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ON THE CALCULATION OF THE MORSE INDEX AND THE EXTENSION OF RAY FORMULAS BEYOND CAUSTICS

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Being the simplest and illustrative from the physical point of view, the ray method is extensively used for computations of short wave fields of a different physical nature: acoustic, electrodynamic, and elastodynamic. However it is not applicable in neighborhoods of caustics, where ray amplitudes get singular. Although caustics may appear in quantities in complex inhomogeneous media, they have zero measure, and therefore there are subdomains free of caustics in inhomogeneous media where ray formulas can be used for computations of wave fields. To this end it is necessary to calculate the phase jumps caused by the transition of rays through caustics. Mathematically, we must calculate the Morse index for a ray, i.e., the number of focal points (counting their multiplicity) on the ray between the source and the observation point. In the article, this problem is considered and a complete solution to it is given in the case of two space variables. Namely, a complex-valued function of the arc length along a ray is constructed and the increment of its argument between the source and the observation point, computed modulo 2π, gives the Morse index for that ray in the two cases where the field of rays is produced by a point source or is generated by an initially given wave front. Bibliography: 8 titles.

1. Introduction

The ray method, which remains the simplest and illustrative from the physical point of view, is used widely in the problems related to the short wave propagation in acoustics, electrodynamics, and the theory of elasticity. Among them there are also quite laborious migration problems, i.e., problems of reconstruction of the reflecting boundaries in inhomogeneous media according to given seismograms on the seismic boundary. However, the ray method is not applicable in neighborhoods of the caustics, because the geometric spreading of a ray tube vanishes on them, which leads to the amplitude singularity in ray formulas on caustics. Although caustics appear in complicated inhomogeneous media in large quantities, they have zero measure, i.e., there are subdomains in an inhomogeneous medium free of caustics. We can use the ray formulas in these subdomains if we correctly calculate the phase jumps in these formulas arisen when a given ray goes through caustics on its way from the wave field source to the observation point.

This article presents a method of calculating phase jumps in ray formulas that are caused by vanishing of the geometric spreading on the caustics. So far we have considered the 2D case, because at present it remains dominating in the migration problems. On the one hand, the proposed method is based in fact on the calculation of the Morse index (see [1], as well as [2]), i.e., of the number of focal points on a ray between the source and observation points. On the other hand, the computations rest upon the method of calculating the geometric spreading on the central ray of a ray tube, suggested for the first time by the author (see [3]), and make use of the techniques of constructing the Gaussian beams. As a result, we were able to find a complex-valued function $Q(s)$ of the arc length s of a ray such that it does not vanish for any s; the argument $\arg Q(s)$ of this function is a monotone function of s, and focal points on this ray exist precisely at the points where either $\arg Q(s) = \pi m,\ m = 0, 1, 2, \ldots$ (in this case the

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ray field is generated by a point source) or $\arg Q(s) = \pi/2 + \pi m,\ m = 0, 1, 2, \ldots$ (in this case the ray field is given initially by a wave front $\tau(x, y) = \text{const}$, where $\tau(x, y)$ is the eikonal).

It is well known that there exist methods of examination and computation of short waves, which have no problems on caustics. For example, the method of Maslov canonical operator ([2]) and the Gaussian beam summation method ([4]) are such methods. In the latter case, we need not calculate phase jumps on caustics, because they just do not appear in this approach. The numerical implementation of both methods, however, proves to be much more complicated in comparison with the ray method (see [5] and [6, 7], for example).

2. Geometric spreading of a ray tube and equations in variations

The starting point of this article is the Fermat functional $\Phi = \int C^{-1}\sqrt{d^2S}$, where C is a wave propagation velocity and d^2S is the square of a length element. It appears naturally in the problems of propagation of short waves of a different physical nature in inhomogeneous isotropic media. The extremals of the Fermat functional are called *rays*. Further we shall consider it in the 2D case, the velocity $C(x, y)$ is regarded as a function smooth enough, and in place of Cartesian coordinates we shall use curvilinear ones associated with the central ray of a ray tube.

We assume that the central ray is given in the form $\vec{r} = \vec{r}_0(s)$, where $\vec{r}$ is a radius vector on the plane x, y, and s is the arc length of this ray. In its neighborhood, we introduce coordinates s and q in accordance with the formula $\vec{r} = \vec{r}_0(s) + q\vec{e}(s)$, where $\vec{e}(s)$ represents the orthogonal vector of unit length to the central ray at every point s. Moreover, obviously, $\frac{d}{ds}\vec{e}(s) = æ(s)\vec{t}_0(s)$, where $\vec{t}_0(s)$ is the unit vector tangent to the central ray $\vec{r}_0(s)$ and $æ(s)$ is the curvature of this ray. The coordinate system s, q is regular in a neighborhood of $\vec{r}_0(s)$ and is suitable for describing rays close to $\vec{r}_0(s)$, i.e., the ones that constitute a ray tube.

The square of the length element $dS^2 = (d\vec{r}, d\vec{r})$ takes the form $dS^2 = h^2 ds^2 + dq^2$, where $h = h(s, q) = 1 + æ(s)q$ is the Lamé coefficient of the s, q coordinate system, and the Fermat functional is described by the formula

$$\Phi = \int \frac{\sqrt{h^2 + \dot{q}^2}}{C(s, q)} ds, \quad \dot{q} = \frac{d}{ds}q, \tag{2.1}$$

where the velocity of wave propagation C is assumed to be given in the coordinates s, q. We point out the relationship between the curvature $æ(s)$ of the central ray $\vec{r}_0(s)$ and the velocity:

$$æ(s) = \left(C^{-1}(s, q) \frac{\partial C(s, q)}{\partial q} \right)\bigg|_{q=0},$$

which stems from the fact that this ray is an extremal of the functional (2.1), on the one hand, and its equation in the coordinates considered has the form $q(s) \equiv 0$, on the other.

Further it is more convenient to use the variation principle in Hamilton form. To this end, we introduce the pulse p that corresponds to the coordinate q,

$$p = C^{-1} \frac{\partial}{\partial \dot{q}} \sqrt{h^2 + (\dot{q})^2}, \tag{2.2}$$

and denote the Hamilton function by $H = H(s, q, p)$. Then the Euler equations for the functional (2.1) in Hamilton form consist of two equations

$$\frac{d}{ds}q = \frac{\partial H}{\partial p}, \quad \frac{d}{ds}p = -\frac{\partial H}{\partial q}; \quad H(s, q, p) = -\frac{h}{C}\sqrt{1 - C^2p^2}. \tag{2.3}$$

In view of formula (2.2), the equations of the central ray $\vec{r}_0(s)$ take the form $q(s) \equiv 0, p(s) \equiv 0$.

The rays forming a narrow ray tube in a neighborhood of the central ray $\vec{r}_0(s)$ can be described in the first approximation (i.e., for small q and p) as solutions of the linearized system of equations (2.3). To this end it is sufficient to expand H in (2.3) in powers of q and p and to leave only quadratic terms. As a result, we obtain a linear system of equations, which is called *equations in variations*. A more accurate derivation of it consists of the following. Denote the parameter fixing a ray of the ray tube by α, and let $\alpha = \alpha_0$ correspond to the central ray $\vec{r}_0(s)$. We suppose that a family of rays $q(s,\alpha), p(s,\alpha)$ is built that depends smoothly on the ray parameter α. Introduce the functions

$$Q(s) = \left.\frac{\partial q(s,\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0}, \quad P(s) = \left.\frac{\partial p(s,\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_0}. \tag{2.4}$$

Differentiating Eqs. (2.3) with respect to α and setting $\alpha = \alpha_0$ (and therefore $q = p = 0$), we obtain the desired linear system of equations in variations for functions (2.4):

$$\frac{d}{ds}Q = C_0(s)P, \quad \frac{d}{ds}P = -C_0^{-2}(s)\left.\frac{\partial^2 C(s,q)}{\partial q^2}\right|_{q=0} Q, \tag{2.5}$$

where $C_0(s) \equiv C(s,0)$ stands for the velocity calculated on the central ray of a ray tube.

We shall use the result announced in [3] for the first time and proved in detail in book [8]. The functional determinant $\frac{D(x,y)}{D(s,\alpha)}$, considered as a function of the arc length on the central ray $\alpha = \alpha_0$ of the ray tube, is a solution of Eqs. (2.5). The initial conditions $Q(0), P(0)$ depend on the way of specifying the ray field. Here two cases are possible: (1) the rays are given by a point source, (2) the rays are given by the position of the wave front $\tau(x,y) = \text{const}$. Thus, the following formula for the geometric spreading J on the central ray holds:

$$J = \left|\frac{D(x,y)}{D(s,\alpha)}\right| = |Q(s)|. \tag{2.6}$$

Here the value of the parameter $\alpha = \alpha_0$ is omitted, because these relations, as well as the previous ones, take place for any ray since each of them can be regarded as the central ray for a certain ray tube.

We now turn to derivation of initial conditions for the functions $Q(s)$ and $P(s)$ necessary for calculating the geometric spreading in the two cases where the ray field is generated (1) by a point source and (2) by an initial wave front $\tau(x,y) = \text{const}$. To do this, we need a more convenient formula for the pulse than (2.2):

$$p(s,\alpha) = \frac{\left(\vec{t}(s,\alpha), \vec{e}(s)\right)}{C\left(s, q(s,\alpha)\right)}, \tag{2.7}$$

where $\vec{t}(s,\alpha)$ stands for the unit tangent vector to the ray from the ray tube, which is fixed by the value of the parameter α.

To prove relation (2.7), we make use of the equation $\vec{r}(s,\alpha) = \vec{r}_0(s) + q(s,\alpha)\vec{e}(s)$ for the rays of the ray tube. Differentiating it with respect to s, we get

$$\frac{d}{ds}\vec{r}(s,\alpha) = \vec{t}_0(s)\left(1 + æ(s)q(s,\alpha)\right) + \dot{q}(s,\alpha)\vec{e}(s) = \vec{t}_0(s)h(s,\alpha) + \dot{q}(s,\alpha)\vec{e}(s).$$

Hence, it follows that

$$\vec{t}(s,\alpha) = \frac{\vec{t}_0(s)h + \dot{q}\vec{e}(s)}{\sqrt{h^2 + \dot{q}^2}}, \tag{2.8}$$

and comparing (2.8) with definition (2.2) of the pulse p, we arrive at (2.7).

(1) A point source.

We suppose that the source is situated at a point $\vec{r}_0(s)$ and the arc length s is measured from the source. The parameter α represents the polar angle under which a ray issues out of the source. The rays form a central field in a neighborhood of the source, and it is obvious that $q(0,\alpha)=0$ and $p(0,\alpha)=C^{-1}(0,0)\left(\vec{t}(0,\alpha),\vec{e}(0)\right)$ for all angles α. Using relations (2.4), we obtain the following initial data $Q_{(0)}^{(ps)}$ and $P_{(0)}^{(ps)}$ to calculate the geometric spreading on a given ray:

$$Q_{(0)}^{(ps)}=0;\quad P_{(0)}^{(ps)}=C^{-1}(0,0)\left(\frac{\partial\vec{t}(0,\alpha)}{\partial\alpha},\vec{e}(0)\right)\Big|_{\alpha_0}. \tag{2.9}$$

The superscript ps (point source) is introduced here to denote the fact that the ray field is generated by a point source. Because of the choice of the orientation of the normal $\vec{e}(0)$ to the ray, we may assume that $\left(\frac{\partial\vec{t}(0,\alpha)}{\partial\alpha},\vec{e}(0)\right)=1$.

(2) A wave front.

Assume that the initially given wave front $\tau(x,y)=\text{const}$ is a curve smooth enough. In this case the ray field is formed by the multitude of rays leaving every point of the front in the direction of the normal to it; the arc length s is counted from the exit point. Now we consider a neighborhood of the exit point $\vec{r}_0(0)$ of the central ray $\vec{r}_0(s)$.

By the ray parameter α we mean the arc length of the front. The vector $\vec{e}(0)$ coincides with the tangent and $\vec{t}_0(0)$ coincides with the unit vector of the normal $\vec{n}(\alpha_0)$ to the front at the point out of which the central ray issues. The equation of the front near $\vec{r}_0(0)$ has the form $n\simeq\frac{1}{2}K_0q^2$ in the first approximation. Here n is the length along the normal $\vec{n}(\alpha_0)$, and K_0 is the front curvature at this point. Taking into account the fact that the length along the tangent to the curve and the arc length of the curve in a neighborhood of the point of contact differ by a quantity of the second order of smallness with respect to the distance to the point of contact, we arrive at the following relations valid in the first approximation:

$$n\simeq s,\quad q(s,\alpha)\simeq\alpha-\alpha_0,\quad s\simeq\frac{1}{2}K_0(\alpha-\alpha_0)^2, \tag{2.10}$$

which are sufficient to calculate the derivatives on the right-hand sides of (2.4). The equation for the pulse $p(s,\alpha)$ takes in this case the form

$$p(s,\alpha)=\frac{(\vec{n}(\alpha),\vec{e}(s))}{C\left(s,q(s,\alpha)\right)}. \tag{2.11}$$

From formulas (2.10), (2.11) we get the following desired initial data $Q_{(0)}^{(wf)}$ and $P_{(0)}^{(wf)}$ for the geometric spreading in the case where the ray field is generated by a wave front (wf):

$$Q_{(0)}^{(wf)}=1,\quad P_{(0)}^{(wf)}=-\frac{K_0}{C(0,0)}. \tag{2.12}$$

We note that the appearance of the minus is a consequence of the Frenet formula $\frac{d}{d\alpha}\vec{n}(\alpha)\big|_{\alpha_0}=-K_0\vec{e}(0)$. Here, $C(0,0)$ represents the velocity at the ray exit point $\vec{r}_0(0)$ if it is given in the coordinates s,q used in our constructions.

3. Complexification of the geometric spreading problem

Further we assume that the aforesaid constructions can be carried out for any ray from the initial ray field and we can calculate the geometric spreading along every ray as a function of its arc length s. Moreover, the geometric spreading remains to be a smooth function of s, provided that the continuity of the coefficients of system (2.5) holds, irrespective of whether or not the ray gets to a caustic and what the geometric structure of this caustic is. The same result takes place for the phase incursion along the ray, i.e., for the eikonal calculated on that ray as a function of s. Thus these two attributes of the ray method can be extended through any caustic uniquely and without problems. Problems arise when we turn to the ray method formula, because the geometric spreading occurs in the amplitude to the power $-\frac{1}{2}$ and vanishes at focal points.

Henceforth the following property of the equations in variations (2.5) will play an essential role.

Let $Q_1(s), P_1(s)$ and $Q_2(s), P_2(s)$ be any couple of solutions for these equations; then the following identity relative to the argument s takes place:

$$Q_1(s)P_2(s) - P_1(s)Q_2(s) = Q_1(0)P_2(0) - P_1(0)Q_2(0). \tag{3.1}$$

In order to prove this identity it is sufficient to differentiate the left-hand side with respect to s and to employ Eqs. (2.5). From this it follows that if the initial data for these solutions at $s = 0$ make for the right-hand side (3.1) nonzero, then none pair of functions $Q_j(s), P_j(s), j = 1, 2$, $Q_1(s), Q_2(s)$, and $P_1(s), P_2(s)$ vanishes simultaneously for any value of s.

To make formulas more compact, we denote solutions of system (2.5) by a column

$$X(s) = \begin{pmatrix} Q(s) \\ P(s) \end{pmatrix}. \tag{3.2}$$

Let $W(s)$ be the fundamental matrix of it, i.e., it is generated by two linearly independent real solutions and $W(0) = E$, where E is the identity matrix.

Now we can represent solutions of (2.5) for a point source $X^{(ps)}(s)$ and for a wave front $X^{(wf)}(s)$ in compact form:

$$X^{(ps)}(s) = W(s)X^{(ps)}(0), \quad X^{(wf)}(s) = W(s)X^{(wf)}(0) \tag{3.3}$$

by means of the initial data (2.9) and (2.12), respectively.

The complexification of the problem concerning the geometric spreading implies that, using two real solutions $X^{(wf)}(s)$ and $X^{(ps)}(s)$, we construct one complex solution $X(s)$ by the formula

$$X(s) = X^{(wf)}(s) + iX^{(ps)}(s), \tag{3.4}$$

where $X(s)$ contains functions $Q(s)$ and $P(s)$ (see relation (3.2)) without additional indices.

Since the right-hand side of (3.1) for the initial data $X^{(wf)}(0)$ and $X^{(ps)}(0)$ is nonzero, we arrive at the following result: both complex-valued functions $Q(s)$ and $P(s)$ do not vanish for any value of s, and therefore $\arg Q(s)$ is well defined.

Let us prove that for $\arg Q(s)$ the following formula holds:

$$\arg Q(s) = \frac{1}{C(0,0)} \int_0^s C(s,0) \frac{1}{|Q(s)|^2}\, ds, \tag{3.5}$$

whence it follows specifically that $\arg Q(s)$ is a monotone increasing function of s along the ray.

Proof. Introduce the additional notation $\Gamma(s) = P(s)Q^{-1}(s)$. The first equation of system (2.5) gives $\frac{d}{ds}Q(s) = C_0(s)P(s) = C_0(s)\Gamma(s)Q(s)$, whence it follows that

$$Q(s) = \exp\Bigl\{ \int_0^s C_0(s)\Gamma(s)\, ds \Bigr\},$$

because $Q(0) = 1$ (see relations (2.12) and (3.3)). This relation implies that $\arg Q(s) = \int_0^s C_0(s)\mathrm{Im}\,\Gamma(s)\, ds$. Now it remains only to calculate $\mathrm{Im}\,\Gamma(s)$. Making use of (2.9), (2.12), and (3.1), we obtain step by step

$$\operatorname{Im}\Gamma(s) = \frac{1}{2i}\left(\frac{P(s)}{Q(s)} - \frac{\bar{P}(s)}{\bar{Q}(s)}\right) = \frac{1}{2i}\frac{P(s)\bar{Q}(s) - Q(s)\bar{P}(s)}{|Q(s)|^2}$$
$$= \frac{1}{2i}\frac{P(0)\bar{Q}(0) - Q(0)\bar{P}(0)}{|Q(s)|^2} = \frac{1}{C(0,0)|Q(s)|^2},$$

which proves formula (3.5). □

A consequence of (3.2), (3.3), and (3.4) is the main result of this article:

$$Q^{(wf)}(s) = |Q(s)|\cos\arg Q(s), \; Q^{(ps)}(s) = |Q(s)|\sin\arg Q(s). \tag{3.6}$$

The following conclusions can be drawn from the above formulas:

(1) The focal points on a given ray (or caustic points) are precisely the values of s where $\arg Q(s) = \frac{\pi}{2} + \pi m,\ m = 0, 1, 2\dots$, if the ray field is generated by a wave front $\tau = \text{const}$, and where $\arg Q(s) = \pi m,\ m = 0, 1, 2\dots$, if the ray field is produced by a point source. We recall that the point source is a caustic point itself that corresponds to $m = 0$.

(2) In the 2D case, all the focal points are simple (not multiple). This means that whatever the caustic on a plane (i.e., of general position or singular, like, for example, a focus where all the rays intersect), the ray formula gains the multiplier $\exp(-i\frac{\pi}{2})$. We recall that exactly this multiplier appears on a simple smooth caustic where the wave field is described by the Airy functions.

(3) In order to calculate the Morse index between the source and the observation points, one need to calculate $\arg Q(s)$ modulo 2π. On every interval of length 2π, two focal points do exist. If the Morse index equals M, then an additional factor $\exp(-i\frac{\pi}{2}M)$ appears in ray formulas.

4. Conclusions

In what we have said above, it is assumed that the velocity of wave propagation in inhomogeneous media is a function smooth enough. However, the results obtained are easily generalized to the case where the medium contains smooth boundaries on which the velocity changes with a jump. Moreover, reflected and refracted rays appear on such boundaries and the geometric spreading can be constructed along these rays by the techniques described above.

At the points of reflection/refraction they match with the spreading on incident rays by means of the linearized Snell's law (see monograph [8, Chap. 5], where this is described in detail). Thus, the complex-valued function $Q(s)$ is extended along the reflected/refracted rays, preserving its own properties. However, an additional phase shift may appear on the medium boundary depending on the way of choosing the orientation of normal vectors $\vec{e}(s)$ on reflected/refracted rays.

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