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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF ESSENTIALLY NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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For a two-term second-order differential equation with regularly and rapidly varying nonlinearities, we study the asymptotic behavior of a class of solutions as $t \uparrow \omega$ $(\omega \le +\infty)$.

1. Statement of the Problem

Consider a differential equation

$$y'' = \alpha_0 p(t)\varphi_1(y)\varphi_2(y') \tag{1.1}$$

in which $\alpha_0 \in \{-1, 1\}$, $p\colon [a, \omega[\longrightarrow]0, +\infty[$ is a continuous function, and $\varphi_i\colon \Delta(Y_i^0) \to]0, +\infty[$, $i = 1, 2$, are twice continuously differential functions (here, $\Delta(Y_i^0)$ is a one-sided neighborhood of the point Y_i^0 and Y_i^0 is equal either to 0 or to $\pm\infty$) satisfying the conditions

$$\lim_{\substack{z \to Y_1^0 \\ z \in \Delta(Y_1^0)}} \frac{z\varphi_1'(z)}{\varphi_1(z)} = \lambda, \quad \lambda \in \mathbb{R}, \tag{1.2}$$

$$\varphi_2'(z) \neq 0 \quad \text{for} \quad z \in \Delta(Y_2^0), \qquad \lim_{\substack{z \to Y_2^0 \\ z \in \Delta(Y_2^0)}} \varphi_2(z) = \Phi_2^0, \quad \Phi_2^0 \in \{0, +\infty\},$$

$$\lim_{\substack{z \to Y_2^0 \\ z \in \Delta(Y_2^0)}} \frac{\varphi_2''(z)\varphi_2(z)}{\left[\varphi_2'(z)\right]^2} = 1. \tag{1.3}$$

By virtue of conditions (1.2) and (1.3), the function $\varphi_1(z)$ is regularly or slowly varying as $z \to Y_1^0$ and $\varphi_2(z)$ is a rapidly varying function as $z \to Y_2^0$ (see [1]).

For power and regularly varying nonlinearities φ_i, $i = 1, 2$, the asymptotic behavior of the solutions of Eq. (1.1) was investigated in [2–10].

In [11], the following sufficiently broad class of monotonic solutions was introduced for equations of the analyzed type (1.1):

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Definition 1.1. *A solution y of Eq. (1.1) is called a $P_\omega(\Lambda_0)$-solution, where $-\infty \le \Lambda_0 \le +\infty$, if it is defined in a certain interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the conditions*

$$\lim_{t\uparrow\omega} y(t) = Y_1^0, \quad \lim_{t\uparrow\omega} \varphi_2(y'(t)) = \Phi_2^0,$$
$$\lim_{t\uparrow\omega} \frac{\varphi_2'(y'(t))}{\varphi_2(y'(t))} \frac{y''(t)y(t)}{y'(t)} = \Lambda_0. \tag{1.4}$$

Moreover, in [11], the asymptotics of the $P_\omega(\Lambda_0)$-solutions of Eq. (1.1) was investigated in the case where $\Lambda_0 \in \mathbb{R} \setminus \{0\}$.

The aim of the present paper is to establish the asymptotic properties of $P_\omega(\Lambda_0)$-solutions of Eq. (1.1) and the conditions for their existence in a special case where $\Lambda_0 = 0$.

2. Auxiliary Results

To establish the main results of the present paper, we need some auxiliary results on the behavior of solutions of the system

$$u_1' = \alpha_1 p_1(t)\psi_2(u_2),$$
$$u_2' = \alpha_2 p_2(t)\psi_1(u_1). \tag{2.1}$$

Here, $\alpha_i \in \{-1, 1\}$, $i = \overline{1,2}$, $p_i\colon [a,\omega[\to]0+\infty[$, $i = \overline{1,2}$, are continuous functions, and $\psi_i\colon \Delta(U_i^0) \to]0;+\infty[$, $i = \overline{1,2}$, are continuously differentiable functions satisfying the conditions

$$\lim_{\substack{z\to U_i^0 \\ z\in\Delta(U_i^0)}} \frac{z\psi_i'(z)}{\psi_i(z)} = \sigma_i, \quad i = \overline{1,2}, \tag{2.2}$$

where $\sigma_i \in \mathbb{R}$ and are such that

$$\sigma_1\sigma_2 \ne 1, \tag{2.3}$$

$-\infty < a < \omega \le +\infty$, U_i^0 is equal either to 0 or to $\pm\infty$, and $\Delta(U_i^0)$ is a one-sided neighborhood of the point U_i^0.

It follows from condition (2.2) that $\psi_i(z)$ are functions regularly varying as $z \to U_i^0$, $i = \overline{1,2}$, and hence, they can be represented in the form

$$\psi_i(z) = |z|^{\sigma_i}\theta_i(z), \tag{2.4}$$

where $\theta_i(z)$ are functions slowly varying as $z \to U_i^0$, $i = \overline{1,2}$.

Definition 2.1 [7]. *We say that a function* $\theta\colon \Delta(U^0) \longrightarrow]0,+\infty[$, $U^0 \in \{0, \pm\infty\}$ *slowly varying as* $z \to U^0$ *satisfies the condition* S *if, for any continuously differentiable function* $l\colon \Delta(U^0) \longrightarrow]0,+\infty[$ *such that*

$$\lim_{\substack{z\to U^0\\ z\in\Delta(U^0)}} \frac{z\,l'(z)}{l(z)} = 0,$$

the asymptotic relation

$$\theta(zl(z)) = \theta(z)[1+o(1)] \quad as \quad z \to U^0 \quad (z \in \Delta\left(U^0\right))$$

is true.

Definition 2.2. *A solution* (u_1, u_2) *of system (2.1) given on the interval a* $[t_0, \omega[\subset [a, \omega[$ *is called a* $\mathcal{P}_\omega(0)$*-solution if it satisfies the conditions*

$$u_i(t) \in \Delta(U_i^0) \quad for \quad t \in [t_0, \omega[, \quad \lim_{t\uparrow\omega} u_i(t) = U_i^0, \quad i = \overline{1,2},$$

$$\lim_{t\uparrow\omega} \frac{u_1(t)u_2'(t)}{u_1'(t)u_2(t)} = 0.$$

We now present two examples of the asymptotic behavior of $\mathcal{P}_\omega(0)$-solutions of system (2.1), which follow from the results established in [12]. To this end, we need the following notation:

$$\mu_i = \begin{cases} 1 & \text{if } \; U_i^0 = +\infty, \\ & \text{or } \; U_i^0 = 0 \text{ and } \Delta(U_i^0) \text{ is a right neighborhood of } 0, \\ -1 & \text{if } \; U_i^0 = -\infty, \\ & \text{or } \; U_i^0 = 0 \text{ and } \Delta(U_i^0) \text{ is a left neighborhood of } 0, \end{cases}$$

$$\beta_1 = 1, \quad \beta_2 = 1 - \sigma_1\sigma_2 \neq 0,$$

$$I_1(t) = \int_{A_1}^{t} p_1(\tau)\,d\tau, \quad I_2(t) = \int_{A_2}^{t} p_2(\tau)\psi_1(\mu_1|I_1(\tau)|)\,d\tau.$$

Here, each limit of integration $A_i \in \{\omega, a\}$ is chosen to guarantee that the corresponding integral I_i with this limit tends either to zero or to ∞ as $t \uparrow \omega$.

In addition, we set

$$A_i^* = \begin{cases} 1 & \text{for } \; A_i = a, \\ -1 & \text{for } \; A_i = \omega, \end{cases} \qquad i = \overline{1,2}.$$

Lemma 2.1. *Let a function $\theta_1(z)$ satisfy condition S. Then, in order that the $\mathcal{P}_\omega(0)$-solutions of the differential equations (2.1) exist, it is necessary and sufficient that*

$$\lim_{t\uparrow\omega}\frac{I_1(t)I_2'(t)}{I_1'(t)I_2(t)}=0 \tag{2.5}$$

and, for each $i\in\{1,2\}$, the following sign conditions be satisfied:

$$A_i^*\beta_i>0 \quad for \quad U_i^0=\pm\infty, \quad A_i^*\beta_i<0 \quad for \quad U_i^0=0, \tag{2.6}$$

$$\operatorname{sign}\left[\alpha_i A_i^*\beta_i\right]=\mu_i. \tag{2.7}$$

Moreover, each solution of this kind admits, as $t\uparrow\omega$, the asymptotic representations

$$\frac{u_1(t)}{\psi_2(u_2(t))}=\alpha_1\beta_1 I_1(t)[1+o(1)], \tag{2.8}$$

$$\frac{u_2(t)}{[\psi_2(u_2(t))]^{\sigma_1}}=\alpha_2\beta_2 I_2(t)[1+o(1)]. \tag{2.9}$$

Furthermore, there exist a one-parameter family of these solutions in the case where one of the numbers A_1^* or A_2^* is positive and a two-parameter family of solutions in the case where both numbers A_1^* and A_2^* are positive.

Lemma 2.2. *Assume that $\theta_i(z)$, $i=1,2$, satisfy condition S. Then each $\mathcal{P}_\omega(0)$-solution of the system of differential equations (2.1) admits (if it exists) the following asymptotic representations as $t\uparrow\omega$:*

$$u_1(t)=\mu_1\left|\beta_1 I_1(t)\theta_2\left(\mu_2|I_2(t)|^{\frac{1}{\beta_2}}\right)\right|\left|\beta_2 I_2(t)\left[\theta_2\left(\mu_2|I_2(t)|^{\frac{1}{\beta_2}}\right)\right]^{\sigma_1}\right|^{\frac{\sigma_2}{1-\sigma_1\sigma_2}},$$

$$u_2(t)=\mu_2\left|\beta_2 I_2(t)\left[\theta_2\left(\mu_2\,|I_2(t)|^{\frac{1}{\beta_2}}\right)\right]^{\sigma_1}\right|^{\frac{1}{1-\sigma_1\sigma_2}}.$$

3. Main Results

We introduce the following numbers:

$$\mu_i^0=\begin{cases}1 & \text{if} \quad Y_i^0=+\infty,\\ & \text{or} \quad Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is a right neighborhood of } 0,\\ -1 & \text{if} \quad Y_i^0=-\infty,\\ & \text{or} \quad Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is a left neighborhood of } 0,\end{cases} \qquad i=1,2.$$

These numbers specify the signs of the $P_\omega(0)$-solutions of Eq. (1.1) and their derivatives in a certain left neighborhood of ω and the functions

$$\pi_\omega(t) = \begin{cases} t & \text{for} \quad \omega = +\infty, \\ t - \omega & \text{for} \quad \omega < +\infty, \end{cases} \qquad J(t) = \int\limits_A^t p(\tau)\varphi_1(\mu_1^0|\pi_\omega(\tau)|)\,d\tau,$$

where the limit of integration $A \in \{\omega, a\}$ is chosen to guarantee that the integral J tends either to zero or to ∞ as $t \uparrow \omega$.

In addition, we set

$$A_1^* = \begin{cases} 1 & \text{for} \quad \omega = \infty, \\ -1 & \text{for} \quad \omega < \infty, \end{cases} \qquad A_2^* = \begin{cases} 1 & \text{for} \quad A = a, \\ -1 & \text{for} \quad A = \omega. \end{cases}$$

Since $\varphi_1(z)$ is a regularly varying function of order λ as $z \to Y_1^0$, it can be represented in the form

$$\varphi_1(z) = |z|^\lambda \theta_1(z), \tag{3.1}$$

where $\theta_1(z)$ is a function slowly varying as $z \to Y_1^0$.

The following assertion is true for Eq. (1.1):

Theorem 3.1. *Assume that a function $\theta_1(z)$ satisfies condition S. Then, in order that the $P_\omega(0)$-solutions of the differential equation (1.1) exist, it is necessary and sufficient that*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J'(t)}{J(t)} = 0 \tag{3.2}$$

and the sign conditions

$$\begin{gathered} A_1^* > 0 \quad \text{for} \quad Y_1^0 = \pm\infty, \qquad A_1^* < 0 \quad \text{for} \quad Y_1^0 = 0, \\ A_2^* > 0 \quad \text{for} \quad \Phi_2^0 = 0, \qquad A_2^* < 0 \quad \text{for} \quad \Phi_2^0 = \pm\infty, \end{gathered} \tag{3.3}$$

$$\mu_1^0\mu_2^0 A_1^* > 0 \quad \text{and} \quad \alpha_0\mu_2^0 A_2^* > 0 \tag{3.4}$$

be satisfied. Moreover, each of these solution admits the following asymptotic representations as $t \uparrow \omega$:

$$\frac{y(t)}{y'(t)} = \pi_\omega(t)[1 + o(1)], \tag{3.5}$$

$$\frac{1}{|y'|^\lambda \varphi_2'(y'(t))} = -\alpha_0 J(t)[1 + o(1)]. \tag{3.6}$$

Furthermore, a one-parameter family of these solutions exists in the case where one of the numbers A_1^* and A_2^* is positive and a two-parameter family of solutions exists in the case where both numbers A_1^* and A_2^* are positive.

Proof. We now show that Eq. (1.1) is reduced to system (2.1). To this end, for a rapidly varying function $\varphi_2(z)$, we introduce a function

$$\psi(z)=\int\limits_B^z \frac{ds}{\varphi_2(s)}, \quad \text{where} \quad B=\begin{cases} Y_2^0 & \text{if} \quad \displaystyle\int_b^{Y_2^0} \frac{ds}{\varphi_2(s)} \quad \text{converges}, \\ b & \text{if} \quad \displaystyle\int_b^{Y_2^0} \frac{ds}{\varphi_2(s)} \quad \text{diverges}, \end{cases} \tag{3.7}$$

and b is any number from the interval $\Delta(Y_2^0)$.

Since $\psi'(z)>0$ for $z\in\Delta(Y_2^0)$, we conclude that $\psi\colon \Delta(Y_2^0)\longrightarrow\Delta(\Psi^0)$ is an increasing function, where

$$\Psi^0=\lim_{z\to Y_2^0}\psi(z).$$

Hence, Ψ^0 is equal either to zero or to $\pm\infty$ and $\Delta(\Psi^0)$ is a one-sided neighborhood of Ψ^0.

Note that the function ψ, just as $\varphi_2(z)$, is rapidly varying as $z\to Y_2^0$.

Indeed, by using (1.3) and the L'Hospital rule, we get

$$\lim_{z\to Y_2^0}\frac{\psi(z)\psi''(z)}{[\psi'(z)]^2}=-\lim_{z\to Y_2^0}\psi(z)\varphi_2'(z)=\lim_{z\to Y_2^0}\frac{\psi'(z)}{\left(\dfrac{-1}{\varphi_2'(z)}\right)'}=\lim_{z\to Y_2^0}\frac{[\varphi_2'(z)]^2}{\varphi_2(z)\varphi_2''(z)}=1. \tag{3.8}$$

In addition, it follows from relations (3.8) that

$$\frac{\psi(z)}{\psi'(z)}\sim\frac{\psi'(z)}{\psi''(z)}=-\frac{\varphi_2(z)}{\varphi_2'(z)} \quad \text{as} \quad z\to Y_2^0, \tag{3.9}$$

$$\psi(z)\sim-\frac{1}{\varphi_2'(z)} \quad \text{as} \quad z\to Y_2^0. \tag{3.10}$$

In view of (3.9), the limit relation (1.4) in the definition of $P_\omega(0)$-solutions can be rewritten in the equivalent form as follows:

$$\lim_{t\uparrow\omega}\frac{\psi'(y'(t))}{\psi(y'(t))}\,\frac{y''(t)y(t)}{y'(t)}=0. \tag{3.11}$$

Moreover, in view of the fact that $\varphi_2(z)$ is a positive and monotone function in $\Delta(Y_2^0)$, it follows from (3.10) that $\Phi_2^0=0$ for $\Psi^0=\infty$ and $\Phi_2^0=\infty$ for $\Psi^0=0$, and vice versa.

As indicated above, the function $\psi(z)$ is increasing and, hence, invertible. Moreover, by the properties of slowly, rapidly, and regularly varying functions, $\psi^{-1}(z)\colon\Delta(\Psi^0)\to\Delta(Y_2^0)$ is a slowly varying function as $z\to\Psi^0$. For this function, we find

$$\lim_{z\to\Phi^0}\frac{z\left(\psi^{-1}(z)\right)'}{\psi^{-1}(z)}=\lim_{z\to\Phi^0}\frac{z\varphi_2\left(\psi^{-1}(z)\right)}{\psi^{-1}(z)}=\lim_{u\to Y_2^0}\frac{\psi(u)\varphi_2(u)}{u}=-\lim_{u\to Y_2^0}\frac{\varphi_2(u)}{u\varphi'(u)}=0. \tag{3.12}$$

By using the transformations

$$y = u_1, \quad \psi(y') = u_2, \tag{3.13}$$

we reduce Eq. (1.1) to the system of differential equations

$$\begin{aligned} u_1' &= \mu_2^0 \left|\psi^{-1}(u_2)\right|, \\ u_2' &= \alpha_0 p(t)\varphi_1(u_1). \end{aligned} \tag{3.14}$$

Since the function $\varphi_1\colon \Delta(Y_1) \to]0, +\infty[$ satisfies condition (1.2) and the function

$$\left|\psi^{-1}(z)\right|: \ \Delta(\Psi^0) \to]\, 0, +\infty\, [$$

satisfies condition (3.12), we conclude that (3.14) is a system of the form (2.1).

Moreover, in view of (3.11), it is easy to see that y is a $P_\omega(0)$-solution of Eq. (1.1) if and only if the solution (u_1, u_2) of system (3.14) corresponding to this equation by virtue of relations (3.13) is a $\mathcal{P}_\omega(0)$-solution of system (3.14). In addition, since the function $\left|\psi^{-1}(z)\right|$ satisfies (3.12), condition (2.3) is clearly satisfied for system (3.14) and, hence, Lemma 2.1 is true for system (3.14).

Hence, the necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(0)$-solutions of system (3.14) in Lemma 2.1 are necessary and sufficient conditions for the existence of $P_\omega(0)$-solutions of Eq. (1.1).

We now specify the definitions introduced in Lemma 2.1 for system (3.14):

$$\alpha_1 = \mu_2^0, \quad p_1(t) \equiv 1, \quad \alpha_2 = \alpha_0, \quad p_2(t) = p(t), \quad \mu_1 = \mu_1^0, \quad \mu_2 = \mu_2^0,$$

$$\sigma_1 = \lambda, \quad \sigma_2 = 0, \quad I_1(t) = \pi_\omega(t), \quad \beta_1 = 1, \quad I_2(t) = J(t) \quad \beta_2 = 1.$$

Thus, writing conditions (2.5)–(2.7) for system (3.14), we arrive at conditions (3.2)–(3.4).

To obtain the asymptotic representations (3.5) and (3.6), it suffices to write the asymptotic representations (2.8) and (2.9) for system (3.14) by using transformations (3.13) and relation (3.10).

The theorem is proved.

It is worth noting that the asymptotic representations for y and y' in relations (3.5) and (3.6) are written in the implicit form. By using Lemma 2.2, we now establish conditions under which the asymptotic representations can be rewritten in a simpler form.

Theorem 3.2. *Assume that the functions $\theta_1(z)$ and $\left|\psi^{-1}(z)\right|$ satisfy the condition S. Then each $P_\omega(0)$-solution of the differential equation (1.1) admits (if it exists) the following asymptotic representations as $t \uparrow \omega$:*

$$y(t) = \mu_1^0 \left|\pi_\omega(t)\psi^{-1}\left(\mu_2^0|J(t)|\right)\right| [1 + o(1)],$$

$$\frac{1}{\varphi_2'(y'(t))} = -\mu_2^0|J(t)| \left|\psi^{-1}\left(\mu_2^0|J(t)|\right)\right|^\lambda [1 + o(1)].$$

4. Application of the Main Results

Consider a class of differential equations of the form

$$y'' = \alpha_0 p(t)|y|^{\lambda}|\ln|y||^{\gamma} e^{-\sigma|y'|^{\delta}}|y'|^{1-\delta}, \tag{4.1}$$

where $\alpha_0 \in \{1, -1\}$, $\delta, \sigma \in \mathbb{R} \setminus \{0\}$, $\lambda, \gamma \in \mathbb{R}$, and $p\colon [a, \omega[\longrightarrow]0, +\infty[$ is a continuous function.

Equation (4.1) is an equation of the form (1.1) with

$$\varphi_1(z) = |z|^{\lambda} \ln^{\gamma}|z| \quad \text{and} \quad \varphi_2(z) = e^{-\sigma|z|^{\delta}}|z|^{1-\delta}.$$

The function $\varphi_1(z)$ is a regularly varying function of order λ as $z \to Y_2^0$ and $\varphi_2(z)$ is a regularly varying function as $z \to \pm\infty$ for $\delta > 0$ and regularly varies as $z \to 0$ for $\delta < 0$.

For the function $\varphi_2(z)$, the function $\psi(z)$ defined in (3.7) takes the form

$$\psi(z) = \frac{1}{\sigma\delta} e^{\sigma|z|^{\delta}} \operatorname{sign} z.$$

Moreover, the function $\theta_1(z)$ defined in (3.1) and the function $\psi^{-1}(z)$ have the form

$$\theta_1(z) = \ln^{\gamma}|z|, \quad \psi^{-1}(z) = \mu_2^0 \left|\frac{1}{\sigma} \ln|\sigma\delta z|\right|^{\frac{1}{\delta}}$$

and satisfy condition S.

For Eq. (4.1), condition (1.4) in the definition of $P_{\omega}(0)$-solutions rewritten in the equivalent form (3.11) takes the form

$$\lim_{t\uparrow\omega} \frac{yy''(t)}{|y'(t)|^{2-\delta}} = 0.$$

Thus, by using Theorems 3.1 and 3.2, we get the following assertion for Eq. (4.1):

Corollary 4.1. *in order that a $P_{\omega}(0)$-solution of the differential equation (4.1) exist, it is necessary and sufficient that conditions (3.2)–(3.4) be satisfied. Each solution of this kind admits the following asymptotic representations as $t \uparrow \omega$:*

$$y(t) = \mu_1^0|\pi_{\omega}(t)| \left|\frac{1}{\sigma} \ln|\sigma\delta J(t)|\right|^{\frac{1}{\delta}} [1 + o(1)],$$

$$y'(t) = \mu_2^0 \left|\frac{1}{\sigma} \ln|\sigma\delta J(t)| + \frac{\lambda}{\sigma\delta} \ln\left|\frac{1}{\sigma} \ln|\sigma\delta J(t)|\right| + o(1)\right|^{\frac{1}{\delta}}.$$

Moreover, there exist a one-parameter family of these solutions in the case where one of the numbers A_1^* or A_2^* is positive and a two-parameter family of solutions in the case where both numbers A_1^* and A_2^* are positive.

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