . \tag{3.3}$$

Using (3.3) and (2.3), we get

$$\|T^h f(x) - f(x)\|_{L_p^\gamma(\Omega_{\delta,N}^+)} \leqslant \|\widetilde{f}(\xi + \eta\; x'' + h'') - \widetilde{f}(\xi, x'')\|_{L_p(\Omega_{\delta,N+m}^+)},$$

where the domain $\Omega_{\delta,N+m}^+ \in \mathbb{R}_{N+m}$ is obtained by rotation of $\Omega_{\delta,N}^+$ about the coordinate hyperplane $x_k = 0$, $k = \overline{1,n}$, $|\eta^{(k)}| = h_k$, and the function $\widetilde{f}$ is radial with respect to each group of variables $\eta^{(k)} \in \mathbb{R}_{m_k}$. However, since any function in the Lebesgue class is globally continuous for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|\widetilde{f}(\xi + \eta, x'' + h'') - \widetilde{f}(\xi, x'')\|_{L_p(\Omega_{\delta,N+m}^+)} < \varepsilon \quad \forall\; |(\eta, h'')| < \delta .$$

Taking into account that $|(\eta, h'')| = |h|$ in this case, we find (3.2).

Let $\{\gamma_k\} \neq 0$. We assume that $f \in L_p^{[\gamma]}(\Omega^+)$, $[\gamma] = m - 1$, $m \geqslant 2$. Taking into account the inequalities

$$x_k - h_k \leqslant \sqrt{x_k^2 + h_k^2 - 2x_kh_k\,\cos\alpha_k} \leqslant x_k + h_k, \tag{3.4}$$

and the first mean-value theorem, for continuous functions $f(x_k, x^k)$ with respect to x_k we get

$$C\mu_k \int_o^\pi \widetilde{f}(\sqrt{x_k^2 + h_k^2 - 2x_kh_k\,\cos\alpha_k}\, x^k)\, \sin^{\{\gamma_k\}}\alpha_k \, d\alpha_k$$

$$= C(\{\gamma_k\})\widetilde{f}(x_k + \widehat{h}_k, x^k) \int_0^\pi \sin^{\{\gamma_k\}}\alpha_k \, d\alpha_k = f(x_k + \widehat{h}_k, x^k)$$

where $x_k + \widehat{h}_k$ is the middle point between x_k and $x_k + h_k$, and, consequently, $\widehat{h}_k < h_k$. Hence

$$\|T^h f(x)-f(x)\|_{L_p^\gamma(\Omega^+_{\delta,N})} = \int\limits_{\mathbb{R}_{N-n}} dy'' \int\limits_{\mathbb{R}^+_{m_1}} \ldots \int\limits_{\mathbb{R}^+_{m_n}} |\,{}^{\{\gamma\}}T_\xi^{h'} \widetilde{f}(\xi, x''+h'') - \widetilde{g}(\xi, x'')\,|^p \prod_{k=1}^n (\xi_1^{(k)})^{\{\gamma_k\}}\, d\eta dx''\,.$$

Using (3.3) and (2.3), we find

$$\|T^h f(x) - f(x)\|_{L_p^\gamma(\Omega^+_{\delta,N})}$$

$$\leqslant C(\{\gamma\}) \int\limits_{\mathbb{R}_{N-n}} dy'' \int\limits_{\mathbb{R}^+_{m_1}} \ldots \int\limits_{\mathbb{R}^+_{m_n}} |\,\widetilde{f}(\xi+\widehat{h}, x''+h'') - \widetilde{f}(\xi, x'')\,|^p \prod_{k=1}^n (\xi_1^{(k)})^{\{\gamma_k\}}\, d\xi dx''\,.$$

Here, powers of the weight are small and, consequently, we can assert that $\widetilde{f} \in L_p(\Omega^+_{N+m})$.

Since $\Omega^+(N)$ is bounded, there exists R such that $\xi_1^{(k)} \leqslant R$ for all $k = \overline{1,n}$. Consequently,

$$\|T^h f(x) - f(x)\|_{L_p^\gamma(\Omega^+_{\delta,N})} \leqslant C(\{\gamma\}) R^{|\gamma|} \int\limits_{\mathbb{R}_{N-n}} dy'' \int\limits_{\mathbb{R}^+_m} |\widetilde{f}(\xi+\widehat{h}, x''+h'') - \widetilde{f}(\xi, x'')\,|^p\, d\xi dx''\,.$$

Thus, the proof of (3.2) is reduced to the theorem on global continuity of functions in the Lebesgue classes without weight for $1 \leqslant p < \infty$.

It remains to consider the case where one or several natural numbers m_k are equal to 1. The case of one weight variable is principal. Assume that $x_1 \in (0, \infty)$, $f(x_1, x'') \in L_p^\gamma(\Omega^+(N))$, $\gamma = \{\gamma\} < 1$. Applying (3.3) and (2.3), we obtain the inequality

$$\int\limits_{\mathbb{R}^+_N} |({}^\gamma T^h f)(x_1, x''+h'') - f(x)|^p\, x^\gamma\, dx$$

$$\leqslant C(\gamma) \int\limits_{\mathbb{R}^+_{N+1}} |f(\sqrt{(z_1-x)^2+z_2^2}\, x''+h'') - f(\sqrt{z_1^2+z_2^2}\, x''+h'')|^p\, z_2^{\gamma-1} dz_1 dz_2\, dx''\,.$$

We see that the weight has negative degree, the singularity of the integrable expressions caused by the weight is concentrated on the hyperplane $z_2 = 0$ and should be weak relative to one-dimensional integration. One can assume (for example, based on the mean-value theorem) that the function is continuous. Introducing the spherical coordinates, we write

$$\int\limits_{\mathbb{R}^+_N} |({}^\gamma T^h f)(x_1, x''+h'') - f(x)|^p\, x^\gamma\, dx \leqslant C(\gamma) \int\limits_0^\infty r^{m+|\gamma|} dr$$

$$\times \int\limits_{S_1^+(m+1)} \big|f(\sqrt{(r\Theta_1+\rho\Theta_1)^2+r^2\theta_2^2}\, r\theta''+\rho\Theta'') - f(\sqrt{r^2\theta_1^2+r^2\Theta_2^2}\, r\theta'')|^p\, \theta_2^{\gamma-1} dS(\theta)\, dr$$

$$= C(\gamma) \int\limits_0^\infty r^{m+|\gamma|} dr \int\limits_{S_1^+(m+1)} |\widetilde{f}((r\widehat{\theta}+\rho\widehat{\Theta}) - \widetilde{f}(r\widehat{\theta})|^p\, \theta_2^{\gamma-1} dS(\widehat{\theta})$$

$$\leqslant C(\gamma)R^{m+|\gamma|}\int\limits_0^\infty r\ \max_{\widehat{\theta},\widehat{\Theta}}|\widetilde{f}((r\widehat{\Theta}+\rho\widehat{\Theta})-\widetilde{f}(r\widehat{\theta})|^p\int\limits_{S_1^+(m+1)}\widehat{\theta}_2^{\gamma-1}dS(\widehat{\theta})$$

$$\leqslant C(\gamma)R^{m+|\gamma|}\,|S_1^+(m+1)|_{\gamma-1}\int\limits_0^\infty\ \max_{\widehat{\theta},\widehat{\Theta}}|\widetilde{f}((r\widehat{\theta}+\rho\widehat{\Theta})-\widetilde{f}(r\widehat{\theta})|^p\,dr\,.$$

By the theorem on global continuity of functions in $L_p(\Omega)$, we arrive at (3.2). □

We formulate a result about the density of the space of x'-even infinitely differentiable compactly supported functions in the space $L_p^\gamma(\Omega^+)$. The Sobolev–Kipriyanov averages in the case $n=1$ were introduced in [6]. In the general case, the average kernel is an infinitely differentiable x'-even function $\psi(t)$ on $\mathbb{R}_N^+$ that vanishes for $|t|\geqslant 1$ and satisfies the condition

$$\int\limits_{\mathbb{R}_N^+}\psi(x)\,(x')^\gamma\,dx=1.$$

We set

$$\omega_\varepsilon(x)=\frac{1}{\varepsilon^{N+|\gamma|}}\,\psi\Big(\frac{x}{\varepsilon}\Big)\,.$$

It is easy to verify that

$$\int\limits_{\mathbb{R}_N^+}\omega_\varepsilon(x)\,(x')^\gamma\,dx=1\,.$$

For an example of ω_ε we can take the same function as in the classical case:

$$\omega_\varepsilon(x)=\begin{cases}\dfrac{1}{\lambda}e^{\frac{|x|^2}{|x|^2-\varepsilon^2}}, & |x|<\varepsilon,\\ 0, & |x|\geqslant\varepsilon,\end{cases}$$

where

$$\lambda=\varepsilon^{N+|\gamma|}\ |S_1^+(N)|_\gamma\int\limits_0^1 e^{\frac{t^2}{t^2-1}}t^{|\gamma|}\,dt\quad |S_1^+(N)|_\gamma=\int\limits_{S_1^+(N)}(x')^\gamma\,dS.$$

Assume that $f\in L_p^\gamma(\Omega^+)$ is extended by zero outside Ω^+. The function

$$f_\varepsilon(x)=(f*\omega_\varepsilon)_\gamma(x)\tag{3.5}$$

is called the *Sobolev–Kipriyanov ε-average* of f. By $({}^\gamma T^y f,g)_\gamma=(f,{}^\gamma T^y g)_\gamma$ (cf. [2]), we have

$$f_\varepsilon(x)-f(x)=\frac{1}{\varepsilon^{N+\gamma}}\int\limits_{R_N^+}({}^\gamma T_x^u\psi)\left(\frac{|x|}{\varepsilon}\right)(f(u)-f(x))(u')^\gamma\,du$$

$$=\int\limits_{R_N^+}\psi(|y|)(T_x^{y\varepsilon}f(x)-f(x))(y')^\gamma\,dy.$$

Using the Minkowsky inequality, we get

$$\|f_\varepsilon - f\|_{L_p^\gamma(\Omega^+)} = \left(\int\limits_{R^+_{N,x}} \left| \int\limits_{R^+_{N,y}} \psi(|y|)(({}^\gamma T_x^{y\varepsilon} f)(x) - f(x))(y')^\gamma \, dy \right|^p (x')^\gamma \, dx \right)^{\frac{1}{p}}$$

$$\leqslant \int\limits_{R^+_{N,y}} \psi(|y|) \|({}^\gamma T_x^{y\varepsilon} f)(x) - f(x)\|_{L_p^\gamma(R^+_{N,x})} (y')^\gamma \, dy \leqslant \sup_{|y| \leqslant \varepsilon} \|({}^\gamma T_x^{y\varepsilon} f)(x) - f(x)\|_{L_p^\gamma(R^+_{N,x})}.$$

By the global continuity of generalized shift, we have

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L_p^\gamma(\Omega^+)} = 0. \tag{3.6}$$

Thus, we proved the following assertion.

Theorem 3.2. *The averaged functions f_ε generated by a mixed generalized shift of order γ strongly converge to a function f in the weighted Lebesgue class L_p^γ.*

Corollary 3.1. *The set $C^\infty_{\mathrm{ev},0}(\mathbb{R}^+_N)$ of functions of the form* (3.5) *is everywhere dense in $L_p^\gamma(\Omega^+)$, $1 \leqslant p < \infty$, $\gamma > 0$.*

Thus, $L_p^\gamma(\Omega^+)$ can be regarded as the closure of $C^\infty_{ev,0}$ in the norm (3.1).

In the case $p = +\infty$, the equality (3.6) fails. However, if $\Omega^+ = \mathbb{R}^+_N$ and $f(x)$ is x'-even and uniformly continuous on $\overline{\mathbb{R}^+_N}$, then

$$\|f_\varepsilon - f\|_{L_\infty^\gamma(\mathbb{R}^+_N)} \leqslant \sup_{|y| < \varepsilon} |T^y f(x) - f(x)| \to 0 \quad (\varepsilon \to 0).$$

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