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Completion and extension of operators in Kreĭn spaces

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Abstract. A generalization of the well-known results of M.G. Kreĭn on the description of the self-adjoint contractive extension of a Hermitian contraction is obtained. This generalization concerns the situation where the self-adjoint operator A and extensions $\widetilde{A}$ belong to a Kreĭn space or a Pontryagin space, and their defect operators are allowed to have a fixed number of negative eigenvalues. A result of Yu. L. Shmul'yan on completions of nonnegative block operators is generalized for block operators with a fixed number of negative eigenvalues in a Kreĭn space.

This paper is a natural continuation of S. Hassi's and author's recent paper [7].

Keywords. Completion, extension of operators, Kreĭn and Pontryagin spaces.

1. Introduction

Let A be a densely defined lower semibounded operator in a separable Hilbert space $\mathfrak{H}$, $A \geq m_A I$. The problem of existence of self-adjoint extensions preserving the lower bound m_A of A was formulated by J. von Neumann [4]. He solved it for the case of an operator with finite deficiency indices. A solution to this problem for operators with arbitrary deficiency indices was obtained by M. Stone, H. Freudental, and K. Friedrichs [4]. M. G. Kreĭn in his seminal paper [19] (see also [1]) described the set $\mathrm{Ext}_A(0,\infty)$ of all nonnegative self-adjoint extensions $\widetilde{A}$ of $A \geq 0$ as follows:

$$(A_F + a)^{-1} \leq (\widetilde{A} + a)^{-1} \leq (A_K + a)^{-1}, \quad a > 0, \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty).$$

Here, A_F and A_K are the Friedrichs (hard) and Kreĭn (soft) extensions of A, respectively.

To obtain such description, he used a special form of the Cayley transform

$$T_1 = (I - A)(I + A)^{-1}, \quad T = (I - \widetilde{A})(I + \widetilde{A})^{-1},$$

to reduce the study of unbounded operators to the study of contractive self-adjoint extensions T of a Hermitian nondensely defined contraction $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$, where $\mathfrak{H}_1 = \mathrm{ran}\,(I + A)$. The set of all self-adjoint contractive extensions of T_1 is denoted by $\mathrm{Ext}_{T_1}(-1,1)$. M.G. Kreĭn proved that the set $\mathrm{Ext}_{T_1}(-1,1)$ forms an operator interval with minimal and maximal entries T_m and T_M, respectively,

$$T_m \leq T \leq T_M, \quad T \in \mathrm{Ext}_{T_1}(-1,1).$$

T. Ando and K. Nishio [2] extended main results of the Kreĭn theory to the case of nondensely defined symmetric operators A. For the case of linear relations (multivalued linear operators) $A \geq 0$, it was done by E.A. Coddington and H.S.V. de Snoo [9].

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With respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, a contraction $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$ admits a block-matrix representation $T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}$. Block matrix representations of the operators T_m and T_M were obtained in [6,18] and [16] (see also [4,12,13,27]. Namely, it is shown that

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & -I + V(I - T_{11})V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & D_{T_{11}} V^* \\ V D_{T_{11}} & I - V(I + T_{11})V^* \end{pmatrix}, \end{aligned} \tag{1.1}$$

where $D_{T_{11}} := (I - T_{11}^2)^{1/2}$, and V is given by $V := \operatorname{clos}(T_{21} D_{T_{11}}^{[-1]})$. Based on these formulas, a complete parametrization of the set $\operatorname{Ext}_{T_1}(-1,1)$ and the main results of the Kreĭn theory have also been obtained there. In turn, the proof of formulas for T_m and T_M was based on a result of Yu. L. Shmul'yan [26] (see also [27]) on nonnegative completions of a nonnegative block operator.

Recently, S. Hassi and the author [7] extended the main result of [16] to the case of "quasicontractive" symmetric operators T_1. Recall that the "quasicontractivity" means that $\nu_-(I - T^*T) < \infty$, where

$$\nu_-(K) = \dim(E_K(-\infty, 0)\mathfrak{H}).$$

For this purpose, the above-mentioned result of Shmul'yan was generalized there. In addition, an analog of block matrix formulas for the operators T_m and T_M was established. The formulas for T_m and T_M look, in this case, similar to (1.1), but the entries $V(I \pm T_{11})V^*$ are replaced by $V(I \pm T_{11})JV^*$, where $J = \operatorname{sign}(I - T_{11}^2)$ and $D_{T_{11}} := |I - T_{11}^2|^{1/2}$.

The first result of the present paper is a further generalization of Shmul'yan's result [26] to the case of block operators acting in a Kreĭn space and having a fixed number of negative eigenvalues.

In Section 4, a first Kreĭn space analog of the completion problem is formulated, and a description of its solutions is found. Namely, we consider classes of "quasicontractive" symmetric operators T_1 in a Kreĭn space with $\nu_-(I - T_1^* T_1) < \infty$ and describe all possible self-adjoint (in the Kreĭn space sense) extensions T of T_1 that preserve the given negative index $\nu_-(I - T^*T) = \nu_-(I - T_1^* T_1)$. This problem is close to the completion problem studied in [7] and has a similar description for its solutions (for related problems, see also [3–5, 10–16, 18, 20, 22–25, 27]).

The main result of the present paper is Theorem 5.7. Namely, we consider the classes of "quasicontractive" symmetric operators T_1 in a Pontryagin space $(\mathfrak{H}, J)$ with

$$\nu_-[I - T_1^{[*]} T_1] := \nu_-(J(I - T_1^{[*]} T_1)) < \infty, \tag{1.2}$$

and we establish a solvability criterion and describe all possible self-adjoint extensions T of T_1 (in the Pontryagin space sense) that preserve the given negative index $\nu_-[I - T^{[*]}T] = \nu_-[I - T_1^{[*]} T_1]$. The formulas for T_m and T_M are also extended in an appropriate manner (see (5.16)). It should be emphasized that, in this more general setting, formulas (5.16) involve the so-called link operator L_T which was introduced by Arsene, Constantintscu, and Gheondea in [5] (see also [4, 10, 11, 21]).

2. The completion problem for block operators in Kreĭn spaces

By definition, the modulus $|C|$ of a closed operator C is the nonnegative self-adjoint operator $|C| = (C^*C)^{1/2}$. Every closed operator admits a polar decomposition $C = U|C|$, where U is a (unique) partial isometry with the initial space $\overline{\operatorname{ran}}\,|C|$ and the final space $\overline{\operatorname{ran}}\,C$, cf. [17]. For a self-adjoint operator $H = \int_{\mathbb{R}} t\, dE_t$ in a Hilbert space $\mathfrak{H}$, the partial isometry U can be identified with the signature operator which can be taken to be unitary: $J = \operatorname{sign}(H) = \int_{\mathbb{R}} \operatorname{sign}(t)\, dE_t$. In this case, one should define $\operatorname{sign}(t) = 1$, if $t \geq 0$, and $\operatorname{sign}(t) = -1$ otherwise.

Let $\mathcal{H}$ be a Hilbert space, and let $J_{\mathcal{H}}$ be a signature operator in it, i.e., $J_{\mathcal{H}} = J_{\mathcal{H}}^* = J_{\mathcal{H}}^{-1}$. We interpret the space $\mathcal{H}$ as a Kreĭn space $(\mathcal{H}, J_{\mathcal{H}})$ (see [6, 8]) in which the indefinite scalar product is defined by the equality

$$[\varphi, \psi]_{\mathcal{H}} = (J_{\mathcal{H}}\varphi, \psi)_{\mathcal{H}}.$$

Let us introduce a partial ordering for self-adjoint Kreĭn space operators. For self-adjoint operators A and B with the same domains, $A \geq_J B$ iff $[(A - B)f, f] \geq 0$ for all $f \in \operatorname{dom} A$. If not otherwise indicated, the word "smallest" means the smallest operator in the sense of this partial ordering.

Consider the bounded incomplete block operator

$$A^0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & * \end{pmatrix} \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix} \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix} \tag{2.1}$$

in the Kreĭn space $\mathfrak{H} = (\mathfrak{H}_1 \oplus \mathfrak{H}_2, J)$, where $(\mathfrak{H}_1, J_1)$ and $(\mathfrak{H}_2, J_2)$ are Kreĭn spaces with fundamental symmetries J_1 and J_2, and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$.

Theorem 2.1. *Let $\mathfrak{H} = (\mathfrak{H}_1 \oplus \mathfrak{H}_2, J)$ be an orthogonal decomposition of the Kreĭn space $\mathfrak{H}$, and let A^0 be an incomplete block operator of the form (2.1). Assume that $A_{11} = A_{11}^{[*]}$ and $A_{21} = A_{12}^{[*]}$ are bounded, the numbers of negative squares of the quadratic form $[A_{11}f, f]$ $(f \in \operatorname{dom} A_{11})$ $\nu_-[A_{11}] := \nu_-(J_1 A_{11}) = \kappa < \infty$, where $\kappa \in \mathbb{Z}_+$, and let us introduce $J_{11} := \operatorname{sign}(J_1 A_{11})$ that is the (unitary) signature operator of $J_1 A_{11}$. Then*

(i) *There exists a completion $A \in [(\mathfrak{H}, J)]$ of A^0 with some operator $A_{22} = A_{22}^{[*]} \in [(\mathfrak{H}_2, J_2)]$ such that $\nu_-[A] = \nu_-[A_{11}] = \kappa$, iff*

$$\operatorname{ran} J_1 A_{12} \subset \operatorname{ran} |A_{11}|^{1/2}.$$

(ii) *In this case, the operator $S = |A_{11}|^{[-1/2]} J_1 A_{12}$, where $|A_{11}|^{[-1/2]}$ denotes the (generalized) Moore–Penrose inverse of $|A_{11}|^{1/2}$, is well defined, and $S \in [(\mathfrak{H}_2, J_2), (\mathfrak{H}_1, J_1)]$. Moreover, $S^{[*]} J_1 J_{11} S$ is the "smallest" operator in the solution set*

$$\mathcal{A} := \left\{ A_{22} = A_{22}^{[*]} \in [(\mathfrak{H}_2, J_2)] : \ A = (A_{ij})_{i,j=1}^2 : \nu_-[A] = \kappa, \right\}$$

and this solution set admits the description

$$\mathcal{A} = \Big\{ A_{22} \in [(\mathfrak{H}_2, J_2)] : \ A_{22} = J_2(S^* J_{11} S + Y) = S^{[*]} J_1 J_{11} S + J_2 Y, \ \text{where } Y = Y^* \geq 0 \Big\}.$$

Proof. Let us introduce a block operator

$$\widetilde{A}^0 = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & * \end{pmatrix} = \begin{pmatrix} J_1 A_{11} & J_1 A_{12} \\ J_2 A_{21} & * \end{pmatrix}.$$

The blocks of this operator satisfy the identities $\widetilde{A}_{11} = \widetilde{A}_{11}^*$, $\widetilde{A}_{21}^* = \widetilde{A}_{12}$ and

$$\begin{aligned} \operatorname{ran} J_1 A_{11} = \operatorname{ran} \widetilde{A}_{11} \subset \operatorname{ran} |\widetilde{A}_{11}|^{1/2} &= \operatorname{ran} (\widetilde{A}_{11}^* \widetilde{A}_{11})^{1/4} \\ &= \operatorname{ran} (A_{11}^* A_{11})^{1/4} = \operatorname{ran} |A_{11}|^{1/2}. \end{aligned}$$

Then, due to [7, Theorem 1], the description of all self-adjoint operator completions of $\widetilde{A}^0$ admits the representation $\widetilde{A} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}$ with $\widetilde{A}_{22} = \widetilde{S}^* J_{11} \widetilde{S} + Y$, where $\widetilde{S} = |\widetilde{A}_{11}|^{[-1/2]} \widetilde{A}_{12}$ and $Y = Y^* \geq 0$.

This yields the description for the solutions of the completion problem. The set of completions has the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$$\begin{aligned} A_{22} = J_2 \widetilde{A}_{22} &= J_2 A_{21} J_1 |A_{11}|^{[-1/2]} J_{11} |A_{11}|^{[-1/2]} J_1 A_{12} + J_2 Y \\ &= J_2 S^* J_{11} S + J_2 Y = S^{[*]} J_1 J_{11} S + J_2 Y. \end{aligned}$$

□

3. Some inertia formulas

Some simple inertia formulas are now recalled. The factorization $H = B^{[*]} E B$ clearly implies that $\nu_\pm[H] \leq \nu_\pm[E]$, cf. (1.2). If H_1 and H_2 are self-adjoint operators in a Kreĭn space, then

$$H_1 + H_2 = \begin{pmatrix} I \\ I \end{pmatrix}^{[*]} \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix}$$

shows that $\nu_\pm[H_1 + H_2] \leq \nu_\pm[H_1] + \nu_\pm[H_2]$. Consider the self-adjoint block operator $H \in [(\mathfrak{H}_1, J_1) \oplus (\mathfrak{H}_2, J_2)]$, where $J_i = J_i^* = J_i^{-1}$, $(i = 1, 2)$, of the form

$$H = H^{[*]} = \begin{pmatrix} A & B^{[*]} \\ B & I \end{pmatrix},$$

By applying the above-mentioned inequalities, we see that

$$\nu_\pm[A] \leq \nu_\pm[A - B^{[*]} B] + \nu_\pm(J_2). \tag{3.1}$$

Assuming that $\nu_-[A - B^* J_2 B]$ and $\nu_-(J_2)$ are finite, the question about when $\nu_-[A]$ attains its maximum in (3.1) or, equivalently, $\nu_-[A - B^* J_2 B] \geq \nu_-[A] - \nu_-(J_2)$ attains its minimum, turns out to be of particular interest. The next result characterizes this situation as an application of Theorem 2.1. Recall that if $J_1 A = J_A |A|$ is the polar decomposition of $J_1 A$, then one can interpret $\mathfrak{H}_A = (\overline{\operatorname{ran}}\, J_1 A, J_A)$ as a Kreĭn space generated on $\overline{\operatorname{ran}}\, J_1 A$ by the fundamental symmetry $J_A = \operatorname{sign}(J_1 A)$.

Theorem 3.1. *Let $A \in [(\mathfrak{H}_1, J_1)]$ be self-adjoint, $B \in [(\mathfrak{H}_1, J_1), (\mathfrak{H}_2, J_2)]$, $J_i = J_i^* = J_i^{-1} \in [\mathfrak{H}_i]$, $(i = 1, 2)$, and let us assume that $\nu_-[A], \nu_-(J_2) < \infty$. If the equality*

$$\nu_-[A] = \nu_-[A - B^{[*]} B] + \nu_-(J_2)$$

holds, then $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$ and $J_1 B^{[*]} = |A|^{1/2} K$ for a unique operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$ which is J-contractive: $J_2 - K^* J_A K \geq 0$.

Conversely, if $B^{[*]} = |A|^{1/2} K$ for some J-contractive operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$, then equality (3.1) is satisfied.

Proof. Assume that (3.1) is satisfied. The factorization

$$H = \begin{pmatrix} A & B^{[*]} \\ B & I \end{pmatrix} = \begin{pmatrix} I & B^{[*]} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - B^{[*]} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$$

shows that $\nu_-[H] = \nu_-[A - B^{[*]}B] + \nu_-(J_2)$. This relation combined with equality (3.1) gives $\nu_-[H] = \nu_-[A]$. Therefore, by Theorem 2.1, one has $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$, and this is equivalent to the existence of a unique operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$ such that $J_1 B^{[*]} = |A|^{1/2} K$; i.e., $K = |A|^{[-1/2]} J_1 B^{[*]}$. Furthermore, $K^{[*]} J_1 J_A K \leq_{J_2} I$ by the minimality property of $K^{[*]} J_1 J_A K$ in Theorem 2.1. In other words, K is a J-contraction.

Conversely, if $J_1 B^{[*]} = |A|^{1/2} K$ for some J-contractive operator $K \in [(\mathfrak{H}_2, J_2), \mathfrak{H}_A]$, then, clearly, $\operatorname{ran} J_1 B^{[*]} \subset \operatorname{ran} |A|^{1/2}$. By Theorem 2.1, the completion problem for H^0 has solutions with the minimal solution $S^{[*]} J_1 J_A S$, where

$$S = |A|^{[-1/2]} J_1 B^{[*]} = |A|^{[-1/2]} |A|^{1/2} K = K.$$

Furthermore, by J-contractivity of K, one has $K^{[*]} J_1 J_A K \leq_{J_2} I$, i.e., I is also a solution, and, thus, $\nu_-[H] = \nu_-[A]$ or, equivalently, equality (3.1) is satisfied. □

4. A pair of completion problems in a Kreĭn space

In this section, we introduce and describe the solutions of a Kreĭn space version of a completion problem that was treated in [7].

Let $(\mathfrak{H}_i, (J_i \cdot, \cdot))$ and $(\mathfrak{H}, (J \cdot, \cdot))$ be Kreĭn spaces, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$, let J_i be fundamental symmetries ($i = 1, 2$), and let $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ be an operator such that $\nu_-(I - T_{11}^* T_{11}) = \kappa < \infty$. Denote $\widetilde{T}_{11} = J_1 T_{11}$. Then $\widetilde{T}_{11} = \widetilde{T}_{11}^*$ in the Hilbert space $\mathfrak{H}_1$. Rewrite $\nu_-(I - T_{11}^* T_{11}) = \nu_-(I - \widetilde{T}_{11}^2)$. Denote

$$J_+ = \operatorname{sign}(I - \widetilde{T}_{11}), \ J_- = \operatorname{sign}(I + \widetilde{T}_{11}), \text{ and } J_{11} = \operatorname{sign}(I - \widetilde{T}_{11}^2),$$

and let $\kappa_+ = \nu_-(J_+)$ and $\kappa_- = \nu_-(J_-)$. It is easy to get that $J_{11} = J_- J_+ = J_+ J_-$. Moreover, we have the equality $\kappa = \kappa_- + \kappa_+$ (see [7, Lemma 5.1]). We recall the results for the operator $\widetilde{T}_{11}$ from work [7] and then reformulate them for the operator T_{11}. We recall the completion problem and its solutions that were investigated in a Hilbert space in [7]. The problem concerns the existence and the description of the self-adjoint operators $\widetilde{T}$ such that $\widetilde{A}_+ = I + \widetilde{T}$ and $\widetilde{A}_- = I - \widetilde{T}$ solve the corresponding completion problems

$$\widetilde{A}_\pm^0 = \begin{pmatrix} I \pm \widetilde{T}_{11} & \pm \widetilde{T}_{21}^* \\ \pm \widetilde{T}_{21} & * \end{pmatrix}, \tag{4.1}$$

under *minimal index conditions* $\nu_-(I + \widetilde{T}) = \nu_-(I + \widetilde{T}_{11})$, $\nu_-(I - \widetilde{T}) = \nu_-(I - \widetilde{T}_{11})$, respectively. The solution set is denoted by $\operatorname{Ext}_{\widetilde{T}_1, \kappa}(-1, 1)$.

The next theorem gives a general solvability criterion for the completion problem (4.1) and describes all solutions to this problem.

Theorem 4.1. *([7, Theorem 5]) Let* $\widetilde{T}_1 = \begin{pmatrix} \widetilde{T}_{11} \\ \widetilde{T}_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}$ *be a symmetric operator with* $\widetilde{T}_{11} = \widetilde{T}_{11}^* \in [\mathfrak{H}_1]$ *and* $\nu_-(I - \widetilde{T}_{11}^2) = \kappa < \infty$, *and let* $J_{11} = \operatorname{sign}(I - \widetilde{T}_{11}^2)$. *Then the completion problem for* $\widetilde{A}_\pm^0$ *in* (4.1) *has a solution* $I \pm \widetilde{T}$ *for some* $\widetilde{T} = \widetilde{T}^*$ *with* $\nu_-(I - \widetilde{T}^2) = \kappa$, *iff the following condition is satisfied:*

$$\nu_-(I - \widetilde{T}_{11}^2) = \nu_-(I - \widetilde{T}_1^* \widetilde{T}_1). \tag{4.2}$$

If this condition is satisfied, then the following facts hold:

(i) *The completion problems for* $\widetilde{A}^0_\pm$ *in* (4.1) *have minimal solutions* $\widetilde{A}_\pm$.

(ii) *The operators* $\widetilde{T}_m := \widetilde{A}_+ - I$ *and* $\widetilde{T}_M := I - \widetilde{A}_- \in \operatorname{Ext}_{\widetilde{T}_1,\kappa}(-1,1)$.

(iii) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *have the block form*

$$\begin{aligned} \widetilde{T}_m &= \begin{pmatrix} \widetilde{T}_{11} & D_{\widetilde{T}_{11}}V^* \\ VD_{\widetilde{T}_{11}} & -I + V(I - \widetilde{T}_{11})J_{11}V^* \end{pmatrix}, \\ \widetilde{T}_M &= \begin{pmatrix} \widetilde{T}_{11} & D_{\widetilde{T}_{11}}V^* \\ VD_{\widetilde{T}_{11}} & I - V(I + \widetilde{T}_{11})J_{11}V^* \end{pmatrix}, \end{aligned} \tag{4.3}$$

where $D_{\widetilde{T}_{11}} := |I - \widetilde{T}_{11}^2|^{1/2}$, *and* V *is given by* $V := \operatorname{clos}(\widetilde{T}_{21}D^{[-1]}_{\widetilde{T}_{11}})$.

(iv) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *are extremal extensions of* $\widetilde{T}_1$:

$$\widetilde{T} \in \operatorname{Ext}_{\widetilde{T}_1,\kappa}(-1,1) \quad \textit{iff} \quad \widetilde{T} = \widetilde{T}^* \in [\mathfrak{H}], \quad \widetilde{T}_m \le \widetilde{T} \le \widetilde{T}_M.$$

(v) *The operators* $\widetilde{T}_m$ *and* $\widetilde{T}_M$ *are connected via*

$$(-\widetilde{T})_m = -\widetilde{T}_M, \quad (-\widetilde{T})_M = -\widetilde{T}_m.$$

In what follows, it is convenient to reformulate the above theorem in the statement with a Kreĭn space. Consider the Kreĭn space $(\mathfrak{H}, J)$ and a self-adjoint operator T in this space. Now, the problem concerns the self-adjoint operators $A_+ = I + T$ and $A_- = I - T$ in the Kreĭn space $(\mathfrak{H}, J)$ that solve the completion problems

$$A^0_\pm = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^{[*]} \\ \pm T_{21} & * \end{pmatrix}, \tag{4.4}$$

under *minimal index conditions* $\nu_-(I + JT) = \nu_-(I + J_1T_{11})$ and $\nu_-(I - JT) = \nu_-(I - J_1T_{11})$, respectively. The set of solutions T to problem (4.4) will be denoted by $\operatorname{Ext}_{J_2T_1,\kappa}(-1,1)$.

Denote

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}, \tag{4.5}$$

so that T_1 is a symmetric (nondensely defined) operator in the Kreĭn space $[(\mathfrak{H}_1, J_1)]$, i.e., $T_{11} = T_{11}^{[*]}$.

Theorem 4.2. *Let* T_1 *be a symmetric operator in the Kreĭn space sense as in* (4.5) *with* $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ *and* $\nu_-(I - T_{11}^*T_{11}) = \kappa < \infty$, *and let* $J = \operatorname{sign}(I - T_{11}^*T_{11})$. *Then the completion problems for* $A^0_\pm$ *in* (4.4) *have a solution* $I \pm T$ *for some* $T = T^{[*]}$ *with* $\nu_-(I - T^*T) = \kappa$, *iff the following condition is satisfied:*

$$\nu_-(I - T_{11}^*T_{11}) = \nu_-(I - T_1^*T_1). \tag{4.6}$$

If this condition is satisfied, then the following facts hold:

(i) *The completion problems for* $A^0_\pm$ *in* (4.4) *have "minimal"* (J_2*-minimal) solutions* $A_\pm$.

(ii) *The operators* $T_m := A_+ - J$ *and* $T_M := J - A_- \in \operatorname{Ext}_{J_2T_1,\kappa}(-1,1)$.

(iii) *The operators* T_m *and* T_M *have the block form*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & -J_2 + J_2 V (I - J_1 T_{11}) J_{11} V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & J_2 - J_2 V (I + J_1 T_{11}) J_{11} V^* \end{pmatrix}, \end{aligned} \tag{4.7}$$

where $D_{T_{11}} := |I - T_{11}^* T_{11}|^{1/2}$, *and* V *is given by* $V := \operatorname{clos}(J_2 T_{21} D_{T_{11}}^{[-1]})$.

(iv) *The operators* T_m *and* T_M *are* J_2*-extremal extensions of* T_1*:*

$$T \in \operatorname{Ext}_{J_2 T_1, \kappa}(-1, 1) \quad \textit{iff} \quad T = T^{[*]} \in [(\mathfrak{H}, J)], \quad T_m \leq_{J_2} T \leq_{J_2} T_M.$$

(v) *The operators* T_m *and* T_M *are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m.$$

Proof. The proof is obtained by the systematic use of the equivalence that T is a self-adjoint operator in a Kreĭn space, iff $\widetilde{T}$ is self-adjoint in a Hilbert space. In particular, T gives solutions to the completion problems (4.4), iff $\widetilde{T}$ solves the completion problems (4.4). In view of

$$I - T_{11}^* T_{11} = I - T_{11}^* J J T_{11} = I - \widetilde{T}_{11}^2,$$

we get formula (4.6) from (4.2). Then formula (4.7) can be obtained by multiplying the operators in (4.3) ij view of the fundamental symmetry. □

5. Completion problem in a Pontryagin space

5.1. Defect operators and link operators

Let $(\mathfrak{H}, (\cdot, \cdot))$ be a Hilbert space, and let J be a symmetry in $\mathfrak{H}$, i.e., $J = J^* = J^{-1}$, so that $(\mathfrak{H}, (J\cdot, \cdot))$ becomes a Pontryagin space. Then we associate the following corresponding defect and signature operators with $T \in [\mathfrak{H}]$:

$$D_T = |J - T^* J T|^{1/2}, \quad J_T = \operatorname{sign}(J - T^* J T), \quad \mathfrak{D}_T = \overline{\operatorname{ran}}\, D_T,$$

where the so-called defect subspace $\mathfrak{D}_T$ can be considered as a Pontryagin space with the fundamental symmetry J_T. Similar notations are used with T^*:

$$D_{T^*} = |J - T J T^*|^{1/2}, \quad J_{T^*} = \operatorname{sign}(J - T J T^*), \quad \mathfrak{D}_{T^*} = \overline{\operatorname{ran}}\, D_{T^*}.$$

By definition, $J_T D_T^2 = J - T^* J T$ and $J_T D_T = D_T J_T$ with analogous identities for D_{T^*} and J_{T^*}. In addition,

$$(J - T^* J T) J T^* = T^* J (J - T J T^*), \; (J - T J T^*) J T = T J (J - T^* J T).$$

Recall that $T \in [\mathfrak{H}]$ is said to be a J-contraction, if $J - T^* J T \geq 0$, i.e., $\nu_-(J - T^* J T) = 0$. If, in addition, T^* is a J-contraction, T is termed as a J-bicontraction.

For the following consideration, an indefinite version of the commutation relation of the form $T D_T = D_{T^*} T$ is needed; it involves the so-called link operators introduced in [5, Section 4] (see also [7]).

Definition 5.1. *There exist the unique operators* $L_T \in [\mathfrak{D}_T, \mathfrak{D}_{T^*}]$ *and* $L_{T^*} \in [\mathfrak{D}_{T^*}, \mathfrak{D}_T]$ *such that*

$$D_{T^*} L_T = TJD_T \upharpoonright \mathfrak{D}_T, \quad D_T L_{T^*} = T^* J D_{T^*} \upharpoonright \mathfrak{D}_{T^*}; \tag{5.1}$$

in fact, $L_T = D_{T^*}^{[-1]} TJD_T \upharpoonright \mathfrak{D}_T$ *and* $L_{T^*} = D_T^{[-1]} T^* J D_{T^*} \upharpoonright \mathfrak{D}_{T^*}$.

The following identities can be obtained by direct calculations (see [5, Section 4]):

$$\begin{gathered} L_T^* J_{T^*} \upharpoonright \mathfrak{D}_{T^*} = J_T L_{T^*}; \\ (J_T - D_T J D_T) \upharpoonright \mathfrak{D}_T = L_T^* J_{T^*} L_T; \\ (J_{T^*} - D_{T^*} J D_{T^*}) \upharpoonright \mathfrak{D}_{T^*} = L_{T^*}^* J_T L_{T^*}. \end{gathered} \tag{5.2}$$

Now, let T be self-adjoint in the Pontryagin space $(\mathfrak{H}, J)$, i.e., $T^* = JTJ$. Then connections between D_{T^*} and D_T, J_{T^*} and J_T, and L_{T^*} and L_T can be established.

Lemma 5.1. *Assume that* $T^* = JTJ$. *Then* $D_T = |I - T^2|^{1/2}$, *and the following equalities hold:*

$$D_{T^*} = J D_T J, \tag{5.3}$$

in particular,

$$\mathfrak{D}_{T^*} = J\mathfrak{D}_T \text{ and } \mathfrak{D}_T = J\mathfrak{D}_{T^*};$$

$$J_{T^*} = J J_T J; \tag{5.4}$$

$$L_{T^*} = J L_T J. \tag{5.5}$$

Proof. The defect operator of T can be calculated by the formula

$$D_T = \left(\left(I - (T^*)^2\right) JJ (I - T^2)\right)^{1/4} = \left(\left(I - (T^*)^2\right)(I - T^2)\right)^{1/4}.$$

Then

$$\begin{aligned} D_{T^*} &= \left(J\left(I - (T^*)^2\right)(I - T^2) J\right)^{1/4} = J\left(\left(I - (T^*)^2\right)(I - T^2)\right)^{1/4} J \\ &= J D_T J, \end{aligned}$$

i.e., (5.3) holds. This yields

$$J\mathfrak{D}_{T^*} \subset \mathfrak{D}_T \text{ and } J\mathfrak{D}_T \subset \mathfrak{D}_{T^*}.$$

Hence, from two last formulas, we get

$$\mathfrak{D}_{T^*} = J(J\mathfrak{D}_{T^*}) \subset J\mathfrak{D}_T \subset \mathfrak{D}_{T^*}$$

and, similarly,

$$\mathfrak{D}_T = J(J\mathfrak{D}_T) \subset J\mathfrak{D}_{T^*} \subset \mathfrak{D}_T.$$

The formula

$$\begin{aligned} J_T D_T^2 &= J - T^* J T = J(J - TJT^*)J = J J_{T^*} D_{T^*}^2 J = J J_{T^*} J D_T^2 J J \\ &= J J_{T^*} J D_T^2 \end{aligned}$$

yields relation (5.4).

Relation (5.5) follows from

$$D_T L_{T^*} = T^* J D_{T^*} \upharpoonright \mathfrak{D}_{T^*} = JTJD_T J \upharpoonright \mathfrak{D}_{T^*} = J D_{T^*} L_T J = D_T J L_T J.$$

□

5.2. Lemmas on negative indices of certain block operators

Two first lemmas are of preparatory nature for two last lemmas, which are used in the proof of the main theorem.

Lemma 5.2. *Let* $\begin{pmatrix} J & T \\ T & J \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H} \end{pmatrix}$ *be a self-adjoint operator in the Hilbert space* $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$. *Then*

$$\left| \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right|^{1/2} = U \begin{pmatrix} |J+T|^{1/2} & 0 \\ 0 & |J-T|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ *is a unitary operator.*

Proof. It is easy to check that

$$\begin{pmatrix} J & T \\ T & J \end{pmatrix} = U \begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix} U^*. \tag{5.6}$$

Then, by taking the modulus, we get

$$\left| \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right|^2 = \left(\begin{pmatrix} J & T \\ T & J \end{pmatrix}^* \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right) = U \begin{pmatrix} |J+T|^2 & 0 \\ 0 & |J-T|^2 \end{pmatrix} U^*.$$

The last step is to extract the square roots (twice) from the both sides of the relation:

$$\left| \begin{pmatrix} J & T \\ T & J \end{pmatrix} \right|^{1/2} = U \begin{pmatrix} |J+T|^{1/2} & 0 \\ 0 & |J-T|^{1/2} \end{pmatrix} U^*.$$

The right-hand side can be written in this form, because U is unitary. □

Lemma 5.3. *Let* $T = T^* \in \mathfrak{H}$ *be a self-adjoint operator in a Hilbert space* $\mathfrak{H}$, *and let* $J = J^* = J^{-1}$ *be a fundamental symmetry in* $\mathfrak{H}$ *with* $\nu_-(J) < \infty$. *Then*

$$\nu_-(J - TJT) + \nu_-(J) = \nu_-(J-T) + \nu_-(J+T). \tag{5.7}$$

In particular, $\nu_-(J - TJT) < \infty$, *iff* $\nu_-(J \pm T) < \infty$.

Proof. Consider the block operators $\begin{pmatrix} J & T \\ T & J \end{pmatrix}$ and $\begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$. Equality (5.6) yields $\nu_- \begin{pmatrix} J & T \\ T & J \end{pmatrix} = \nu_- \begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$. The negative index of $\begin{pmatrix} J+T & 0 \\ 0 & J-T \end{pmatrix}$ equals $\nu_-(J-T) + \nu_-(J+T)$, and the negative index of $\begin{pmatrix} J & T \\ T & J \end{pmatrix}$ can be easily found, by using the equality

$$\begin{pmatrix} J & T \\ T & J \end{pmatrix} = \begin{pmatrix} I & 0 \\ TJ & I \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J - TJT \end{pmatrix} \begin{pmatrix} I & JT \\ 0 & I \end{pmatrix}. \tag{5.8}$$

Then we get (5.7). □

Let $(\mathfrak{H}_i, (J_i\cdot,\cdot))$ $(i=1,2)$ and $(\mathfrak{H}, (J\cdot,\cdot))$ be Pontryagin spaces, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$. Consider the operator $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ such that $\nu_-[I - T_{11}^2] = \kappa < \infty$ (see (1.2)). Denote $\widetilde{T}_{11} = J_1 T_{11}$. Then $\widetilde{T}_{11} = \widetilde{T}_{11}^*$ in the Hilbert space $\mathfrak{H}_1$. Rewrite

$$\begin{aligned}\nu_-[I - T_{11}^2] &= \nu_-(J_1(I - T_{11}^2)) = \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) \\ &= \nu_-((J_1 - \widetilde{T}_{11}) J_1 (J_1 + \widetilde{T}_{11})).\end{aligned}$$

Furthermore, we denote

$$\begin{aligned} J_+ &= \operatorname{sign}(J_1(I - T_{11})) = \operatorname{sign}(J_1 - \widetilde{T}_{11}), \\ J_- &= \operatorname{sign}(J_1(I + T_{11})) = \operatorname{sign}(J_1 + \widetilde{T}_{11}), \\ J_{11} &= \operatorname{sign}(J_1(I - T_{11}^2)) \end{aligned} \tag{5.9}$$

Let $\kappa_+ = \nu_-[I - T_{11}]$ and $\kappa_- = \nu_-[I + T_{11}]$. Note that $|I \mp T_{11}| = |J_1 \mp \widetilde{T}_{11}|$. Then we have the polar decompositions

$$I \mp T_{11} = J_1 J_\pm |I \mp T_{11}|. \tag{5.10}$$

Lemma 5.4. *Let $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$, and let $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in [(\mathfrak{H}, J)]$ be a self-adjoint extension of the operator T_{11} with $\nu_-[I \pm T_{11}] < \infty$ and $\nu_-(J) < \infty$. Then the statements*

(i) $\nu_-[I \pm T_{11}] = \nu_-[I \pm T]$;

(ii) $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$;

(iii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$

are connected by the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

Proof. The Lemma can be formulated in an equivalent way for the Hilbert space operators: the block operator $\widetilde{T} = JT = \begin{pmatrix} \widetilde{T}_{11} & \widetilde{T}_{12} \\ \widetilde{T}_{21} & \widetilde{T}_{22} \end{pmatrix}$ is a self-adjoint extension of $\widetilde{T}_{11} = \widetilde{T}_{11}^* \in [\mathfrak{H}_1]$. Then the statements

(i') $\nu_-(J_1 \pm \widetilde{T}_{11}) = \nu_-(J \pm \widetilde{T})$

(ii') $\nu_-(J - \widetilde{T} J \widetilde{T}) = \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) - \nu_-(J_2)$;

(iii') $\operatorname{ran} \widetilde{T}_{12} \subset \operatorname{ran} |J_1 \pm \widetilde{T}_{11}|^{1/2}$

are connected by the implications $(i') \Leftrightarrow (ii') \Rightarrow (iii')$.

Hence, it is sufficient to prove this form of the Lemma.

Let us prove the equivalence $(i') \Leftrightarrow (ii')$. Condition (ii') is equivalent to

$$\nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix}. \tag{5.11}$$

Indeed, in view of (5.8),

$$\nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_-(J_1) + \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11})$$

and

$$\nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix} = \nu_-(J) + \nu_-(J - \widetilde{T}J\widetilde{T})$$
$$= \nu_-(J_1) + \nu_-(J_2) + \nu_-(J - \widetilde{T}J\widetilde{T}).$$

By using Lemma 5.3, equality (5.11) is equivalent to

$$\nu_-(J_1 - \widetilde{T}_{11}) + \nu_-(J_1 + \widetilde{T}_{11}) = \nu_-(J - \widetilde{T}) + \nu_-(J + \widetilde{T}). \tag{5.12}$$

Hence, $(i') \Rightarrow (ii')$.

Because $\nu_-(J_1 \pm \widetilde{T}_{11}) \leq \nu_-(J \pm \widetilde{T})$, relation (5.12) shows that $(ii') \Rightarrow (i')$.

Now, we prove implication $(ii') \Rightarrow (iii')$; the arguments will be useful also in the proof of Lemma 5.5 below. Let us use a permutation to transform the matrix on the right-hand side of (5.11):

$$\nu_-\begin{pmatrix} J & \widetilde{T} \\ \widetilde{T} & J \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & 0 & \widetilde{T}_{11} & \widetilde{T}_{12} \\ 0 & J_2 & \widetilde{T}_{21} & \widetilde{T}_{22} \\ \widetilde{T}_{11} & \widetilde{T}_{12} & J_1 & 0 \\ \widetilde{T}_{21} & \widetilde{T}_{22} & 0 & J_2 \end{pmatrix} = \nu_-\begin{pmatrix} J_1 & \widetilde{T}_{11} & 0 & \widetilde{T}_{12} \\ \widetilde{T}_{11} & J_1 & \widetilde{T}_{12} & 0 \\ 0 & \widetilde{T}_{21} & J_2 & \widetilde{T}_{22} \\ \widetilde{T}_{21} & 0 & \widetilde{T}_{22} & J_2 \end{pmatrix}.$$

Then condition (5.11) yields the condition

$$\operatorname{ran}\begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} \subset \operatorname{ran}\left|\begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix}\right|^{1/2}$$

(see Theorem 2.1). By Lemma 5.2, the last inclusion can be rewritten as

$$\operatorname{ran}\begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} \subset \operatorname{ran} U\begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is a unitary operator. This inclusion is equivalent to

$$\operatorname{ran} U^*\begin{pmatrix} 0 & \widetilde{T}_{12} \\ \widetilde{T}_{12} & 0 \end{pmatrix} U = \operatorname{ran}\begin{pmatrix} \widetilde{T}_{12} & 0 \\ 0 & -\widetilde{T}_{12} \end{pmatrix}$$
$$\subset \operatorname{ran}\begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix}.$$

This is clearly equivalent to condition (iii').

Note that if $\widetilde{T}_{11}$ has a self-adjoint extension $\widetilde{T}$ satisfying (i'). Then, by applying Theorem 2.1 (or [7, Theorem 1]), it yields (iii'). ☐

Lemma 5.5. *Let $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ be an operator, and let*

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix} (\mathfrak{H}_1, J_1) \\ (\mathfrak{H}_2, J_2) \end{pmatrix}$$

be an extension of T_{11} with $\nu_-[I - T_{11}^2] < \infty$, $\nu_-(J_1) < \infty$, and $\nu_-(J_2) < \infty$. Then, under the conditions *(i)* $\nu_-[I_1 - T_{11}^2] = \nu_-[I_1 - T_1^{[*]}T_1] + \nu_-(J_2)$;

(ii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}^2|^{1/2}$;

(iii) $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$,

implications (i) ⇒ (ii) and (i) ⇒ (iii) hold.

Proof. First, we prove that (i)⇒(ii). In fact, this follows from Theorem 3.1, by taking $A = I - T_{11}^2$ and $B = T_{21}$.

The proof of (i)⇒(iii) is quite similar to the proof used in Lemma 5.4. Statement (i) is equivalent to the following relation:

$$\nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix}.$$

Indeed,

$$\nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J_1 & 0 \\ 0 & J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11} \end{pmatrix} = \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11}) + \nu_-(J_1) < \infty$$

and

$$\nu_- \begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J & 0 \\ 0 & J_1 - \widetilde{T}_1^* J \widetilde{T}_1 \end{pmatrix} = \nu_-(J_1 - \widetilde{T}_{11} J_1 \widetilde{T}_{11} - \widetilde{T}_{21}^* J_2 \widetilde{T}_{21}) + \nu_-(J_1) + \nu_-(J_2).$$

Due to (i), the right-hand sides coincide. Then the left-hand sides coincide as well.

Now, let us rearrange the matrix in the latter relation:

$$\nu_- \begin{pmatrix} J & \widetilde{T}_1 \\ \widetilde{T}_1^* & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J_1 & 0 & \widetilde{T}_{11} \\ 0 & J_2 & \widetilde{T}_{21} \\ \widetilde{T}_{11} & \widetilde{T}_{21}^* & J_1 \end{pmatrix} = \nu_- \begin{pmatrix} J_1 & \widetilde{T}_{11} & 0 \\ \widetilde{T}_{11} & J_1 & \widetilde{T}_{21}^* \\ 0 & \widetilde{T}_{21} & J_2 \end{pmatrix}.$$

It follows from [7, Theorem 1] that condition (i) yields the condition

$$\operatorname{ran} \begin{pmatrix} 0 \\ \widetilde{T}_{21}^* \end{pmatrix} \subset \operatorname{ran} \left| \begin{pmatrix} J_1 & \widetilde{T}_{11} \\ \widetilde{T}_{11} & J_1 \end{pmatrix} \right|^{1/2} = \operatorname{ran} U \begin{pmatrix} |J_1 + \widetilde{T}_{11}|^{1/2} & 0 \\ 0 & |J_1 - \widetilde{T}_{11}|^{1/2} \end{pmatrix} U^*,$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ is a unitary operator (see Lemma 5.2). Then, equivalently,

$$\operatorname{ran} \widetilde{T}_{21}^* \subset \operatorname{ran} |J_1 \pm \widetilde{T}_{11}|^{1/2}.$$

□

5.3. Contractive extensions of contractions with minimal negative indices

Following [7, 16, 18], we consider the problem of existence and description of the self-adjoint operators T in the Pontryagin space $\begin{pmatrix}(\mathfrak{H}_1, J_1)\\(\mathfrak{H}_2, J_2)\end{pmatrix}$ such that $A_+ = I + T$ and $A_- = I - T$ solve the corresponding completion problems

$$A_{\pm}^0 = \begin{pmatrix} I \pm T_{11} & \pm T_{21}^{[*]} \\ \pm T_{21} & * \end{pmatrix}, \tag{5.13}$$

under *minimal index conditions* $\nu_-[I+T] = \nu_-[I+T_{11}]$, $\nu_-[I-T] = \nu_-[I-T_{11}]$, respectively. Observe that, by Lemma 5.4, the two minimal index conditions above are equivalent to the single condition $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$.

It is clear from Theorem 2.1 that the conditions $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}|^{1/2}$ and $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I + T_{11}|^{1/2}$ are necessary for the existence of solutions; however, as noted already in [7], they are not sufficient even in the statement with a Hilbert space.

The next theorem gives a general solvability criterion for the completion problem (5.13) and describes all solutions to this problem. As in the definite case, there are minimal solutions A_+ and A_- which are connected to two extreme self-adjoint extensions T of

$$T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix} : (\mathfrak{H}_1, J_1) \to \begin{pmatrix}(\mathfrak{H}_1, J_1)\\(\mathfrak{H}_2, J_2)\end{pmatrix}, \tag{5.14}$$

now with finite negative index $\nu_-[I - T^2] = \nu_-[I - T_{11}^2] - \nu_-(J_2) > 0$. The set of solutions T to problem (5.13) will be denoted by $\operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$.

Theorem 5.1. *Let T_1 be a symmetric operator, as in* (5.14)*, with $T_{11} = T_{11}^{[*]} \in [(\mathfrak{H}_1, J_1)]$ and $\nu_-[I - T_{11}^2] = \kappa < \infty$, and let $J_{T_{11}} = \operatorname{sign}(J_1(I - T_{11}^2))$. Then the completion problem for $A_{\pm}^0$ in* (5.13) *has a solution $I \pm T$ for some $T = T^{[*]}$ with $\nu_-[I - T^2] = \kappa - \nu_-(J_2)$, iff the following condition is satisfied:*

$$\nu_-[I - T_{11}^2] = \nu_-[I - T_1^{[*]} T_1] + \nu_-(J_2). \tag{5.15}$$

If this condition is satisfied, then the following facts hold:

(i) *The completion problems for $A_{\pm}^0$ in* (5.13) *have "minimal" solutions $A_{\pm}$ (for the partial ordering introduced in the first section).*

(ii) *The operators $T_m := A_+ - I$ and $T_M := I - A_- \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$.*

(iii) *The operators T_m and T_M have the block form*

$$\begin{aligned} T_m &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & -I + J_2 V (I - L_T^* J_1) J_{11} V^* \end{pmatrix}, \\ T_M &= \begin{pmatrix} T_{11} & J_1 D_{T_{11}} V^* \\ J_2 V D_{T_{11}} & I - J_2 V (I + L_T^* J_1) J_{11} V^* \end{pmatrix}, \end{aligned} \tag{5.16}$$

where $D_{T_{11}} := |I - T_{11}^2|^{1/2}$, and V is given by $V := \operatorname{clos}(J_2 T_{21} D_{T_{11}}^{[-1]})$.

(iv) *The operators T_m and T_M are "extremal" extensions of T_1:*

$$T \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2} \quad \textit{iff} \quad T = T^{[*]} \in [(\mathfrak{H}, J)], \quad T_m \le_{J_2} T \le_{J_2} T_M. \tag{5.17}$$

(v) *The operators T_m and T_M are connected via*

$$(-T)_m = -T_M, \quad (-T)_M = -T_m. \tag{5.18}$$

Proof. It is easy to see by (3.1) that $\kappa = \nu_-[I - T_{11}^2] \le \nu_-[I - T_1^{[*]}T_1] + \nu_-(J_2) \le \nu_-[I - T^2] + \nu_-(J_2)$. Hence, the condition $\nu_-[I-T^2] = \kappa - \nu_-(J_2)$ yields (5.15). The sufficiency of this condition is obtained, when proving assertions (i)–(iii) below.

(i) If condition (5.15) is satisfied, then, by using Lemma 5.5, we get the inclusions $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I \pm T_{11}|^{1/2}$, which means, by Theorem 2.1, that each of the completion problems, $A_\pm^0$ in (5.13), is solvable. It follows that the operators

$$S_- = |I + T_{11}|^{[-1/2]} J_1 T_{21}^{[*]}, \quad S_+ = |I - T_{11}|^{[-1/2]} J_1 T_{21}^{[*]} \tag{5.19}$$

are well defined and provide the minimal solutions $A_\pm$ to the completion problems for $A_\pm^0$ in (5.13).

(ii) & (iii) By Lemma 5.5, the inclusion $\operatorname{ran} J_1 T_{21}^{[*]} \subset \operatorname{ran} |I - T_{11}^2|^{1/2}$ holds. This inclusion is equivalent to the existence of a (unique) bounded operator $V^* = D_{T_{11}}^{[-1]} J_1 T_{21}^{[*]}$ with $\ker V \supset \ker D_{T_{11}}$, such that $J_1 T_{21}^{[*]} = D_{T_{11}} V^*$. By using (5.1), (5.2), and 5.1, the operators $T_m := A_+ - I$ and $T_M := I - A_-$ (see the proof of (i)) can be now rewritten as in (5.16). Indeed, observe that (see Theorem 2.1, (5.9), and (5.10))

$$\begin{aligned}
J_2 S_-^* J_- S_- &= J_2 V D_{T_{11}} |I + T_{11}|^{[-1/2]} J_- |I + T_{11}|^{[-1/2]} D_{T_{11}} V^* \\
&= J_2 V D_{T_{11}} (J_1 (I + T_{11}))^{[-1]} D_{T_{11}} V^* \\
&= J_2 V D_{T_{11}} D_{T_{11}}^{[-1]} (I + L_{T_{11}}^* J_1)^{[-1]} D_{T_{11}} J_1 D_{T_{11}} V^* \\
&= J_2 V (I + L_{T_{11}}^* J_1)^{[-1]} (J_{11} - L_{T_{11}}^* J_{T_{11}^*} L_{T_{11}}) V^* \\
&= J_2 V (I + L_{T_{11}}^* J_1)^{[-1]} (J_{11} - (L_{T_{11}}^* J_1)^2 J_{11}) V^* \\
&= J_2 V (I + L_{T_{11}}^* J_1)^{[-1]} (I + L_{T_{11}}^* J_1)(I - L_{T_{11}}^* J_1) J_{11} V^* \\
&= J_2 V (I - L_{T_{11}}^* J_1) J_{11} V^*,
\end{aligned}$$

where the third equality follows from (5.1) and the fourth from (5.2).

Similarly,

$$\begin{aligned}
J_2 S_+^* J_+ S_+ &= J_2 V D_{T_{11}} |I - T_{11}|^{[-1/2]} J_+ |I - T_{11}|^{[-1/2]} D_{T_{11}} V^* \\
&= J_2 V D_{T_{11}} (J_1 (I - T_{11}))^{[-1]} D_{T_{11}} V^* \\
&= J_2 V D_{T_{11}} D_{T_{11}}^{[-1]} (I - L_{T_{11}}^* J_1)^{[-1]} D_{T_{11}} J_1 D_{T_{11}} V^* \\
&= J_2 V (I - L_{T_{11}}^* J_1)^{[-1]} (J_{11} - L_{T_{11}}^* J_{T_{11}^*} L_{T_{11}}) V^* \\
&= J_2 V (I - L_{T_{11}}^* J_1)^{[-1]} (J_{11} - (L_{T_{11}}^* J_1)^2 J_{11}) V^* \\
&= J_2 V (I - L_{T_{11}}^* J_1)^{[-1]} (I - L_{T_{11}}^* J_1)(I + L_{T_{11}}^* J_1) J_{11} V^* \\
&= J_2 V (I + L_{T_{11}}^* J_1) J_{11} V^*,
\end{aligned}$$

which implies the representations for T_m and T_M in (5.16). Clearly, T_m and T_M are self-adjoint extensions of T_1, which satisfy the equalities

$$\nu_-[I + T_m] = \kappa_-, \quad \nu_-[I - T_M] = \kappa_+.$$

Moreover, it follows from (5.16) that

$$T_M - T_m = \begin{pmatrix} 0 & 0 \\ 0 & 2(I - J_2 V J_{11} V^*) \end{pmatrix}. \tag{5.20}$$

Now, assumption (5.15) will be used again. Since $\nu_-[I - T_1^{[*]} T_1] = \nu_-[I - T_{11}^2] - \nu_-(J_2)$ and $T_{21} = J_2 V D_{T_{11}}$, it follows from Theorem 3.1 that $V^* \in [\mathfrak{H}_2, \mathfrak{D}_{T_{11}}]$ is J-contractive: $J_2 - V J_{11} V^* \geq 0$. Therefore, (5.20) shows that $T_M \geq_{J_2} T_m$ and $I + T_M \geq_{J_2} I + T_m$. Hence, in addition to $I + T_m$, $I + T_M$ is also a solution to the problem A_+^0. In particular, $\nu_-[I + T_M] = \kappa_- = \nu_-[I + T_m]$. Similarly, $I - T_M \leq_{J_2} I - T_m$, which implies that $I - T_m$ is also a solution to the problem A_-^0. In particular, $\nu_-[I - T_m] = \kappa_+ = \nu_-[I - T_M]$. Now, by applying Lemma 5.4, we get

$$\nu_-[I - T_m^2] = \kappa - \nu_-(J_2),$$

$$\nu_-[I - T_M^2] = \kappa - \nu_-(J_2).$$

Therefore, $T_m, T_M \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$, which proves, in particular, that condition (5.15) is sufficient for the solvability of the completion problem (5.13).

(iv) Observe that $T \in \operatorname{Ext}_{T_1,\kappa}(-1,1)_{J_2}$, iff $T = T^{[*]} \supset T_1$ and $\nu_-[I \pm T] = \kappa_\mp$. By Theorem 2.1, this is equivalent to

$$J_2 S_-^* J_- S_- - I \leq_{J_2} T_{22} \leq_{J_2} I - J_2 S_+^* J_+ S_+. \tag{5.21}$$

Inequalities (5.21) are equivalent to (5.17).

(v) Relations (5.18) follow from (5.19) and (5.16). □

Remark 5.1. This result coincides with the main result of [16] in the case of a contraction operator T_1 and with the result of [7, Theorem 5] in the case of a "quasicontraction" operator T_1 with finite negative index.

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