DOI 10.1007/s10958-017-3437-x
Journal of Mathematical Sciences, Vol. 224, No. 4, July, 2017

Exact constants in Jackson-type inequalities for the best mean square approximation in $L_2(\mathbb{R})$ and exact values of mean ν-widths of the classes of functions

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Presented by R. M. Trigub

Abstract. On the classes of functions $L_2^r(\mathbb{R})$, where $r \in \mathbb{Z}_+$, for the characteristics of smoothness $\Lambda_k(f,t) = \{(1/t)\int_0^t \|\Delta_h^k(f)\|^2 dh\}^{1/2}$, $t \in (0,\infty)$, $k \in \mathbb{N}$, the exact constants in the Jackson-type inequalities have been obtained in the case of the best mean square approximation by entire functions of the exponential type in the space $L_2(\mathbb{R})$. The exact values of mean ν-widths of the classes of functions defined by $\Lambda_k(f)$ and the majorants Ψ satisfying some conditions are calculated.

Keywords. Entire function, best mean square approximation, characteristic of smoothness of a function, majorant, mean ν-width.

1. Introduction

The theory of approximation of functions by various aggregates such as polynomials, splines, entire functions, splashes, linear operators, etc. is one of the most succefully developed trends of modern mathematics and is of significant importance for its various fields. As for the approximation of functions given on the whole real axis, we note that the start of studies in this direction was given by S. N. Bernshtein in [1]. In this case, the entire functions of the finite exponential type serve as a tool of approximation. The indicated space was established by S.N. Bernshtein with the help of a definite limiting process involving the algebraic polynomials. In what follows, the various aspects of the theory of approximation of functions on the real axis by entire functions of the exponential type were analyzed by N. I. Achiezer, A. F. Timan, M. F. Timan, S. M. Nikol'skii, I. I. Ibragimov, F. G. Nasibov, V. Yu. Popov, A. G. Babenko, V. V. Arestov, A. I. Stepanets, S. Ya. Yanchenko, and others (see, e.g., [2]– [17]). The list of some final results related to the calculation of exact constants in the Jackson inequalities in the mean square approximation by entire functions of the exponential type can be found, for example, in [10, 14, 17]. As for the solution of an analogous problem for other characteristics of smoothness distinct from the classical modulus of continuity, it is worth to note the results obtained in works [12–14, 16]. In this case, the Jackson-type inequalities were considered.

The present work continues the indicated direction for the characteristics of smoothness that were got by means of the averaging of the squared norms of finite differences of functions. Recall that the similar characteristics of smoothness were considered by L. Leindler, R. M. Trigub, K. V. Runovskii, N. P. Pustovoitov and others (see, e.g., [18]– [24]) in the 2π-periodic case, while solving a number of problems of the theory of approximation of functions.

Let $L_2(\mathbb{R})$, where $\mathbb{R} := \{x : -\infty < x < \infty\}$, be a space of all functions f measurable on the real axis. Their squared modulus is integrable by Lebesgue on any finite segment, and the norm

Translated from Ukrains'kiĭ Matematychnyĭ Visnyk, Vol. 13, No. 4, pp. 543–569 October–December, 2016.
Original article submitted August 05, 2016

$\|f\| := \{\int\limits_{-\infty}^{\infty} |f(x)|^2 dx\}^{1/2} < \infty$. For any function $f \in L_2(\mathbb{R})$, there exists a finite difference of the k-th order $\Delta_h^k(f,x) := \sum\limits_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f(x+jh)$ almost everywhere on $\mathbb{R}$, where $h \in \mathbb{R}$, $k \in \mathbb{N}$. Consider the following characteristic of smoothness:

$$\Lambda_k(f,t) := \Big\{\frac{1}{t}\int\limits_0^t \|\Delta_h^k(f)\|^2 dh\Big\}^{1/2}, \tag{1.1}$$

where $t > 0$. The comparison of quantity (1.1) with the customary modulus of continuity of the k-th order $\omega_k(f,t) := \sup\{\|\Delta_h^k(f)\| : |h| \leqslant t\}$ implies that $\Lambda_k(f,t) \leqslant \omega_k(f,t)$ for any $t > 0$. We note that, in the 2π-periodic case, the properties of $\Lambda_k(f)$ were analyzed in work [24].

The characteristic of smoothness (1.1) arises naturally, if we consider the τ-moduli of smoothness of the k-th order in $L_2(\mathbb{R})$ that were introduced by K. Ivanov in [25, 26], while solving a number of problems of the theory of approximation of functions in the weight spaces $L_{p,w}[a,b]$. By setting that the weight function $w \equiv 1$ and $p = p' = 2$ in $L_2(\mathbb{R})$ [25,26], we have $\tau_k(f,1;\lambda(x))_{2,2} = \|\omega_k(f,\cdot;\lambda(\cdot))_2\|$, where $\lambda(x)$ is any positive function, and

$$\omega_k(f,x;\lambda(x))_2 = \Big\{\frac{1}{2\lambda(x)}\int\limits_{-\lambda(x)}^{\lambda(x)} |\Delta_h^k(f,x)|^2 dh\Big\}^{1/2}.$$

Taking $\lambda(x) \equiv t$, where t is any positive constant, we have

$$\tau_k^2(f,1;t)_{2,2} = \frac{1}{2t}\int\limits_{-\infty}^{\infty} dx \int\limits_{-t}^{t} |\Delta_h^k(f,x)|^2 dh = \frac{1}{2t}\int\limits_{-t}^{t} \|\Delta_h^k(f)\|^2 dh. \tag{1.2}$$

Since

$$\|\Delta_h^k(f)\|^2 = 2^k \int\limits_{-\infty}^{\infty} (1-\cos(hu))^k |\mathcal{F}(f,u)|^2 du, \tag{1.3}$$

where $\mathcal{F}(f)$ is the Fourier transform of the function $f \in L_2(\mathbb{R})$ (see, e.g., [17]), we have $\|\Delta_h^k(f)\| = \|\Delta_{-h}^k(f)\|, h > 0$. With regard for this result and relations (1.1)–(1.2), we get $\tau_k(f,1;t)_{2,2} = \Lambda_k(f,t)$, where $t > 0$.

The characteristic of smoothness (1.1) for functions from $L_2(\mathbb{R})$ can be also obtained on the basis of a different reasoning. Let $D := (a,b)$, where a and b can take not only finite, but also infinite values $-\infty$ and ∞, respectively, i.e., the interval (a,b) can be finite or infinite. For the functions $f \in L_p(D)$, $1 \leqslant p < \infty$, the following characteristic of smoothness was considered in [27, p. 26]:

$$\overline{\omega}_\varphi^{*k}(f,t)_p = \Big\{\frac{1}{t}\int\limits_0^t \int\limits_D |\overline{\Delta}_{h\varphi(x)}^k f(x)|^p dx dh\Big\}^{1/p}. \tag{1.4}$$

Here, $t > 0$. The function φ defined on the set D is positive and satisfies a number of requirements presented in item 1.2 in [27]; $\overline{\Delta}_{h\varphi(x)}^k f(x)$ is the direct or inverse finite difference of the k-th order for the function f such that it exists almost everywhere on D, i.e.,

$$\overline{\Delta}_{h\varphi(x)}^k f(x) := \overrightarrow{\Delta}_{h\varphi(x)}^k f(x) = \sum_{j=0}^{k}(-1)^j\binom{k}{j} f(x+(k-j)h\varphi(x))$$

or

$$\overline{\Delta}^k_{h\varphi(x)} f(x) := \overleftarrow{\Delta}^k_{h\varphi(x)} f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x - jh\varphi(x)).$$

In this case, we set $\overrightarrow{\Delta}^k_{h\varphi(x)} f(x) = 0$ or $\overleftarrow{\Delta}^k_{h\varphi(x)} f(x) = 0$, if the segment $[x, x+kh\varphi(x)]$ or $[x-kh\varphi(x), x]$, respectively, does not belong to the set D. Let, for example, $D = (-\infty, \infty)$, $\widetilde{\varphi}(x) \equiv 1$, $p = 2$, $\overline{\Delta}^k_{h\widetilde{\varphi}(x)} f(x) = \overrightarrow{\Delta}^k_{h\widetilde{\varphi}(x)} f(x) = \Delta^k_h(f, x)$ in formula (1.4). Then relations (1.1) and (1.4) for $f \in L_2(\mathbb{R})$ yield $\overline{\omega}^{*k}_{\widetilde{\varphi}}(f, t)_2 = \Lambda_k(f, t)$, $t > 0$.

From our viewpoint, the above results confirm the self-sufficiency of the characteristic of smoothness (1.1), which can be used in the study of the behavior of exact constants in the Jackson-type inequalities. On this basis in the subsequent sections, we will get exact Jackson-type inequalities in the space $L_2(\mathbb{R})$ and will calculate the exact values of a number of mean ν-widths of the classes of functions defined by means of $\Lambda_k(f)$.

2. The best mean square approximation by entire functions of the exponential type on the whole real axis

By the symbol $\mathbb{B}_{\sigma,2}$, where $0 < \sigma < \infty$, we denote the set of restrictions of all entire functions of the exponential type σ which belong to the space $L_2(\mathbb{R})$ on $\mathbb{R}$. For any function $f \in L_2(\mathbb{R})$, the quantity $\mathcal{A}_\sigma(f) := \inf\{\|f - g\| : g \in \mathbb{B}_{\sigma,2}\}$ is called the best approximation of f by elements of the subspace $\mathbb{B}_{\sigma,2}$ in the metric of $L_2(\mathbb{R})$. For any class $\mathfrak{M} \subset L_2(\mathbb{R})$, we set $\mathcal{A}_\sigma(\mathfrak{M}) := \sup\{\mathcal{A}_\sigma(f) : f \in \mathfrak{M}\}$. We introduce the notation

$$\eta_k(t; u) := \frac{1}{t} \int_0^t (1 - \cos(hu))^k dh, \tag{2.1}$$

where $k \in \mathbb{N}$, $t \in (0, \infty)$, and $u \in \mathbb{R}$. We note that $\lim_{t \to 0+} \eta_k(t; u) = 0$, and $\eta_k(t; 1)$, as a function of the variable t varying on the set $(0, \pi]$, increases, because

$$\frac{d\eta_k(t; 1)}{dt} = \frac{d}{dt}\Big\{\frac{1}{t}\int_0^t (1 - \cos h)^k dh\Big\} = \frac{1}{t}\Big\{(1 - \cos t)^k - \frac{1}{t}\int_0^t (1 - \cos h)^k dh\Big\}$$

$$> \frac{1}{t}\Big\{(1 - \cos t)^k - (1 - \cos t)^k\Big\} = 0.$$

Since (see, e.g., [28, item 1.320, formula 1])

$$2^k (1 - \cos(hu))^k = \Big(2 \sin\Big(\frac{hu}{2}\Big)\Big)^k$$

$$= \binom{2k}{k} + 2\sum_{j=0}^{k-1} (-1)^{k-j} \binom{2k}{j} \cos((k - j)hu)$$

$$= \binom{2k}{k} - 2\sum_{j=1}^{k} (-1)^{j+1} \binom{2k}{k - j} \cos(jhu),$$

we set sinc (t) is equal to $\sin(t)/t$ for $t \neq 0$ and to 1 for $t = 0$. We consider also that $u := 1$. By virtue of formula (2.1) and the last inequality, we have

$$2^k \eta_k(t; 1) = \binom{2k}{k} - 2\sum_{j=1}^{k}(-1)^{j+1}\binom{2k}{k-j}\operatorname{sinc}(jt). \tag{2.2}$$

Theorem 1. *Let $0 < t \leqslant 3\pi/4$ and $0 < \sigma < \infty$. Then the equality*

$$\sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\Lambda_1(f, t/\sigma)} = \frac{1}{\sqrt{2(1 - \operatorname{sinc}(t))}} \tag{2.3}$$

holds. Here, the upper bound on the left-hand side of relation (2.3) is calculated over all functions f from $L_2(\mathbb{R})$ that are not equivalent to zero.

Proof. It is well known [7] that, for any function $f \in L_2(\mathbb{R})$, there exists a unique function $\mathcal{L}_\sigma(f) \in \mathbb{B}_{\sigma,2}$ that has the least deviation from f in the metric of the space $L_2(\mathbb{R})$ and takes the form

$$\mathcal{L}_\sigma(f, x) := \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \mathcal{F}(f, u) e^{ixu} du, \tag{2.4}$$

where $\mathcal{F}(f)$ is the Fourier transform of the function f in $L_2(\mathbb{R})$. In this case,

$$\mathcal{A}_\sigma^2(f) = \|f - \mathcal{L}_\sigma(f)\|^2 = \int_{|u| \geqslant \sigma} |\mathcal{F}(f, u)|^2 du. \tag{2.5}$$

By formulas (1.1), (1.3), and (2.1), we get

$$\Lambda_1^2(f, \tau) = 2\int_{-\infty}^{\infty} \eta_1(\tau; u)|\mathcal{F}(f, u)|^2 du = 2\int_{-\infty}^{\infty} (1 - \operatorname{sinc}(\tau u))|\mathcal{F}(f, u)|^2 du. \tag{2.6}$$

Consider the function that is defined as $(1 - \operatorname{sinc}(x))_0 := \{1 - \operatorname{sinc}(x)$, if $0 < x \leqslant 3\pi/4$, and $1 - 2\sqrt{2}/(3\pi)$, if $3\pi/4 \leqslant x < \infty\}$. In view of the geometric reasoning related to the behavior of sinc (x), this function satisfies the inequality

$$(1 - \operatorname{sinc}(ax))_0 \geqslant (1 - \operatorname{sinc}(bx))_0, \tag{2.7}$$

where $|b| \leqslant |a| < \infty$ and $x \in \mathbb{R}$. With regard for relations (2.5)–(2.7), we have

$$\Lambda_1^2(f, \tau) \geqslant 2\int_{|u| \geqslant \sigma} (1 - \operatorname{sinc}(\tau u))|\mathcal{F}(f, u)|^2 du$$

$$\geqslant 2\int_{|u| \geqslant \sigma} (1 - \operatorname{sinc}(\tau u))_0 |\mathcal{F}(f, u)|^2 du \geqslant 2\int_{|u| \geqslant \sigma} (1 - \operatorname{sinc}(\tau\sigma))_0 |\mathcal{F}(f, u)|^2 du$$

$$= 2(1 - \operatorname{sinc}(\tau\sigma))_0 \mathcal{A}_\sigma^2(f). \tag{2.8}$$

Let $\tau = t/\sigma$, where $0 < t \leqslant 3\pi/4$. Then formula (2.8) yields the inequality

$$\sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\Lambda_1(f, t/\sigma)} \leqslant \frac{1}{\sqrt{2(1 - \operatorname{sinc}(t))}}. \tag{2.9}$$

In order to get the lower bound of the extreme characteristic on the left-hand side of inequality (2.9), we consider, like in works [12]– [17], the entire function $\lambda_\varepsilon(x) := \sqrt{2/\pi}(q_{\sigma+\varepsilon}(x) - q_\sigma(x))$ of the exponential type $\sigma + \varepsilon$, where $q_a(x) := a\ \mathrm{sinc}\,(ax)$, $a > 0$; $\varepsilon > 0$. In this case, the Fourier transform of the function q_a takes the form $\mathcal{F}(q_a, x) = \{\sqrt{\pi/2}$, if $|x| < a$, $\sqrt{\pi/8}$, if $|x| = a$, and 0, if $|x| > a\}$. Then $\mathcal{F}(\lambda_\varepsilon, x) = \{1$, if $\sigma < |x| < \sigma + \varepsilon$, $1/2$, if $|x| = \sigma + \varepsilon$ or $|x| = \sigma$, and 0, if $|x| > \sigma + \varepsilon$ or $|x| < \sigma\}$, i.e., $\lambda_\varepsilon \in L_2(\mathbb{R})$. By virtue of formula (2.4), we get

$$\mathcal{A}_\sigma(\lambda_\varepsilon) = \sqrt{2\varepsilon}. \tag{2.10}$$

Let

$$\eta_k^*(t; u) := \frac{1}{t}\int_0^t (1 - \cos(hu))_*^k dh, \tag{2.11}$$

where $(1 - \cos u)_* := \{1 - \cos u, \text{if } 0 \leqslant u \leqslant \pi;\ 2,\ \text{if } u \geqslant \pi\}$, $k \in \mathbb{N}$.

Relation (1.3) implies that $\|\Delta_h^1(\lambda_\varepsilon)\|^2 \leqslant 4\varepsilon(1 - \cos((\sigma + \varepsilon)h))_*$. From whence and formulas (1.1) and (2.11), we have

$$\Lambda_1(\lambda_\varepsilon, t/\sigma) \leqslant 2\sqrt{\varepsilon\eta_1^*(t/\sigma; \sigma + \varepsilon)}. \tag{2.12}$$

With regard for (2.10) and (2.12), we get

$$\sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\Lambda_1(f, t/\sigma)} \geqslant \frac{\mathcal{A}_\sigma(\lambda_\varepsilon)}{\Lambda_1(\lambda_\varepsilon, t/\sigma)} = \frac{1}{\sqrt{2\eta_1^*(t/\sigma; \sigma + \varepsilon)}}. \tag{2.13}$$

Relation (2.11) implies that the quantity $\eta_1^*(t/\sigma; \sigma+\varepsilon)$ does not increase as $\varepsilon \to 0+$ for constant values of t and σ. We note that $\lim\limits_{\varepsilon \to 0+} \eta_1^*(t/\sigma; \sigma + \varepsilon) = \eta_1(t/\sigma; \sigma) = 1 - \mathrm{sinc}\,(t)$, where $0 < t \leqslant \pi$. Therefore, for any arbitrarily small number $\delta > 0$, we can indicate such value of $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\delta) \in (0, \delta)$ for which

$$\frac{1}{\eta_1^*(t/\sigma; \sigma + \widetilde{\varepsilon})} > \frac{1}{1 - \mathrm{sinc}\,(t)} - \delta.$$

Using the definition of the upper bound of a number set, we get

$$\sup_{\varepsilon \in (0, \widetilde{\sigma})} \frac{1}{\eta_1^*(t/\sigma; \sigma + \widetilde{\varepsilon})} = \frac{1}{1 - \mathrm{sinc}\,(t)}, \tag{2.14}$$

where $\widetilde{\sigma} := \min(\sigma, 1/\sigma)$. We note that the left-hand side of relation (2.13) is independent of ε. Calculating the upper bound in $\varepsilon \in (0, \widetilde{\sigma})$ for its right-hand side and considering equality (2.14), we have

$$\sup_{f \in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\Lambda_1(f, t/\sigma)} \geqslant \frac{1}{\sqrt{2(1 - \mathrm{sinc}\,(t))}}. \tag{2.15}$$

The required relation (2.3) follows from inequalities (2.9) and (2.15) for $0 < t \leqslant 3\pi/4$. Theorem 1 is proved. □

By $L_2^r(\mathbb{R})$, where $r \in \mathbb{N}$, we denote the class of functions $f \in L_2(\mathbb{R})$ for which the derivatives of the $(r-1)$-th order $f^{(r-1)}$ ($f^{(0)} \equiv f$) are locally absolutely continuous, and the derivatives of the r-th order $f^{(r)}$ belong to the space $L_2(\mathbb{R})$. We note that $L_2^r(\mathbb{R})$ is a Banach space with the norm $\|f\| + \|f^{(r)}\|$.

Theorem 2. *Let $\sigma \in (0,\infty)$; $r,k \in \mathbb{N}$; $t \in (0,\pi]$. Then the equality*

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\Lambda_k(f^{(r)}, t/\sigma)} = \left\{ \binom{2k}{k} - 2\sum_{j=1}^{k} (-1)^{j+1} \binom{2k}{k-j} \operatorname{sinc}(jt) \right\}^{-1/2} \tag{2.16}$$

holds.

Proof. Let f be any function from the class $L_2^r(\mathbb{R})$. Since, almost everywhere on $\mathbb{R}$,

$$\mathcal{F}(f^{(r)}, x) = (ix)^r \mathcal{F}(f, x), \tag{2.17}$$

we can write

$$\|\Delta_h^k(f^{(r)})\|^2 = 2^k \int_{-\infty}^{\infty} (1-\cos(hu))^k u^{2r} |\mathcal{F}(f,u)|^2 du$$

in view of formulas (1.3) and (2.17). From whence with the use of relation (1.1) and notation (2.1), we have

$$\Lambda_k^2(f^{(r)}, \tau) = \frac{2^k}{\tau} \int_0^\tau dh \int_{-\infty}^{\infty} (1-\cos(hu))^k u^{2r} |\mathcal{F}(f,u)|^2 du$$

$$\geqslant 2^k \int_{|u|\geqslant\sigma} u^{2r} \eta_k(\tau; u) |\mathcal{F}(f,u)|^2 du. \tag{2.18}$$

Starting from (2.18), we consider the auxiliary function

$$G_k(\tau, u) := u^{2r} \eta_k(\tau; u) = \frac{u^{2r-1}}{\tau} \int_0^{u\tau} (1-\cos h)^k dh, \tag{2.19}$$

where $u \in \mathbb{R}$, $\tau \in (0,\infty)$. Formulas (2.1) and (2.19) imply that, for any fixed τ, the function G_k, as a function of the variable u, is nonnegative and even on the real axis $\mathbb{R}$ and increases monotonically on the semiaxis $\mathbb{R}_+ := \{x : 0 \leqslant x < \infty\}$. Hence,

$$\inf\{G_k(\tau, u) : |u| \geqslant \sigma\} = G_k(\tau, \sigma), \tag{2.20}$$

where $\tau \in (0,\infty)$ is any constant. Using relations (2.18)–(2.20) and (2.5), we have

$$\Lambda_k^2(f^{(r)}, \tau) \geqslant 2^k \int_{|u|\geqslant\sigma} G_k(\tau, u) |\mathcal{F}(f,u)|^2 du \geqslant 2^k G_k(\tau, \sigma) \mathcal{A}_\sigma^2(f)$$

$$= 2^k \sigma^{2r} \eta_k(\tau; \sigma) \mathcal{A}_\sigma^2(f).$$

From whence, by setting $\tau = t/\sigma$, where $t \in (0,\pi]$, we get

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\Lambda_k(f^{(r)}, t/\sigma)} \leqslant \frac{1}{\sqrt{2^k \eta_k(t/\sigma; \sigma)}} = \frac{1}{\sqrt{2^k \eta_k(t; 1)}}. \tag{2.21}$$

In order to calculate the lower bound of the extreme characteristic on the left-hand side of relation (2.21), we consider again the function $\lambda_\varepsilon \in L_2^r(\mathbb{R})$, where $\varepsilon > 0$, which was introduced in the proof of Theorem 1. Using formula (2.17), we have

$$\|\Delta_h^k(\lambda_\varepsilon^r)\|^2 = 2^{k+1} \int\limits_{\sigma}^{\sigma+\varepsilon} u^{2r}(1-\cos(hu))^k du$$

$$\leqslant 2^{k+1}\varepsilon(\sigma+\varepsilon)^{2r}(1-\cos(h(\sigma+\varepsilon)))_*^k.$$

Integrating the given relation over the variable h in the limits from 0 to t/σ and multiplying the obtained result by σ/t, we get

$$\Lambda_k(\lambda_\varepsilon^{(r)}, t/\sigma) \leqslant \sqrt{2\varepsilon}(\sigma+\varepsilon)^r \sqrt{2^k \eta_k^*(t/\sigma; \sigma+\varepsilon)}$$

by virtue of (1.1) and (2.11). Hence,

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\Lambda_k(f^{(r)}, t/\sigma)} \geqslant \frac{\sigma^r \mathcal{A}_\sigma(\lambda_\varepsilon)}{\Lambda_k(\lambda_\varepsilon^{(r)}, t/\sigma)} \geqslant \frac{1}{(1+\varepsilon/\sigma)^r \sqrt{2^k \eta_k^*(t/\sigma; \sigma+\varepsilon)}}. \tag{2.22}$$

In view of (2.11), the denominator of the right-hand side of inequality (2.22) is a positive monotonically decreasing function of $\varepsilon > 0$ for constant values of σ, k, t. Using the definition of the upper bound of a number set, as in the proof of Theorem 1, and formula (2.1), we have

$$\sup_{\varepsilon\in(0,\widetilde{\sigma})} \frac{1}{(1+\varepsilon/\sigma)^r \sqrt{2^k \eta_k^*(t/\sigma; \sigma+\varepsilon)}} = \frac{1}{\sqrt{2^k \eta_k(t;1)}}, \tag{2.23}$$

where $\widetilde{\sigma} = \min(\sigma, 1/\sigma)$. We note that the left-hand side of relation (2.22) is independent of ε. Therefore, by calculating the upper bound in $\varepsilon \in (0, \widetilde{\sigma})$ for its right-hand side and using equality (2.23), we get

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\Lambda_k(f^{(r)}, t/\sigma)} \geqslant \frac{1}{\sqrt{2^k \eta_k(t;1)}}. \tag{2.24}$$

The required equality (2.16) follows from inequalities (2.21) and (2.24) and relation (2.2), which completes the proof of Theorem 2. □

Let us set

$$\gamma_{k,r,p,x}(\varphi, u) := u^{2r}\Big\{ \int\limits_0^x \eta_k^{p/2}(t;u)\varphi(t)dt \Big\}^{2/p}. \tag{2.25}$$

Relations (2.1) and (2.25) imply that $\gamma_{k,r,p,x}(\varphi, u)$, as a function of u for fixed values of the remaining parameters, is positive and even.

The following proposition is valid.

Theorem 3. *Let $k \in \mathbb{N}$; $r \in \mathbb{Z}_+$; $p \in (0, 2]$; $\sigma \in (0, \infty)$; x be a finite positive number, and let φ be a nonnegative function summable on the segment $[0, x]$, which is not equivalent to zero. Then the equality*

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\{\int_0^x \Lambda_k^p(f^{(r)}, t)\varphi(t)dt\}^{1/p}} = 2^{-k/2}\{\inf_{u\geqslant\sigma} \gamma_{k,r,p,x}(\varphi, u)\}^{-1/2} \tag{2.26}$$

holds. In the case $r = 0$, we set $L_2^0(\mathbb{R}) \equiv L_2(\mathbb{R})$ and calculate the upper bound in relation (2.26) over all functions $f \in L_2(\mathbb{R})$, which are not equivalent to zero.

Proof. Setting

$$\mathcal{H}(f;t,u) := 2^{kp/2}\ u^{rp}\ \eta_k^{p/2}(t;u)\ |\mathcal{F}(f,u)|^p\ \varphi(t),$$

using relations (2.18) and (2.5) and the generalized Minkowski inequality (see, e.g., [5, Chapt. 1, i. 1.3]), and considering notation (2.25), we get

$$\Big\{\int\limits_0^x \Lambda_k^p(f^{(r)},t)\varphi(t)dt\Big\}^{1/p}$$

$$\geqslant \Big\{\int\limits_0^x \Big[2^k \int\limits_{|u|\geqslant\sigma} u^{2r}\eta_k(t;u)|\mathcal{F}(f,u)|^2du\Big]^{p/2}\varphi(t)dt\Big\}^{1/p}$$

$$= \Big\{\int\limits_0^x \Big[\int\limits_{|u|\geqslant\sigma} \mathcal{H}^{2/p}(f;t,u)du\Big]^{p/2}dt\Big\}^{1/p}$$

$$\geqslant \Big\{\int\limits_{|u|\geqslant\sigma} \Big[\int\limits_0^x \mathcal{H}(f;t,u)dt\Big]^{2/p}du\Big\}^{\frac{p}{2}\cdot\frac{1}{p}}$$

$$= 2^{k/2}\Big\{\int\limits_{|u|\geqslant\sigma} |\mathcal{F}(f,u)|^2\Big[u^{rp}\int\limits_0^x \eta_k^{p/2}(t,u)\varphi(t)dt\Big]^{2/p}du\Big\}^{1/2}$$

$$= 2^{k/2}\Big\{\int\limits_{|u|\geqslant\sigma} |\mathcal{F}(f,u)|^2\gamma_{k,r,p,x}(\varphi,u)du\Big\}^{1/2}$$

$$\geqslant 2^{k/2}\mathcal{A}_\sigma(f)\Big\{\inf_{u\geqslant\sigma}\gamma_{k,r,p,x}(\varphi,u)\Big\}^{1/2}.$$

Hence,

$$\sup_{f\in L_2^r(\mathbb{R})}\frac{\mathcal{A}_\sigma(f)}{\big\{\int_0^x \Lambda_k^p(f^{(r)},t)\varphi(t)dt\big\}^{1/p}} \leqslant 2^{-k/2}\Big\{\inf_{u\geqslant\sigma}\gamma_{k,r,p,x}(\varphi,u)\Big\}^{-1/2}. \tag{2.27}$$

We now get the lower bound of the extreme characteristic on the left-hand side of inequality (2.27). Let $u\in\mathbb{R}$ be any number satisfying the condition $|u|\geqslant\sigma$. Consider the function $\widetilde{\lambda}_{\varepsilon,u}(x) := \sqrt{2/\pi}\ \big(q_{|u|+\varepsilon}(x) - q_{|u|}(x)\big)$, where $\varepsilon\in(0,\widetilde{u})$, $\widetilde{u} := \min(|u|,1/|u|)$; $q_a(x) := a\ \mathrm{sinc}\,(ax)$, $a>0$, that is an entire function of the exponential type, $|u|+\varepsilon$. For the Fourier transform of the function $\widetilde{\lambda}_{\varepsilon,u}$, we have $\mathcal{F}(\widetilde{\lambda}_{\varepsilon,u};x) = \{1$, if $|u|<|x|<|u|+\varepsilon$; $1/2$, if $|x|=|u|$ or $|x|=|u|+\varepsilon$; and 0, if $|x|<|u|$ or $|x|>|u|+\varepsilon\}$. Therefore, by virtue of formula (2.17) and the equality $\|\widetilde{\lambda}_{\varepsilon,u}^{(r)}(\cdot)\| = \|(\cdot)^r\mathcal{F}(\widetilde{\lambda}_{\varepsilon,u};\cdot)\|$ we get $\widetilde{\lambda}_{\varepsilon,u}\in L_2^r(\mathbb{R})$ for any $u\in\mathbb{R}$ and $|u|\geqslant\sigma$. We note that

$$\|\Delta_h^k(\widetilde{\lambda}_{\varepsilon,u}^{(r)})\|^2 = 2^{k+1}\int\limits_{|u|}^{|u|+\varepsilon}(1-\cos(hv))^k v^{2r}dv \leqslant 2^{k+1}\varepsilon(|u|+\varepsilon)^{2r}\big(1-\cos(h(|u|+\varepsilon))\big)_*^k.$$

Integrating the given relation over h in the limits from 0 to t, multiplying the obtained inequality by $1/t$, and using formulas (1.1) and (2.11), we have

$$\Lambda_k\big(\widetilde{\lambda}_{\varepsilon,u}^{(r)};t\big) \leqslant 2^{k/2}\ \sqrt{2\varepsilon}\ (|u|+\varepsilon)^r\sqrt{\eta_k^*(t;|u|+\varepsilon)}. \tag{2.28}$$

Raising both sides of inequality (2.28) to the power $p \in (0,2]$, multiplying then by the function $\varphi(t)$, integrating over the variable t in the limits from 0 to x, and taking into account that, by virtue of (2.5), $\mathcal{A}_\sigma(\widetilde{\lambda}_{\varepsilon,u}) = \sqrt{2\varepsilon}$, we get

$$\int_0^x \Lambda_k^p\big(\widetilde{\lambda}_{\varepsilon,u}^{(r)};t\big)\varphi(t)dt \leqslant 2^{pk/2}\mathcal{A}_\sigma^p(\widetilde{\lambda}_{\varepsilon,u})(|u|+\varepsilon)^{pr}\int_0^x \big(\eta_k^*(t;|u|+\varepsilon)\big)^{p/2}\varphi(t)dt. \tag{2.29}$$

Using (2.29), we have

$$\sup_{f\in L_2^r(\mathbb{R})}\frac{\mathcal{A}_\sigma(f)}{\big\{\int_0^x \Lambda_k^p(f^{(r)},t)\varphi(t)dt\big\}^{1/p}} \geqslant \frac{\mathcal{A}_\sigma(\widetilde{\lambda}_{\varepsilon,u})}{\big\{\int_0^x \Lambda_k^p(\widetilde{\lambda}_{\varepsilon,u}^{(r)};t)\varphi(t)dt\big\}^{1/p}}$$

$$\geqslant 2^{-k/2}(|u|+\varepsilon)^{-r}\Big\{\int_0^x \big(\eta_k^*(t;|u|+\varepsilon)\big)^{p/2}\varphi(t)dt\Big\}^{-1/p}. \tag{2.30}$$

We note that

$$\sup_{\varepsilon\in(0,\widetilde{u})}(|u|+\varepsilon)^{-r}\Big\{\int_0^x \big(\eta_k^*(t;|u|+\varepsilon)\big)^{p/2}\varphi(t)dt\Big\}^{-1/p} = \big\{\gamma_{k,r,p,x}(\varphi,u)\big\}^{-1/2}.$$

Calculating the upper bound in $\varepsilon \in (0,\widetilde{u})$ for the right-hand side of relation (2.30), we have

$$\sup_{f\in L_2^r(\mathbb{R})}\frac{\mathcal{A}_\sigma(f)}{\big\{\int_0^x \Lambda_k^p(f^{(r)},t)\varphi(t)dt\big\}^{1/p}} \geqslant 2^{-k/2}\big\{\gamma_{k,r,p,x}(\varphi,u)\big\}^{-1/2},$$

where $u \geqslant \sigma$. Using the definitions and properties of exact upper and lower bounds of a number set, we get

$$\sup_{f\in L_2^r(\mathbb{R})}\frac{\mathcal{A}_\sigma(f)}{\big\{\int_0^x \Lambda_k^p(f^{(r)},t)\varphi(t)dt\big\}^{1/p}}$$

$$\geqslant 2^{-k/2}\sup_{u\geqslant\sigma}\big\{\gamma_{k,r,p,x}(\varphi,u)\big\}^{-1/2} \geqslant 2^{-k/2}\big\{\inf_{u\geqslant\sigma}\gamma_{k,r,p,x}(\varphi,u)\big\}^{-1/2}. \tag{2.31}$$

The required equality (2.26) follows from relations (2.27) and (2.31). Theorem 3 is proved. □

Let us set, for example, $k=1$ in the conditions of Theorem 3. Then equality (2.26) yields

$$\sup_{f\in L_2^r(\mathbb{R})}\frac{\mathcal{A}_\sigma(f)}{\big\{\int_0^x \Lambda_1^p(f^{(r)},t)\varphi(t)dt\big\}^{1/p}}$$

$$= \frac{1}{\sqrt{2}\Big\{\inf_{u\geqslant\sigma} u^r\Big[\int_0^x(1-\operatorname{sinc}(ut))^{p/2}\varphi(t)dt\Big]^{1/p}\Big\}}. \tag{2.32}$$

3. Some corollaries of Theorem 3

In the first turn, we consider the conditions under which the lower bound for the right-hand side of equality (2.32) can be calculated.

Corollary 1. *Let $\sigma \in (0,\infty)$; $r \in \mathbb{Z}_+$; $p \in (0,2]$; $x \in (0, 3\pi/(4\sigma)]$; φ be a nonnegative function summable on the segment $[0,x]$, which is not equivalent to zero. Then the equality*

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\big\{\int_0^x \Lambda_1^p(f^{(r)},t)\varphi(t)dt\big\}^{1/p}} = \frac{1}{\sqrt{2}\Big\{\int_0^x (1-\operatorname{sinc}(\sigma t))^{p/2}\varphi(t)dt\Big\}^{1/p}} \tag{3.1}$$

holds. In the case $r=0$, the upper bound in (3.1) is calculated over all functions $f \in L_2(\mathbb{R})$, which are not equivalent to zero.

Proof. In view of the behavior of the function $\operatorname{sinc}(t)$ (see, e.g., [29, pp. 129, 132]), we have $\operatorname{sinc}(z) \geqslant \operatorname{sinc}(vz)$ for arbitrary values of $v \in [1,\infty)$ and $z \in (0, 3\pi/4]$. Hence, the inequality

$$v^\nu (1-\operatorname{sinc}(vz))^\alpha \geqslant (1-\operatorname{sinc}(z))^\alpha \tag{3.2}$$

holds. Here, $\nu, \alpha \in [0,\infty)$ are any numbers. Let us set $v := u/\sigma$, where $u \in [\sigma,\infty)$; $z := \sigma t$, where $t \in (0,x]$; $\nu := rp$; $\alpha := p/2$. Then relation (3.2) yields

$$u^{rp}(1-\operatorname{sinc}(ut))^{p/2} \geqslant \sigma^{rp}(1-\operatorname{sinc}(\sigma t))^{p/2}.$$

Multiplying both sides of the given inequality by the function $\varphi(t)$ and integrating them over the variable t in the limits from 0 to x, we have

$$u^{rp}\int_0^x (1-\operatorname{sinc}(ut))^{p/2}\varphi(t)dt \geqslant \sigma^{rp}\int_0^x (1-\operatorname{sinc}(\sigma t))^{p/2}\varphi(t)dt, \tag{3.3}$$

where $u \in [\sigma,\infty)$ is any number. We note that relation (3.3) yields

$$\inf_{u\geqslant\sigma} u^r\Big\{\int_0^x (1-\operatorname{sinc}(ut))^{p/2}\varphi(t)dt\Big\}^{1/p} = \sigma^r\Big\{\int_0^x (1-\operatorname{sinc}(\sigma t))^{p/2}\varphi(t)dt\Big\}^{1/p}.$$

From whence with regard for formula (2.32), we get the required equality (3.1). Corollary 1 is proved. □

Let us set, for example, $p=2$, $\varphi(t)\equiv 1$, and $h=\sigma x$, where $h \in (0, 3\pi/4]$. Then equality (3.1) yields

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^{r-1/2} \mathcal{A}_\sigma(f)}{\big\{\int_0^{h/\sigma} \Lambda_1^2(f^{(r)},t)dt\big\}^{1/2}} = \frac{1}{\sqrt{2(h-Si(h))}}, \tag{3.4}$$

where $Si(x) := \int_0^x \operatorname{sinc}(t)dt$ is the integral sine. In the case $r=0$, relation (3.4) yields, in particular,

$$\sup_{f\in L_2(\mathbb{R})} \frac{\mathcal{A}_\sigma(f)}{\big\{\sigma\int_0^{h/\sigma} \Lambda_1^2(f,t)dt\big\}^{1/2}} = \frac{1}{\sqrt{2(h-Si(h))}}.$$

Corollary 2. *Let* $\sigma \in (0,\infty)$, $r,k \in \mathbb{N}$, $p \in [1/r, 2]$, *let* x *be a finite positive number; and let* φ *be a nonnegative function differentiable almost everywhere on the interval* $(0,x)$, *which is not equivalent to zero and, almost for all* $t \in (0,x)$, *satisfies the condition*

$$\varphi(t)(pr-1) - t\varphi'(t) \geqslant 0. \tag{3.5}$$

Then the equality

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\left\{\int_0^x \Lambda_k^p(f^{(r)}, t)\varphi(t)dt\right\}^{1/p}}$$

$$= \left\{\int_0^x \left[\binom{2k}{k} - 2\sum_{j=1}^{k}(-1)^{j+1}\binom{2k}{k-j}\operatorname{sinc}(j\sigma t)\right]^{p/2}\varphi(t)dt\right\}^{-1/p} \tag{3.6}$$

holds.

Proof. Consider the auxiliary function

$$Z(u) := \gamma_{k,r,p,x}^{p/2}(\varphi, u) = u^{pr}\int_0^x \eta_k^{p/2}(t;u)\varphi(t)dt, \tag{3.7}$$

where $u \geqslant \sigma$, and calculate its derivative of the first order

$$Z'(u) = pru^{pr-1}\int_0^x \eta_k^{p/2}(t;u)\varphi(t)dt + u^{pr}\int_0^x \varphi(t)\frac{\partial}{\partial u}\big(\eta_k^{p/2}(t;u)\big)dt. \tag{3.8}$$

We note that relation (2.1) yields

$$\eta_k(t;u) = \eta_k(tu;1). \tag{3.9}$$

Therefore, setting $\lambda(x) := \eta_k^{p/2}(x;1)$, we can verify that

$$\frac{1}{t}\frac{\partial}{\partial u}(\lambda(tu)) = \frac{1}{u}\frac{\partial}{\partial t}(\lambda(tu)),$$

where t and u take nonzero values. From whence with regard for (3.9), we have

$$\frac{1}{t}\frac{\partial}{\partial u}(\eta_k^{p/2}(t;u)) = \frac{1}{u}\frac{\partial}{\partial t}(\eta_k^{p/2}(t;u)). \tag{3.10}$$

Using relation (3.10) in the course of the integration by parts of the second integral in formula (3.8), we get

$$Z'(u) = u^{pr-1}\Big\{pr\int_0^x \eta_k^{p/2}(t;u)\varphi(t)dt + \int_0^x t\varphi(t)\frac{\partial}{\partial t}\big(\eta_k^{p/2}(t;u)\big)dt\Big\}$$

$$= u^{pr-1}\Big\{x\varphi(x)\eta_k^{p/2}(x;u) + \int_0^x \eta_k^{p/2}(t;u)\big[\varphi(t)(pr-1) - t\varphi'(t)\big]dt\Big\}. \tag{3.11}$$

With regard for formula (3.5) relation (3.11) yields $Z'(u) \geqslant 0$ for all $u \geqslant \sigma$. Hence, the function Z is nondecreasing on the set $[\sigma,\infty)$. In view of (3.7), (3.9), and (2.25), we get

$$\inf_{\sigma \leqslant u < \infty} \gamma_{k,r,p,x}(\varphi,u) = \gamma_{k,r,p,x}(\varphi,\sigma) = \sigma^{2r}\Big\{\int_0^x \eta_k^{p/2}(t\sigma;1)\varphi(t)dt\Big\}^{2/p}. \tag{3.12}$$

The required equality (3.6) follows from relations (2.26), (3.12), and (2.2). Corollary 2 is proved. □

Let us compare the results of Corollaries 1 and 2 in the case where $k = 1$. We note that relation (3.6) from Corollary 2 holds, if $r \in \mathbb{N}$, $p \in [1/r, 2]$, and the function φ satisfies limitation (3.5). In this case, x is any positive number. As for Corollary 1, the value of r can take also the zero value for $p \in (0, 2]$ (as distinct from Corollary 2), and the function φ satisfies weaker conditions. However, the positive number x is already bounded from above by the number $3\pi/(4\sigma)$.

Let us set, for example, $p = 2$ and $\varphi(t) \equiv 1$. In this case, inequality (3.5) is automatically satisfied. Then formula (3.6) yields

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\left\{ \int_0^x \Lambda_k^2(f^{(r)}, t)dt \right\}^{1/2}} = \left\{ x \left[\binom{2k}{k} - 2 \sum_{j=1}^{k} (-1)^{j+1} \binom{2k}{k-j} \frac{Si(j\sigma x)}{j\sigma x} \right] \right\}^{-1/2}. \tag{3.13}$$

We note that, for $k = 1$ and $r \in \mathbb{N}$, the right- and left-hand sides of equalities (3.4) and (3.13) coincide, if we set $x = h/\sigma$, where $0 < h \leqslant 3\pi/4$, in (3.13).

Let now $\varphi(t) := t^m$, where $m \in (0, \infty)$ is any number. Then relation (3.5) takes the form $p \geqslant (1 + m)/r$. By the condition of Corollary 3, we have $p \in (0, 2]$. Therefore, we get the double inequality $(1 + m)/r \leqslant p \leqslant 2$, which should be satisfied by p. In this case, the numbers $r \in \mathbb{N}$ must satisfy the limitation from below $r \geqslant (1 + m)/2$. If p and r satisfy the indicated limitations, formula (3.6) yields

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\left\{ \int_0^x \Lambda_k^p(f^{(r)}, t) t^m dt \right\}^{1/p}}$$

$$= \left\{ \int_0^x \left[\binom{2k}{k} - 2 \sum_{j=1}^{k} (-1)^{j+1} \binom{2k}{k-j} \operatorname{sinc}(j\sigma t) \right]^{p/2} t^m dt \right\}^{-1/p}.$$

In the given formula, let us set, for example, $k = 1$, $m = 1$, and $p = 2$. Then, for any $r \in \mathbb{N}$, we have

$$\sup_{f \in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\left\{ \int_0^x \Lambda_1^2(f^{(r)}, t) t dt \right\}^{1/2}} = \frac{1}{x\sqrt{1 - \operatorname{sinc}^2(\sigma x/2)}},$$

where $x \in (0, \infty)$ is any number.

In formula (2.26), let $x := x_* = \alpha/\sigma$, where $\alpha \in (0, \infty)$ and $\varphi(t) := \varphi_*(t) = \psi(\sigma t)$. Setting

$$\theta_{k,r,p,\alpha}(\psi, v) := v^{2r} \left\{ \int_0^\alpha \eta_k^{p/2}(vt; 1)\psi(t)dt \right\}^{2/p} \tag{3.14}$$

and considering relation (3.9), we get

$$\gamma_{k,r,p,x_*}(\varphi_*, u) = \left\{ u^{rp} \int_0^{\alpha/\sigma} \eta_k^{p/2}(ut; 1)\psi(\sigma t)dt \right\}^{2/p}$$

$$= \sigma^{2(r-1/p)} \left\{ \left(\frac{u}{\sigma} \right)^{rp} \int_0^\alpha \eta_k^{p/2}(ut/\sigma; 1)\psi(t)dt \right\}^{2/p},$$

where $u \geqslant \sigma$. Using notation (3.14), we get

$$\sigma^{2(r-1/p)} \inf_{1 \leqslant v < \infty} \theta_{k,r,p,\alpha}(\psi, v) \leqslant \inf_{\sigma \leqslant u < \infty} \gamma_{k,r,p,x_*}(\varphi_*, u)$$

$$\leqslant \gamma_{k,r,p,x_*}(\varphi_*, \sigma) = \sigma^{2(r-1/p)}\, \theta_{k,r,p,\alpha}(\psi, 1). \tag{3.15}$$

Then Theorem 3 and relation (3.15) yield the following proposition.

Corollary 3. *Let $k, r \in \mathbb{N}$, $p \in (0, 2]$, $\sigma, \alpha \in (0, \infty)$, and let ψ be a nonnegative function summable on the segment $[0, \alpha]$, which is not equivalent to zero. Then the following double inequality holds:*

$$2^{-k/2}\big\{\theta_{k,r,p,\alpha}(\psi,1)\big\}^{-1/2} \leqslant \sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\big\{\int_0^\alpha \Lambda_k^p(f^{(r)}, t/\sigma)\psi(t)dt\big\}^{1/p}}$$

$$\leqslant 2^{-k/2}\big\{\inf_{1\leqslant v<\infty} \theta_{k,r,p,\alpha}(\psi, v)\big\}^{-1/2}.$$

But if the function ψ satisfies the condition

$$\inf_{1\leqslant v<\infty} \theta_{k,r,p,\alpha}(\psi, v) = \theta_{k,r,p,\alpha}(\psi, 1), \tag{3.16}$$

then the equality

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\big\{\int_0^\alpha \Lambda_k^p(f^{(r)}, t/\sigma)\psi(t)dt\big\}^{1/p}} = 2^{-k/2}\big\{\theta_{k,r,p,\alpha}(\psi,1)\big\}^{-1/2}. \tag{3.17}$$

holds.

The following proposition is devoted to the study of the conditions necessary for equality (3.16) to be held.

Corollary 4. *Let all conditions of Corollary 3 be satisfied, and let the function $\psi(t) := \widetilde{\psi}(t) = t^{rp-1}\widetilde{\psi}_1(t)$, where $\widetilde{\psi}_1$ is a nonnegative nonincreasing function defined and summable on the segment $[0, \alpha]$, which is not equivalent to zero. Then the function ψ defined in such way satisfies equality* (3.16), *and, hence, relation* (3.17) *is valid.*

Proof. Consider the auxiliary function $\widetilde{\psi}_*(t) := \{\widetilde{\psi}_1(t)$, if $0 \leqslant t \leqslant \alpha$, and $\widetilde{\psi}_1(\alpha)$, if $\alpha \leqslant t < \infty\}$. Then, on the basis of (3.14), we get

$$\theta_{k,r,p,\alpha}(\widetilde{\psi}, v) = v^{2r}\Big\{\int_0^\alpha \eta_k^{p/2}(vt;1)t^{rp-1}\widetilde{\psi}_1(t)dt\Big\}^{2/p}$$

$$= \Big\{\int_0^{\alpha v} \eta_k^{p/2}(t;1)t^{rp-1}\widetilde{\psi}_*(t/\sigma)dt\Big\}^{2/p} \geqslant \Big\{\int_0^{\alpha v} \eta_k^{p/2}(t;1)t^{rp-1}\widetilde{\psi}_*(t)dt\Big\}^{2/p}$$

$$\geqslant \Big\{\int_0^{\alpha} \eta_k^{p/2}(t;1)t^{rp-1}\widetilde{\psi}_1(t)dt\Big\}^{2/p} = \theta_{k,r,p,\alpha}(\widetilde{\psi}, 1)$$

for any value of $v \in [1, \infty)$. Hence, condition (3.16) is satisfied. This means the validity of equality (3.17) for the function $\widetilde{\psi}$ by virtue of Corollary 3. Then, with regard for formulas (3.14) and (2.2), we get

$$\sup_{f\in L_2^r(\mathbb{R})} \frac{\sigma^r \mathcal{A}_\sigma(f)}{\big\{\int_0^\alpha \Lambda_k^p(f^{(r)}, t/\sigma)t^{rp-1}\widetilde{\psi}_1(t)dt\big\}^{1/p}} = 2^{-k/2}\big\{\theta_{k,r,p,\alpha}(\widetilde{\psi},1)\big\}^{-1/2}$$

$$= \Big\{\int_0^\alpha \Big[\binom{2k}{k} - 2\sum_{j=1}^{k}(-1)^{j+1}\binom{2k}{k-j}\operatorname{sinc}(jt)\Big]^{p/2} t^{rp-1}\widetilde{\psi}_1(t)dt\Big\}^{-1/2}.$$

Corollary 4 is proved. □

4. Exact values of mean ν-widths of the classes of functions defined by means of the characteristic of smoothness Λ_k in the space $L_2(\mathbb{R})$

In works [30, 31], the definition of mean dimension, being some modification of the corresponding notion given in [32], was introduced. This allowed one to define asymptotic extreme characteristics similar to the widths, where the role of dimension was played by the mean dimension. As a result, we can compare the approximative properties of the subspaces $\mathbb{B}_{\sigma,2}$, where $\sigma \in (0,\infty)$, with analogous properties of other subspaces from $L_2(\mathbb{R})$ that have the same mean dimension, and solve a number of extreme optimization problems of the theory of approximation of functions in $L_2(\mathbb{R})$.

Recall the necessary notions and definitions from [30, 31]. Let $BL_2(\mathbb{R})$ be a unit ball in $L_2(\mathbb{R})$, let $Lin(L_2(\mathbb{R}))$ be a collection of all linear subspaces in $L_2(\mathbb{R})$, and let

$$Lin_n(L_2(\mathbb{R})) := \{\mathcal{L} \in Lin(L_2(\mathbb{R})) : dim\mathcal{L} \leqslant n\},\ n \in \mathbb{Z}_+,$$

$$d(\mathfrak{M}, A, L_2(\mathbb{R})) := \sup\{\inf\{\|x-y\| : y \in A\} : x \in \mathfrak{M}\}$$

be the best approximation of the set $\mathfrak{M} \subset L_2(\mathbb{R})$ by the set $A \subset L_2(\mathbb{R})$. The symbol A_T, where $T > 0$, stands for a restriction of the set $A \subset L_2(\mathbb{R})$ on the segment $[-T,T]$, and $Lin_C L_2(\mathbb{R})$ denotes a collection of subspaces $\mathcal{L} \in Lin(L_2(\mathbb{R}))$ such that the set $(\mathcal{L} \cap BL_2(\mathbb{R}))_T$ is precompact in $L_2([-T,T])$ for any $T > 0$.

If $\mathcal{L} \in Lin_C(L_2(\mathbb{R}))$ and $T, \varepsilon > 0$, then there exist $n \in \mathbb{Z}_+$ and $\mathcal{M} \in Lin_n(L_2(\mathbb{R}))$ such that $d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T,T])) < \varepsilon$. Let

$$D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R})) := \min\{n \in \mathbb{Z}_+ : \exists \mathcal{M} \in Lin_n(L_2([-T,T])),$$
$$d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T,T])) < \varepsilon\}.$$

This quantity does not decrease in T and does not increase in ε. The quantity

$$\overline{dim}(\mathcal{L}, L_2(\mathbb{R})) := \lim\{\liminf\{D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R}))/(2T) : T \to \infty\} : \varepsilon \to 0\},$$

where $\mathcal{L} \in Lin_C(L_2(\mathbb{R}))$, is called the mean dimension of a subspace $\mathcal{L}$ in $L_2(\mathbb{R})$. It was shown [30] that

$$\overline{dim}(\mathbb{B}_{\sigma,2}; L_2(\mathbb{R})) = \sigma/\pi. \tag{4.1}$$

Let $\mathfrak{M}$ be a centrally symmetric subset from $L_2(\mathbb{R})$, and let $\nu > 0$ be any finite number. Then the Kolmogorov mean ν-width of a set $\mathfrak{M}$ in $L_2(\mathbb{R})$ is the quantity

$$\overline{d}_\nu(\mathfrak{M}, L_2(\mathbb{R})) := \inf\{\sup\{\inf\{\|f-\varphi\| : \varphi \in \mathcal{L}\} : f \in \mathfrak{M}\} : \mathcal{L} \in Lin_C(L_2(\mathbb{R})), \overline{dim}(\mathcal{L}, L_2(\mathbb{R})) \leqslant \nu\}.$$

The subspace on which the external lower bound is attained is called extreme.

The mean linear ν-width of of a set $\mathfrak{M}$ in $L_2(\mathbb{R})$ is

$$\overline{\delta}_\nu(\mathfrak{M}, L_2(\mathbb{R})) := \inf\{\sup\{\|f - V(f)\| : f \in \mathfrak{M}\} : (X, V)\},$$

where the lower bound is taken over all pairs (X, V) such that X is a normed space immediately embedded in $L_2(\mathbb{R})$, and $V : X \to L_2(\mathbb{R})$ is a continuous linear operator for which $ImV \subset Lin_C(L_2(\mathbb{R}))$, and the inequality $\overline{dim}(ImV, L_2(\mathbb{R})) \leqslant \nu$; $\mathfrak{M} \subset X$ holds. Here, ImV is the image of the operator V. The pair on which the lower bound is attained is called extreme.

The quantity

$$\overline{b}_\nu(\mathfrak{M}, L_2(\mathbb{R})) := \sup\{\sup\{\rho > 0 : \mathcal{L} \cap \rho BL_2(\mathbb{R}) \subset \mathfrak{M}\}$$
$$: \mathcal{L} \in Lin_C(L_2(\mathbb{R})), \overline{dim}(\mathcal{L}, L_2(\mathbb{R})) > \nu, \overline{d}_\nu(\mathcal{L} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1\}$$

is called the Bernshtein mean ν-width of a set $\mathfrak{M}$ in $L_2(\mathbb{R})$. The last condition imposed onto $\mathcal{L}$ in the calculation of the external upper bound means that only those subspaces, for which an analog of the Tikhomirov theorem on the width of a ball is true, are considered. This requirement is satisfied, for example, by the subspace $\mathbb{B}_{\sigma,2}$, if $\sigma > \nu\pi$, i.e., $\overline{d}_\nu(\mathbb{B}_{\sigma,2} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1$.

For the set $\mathfrak{M} \subset L_2(\mathbb{R})$, the inequalities

$$\overline{b}_\nu(\mathfrak{M}, L_2(\mathbb{R})) \leqslant \overline{d}_\nu(\mathfrak{M}, L_2(\mathbb{R})) \leqslant \overline{\delta}_\nu(\mathfrak{M}, L_2(\mathbb{R})) \tag{4.2}$$

hold between its above-presented extreme characteristics.

We note that the exact values of mean ν-widths of some classes of functions were first obtained in [30, 31]. Then this theme was developed by other researchers (see, e.g., [12, 13, 15–17]). The short review concerning the calculation of exact values of the indicated extreme characteristics can be found in [33].

We now use the characteristic of smoothness (1.1) to define the classes of functions в $L_2(\mathbb{R})$. Let $\Psi(t)$, where $t \in [0, \infty)$, be a continuous increasing function such that $\Psi(0) = 0$. We call it a majorant. By $\mathcal{W}(\Lambda_1, \Psi)$, we denote the class of functions $f \in L_2(\mathbb{R})$ such that every function satisfies the inequality $\Lambda_1(f, t) \leqslant \Psi(t)$ for any $t \in (0, \infty)$. By t_*, we denote the value of argument of the function $\operatorname{sinc}(t)$, at which it attains the least value on the set $0 < t < \infty$. We note (see, e.g., [12, 13, 33]) that t_* is the least positive root of the equation $t = \operatorname{tg} t$ and $4.49 < t_* < 4.51$. Then we set $(1 - \operatorname{sinc}(t))_* := \{1 - \operatorname{sinc}(t)$, if $0 < t \leqslant t_*$, and $1 - \operatorname{sinc}(t_*)$, if $t_* \leqslant t < \infty\}$.

Theorem 4. *Let $\nu \in (0, \infty)$ be any number, and let Ψ be a majorant that satisfies the condition*

$$\frac{\Psi^2(t)}{\Psi^2(\pi/(2\sigma))} \geqslant \frac{\pi}{\pi - 2}\big(1 - \operatorname{sinc}(\sigma t)\big)_* \tag{4.3}$$

for any $t \in (0, \infty)$ and $\sigma \in (\nu\pi, \infty)$. Then the equalities

$$\begin{aligned} \overline{\Pi}_\nu(\mathcal{W}(\Lambda_1, \Psi); L_2(\mathbb{R})) &= \mathcal{A}_{\nu\pi}(\mathcal{W}(\Lambda_1, \Psi)) = \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in \mathcal{W}(\Lambda_1, \Psi)\} \\ &= \sqrt{\frac{\pi}{2(\pi - 2)}}\, \Psi\Big(\frac{1}{2\nu}\Big) \end{aligned} \tag{4.4}$$

hold. Here, $\overline{\Pi}_\nu(\cdot)$ is any of the above-considered mean ν-widths, and the operator $\mathcal{L}_{\nu\pi}$ is defined by formula (2.3) *for $\sigma = \nu\pi$. In this case, the pair [4] $(L_2(\mathbb{R}), \mathcal{L}_{\nu\pi})$ is extreme for the mean linear ν-width $\overline{\delta}_\nu(\mathcal{W}(\Lambda_1, \Psi); L_2(\mathbb{R}))$, and the subspace $\mathbb{B}_{\nu\pi,2}$ is extreme for the Kolmogorov mean ν-width $\overline{d}_\nu(\mathcal{W}(\Lambda_1, \Psi); L_2(\mathbb{R}))$. The set of majorants satisfying condition* (4.3) *is not empty.*

Proof. Using relation (2.3), where we set $t = \pi/2$, for any function $f \in L_2(\mathbb{R})$, we get

$$\mathcal{A}_\sigma(f) \leqslant \sqrt{\frac{\pi}{2(\pi - 2)}}\, \Lambda_1\Big(f, \frac{\pi}{2\sigma}\Big). \tag{4.5}$$

Let $\sigma = \nu\pi$. Then, by virtue of formula (4.1), the mean dimension of the subspace $\mathbb{B}_{\nu\pi,2}$ reads $\overline{dim}\ \mathbb{B}_{\nu\pi,2} = \nu$. In view of this fact and the definition of the class of functions $\mathcal{W}(\Lambda_1, \Psi)$, we get the following upper bounds from relations (2.4)–(2.5), (4.2) and (4.5):

$$\begin{aligned} \overline{\Pi}_\nu(\mathcal{W}(\Lambda_1, \Psi); L_2(\mathbb{R})) &\leqslant \overline{\delta}_\nu(\mathcal{W}(\Lambda_1, \Psi); L_2(\mathbb{R})) \leqslant \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in \mathcal{W}(\Lambda_1, \Psi)\} = \mathcal{A}_{\nu\pi}(\mathcal{W}(\Lambda_1, \Psi)) \\ &\leqslant \sqrt{\frac{\pi}{2(\pi - 2)}}\, \Psi\Big(\frac{1}{2\nu}\Big). \end{aligned} \tag{4.6}$$

To obtain the lower bounds for the studied mean ν-widths, it is necessary to find the lower bound of the mean Bernshtein ν-width $\overline{b}_\nu(\mathcal{W}(\Lambda_1,\Psi); L_2(\mathbb{R}))$ by formula (4.2). For this purpose, we set $\widehat{\sigma} := \nu\pi(1+\varepsilon)$, where $\varepsilon \in (0, \widetilde{\nu})$ is any number, and $\widetilde{\nu} := \min(\nu, 1/\nu)$. By (4.1), $\overline{dim}\ \mathbb{B}_{\widehat{\sigma},2} = \nu(1+\varepsilon)$ and $\overline{d}_\nu(\mathbb{B}_{\widehat{\sigma},2} \cap BL_2(\mathbb{R}); L_2(\mathbb{R})) = 1$, since $\widehat{\sigma} > \nu\pi$. Therefore, we can consider the subspace $\mathbb{B}_{\widehat{\sigma},2}$ as one of the realizations of the subspace $\mathcal{L} \in Lin_C(L_2(\mathbb{R}))$ entering the definition of the quantity $\overline{b}_\nu(\cdot)$. In this connection, we consider the set of entire functions $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon) := \mathbb{B}_{\widehat{\sigma},2} \cap \rho_\varepsilon BL_2(\mathbb{R}) = \{g \in \mathbb{B}_{\widehat{\sigma},2} : \|g\| \leqslant \rho_\varepsilon\}$, where

$$\rho_\varepsilon := \sqrt{\frac{\pi}{2(\pi-2)}}\ \Psi\Big(\frac{1}{2\nu(1+\varepsilon)}\Big). \tag{4.7}$$

We now show that $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon) \subset \mathcal{W}(\Lambda_1,\Psi)$. Recall that, by the Wiener–Paley fundamental theorem (see, e.g., [3, Chapt. 4, Sect. 4.6]), any element $g \in \mathbb{B}_{\widehat{\sigma},2}$ can be represent as follows:

$$g(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} e^{ixu}\mu(u)du. \tag{4.8}$$

Here, μ is some function with the squared modulus integrable by Lebesgue on the segment $[-\widehat{\sigma}, \widehat{\sigma}]$. We note that, for the Fourier transform of the function g, we have $\mathcal{F}(g,x) = \{\mu(x)$, if $|x| \leqslant \widehat{\sigma}$, and 0, if $|x| > \widehat{\sigma}\}$, as well as

$$\int\limits_{-\infty}^{\infty} |g(x)|^2 dx = \int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} |\mu(u)|^2 du. \tag{4.9}$$

Based on formula (2.6), we can write

$$\Lambda_1^2(g,t) = 2\int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} (1-\mathrm{sinc}\,(tu))|\mu(u)|^2 du \leqslant (1-\mathrm{sinc}\,(t\widehat{\sigma}))_* \|g\|^2. \tag{4.10}$$

We now use relations (4.7)–(4.10) and condition (4.3) for the majorant Ψ, where we set $\sigma := \widehat{\sigma}$. Then, for any function $g \in \mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon)$ and any $t \in (0,\infty)$, we get

$$\Lambda_1(g,t) \leqslant \sqrt{\frac{\pi}{\pi-2}\ (1-\mathrm{sinc}\,(t\widehat{\sigma}))_*}\ \Psi\Big(\frac{1}{2\nu(1+\varepsilon)}\Big) \leqslant \Psi(t).$$

Hence, $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon) \subset \mathcal{W}(\Lambda_1,\Psi)$. Considering this fact and the definition of Bernshtein mean ν-width, we get

$$\overline{b}_\nu(\mathcal{W}(\Lambda_1,\Psi); L_2(\mathbb{R})) \geqslant \overline{b}_\nu(\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon); L_2(\mathbb{R})) \geqslant \rho_\varepsilon.$$

This relation and formulas (4.2) and (4.7) yield

$$\overline{\Pi}_\nu(\mathcal{W}(\Lambda_1,\Psi); L_2(\mathbb{R})) \geqslant \sqrt{\frac{\pi}{2(\pi-2)}}\ \Psi\Big(\frac{1}{2\nu(1+\varepsilon)}\Big). \tag{4.11}$$

We note that the left-hand side of inequality (4.11) is independent of ε. By calculating the upper bound in $\varepsilon \in (0,\widetilde{\nu})$ for its right-hand side, we have

$$\overline{\Pi}_\nu(\mathcal{W}(\Lambda_1,\Psi); L_2(\mathbb{R})) \geqslant \sqrt{\frac{\pi}{2(\pi-2)}}\ \Psi\Big(\frac{1}{2\nu}\Big). \tag{4.12}$$

The required equalities (4.4) follow from relations (4.6) and (4.12). Further, we can verify that the set of majorants satisfying condition (4.3) is not empty. For this purpose, we consider the majorant $\widetilde{\Psi}(t) := t^{\beta/2}$ for which relation (4.3) takes the form

$$(t\sigma)^{\beta} \geqslant \frac{\pi^{1+\beta}}{2^{\beta}(\pi-2)}(1-\operatorname{sinc}(t\sigma))_{*}, \tag{4.13}$$

where $t \in (0,\infty)$. Using the course of reasonings used in the proof of the main theorem in [34], we can verify that inequality (4.13) holds for $\beta = 2/(\pi-2)$. Hence, the majorant $\widetilde{\Psi}(t) = t^{1/(\pi-2)}$ satisfies condition (4.3). Theorem 4 is proved. □

By $\mathcal{W}^{r}(\Lambda_k,\Psi)$, where $r,k \in \mathbb{N}$, and the function Ψ is a majorant, we denote a class of functions $f \in L_2^r(\mathbb{R})$ such that every function satisfies the inequality $\Lambda_k(f^{(r)},t) \leqslant \Psi(t)$ for any $t \in (0,\infty)$.

Theorem 5. *Let $r,k \in \mathbb{N}$, let $\nu \in (0,\infty)$ be any number, and Ψ be a majorant satisfying the condition*

$$\frac{\Psi^2(t)}{\Psi^2(\pi/\sigma)} \geqslant \frac{2^k}{C_{2k}^k \sigma t}\int_0^{\sigma t}(1-\cos h)^k dh \tag{4.14}$$

for any values of $t \in (0,\infty)$ and $\sigma \in (\nu\pi,\infty)$. Then the following equalities are valid:

$$\overline{\Pi}_{\nu}(\mathcal{W}^{r}(\Lambda_k,\Psi);L_2(\mathbb{R}))=\mathcal{A}_{\nu\pi}(\mathcal{W}^{r}(\Lambda_k,\Psi))=\sup\{\|f-\mathcal{L}_{\nu\pi}(f)\|:f\in\mathcal{W}^{r}(\Lambda_k,\Psi)\} = \frac{1}{\sqrt{C_{2k}^k}\ \pi^r\nu^r}\ \Psi\Big(\frac{1}{\nu}\Big), \tag{4.15}$$

where $\overline{\Pi}_{\nu}(\cdot)$ is any of the mean ν-widths considered above. In this case, the pair $(L_2^r(\mathbb{R}),\mathcal{L}_{\nu\pi})$ is extreme for a mean linear ν-width $\overline{\delta}_{\nu}(\mathcal{W}^{r}(\Lambda_k,\Psi),L_2(\mathbb{R}))$, the subspace $\mathbb{B}_{\nu\pi,2}$ is extreme for the Kolmogorov mean ν-width $\overline{d}_{\nu}(\mathcal{W}^{r}(\Lambda_k,\Psi);L_2(\mathbb{R}))$. The set of majorants satisfying condition (4.14) is not empty.

Proof. Using formula (2.16), where we set $\sigma = \nu\pi$ and $t=\pi$, we have

$$\mathcal{A}_{\nu\pi}(f) \leqslant \frac{1}{\sqrt{C_{2k}^k}\ \pi^r\nu^r}\ \Lambda_k\Big(f^{(r)},\frac{1}{\nu}\Big) \tag{4.16}$$

for any function $f \in L_2^r(\mathbb{R})$. From relations (4.2) and (4.16) and by virtue of the definition of the class $\mathcal{W}^{r}(\Lambda_k,\Psi)$, we get the following upper bounds:

$$\overline{\Pi}_{\nu}(\mathcal{W}^{r}(\Lambda_k,\Psi);L_2(\mathbb{R})) \leqslant \overline{\delta}_{\nu}(\mathcal{W}^{r}(\Lambda_k,\Psi);L_2(\mathbb{R}))$$

$$\leqslant \sup\{\|f-\mathcal{L}_{\nu\pi}(f)\| : f \in \mathcal{W}^{r}(\Lambda_k,\Psi)\} = \mathcal{A}_{\nu\pi}(\mathcal{W}^{r}(\Lambda_k,\Psi)) \leqslant \frac{1}{\sqrt{C_{2k}^k}\ \pi^r\nu^r}\ \Psi\Big(\frac{1}{\nu}\Big). \tag{4.17}$$

To obtain the lower bounds for the considered extreme characteristics of the class $\mathcal{W}^{r}(\Lambda_k,\Psi)$, we set $\widehat{\sigma} := \nu\pi(1+\varepsilon)$, where $\varepsilon \in (0,\widetilde{\nu})$, $\widetilde{\nu} := \min(\nu,1/\nu)$. Consider the set of functions $\mathcal{B}_{\widehat{\sigma}}(\widetilde{\rho}_{\varepsilon}) := \mathbb{B}_{\widehat{\sigma},2} \cap \widetilde{\rho}_{\varepsilon} BL_2(\mathbb{R}) = \{g \in \mathbb{B}_{\widehat{\sigma},2} : \|g\| \leqslant \widetilde{\rho}_{\varepsilon}\}$. Here,

$$\widetilde{\rho}_{\varepsilon} := \frac{1}{\sqrt{C_{2k}^k}\ (\widehat{\sigma})^r}\ \Psi\Big(\frac{1}{\nu(1+\varepsilon)}\Big). \tag{4.18}$$

Using formulas (2.18)–(2.19) and (4.8) for any element $g \in \mathbb{B}_{\widehat{\sigma},2}$, we get

$$\Lambda_k^2(g^{(r)}, t) = \frac{2^k}{t} \int\limits_0^t dh \int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} (1 - \cos(hu))^k u^{2r} |\mu(u)|^2 du$$

$$= 2^k \int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} |\mu(u)|^2 \Big\{ \frac{u^{2r-1}}{t} \int\limits_0^{ut} (1 - \cos h)^k dh \Big\} du = 2^k \int\limits_{-\widehat{\sigma}}^{\widehat{\sigma}} |\mu(u)|^2 G_k(t, u) du. \tag{4.19}$$

As was mentioned in the proof of Theorem 2, the function G_k, as a function of u, is even and nonnegative on $\mathbb{R}$ for any fixed $t \in (0, \infty)$ and is monotonically increasing on the set $\mathbb{R}_+$. Therefore, from (4.19) with regard for formula (4.9), we get

$$\Lambda_k^2(g^{(r)}, t) \leqslant 2^k G_k(t, \widehat{\sigma}) \|g\|^2. \tag{4.20}$$

We now use formula (2.19) for the function G_k and condition (4.14) for the majorant Ψ. For any element $g \in \mathcal{B}_{\widehat{\sigma}}(\widetilde{\rho}_\varepsilon)$, inequality (4.20) yields

$$\Lambda_k^2(g^{(r)}, t) \leqslant 2^k (\widehat{\sigma})^{2r} \frac{\|g\|^2}{\widehat{\sigma} t} \int\limits_0^{\widehat{\sigma} t} (1 - \cos h)^k dh$$

$$\leqslant \frac{2^k}{C_{2k}^k \widehat{\sigma} t} \Big\{ \int\limits_0^{\widehat{\sigma} t} (1 - \cos h)^k dh \Big\} \Psi^2\Big(\frac{\pi}{\widehat{\sigma}}\Big) \leqslant \Psi^2(t),$$

where $t \in (0, \infty)$. Hence, $\mathcal{B}_{\widehat{\sigma}}(\widetilde{\rho}_\varepsilon) \subset \mathcal{W}^r(\Lambda_k, \Psi)$. Since

$$\overline{b}_\nu(\mathcal{W}^r(\Lambda_k, \Psi), L_2(\mathbb{R})) \geqslant \overline{b}_\nu(\mathcal{B}_{\widehat{\sigma}}(\widetilde{\rho}_\varepsilon), L_2(\mathbb{R})) \geqslant \widetilde{\rho}_\varepsilon,$$

with regard for formulas (4.18) and (4.2), we get

$$\overline{\Pi}_\nu(\mathcal{W}^r(\Lambda_k, \Psi); L_2(\mathbb{R})) \geqslant \frac{1}{\sqrt{C_{2k}^k}(\pi\nu(1+\varepsilon))^r} \Psi\Big(\frac{1}{\nu(1+\varepsilon)}\Big). \tag{4.21}$$

We note that the left-hand side of inequality (4.21) is independent of ε. Calculating the upper bound in $\varepsilon \in (0, \widetilde{\nu})$ for its right-hand side, we have

$$\overline{\Pi}_\nu(\mathcal{W}^r(\Lambda_k, \Psi); L_2(\mathbb{R})) \geqslant \frac{1}{\sqrt{C_{2k}^k}\, \pi^r \nu^r} \Psi\Big(\frac{1}{\nu}\Big). \tag{4.22}$$

The required equalities (4.15) follow from relations (4.17) and (4.22). In conclusion, we will verify that the set of majorants satisfying condition (4.14) is not empty. Using the reasoning in the end of the proof of Theorem 5 in work [24], we can show that relation (4.14) holds for the majorant $\widehat{\Psi}(t) := t^\xi$, where $\xi = 2^{2k}/C_{2k}^k - 1$. Theorem 5 is proved. □

By $\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi)$, where $r \in \mathbb{N}$, and $p \in (0, 2]$; Ψ is a majorant, we denote the class of functions $f \in L_2^r(\mathbb{R})$ whose derivatives of the r-th order $f^{(r)}$ satisfy the condition $\int_0^x \Lambda_1^p(f^{(r)}, t) dt \leqslant \Psi^p(x)$ for any $x \in (0, \infty)$.

Theorem 6. *Let $r \in \mathbb{N}$, let $\nu \in (0, \infty)$ be any number, $p \in [1/r, 2]$, and let Ψ be a majorant satisfying the condition*

$$\frac{\Psi^p(x)}{\Psi^p(\pi/\sigma)} \geqslant \frac{\int_0^{\sigma x}(1 - \operatorname{sinc}(t))^{p/2}dt}{\int_0^{\pi}(1 - \operatorname{sinc}(t))^{p/2}dt} \tag{4.23}$$

for arbitrary values of $x \in (0, \infty)$ and $\sigma \in (\nu\pi, \infty)$. Then the following equalities hold:

$$\overline{\Pi}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi); L_2(\mathbb{R})) = \mathcal{A}_{\nu\pi}(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi))$$

$$= \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in \widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi)\}$$

$$= \frac{(\nu\pi)^{1/p-r}}{\sqrt{2}}\Big\{\int_0^\pi (1 - \operatorname{sinc}(t))^{p/2}dt\Big\}^{-1/p}\Psi\Big(\frac{1}{\nu}\Big), \tag{4.24}$$

where $\overline{\Pi}_\nu(\cdot)$ is any of the above-considered mean ν-widths. In this case, the pair $(L_2^r(\mathbb{R}), \mathcal{L}_{\nu\pi})$ is extreme for $\overline{\delta}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi), L_2(\mathbb{R}))$, the subspace $\mathbb{B}_{\nu\pi,2}$ is extreme for $\overline{d}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi), L_2(\mathbb{R}))$, and the set of majorants satisfying condition (4.23) is not empty.

Proof. Like in two previous theorems, we set $\sigma = \nu\pi$. Using Corollary 2, in which we set $\varphi \equiv 1$, $x = 1/\nu$, and $k = 1$, we get

$$\mathcal{A}_{\nu\pi}(f) \leqslant \frac{(\nu\pi)^{1/p-r}}{\sqrt{2}}\Big\{\int_0^\pi (1 - \operatorname{sinc}(t))^{p/2}dt\Big\}^{-1/p}\Big\{\int_0^{1/\nu} \Lambda_1^p(f^{(r)}, t)dt\Big\}^{1/p}.$$

This inequality, the definition of the class $\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi)$, and relation (4.2) yield the following upper bounds:

$$\overline{\Pi}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi); L_2(\mathbb{R})) \leqslant \overline{\delta}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi); L_2(\mathbb{R}))$$

$$\leqslant \sup\{\|f - \mathcal{L}_{\nu\pi}(f)\| : f \in \widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi)\} = \mathcal{A}_{\nu\pi}(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi))$$

$$\leqslant \frac{(\nu\pi)^{1/p-r}}{\sqrt{2}}\Big\{\int_0^\pi (1 - \operatorname{sinc}(t))^{p/2}dt\Big\}^{-1/p}\Psi\Big(\frac{1}{\nu}\Big). \tag{4.25}$$

We now get the lower bounds for the extreme characteristics of the class $\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi)$ by analogy with the reasoning in the proof of Theorems 4 and 5. Let us consider the set of entire functions [4] $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon^*) := \mathbb{B}_{\widehat{\sigma},2} \cap \rho_\varepsilon^* BL_2(\mathbb{R}) = \{g \in \mathbb{B}_{\widehat{\sigma},2} : \|g\| \leqslant \rho_\varepsilon^*\}$, where $\widehat{\sigma} := \nu\pi(1+\varepsilon)$,

$$\rho_\varepsilon^* := \frac{(\widehat{\sigma})^{1/p-r}}{\sqrt{2}}\Big\{\int_0^\pi (1 - \operatorname{sinc}(t))^{p/2}dt\Big\}^{-1/p}\Psi\Big(\frac{1}{\nu(1+\varepsilon)}\Big), \tag{4.26}$$

$\varepsilon \in (0, \widetilde{\nu})$; $\widetilde{\nu} := \min(\nu, 1/\nu)$. Considering the form of the function G_k given in formula (2.19) for $k = 1$ and raising both sides of inequality (4.20) to the power $p/2$, we have

$$\Lambda_1^p(g^{(r)}, t) \leqslant 2^{p/2}(\widehat{\sigma})^{rp}(1 - \operatorname{sinc}(\widehat{\sigma}t)^{p/2}\|g\|^p$$

for any function $g \in \mathbb{B}_{\widehat{\sigma},2}$. Let us integrate both sides of the last inequality over the variable t in the limits from 0 to x, where $x \in (0, \infty)$ is any number. For any element $g \in \mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon^*)$ by virtue of limitation (4.23) and relation (4.26), we get

$$\int\limits_0^x \Lambda_1^p(g^{(r)}, t)dt \leqslant \frac{\int_0^{\widehat{\sigma}x}(1 - \operatorname{sinc}(t))^{p/2}dt}{\int_0^{\pi}(1 - \operatorname{sinc}(t))^{p/2}dt}\ \Psi^p\Big(\frac{\pi}{\widehat{\sigma}}\Big) \leqslant \Psi^p(x).$$

Hence, $\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon^*) \subset \widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi)$. Using the definition of Bernshtein mean ν-width, we write $\overline{b}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi); L_2(\mathbb{R})) \geqslant \overline{b}_\nu(\mathcal{B}_{\widehat{\sigma}}(\rho_\varepsilon^*), L_2(\mathbb{R})) \geqslant \rho_\varepsilon^*$. Using formulas (4.2) and (4.26), we have

$$\overline{\Pi}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi), L_2(\mathbb{R})) \geqslant \frac{(\nu\pi(1+\varepsilon))^{1/p-r}}{\sqrt{2}}\Big\{\int\limits_0^{\pi}(1 - \operatorname{sinc}(t))^{p/2}dt\Big\}^{-1/p}\ \Psi\Big(\frac{1}{\nu(1+\varepsilon)}\Big). \tag{4.27}$$

Calculating the upper bound in $\varepsilon \in (0, \widetilde{\nu})$ for the right-hand side of inequality (4.27), we get

$$\overline{\Pi}_\nu(\widehat{\mathcal{W}}_p^r(\Lambda_1, \Psi); L_2(\mathbb{R})) \geqslant \frac{(\nu\pi)^{1/p-r}}{\sqrt{2}}\Big\{\int\limits_0^{\pi}(1 - \operatorname{sinc}(t))^{p/2}dt\Big\}^{-1/p}\Psi\Big(\frac{1}{\nu}\Big). \tag{4.28}$$

The required equalities (4.24) follow from relations (4.25) and (4.28).

Practically analogously to the proof of Theorem 6 in [24], we can show that the majorant $\Psi^*(x) := x^\xi$, where $\xi = \pi/(p\int_0^\pi(1 - \operatorname{sinc}(t))^{p/2}dt)$, satisfies condition (4.23). This completes the proof of Theorem 6. □

In conclusion, we note that the results of Theorems 2, 3, 5, and 6 were announced without proofs in the Proceedings of the International Stechkin summer mathematical school-conference [35].

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