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GENERAL ELASTIC SURFACE WAVES IN ANISOTROPIC LAYERED STRUCTURES

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A solution of homogeneous equations of elasticity equations, which describes surface waves and is based on the summation of plane waves, is presented. Bibliography: 15 *titles.*

1. Introduction

Until very recently, explicit solutions for surface waves in layered elastic structures have been essentially confined to plane waves. In the present century, novel exact solutions, which generalize plane waves and describe more complicated wave phenomena, have evolved. We briefly list the main approaches and results.

1.1. Constructions based on the superposition of surface plane waves. Simple solutions for surface waves with plane wave fronts but amplitudes linearly varying along the wave fronts have been found for the classical Rayleigh wave [1], for Love waves in isotropic layered structures [2], and for bulk waves in media with arbitrary anisotropy [3]. These results grew out of a surmise grounded on the analysis of results of the asymptotic ray theory (e.g., see [4]). Further, more general solutions were presented which have plane wave fronts and amplitudes polynomial with respect to lateral variables in the case of a homogeneous anisotropic half-space [5]. The approach applied in [3,5] employed the differentiation of the corresponding classical plane waves with respect to the wave vector $\boldsymbol{k}$ (or techniques, equivalent to this in its essence). More sophisticated solutions describing Gaussian (and not necessarily Gaussian) beams in layered structures (which were earlier known only as asymptotic approximate solutions [4,8]) have been found in [6,7]. In those papers, the standard plane waves were regarded as known solutions.

1.2. Reduction to a couple of functions that satisfy PDEs. We mention a line of investigation, in which the solution for a surface wave is expressed in terms of a couple of functions that satisfy certain partial derivative equations (e.g., see [9,10]).

1.3. The carrier equation. In Achenbach's papers it was shown with particular examples that the construction of a general solution for elastic media that possess an axial symmetry can be reduced to a certain scalar equation, which appeared to be the membrane equation. Achenbach called this equation the *carrier equation*. One of the Achenbach's examples was the Love wave in a transversely isotropic plate of finite width [12], and the other was the Rayleigh wave in a homogeneous isotropic half-space [13]. In Achenbach's approach, an explicit form of the exact solution was used and the depth homogeneity of the structure seemed to be of crucial importance. Analogs of the Achenbach's *carrier equation* have been derived for transient waves in the nondispersive case (it proved to be the 2D wave equation in which the propagation velocity was that of the Raylegh wave) [11], for elastostatics [14], for the theory of thin plates (where, in the static case, it turned out to be biharmonic [14]), for piezoelastic structures [9], and so on.

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1.4. The carrier equation and an integral superposition of plane waves. In papers [6, 7], an alternative derivation of the *carrier equation* has been accompanied with the observation that its general solution (assumed to be not growing too rapidly at infinity) can be represented as a superposition of standard plane waves. For an isotropic (as well as for a transversely isotropic) structure, the possibility of representation of a solution by an integral superposition of plane waves implies the *carrier equation* for it, and vice versa. However, as we show in this note, an integral superposition of plane waves admits a generalization to a rather arbitrary anisotropy. In general, the integral solution does not satisfy any partial differential equation.

The analysis that follows is based on a superposition of plane waves.

2. Equations and boundary conditions

We deal with elastic waves harmonic in time in a layered half-space. We divide Cartesian coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ into lateral ones $\boldsymbol{x}_\perp = (x_1, x_2)$ and the depth $z \equiv x_3$. The layered structure is characterized by elastic stiffnesses c_{ijkl} and a volume density $\rho > 0$, dependent only on the depth $c_{ijkl} = c_{ijkl}(z)$, $\rho = \rho(z)$. For $z > 0$, the components u_p of the displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)$ satisfy the elastodynamics equations

$$\partial_j c_{ijkl} \partial_k u_l + \rho\omega^2 u_i = 0, \;\; i, j, k, l = 1, 2, 3, \tag{1}$$

where $\omega > 0$ is the angular frequency. The surface $z = 0$ of the half-space is traction-free, i.e.,

$$c_{3jkl} \partial_k u_l \big|_{z=0} = 0, \;\; j, k, l = 1, 2, 3. \tag{2}$$

Jumps of stiffnesses at the planes $z = z_1, \ldots, z_N$ are allowed, and then conditions of welded contact are assumed:

$$[\boldsymbol{u}]\big|_{z=z_p} = 0, \quad [c_{3jkl} \partial_k u_l]\big|_{z=z_p} = 0, \tag{3}$$

$p = 1, \ldots, N$, where [ ] stands for a jump. Since we are aimed at describing surface waves, we require that $\boldsymbol{u} \to 0$ as $z \to \infty$.

3. Plane waves

The displacement vector in the standard time-harmonic plane wave can be represented in the form (see, e.g., [15])

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}_\perp, z) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}_\perp} \boldsymbol{W}(z; \boldsymbol{k}). \tag{4}$$

Here $\boldsymbol{k} = (k_1, k_2)$ is a wave vector lying in the lateral plane, which is assumed to be real, $\boldsymbol{W}(z; \boldsymbol{k})$ is the depth dependence of the wave field, vanishing as $z \to \infty$. Typically, for simplicity, the wave propagation is considered along the x_1-axis, i.e., $k_2 = 0$. Generally speaking, the length of $\boldsymbol{k}$ and the direction of $\boldsymbol{W}(\boldsymbol{k})$ may depend on the direction of $\boldsymbol{x}_\perp$.

4. Integral superposition of plane waves

Let us parametrize $\boldsymbol{x}_\perp$ by polar coordinates

$$x_1 = r\cos\varphi, \;\; x_2 = r\sin\varphi, \;\; 0 \le r, \;\; 0 \le \varphi < 2\pi, \tag{5}$$

so that $\boldsymbol{x}_\perp = \boldsymbol{x}_\perp(\varphi)$, $\boldsymbol{k}(\varphi)$, and $\boldsymbol{W}(z, \boldsymbol{k}(\varphi))$. Assume that for a given φ, a solution (4) exists for all values of φ and that it is a single-valued function of φ (which holds, e.g., for a homogeneous half-space with axial symmetry of elastic stiffnesses). We assume that the vector $\boldsymbol{W}$ is somehow normalized, e.g.,

$$|\,\boldsymbol{W}(0, \boldsymbol{k}(\varphi))| \equiv 1. \tag{6}$$

To generalize the constructions of papers [6,7] to the anisotropic case, consider plane waves (4), propagating in different directions. The direction will be parametrized by a polar angle φ'.

Multiplying (4) by an arbitrary generalized function $A(\varphi - \varphi')$ of one variable and integrating over the angle, we arrive at the expression

$$\boldsymbol{U}(\boldsymbol{x}_\perp(\varphi), z) = \int\limits_0^{2\pi} A(\varphi - \varphi') e^{i\boldsymbol{k}(\varphi')\cdot \boldsymbol{x}_\perp(\varphi')} \boldsymbol{W}(z, \boldsymbol{k}(\varphi'))\, d\varphi'. \tag{7}$$

Obviously, (7) satisfies Eq. (1), boundary conditions (2) and (3) and tends to zero as $z \to \infty$. In fact, the above construction requires neither that the plane wave exist for all $0 \le \varphi < 2\pi$ nor that the dependence $\boldsymbol{k}(\varphi')$ be single-valued. The matter is that we can take in (7) the density as a compactly supported function localized on an the interval where the single-valuedness takes place.

5. Examples

5.1. Solutions polynomial in lateral variables. Putting in (7) $A(\psi) = \delta(\psi)$ yields a standard plane wave, and putting $A(\psi) = \delta'(\psi)$ yields a plane wave with a linear dependence of its amplitude on $\boldsymbol{x}_\perp$. For $A(\psi) = \frac{d^n}{d\psi^n}\delta(\psi)$ we get solutions with plane-wave phases and amplitudes polynomial with respect to $\boldsymbol{x}_\perp$. Such solutions were found for a homogeneous anisotropic half-space by another method in [5].

5.2. Surface wave beams. Modification of the approach of papers [6,7] allows a construction of beams of surface waves, including Gaussian beams. Such considerations are intended in another publication.

Also we are planning to generalize the above construction to structures in which the dependence $\boldsymbol{k}(\varphi)$ is not single-valued and to consider other substantial examples of the choice of amplitudes A in (7).

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