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THE WHISHART–PICKRELL DISTRIBUTIONS AND CLOSURES OF GROUP ACTIONS

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Consider probability distributions on the space of infinite Hermitian matrices Herm(∞) *invariant with respect to the unitary group* U(∞). *We describe the closure of* U(∞) *in the semigroup of spreading maps (polymorphisms) of* Herm(∞)*; this closure is a semigroup isomorphic to the semigroup of all contractive operators. Bibliography:* 11 *titles.*

1. The assertion

1.1. Notation. Denote by Herm_∞ the space of all infinite Hermitian matrices. By Herm^0_∞ we denote the space of all infinite Hermitian matrices with finitely many nonzero matrix elements.

By $\mathrm{U}(\infty)$ we denote the group of infinite unitary matrices g such that $g-1$ has only finitely many nonzero matrix elements. This group acts on Herm_∞ by conjugations,

$$U: X \mapsto U^{-1}XU. \tag{1.1}$$

By $\overline{\mathrm{U}}(\infty)$ we denote the full unitary group in ℓ_2, equiped with the weak operator topology. By $\mathcal{B}(\infty)$ we denote the semigroup of all linear operators in ℓ_2 with norm ≤ 1, also equipped with the weak operator topology.

1.2. The Whishart–Pickrell distributions. For any probability measure μ on Herm_∞, we define its characteristic function on Herm^0_∞ by the formula

$$\chi(\mu|A) = \int\limits_{\mathrm{Herm}_\infty} e^{i\,\mathrm{tr}\,AX}\,d\mu(X).$$

Obviously, this function uniquelly determines the measure.

The following theorem, in the spirit of de Finetti's theorem, is due to Pickrell [11] (see also another proof with additional details in [9]).

Any $\mathrm{U}(\infty)$*-invariant measure on* Herm_∞ *can be uniquely decomposed into ergodic measures. An ergodic* $\mathrm{U}(\infty)$*-invariant measure has a characteristic function of the form*

$$\chi_{\gamma_1,\gamma_2,\lambda}(A) = e^{-\frac{\gamma_1}{2}\,\mathrm{tr}\,A^2 + i\gamma_2\,\mathrm{tr}\,A}\prod_{k=1}^{\infty}\Bigl(\det\frac{e^{-i\lambda_k A}}{1-i\lambda_k A}\Bigr), \tag{1.2}$$

where $\gamma_1 \ge 0$*,* γ_2*, and* λ_1*,* λ_2*, ... are real numbers and* $\sum \lambda_k^2 < \infty$*.*

Denote this measure by $\mu_{\gamma_1,\gamma_2,\lambda}$.

The characteristic function is a product, hence the corresponding measure can be decomposed into an (infinite) convolution. The factor $e^{-\frac{\gamma_1}{2}\,\mathrm{tr}\,A^2+i\gamma_2\,\mathrm{tr}\,A}$ corresponds to a Gaussian measure on Herm_∞. Let us explain the meaning of the factors $\det(1-i\lambda_k A)^{-1}$. Consider the complex plane $\mathbb{C}$ equipped with the Gaussian measure $\pi^{-1}e^{-|u|^2}\,d\,\mathrm{Re}\,u\,d\,\mathrm{Im}\,u$. Consider the space $\mathbb{C}^\infty$ equipped with a product measure ν. Consider the map $\mathbb{C}^\infty \to \mathrm{Herm}_\infty$ given by

$$u \mapsto \lambda_k u^* u.$$

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Consider the image μ of ν under this map. The characteristic function of μ is

$$\int\limits_{\mathrm{Herm}_\infty} e^{i \operatorname{tr} AX} d\mu(X) = \int\limits_{\mathbb{C}^\infty} e^{i \operatorname{tr} \lambda_k A u^* u}\, d\nu(u) = \int\limits_{\mathbb{C}^\infty} e^{i\lambda_k u A u^*}\, d\nu(u) = \det(1 - i\lambda_k A)^{-1}.$$

If $\sum |\lambda_k| < \infty$, then we can transform the expression (1.2) into

$$\chi_{\gamma_1,\gamma_2,\lambda}(A) = e^{-\frac{\gamma_1}{2} \operatorname{tr} A^2 + i(\gamma_2 - \sum \lambda_k) \operatorname{tr} A} \prod_{k=1}^{\infty} \det(1 - i\lambda_k A)^{-1}.$$

If the series $\sum \lambda_k$ diverges, we obtain a divergent series in the exponent and a divergent product.

1.3. Polymorphisms. See [3, 10], [5, Sec. VIII.4]. Consider a Lebesgue measure space M with a nonatomic probability measure μ. Denote by $\mathrm{Ams}(M)$ the group of measure-preserving bijective a.e. transformations of M.

A *polymorphism* of M is a measure $\varkappa$ on $M \times M$ whose pushforwards to M with respect to both projections $M \times M \to M$ coincide with μ. Denote by $\mathrm{Pol}(M)$ the set of all polymorphisms of M. We say that a sequence $\pi_j \in \mathrm{Pol}(M)$ converges to π if for any measurable sets $A, B \subset M$ we have the convergence $\pi_j(A \times B) \to \pi(A \times B)$. The space $\mathrm{Pol}(M)$ is compact, and the group $\mathrm{Ams}(M)$ is dense in $\mathrm{Pol}(M)$.

Polymorphisms can be regarded as maps spreading points of M to probability measures on M. Namely, for $\pi \in \mathrm{Pol}(M)$ consider the system of conditional measures π_m on the sets $m \times M \subset M \times M$, where m ranges over M. We declare that the "map" π sends each point m to the measure π_m. If π, $\varkappa \in \mathrm{Pol}(M)$, then the product $\rho = \varkappa \circ \pi$ is defined from the condition

$$\rho_m = \int\limits_M \varkappa_n \, d\pi_m(n).$$

We obtain a semigroup with a separately continuous product.

For any $g \in \mathrm{Ams}(M)$, consider the map from M to $M \times M$ defined by $m \mapsto (m, g(m))$ and take the pushforward of the measure μ. Thus we obtain a polymorphism supported by the graph of g. The group $\mathrm{Ams}(M)$ is dense in $\mathrm{Pol}(M)$.

A *Markov operator* R in $L^2(M)$ is a bounded operator satisfying the following properties:

- for any function $f \geq 0$, we have $Rf \geq 0$;
- $R \cdot 1 = 1$, $R^* \cdot 1 = 1$.

Recall that, automatically, $\|R\| = 1$. There is a one-to-one correspondence between the set of Markov operators $\mathrm{Mar}(M)$ and $\mathrm{Pol}(M)$. Namely, let R be a Markov operator; then we define a polymorphism π by the formula

$$\pi(A \times B) = \langle R I_A, I_B \rangle_{L^2(M)},$$

where A, $B \subset M$ are measurable sets and I_A, I_B are their indicator functions. The weak convergence in $\mathrm{Mar}(M)$ corresponds to the convergence in $\mathrm{Pol}(M)$, the product of Markov operators corresponds to the product of polymorphisms.

1.4. Closures of actions. Let a group G act on M by measure-preserving transformations. That is, we have a homomorphism $G \to \mathrm{Ams}(M)$; for definiteness, assume that this is an embedding. Then the closure of G in $\mathrm{Pol}(M)$ is a compact semigroup $\Delta \supset G$. The description of the closure is not very interesting for connected Lie groups (for instance, for semisimple linear Lie groups, we get a one-point compactification, this follows from [2, Theorem 5.3]). However, the description of such a closure is interesting for infinite-dimensional groups.

The first result of this type was obtained by Nelson [4] in 1973. He showed that the standard action of the infinite-dimensional orthogonal group on the space with a Gaussian measure admits an extension to an action of the semigroup of all contractive operators by polymorphisms and obtained formulas for the corresponding measures. Now, a large collection of actions of infinite-dimensional groups on measure spaces is known, however, the closures are found only in few cases, see [6, 7]. In this paper, we describe a new (relatively simple) example.

1.5. The assertion. For any polymorphism π on Herm_∞, we define a characteristic function on $\mathrm{Herm}^0_\infty \times \mathrm{Herm}^0_\infty$ by the formula

$$F(\pi|A,B) := \int\limits_{\mathrm{Herm}_\infty \times \mathrm{Herm}_\infty} e^{i \operatorname{tr} AX+BY}\, d\pi(X,Y).$$

Our purpose is the following assertion.

Theorem 1.1. *For any ergodic measure $\mu_{\gamma_1,\gamma_2,\lambda}$ on Herm_∞, the action (1.1) of $\mathrm{U}(\infty)$ admits a continuous extension to an action of the semigroup $\mathcal{B}(\infty)$ by polymorphisms of M. The closure of $\mathrm{U}(\infty)$ in $\mathrm{Pol}(\mathrm{Herm}_\infty)$ coincides with the image of $\mathcal{B}(\infty)$. If $S \in \mathcal{B}(\infty)$, then the characteristic function of the corresponding polymorphism π_S is given by*

$$F(\pi_S|A,B) = \exp\Big\{-\frac{\gamma_1}{2}\operatorname{tr}\Big[\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\begin{pmatrix} 1 & S \\ S^* & 1\end{pmatrix}\Big]^2 + i\gamma_2(\operatorname{tr} A + \operatorname{tr} B)\Big\} \times \prod_k \frac{e^{-i\lambda_k(\operatorname{tr} A + \operatorname{tr} B)}}{\det\Big[1 - i\lambda_k\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\begin{pmatrix} 1 & S \\ S^* & 1\end{pmatrix}\Big]}. \tag{1.3}$$

Since the characteristic function is a product, the measure π_S can be decomposed into a convolution of measures. The exponential factor corresponds to a Gaussian measure; let us explain the meaning of the other factors in the product. Consider a measure ν_S on $\mathbb{C}^\infty \times \mathbb{C}^\infty$ defined in the following way in terms of its characteristic function:

$$\int\limits_{\mathbb{C}^\infty\times\mathbb{C}^\infty} e^{i\operatorname{Re} u\bar z_1 + i \operatorname{Re} v\bar z_2} d\nu_S(z_1,z_2) := \exp\Big\{-\frac{1}{2}\begin{pmatrix} u & v\end{pmatrix}\begin{pmatrix} 1 & S \\ S^* & 1\end{pmatrix}\begin{pmatrix} u^* \\ v^*\end{pmatrix}\Big\}.$$

Consider the map $\mathbb{C}^\infty \times \mathbb{C}^\infty \to \mathrm{Herm}_\infty \times \mathrm{Herm}_\infty$ given by

$$(u,v) \mapsto (\lambda u^* u, \lambda v^* v).$$

Then the image of ν_S under this map is the measure whose characteristic function is

$$\det\Big[1 - i\lambda_k\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\begin{pmatrix} 1 & S \\ S^* & 1\end{pmatrix}\Big]^{-1}.$$

2. The proof

2.1. A priori remarks

Theorem 2.1. *Let the full unitary group $\overline{\mathrm{U}}(\infty)$ act by measure-preserving transformations*[1] *on a Lebesgue space M with a probability measure. Then this action admits a continuous extension to an action of $\mathcal{B}(\infty)$ by polymorphisms of M. The closure of $\overline{\mathrm{U}}(\infty)$ in $\mathrm{Pol}(M)$ coincides with the image of $\mathcal{B}(\infty)$.*

[1] Such an action cannot be pointwise; the transformations are defined only almost everywhere, and the equalities $g_1(g_2 m) = (g_1 g_2)m$ are fulfilled only a.e., see [1].

Proof. Let ρ be a unitary representation of $\overline{\mathrm{U}}(\infty)$ in a Hilbert space H. The closure of the group $\rho(\overline{\mathrm{U}}(\infty))$ in the group of all bounded operators in H with respect to the weak operator topology is a semigroup isomorphic to $\mathcal{B}(\infty)$ (this follows from the Kirillov–Olshanski classification of unitary representations of $\overline{\mathrm{U}}(\infty)$, see [8, Theorem 1.2]).

We apply this to the action of $\overline{\mathrm{U}}(\infty)$ in $L^2(M)$. The group $\overline{\mathrm{U}}(\infty)$ acts by Markov operators, and weak limits of Markov operators are Markov operators. Therefore, the semigroup $\mathcal{B}(\infty)$ also acts by Markov operators. □

Lemma 2.2. *For any* $\mathrm{U}(\infty)$*-ergodic measure on* Herm_∞*, the action of* $\mathrm{U}(\infty)$ *admits a continuous extension to an action of the full unitary group* $\overline{\mathrm{U}}(\infty)$*.*

Proof. According to [9, Corollary 2.14], the representation of $\mathrm{U}(\infty)$ in the space $L^2(\mathrm{Herm}_\infty, \mu)$ has a continuous extension to $\overline{\mathrm{U}}(\infty)$. By continuity arguments, the group $\overline{\mathrm{U}}(\infty)$ acts by Markov operators $\rho(g)$. We have $\|\rho(g)\| \le 1$, $\|\rho(g)^{-1}\| \le 1$. Therefore, $\rho(g)$ is a unitary operator. Hence it corresponds to a measure-preserving transformation. □

2.2. The calculation. Consider a measure μ with a characteristic function of the form (1.2). For $g \in \mathrm{U}(\infty)$, consider the corresponding polymorphism π_g of Herm_∞. The characteristic function of π_g is

$$F(\pi_g|A,B) = \exp\Big\{-\frac{\gamma_1}{2}\operatorname{tr}(A+UBU^{-1})^2 + i\gamma_2(\operatorname{tr}A + \operatorname{tr}UBU^{-1})\Big\} \times \prod_{k=1}^{\infty} \frac{e^{-i\lambda_k \operatorname{tr}(A+UBU^{-1})}}{\det\big[1 - i\lambda_k(A+UBU^{-1})\big]}. \tag{2.4}$$

By Theorem 2.1, for any $R \in \mathcal{B}(\infty)$ we have a polymorphism π_R of the space $(\mathrm{Herm}_\infty, \mu)$. If a sequence $R_j \in \mathcal{B}(\infty)$ weakly converges to R, then we have the weak convergence of the corresponding polymorphisms, $\pi_{R_j} \to \pi_R$. It is equivalent to the pointwise convergence of the characteristic functions. Now, for $R \in \mathcal{B}(\infty)$ and a sequence $g_j \in \mathrm{U}(\infty)$ weakly converging to R, we can find $F(\pi_R|A,B)$ as the pointwise limit of $F(\pi_{g_j}|A,B)$.

Let S be a finitary operator with norm ≤ 1, and let S be represented as a $(\alpha+\infty)$-block matrix of the form

$$S = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we can build u as a block into a unitary $(\alpha+\alpha)$-block matrix $\begin{pmatrix} u & v \\ w & z \end{pmatrix}$ (see, e.g., [5, Theorem VIII.3.2]). Let $U = U_m$ be a block unitary $(\alpha+m+m+\alpha+\infty)$-matrix of the form

$$U_m = \begin{pmatrix} u & 0 & 0 & v & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ w & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly, we have the weak convergence $U_m \to S$, and we want to trace the convergence of the characteristic functions $F(\pi_{U_m}|A,B)$. In fact, we will show that this sequence is eventually constant for any fixed A, B.

Fix A, $B \in \mathrm{Herm}^0_\infty$. Assume that actually A, $B \in \mathrm{Herm}_{\alpha+\beta}$. Let m be sufficiently large (in fact, we need that $m \ge \beta$). We represent the matrices A, B as block matrices of size

$\alpha+\beta+(m-\beta)+\beta+(m-\beta)+\infty$:

$$A=\begin{pmatrix} a_{11} & a_{12} & 0 & \dots & 0\\ a_{21} & a_{22} & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix}, \qquad B=\begin{pmatrix} b_{11} & b_{12} & 0 & \dots & 0\\ b_{21} & b_{22} & 0 & \dots & 0\\ 0 & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \dots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix}.$$

We want to evaluate

$$\det\bigl[1-i\lambda_k(A+U_mBU_m^{-1})\bigr] \qquad \text{and} \qquad \operatorname{tr}(A+U_mBU_m^{-1})^2.$$

A straightforward calculation gives

$$A+U_mBU_m^{-1}=\begin{pmatrix} a_{11}+ub_{11}u^* & a_{12} & 0 & ub_{12} & ub_{11}w^* & 0 & 0\\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ b_{21}u^* & 0 & 0 & b_{22} & b_{21}w^* & 0 & 0\\ wb_{11}u^* & 0 & 0 & wb_{12} & wb_{11}w^* & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

Clearly, we can remove zero columns and zero rows from this matrix. Formally,

$$\det\bigl(1-i\lambda_k(A+U_mBU_m^{-1})\bigr)=\det(1-i\lambda_kH),$$

where

$$H=\begin{pmatrix} a_{11}+ub_{11}u^* & a_{12} & ub_{12} & ub_{11}w^*\\ a_{21} & a_{22} & 0 & 0\\ b_{21}u^* & 0 & b_{22} & b_{21}w^*\\ wb_{11}u^* & 0 & wb_{12} & wb_{11}w^*\end{pmatrix}.$$

Denote by Δ the block diagonal matrix with blocks 1, 1, 1, w. Represent H as $H=\Delta Z$ (where the expression for Z is clear). We apply the formula

$$\det(1-i\lambda_k\Delta Z)=\det(1-i\lambda_kZ\Delta).$$

Denote $H':=Z\Delta$,

$$H'=\begin{pmatrix} a_{11}+ub_{11}u^* & a_{12} & ub_{12} & ub_{11}w^*w\\ a_{21} & a_{22} & 0 & 0\\ b_{21}u^* & 0 & b_{22} & b_{21}w^*w\\ b_{11}u^* & 0 & b_{12} & b_{11}w^*w\end{pmatrix}. \tag{2.5}$$

Since the matrix $\begin{pmatrix} u & v\\ w & z\end{pmatrix}$ is unitary, we have $w^*w=1-uu^*$. We substitute this into the 4th column of (2.5) and, keeping the result in mind, continue our calculations. Denote

$$T=\begin{pmatrix} 1 & 0 & 0 & u\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Then

$$H'=\begin{pmatrix} a_{11}+ub_{11}u^* & a_{12} & ub_{12} & a_{11}u+ub_{11}\\ a_{21} & a_{22} & 0 & a_{21}u\\ b_{21}u^* & 0 & b_{22} & b_{21}\\ b_{11}u^* & 0 & b_{12} & b_{11}\end{pmatrix}T^{-1}=T\begin{pmatrix} a_{11} & a_{12} & 0 & a_{11}u\\ a_{21} & a_{22} & 0 & a_{21}u\\ b_{21}u^* & 0 & b_{22} & b_{21}\\ b_{11}u^* & 0 & b_{12} & b_{11}\end{pmatrix}T^{-1}.$$

Hence

$$\det\big(1 - i\lambda_k H'\big) = \det\left[1 - i\lambda_k \begin{pmatrix} a_{11} & a_{12} & 0 & a_{11}u \\ a_{21} & a_{22} & 0 & a_{21}u \\ b_{21}u^* & 0 & b_{22} & b_{21} \\ b_{11}u^* & 0 & b_{12} & b_{11} \end{pmatrix}\right]$$

$$= \det\left[1 - i\lambda \begin{pmatrix} a_{11} & a_{12} & a_{11}u & 0 \\ a_{21} & a_{22} & a_{21}u & 0 \\ b_{11}u^* & 0 & b_{11} & b_{12} \\ b_{21}u^* & 0 & b_{21} & b_{22} \end{pmatrix}\right] \det\left[1 - i\lambda \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & S \\ S^* & 1 \end{pmatrix}\right], \quad (2.6)$$

and we get the desired expression.

Next,

$$\operatorname{tr}(A + U_m B U_m^{-1})^2 = \operatorname{tr} A^2 + \operatorname{tr} B^2 + 2 \operatorname{tr} A U_m B U_m^{-1}.$$

Multiplying the matrices, we observe that $AU_mBU_m^{-1}$ has a unique nonzero diagonal block, $a_{11}ub_{11}u^*$. Thus,

$$\operatorname{tr} AU_mBU_m^{-1} = \operatorname{tr} a_{11}u\, b_{11}u^* = \operatorname{tr} ASBS^*,$$

and this implies that

$$\operatorname{tr}(A + U_m B U_m^{-1})^2 = \operatorname{tr}\left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & S \\ S^* & 1 \end{pmatrix}\right]^2.$$

Thus, for $m \geq \beta$ the value of $F(\pi_{U_m}|A, B)$ is given by formula (1.3).

So, the theorem holds for finitary matrices S. Consider an arbitrary operator $S \in \mathcal{B}(\infty)$. Denote by $S[m]$ the left upper corner of S of size m. Denote

$$S_m := \begin{pmatrix} S[m] & 0 \\ 0 & 0 \end{pmatrix}.$$

We have the weak convergence $S_m \to S$. On the other hand, we have the pointwise convergence

$$F(\pi_{S_m}|A, B) \to F(\pi_S|A, B)$$

(in fact, this sequence is eventually constant for any fixed A, B).

This completes the proof of the theorem.

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