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A CRACK IN THE FORM OF A THREE-LINK BROKEN LINE UNDER THE ACTION OF LONGITUDINAL SHEAR WAVES

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We solve the problem of determination of the dynamic stress intensity factors for a crack in the form of a three-link broken line. The crack is located in an infinite elastic medium with propagating harmonic longitudinal shear waves. The initial problem is reduced to a system of three singular integrodifferential equations with fixed singularities. A numerical method is proposed for the solution of this system with regard for the true asymptotics of the unknown functions.

At present, there are numerous solutions of two-dimensional dynamical problems of the theory of elasticity for bodies with cracks in the form of a segment of the straight line or an arc of a smooth curve. However, the actual cracks may have corner points, can be piecewise smooth, can cross each other and bifurcate. The problems of determination of the dynamic stressed state in the vicinities of cracks of this kind have, in fact, not been studied yet. This is explained by the difficulties encountered in their solution by the method of boundary integral equations extensively applied in recent years and in the reduction of the analyzed problems to singular integrodifferential or hypersingular equations with fixed singularities.

As the most studied, we can mention the static problems for bodies containing cracks with corner points. We especially mention the works [2, 13] in which the exact solution was obtained by the Wiener–Hopf method and the exact values of the stress intensity factors were determined. These solutions and the results obtained in [3] show that the presence of kernels with fixed singularities affects the singularities of solutions in the vicinities of the ends of the intervals of integration. The stressed states near branched, broken, and edge cracks were also studied in [5, 10, 12, 14]. In these works, the integral equations were numerically solved by the method of mechanical quadratures. This method is based on the application of the Gauss–Chebyshev quadrature formulas specifying the root singularities of the solutions. In this case, the true asymptotics of the solutions are not taken into account or an additional condition leading to a singularity weaker than the root singularity is imposed on the solution.

Another drawback of the numerical methods used in these works is connected with the formal application of the Gauss–Chebyshev quadrature formulas to integrals with fixed singularities. As a result, the convergence is very slow (in order to get the results with an error smaller than 0.1%, it is necessary to use several tens of collocation points).

In the present work, we solve the problem of determination of the dynamical stress intensity factors for cracks in the form of three-link broken lines interacting with harmonic longitudinal shear waves. The problem is reduced to a system of three singular integrodifferential equations solved by the method of collocations. In this method, we take into account the true singularity of the solution and use special quadrature formulas for finding the integrals with fixed singularities.

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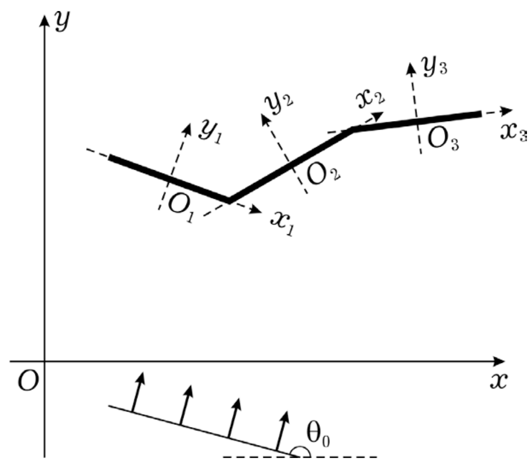

Fig. 1

Statement of the Problem and Its Reduction to a System of Singular Integral Equations

We consider an unbounded isotropic elastic medium under the conditions of antiplane deformation with a through crack in the form of a three-link broken line in the plane Oxy (Fig. 1). The crack interacts with plane longitudinal shear waves. Its front forms an angle θ_0 with the Ox-axis and causes the displacements

$$W_0^{\text{in}}(x,y) = A_0 \exp\left(i\kappa_2(x\sin\theta_0 + y\cos\theta_0)\right), \qquad \kappa_2^2 = \frac{\rho\omega^2}{G}, \tag{1}$$

in the medium along the Oz-axis. In (1), the symbols G and ρ stand, respectively, for the shear modulus and the density of the medium and ω is the frequency of oscillations. The time dependence is specified by the factor $e^{-i\omega t}$. Here and in what follows, this factor is omitted.

Assume that the vector of displacements have a single component $W(x,y)$ nonzero under the conditions of antiplane deformation. In the coordinate system Oxy, it satisfies the Helmholtz equation

$$\Delta W + \kappa_2^2 W = 0. \tag{2}$$

To specify the boundary conditions on the crack, we associate a coordinate system with each link of the crack. Assume that these coordinate systems $O_\ell x_\ell y_\ell$, $\ell = 1, 2, 3$, are centered at the midpoints of the links. The relationship between the coordinate systems is given by the formulas

$$\begin{aligned} x_\ell &= (x - a_\ell)\cos\alpha_\ell + (y - b_\ell)\sin\alpha_\ell, \qquad & x &= a_\ell + x_\ell\cos\alpha_\ell - y_\ell\sin\alpha_\ell, \\ y_\ell &= -(x - a_\ell)\sin\alpha_\ell + (y - b_\ell)\cos\alpha_\ell, \qquad & y &= b_\ell + x_\ell\sin\alpha_\ell + y_\ell\cos\alpha_\ell. \end{aligned} \tag{3}$$

Let

$$W_\ell(x_\ell, y_\ell) = W(a_\ell + x_\ell\cos\alpha_\ell - y_\ell\sin\alpha_\ell, b_\ell + x_\ell\sin\alpha_\ell + y_\ell\cos\alpha_\ell)$$

be the displacements in the coordinate system connected with the ℓth link of the crack. Then, in the absence of loading applied to the crack edges, the following equalities must be true:

$$\tau_{zy_\ell}(x_\ell,0) = -\tau_{zy}^{\text{in}}(x_\ell,0), \quad x_\ell \in [-d_\ell, d_\ell], \tag{4}$$

where

$$\tau_{zy_\ell} = G\frac{\partial W_\ell}{\partial y_\ell}, \qquad \tau_{zy_\ell}^{\text{in}} = G\frac{\partial W_\ell{}^{\text{in}}}{\partial y_\ell}.$$

The presence of the crack leads to the discontinuity of displacements. For their jumps, we introduce the following notation:

$$W_\ell(x_\ell,+0) - W_\ell(x_\ell,+0) = \chi_\ell(x_\ell), \qquad x_\ell \in [-d_\ell, d_\ell], \quad \ell = 1,2,3. \tag{5}$$

In addition, the continuity of displacements along the crack edge implies that

$$\begin{aligned} W_\ell(d_\ell,+0) &= W_{\ell+1}(-d_{\ell+1},+0), \\ W_\ell(d_\ell,-0) &= W_{\ell+1}(-d_{\ell+1},-0), \quad \ell = 1,2. \end{aligned} \tag{6}$$

We start the solution of the posed problem (2)–(5) from the construction of a discontinuous solution of the Helmholtz equation in the coordinate system $O_k x_k y_k$ for each link of the crack [8]

$$W_k^d(x_k,y_k) = \frac{\partial}{\partial y_k}\int_{-d_k}^{d_k} \chi_k(\eta)\, r_2(\eta - x_k, y_k)\, d\eta, \tag{7}$$

where

$$r_2(\eta - x_k, y_k) = -\frac{i}{4} H_0^{(1)}\left(\kappa_2\sqrt{(\eta - x_k)^2 + y_k^2}\right).$$

The corresponding stresses are given by the formulas

$$\begin{aligned} \tau_{zy_k}^d &= -G\int_{-d_k}^{d_k} \chi_k(\eta)\frac{\partial^2}{\partial \eta^2} r_2(\eta - x_k, y_k)\, d\eta \\ &\quad - G\kappa_2^2\int_{-d_k}^{d_k} \chi_k(\eta)\, r_2(\eta - x_k, y_k)\, d\eta, \\ \tau_{zx_k}^d &= -G\frac{\partial}{\partial y_k}\int_{-d_k}^{d_k} \chi_k(\eta)\frac{\partial^2 r_2}{\partial \eta^2}(\eta - x_k, y_k)\, d\eta, \quad k = 1,2. \end{aligned} \tag{8}$$

Then the displacements of the reflected wave field can be represented in the form

$$W(x,y) = \sum_{k=1}^{3} W_k^g(x,y), \tag{9}$$

where

$$W_k^g(x,y) = W_k^d((x-a_k)\cos\alpha_k + (y-b_k)\sin\alpha_k, -(x-a_k)\sin\alpha_k + (y-b_k)\cos\alpha_k).$$

In order to finally determine the displacements of the diffraction field, it is necessary to find the unknown jumps (5). To this end, we use condition (4). By using relations (8) and (9), we preliminarily get

$$\tau_{zy_\ell} = \sum_{k=1}^{2} \tau_{zy_\ell}^k, \quad \tau_{zy_\ell}^k = -\tau_{zx_k}^d \sin\alpha_{\ell k} + \tau_{zy_k}^d \cos\alpha_{\ell k}, \quad \alpha_{\ell k} = \alpha_\ell - \alpha_k, \quad \ell = 1,2,3. \tag{10}$$

Further, we pass to the coordinate system connected with the ℓth link. Then, according to (8), the terms in (10) are given by the formulas

$$\tau_{zy_\ell}^k(x_\ell, y_\ell) = G\int_{-d_k}^{d_k} \chi_k'(\eta) F_{\ell k}(\eta, x_\ell, y_\ell)\, d\eta + G\frac{i\kappa_2^2}{4}\cos\alpha_{\ell k}\int_{-d_k}^{d_k} \chi_k(\eta) H_0^{(1)}\left(\kappa_2\sqrt{Z_{\ell k}}\right) d\eta, \tag{11}$$

where

$$F_{\ell k}(\eta, x_\ell, y_\ell) = \frac{i\kappa_2}{4}(\eta\cos\alpha_{\ell k} - x_\ell - (a_\ell - a_k)\cos\alpha_\ell - (b_\ell - b_k)\sin\alpha_\ell)\frac{H_1^{(1)}\left(\kappa_2\sqrt{Z_{\ell k}}\right)}{Z_{\ell k}},$$

$$Z_{\ell k} = (A_{\ell k} + x_\ell\cos\alpha_{\ell k} - y_\ell\sin\alpha_{\ell k})^2 + (B_{\ell k} + x_\ell\sin\alpha_{\ell k} + y_\ell\cos\alpha_{\ell k})^2,$$

$$A_{\ell k} = (a_\ell - a_k)\cos\alpha_{\ell k} + (b_\ell - b_k)\sin\alpha_{\ell k},$$

$$B_{\ell k} = -(a_\ell - a_k)\sin\alpha_{\ell k} + (b_\ell - b_k)\cos\alpha_{\ell k}.$$

In deducing relation (11), it is necessary to perform the integration by parts in the integrals containing the second-order derivatives and take into account the fact that the terms outside the integral are equal to zero in view of conditions (6).

Substituting (10) and (11) in (4), we get a system of singular integrodifferential equations for jumps (5) and their derivatives. We now remove singular components of the kernels and, after necessary transformations, arrive at the following system:

$$\frac{1}{2\pi}\int_{-1}^{1}\left(\frac{E}{\tau-\zeta}+G(\tau,\zeta)+R(\tau,\zeta)\right)\Phi'(\tau)\,d\tau$$

$$+\frac{1}{2\pi}\int_{-1}^{1}(-\kappa_0^2\ln|\tau-\zeta|+U(\tau,\zeta))\Phi(\tau)\,d\tau = F(\zeta), \quad -1<\zeta<1, \tag{12}$$

$$\Phi(\tau) = \begin{pmatrix}\varphi_1(\tau)\\ \varphi_2(\tau)\\ \varphi_3(\tau)\end{pmatrix}, \quad \Phi'(\tau) = \begin{pmatrix}\varphi_1'(\tau)\\ \varphi_2'(\tau)\\ \varphi_3'(\tau)\end{pmatrix},$$

$$F(\tau) = \begin{pmatrix}f_1(\tau)\\ f_2(\tau)\\ f_3(\tau)\end{pmatrix}, \quad E = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$G(\tau,\zeta) = \begin{pmatrix}0 & g_{12}(\tau,\zeta) & 0\\ -g_{21}(\tau,\zeta) & 0 & g_{23}(\tau,\zeta)\\ 0 & -g_{32}(\tau,\zeta) & 0\end{pmatrix},$$

$$R(\tau,\zeta) = \{R_{\ell k}(\tau,\zeta)\}, \quad U(\tau,\zeta) = \{U_{\ell k}(\tau,\zeta)\}, \quad \varphi_\ell(\tau) = d_\ell\chi(d_\ell\tau),$$

$$f_\ell(\zeta) = -i\kappa_0 C_0\cos(\theta_0)\exp(i\kappa_0(\varepsilon_\ell\sin\theta_0+\delta_\ell\cos\theta_0+\gamma_\ell\zeta\sin\theta_{0\ell})),$$

$$\varepsilon_\ell = a_\ell d^{-1}, \quad \delta_\ell = b_\ell d^{-1}, \quad \gamma_\ell = d_\ell d^{-1}, \quad d = \max(d_\ell), \quad \ell,k=1,2,3,$$

$$C_0 = d^{-1}A_0, \quad \kappa_0 = \kappa_2 d.$$

The nonzero elements of the matrix $G(\tau,\zeta)$ are the functions

$$g_{\ell k}(\tau,\zeta) = -\frac{s_{\ell k}\gamma_k((1-s_{\ell k}\tau)\cos\alpha_{\ell k}+\gamma_k(1+s_{\ell k}\zeta))}{Q_{\ell k}(\tau,\zeta)}, \quad s_{\ell k} = \operatorname{sgn}(\ell-k), \tag{13}$$

where

$$Q_{\ell k}(\tau,\zeta) = \gamma_k^2(1-s_{\ell k}\tau)^2+\gamma_\ell^2(1+s_{\ell k}\zeta)^2+2\gamma_k\gamma_\ell(1-s_{\ell k}\tau)(1+s_{\ell k}\zeta)\cos\alpha_{\ell k},$$

$$k,\ \ell=1,2,3, \quad k\neq\ell.$$

It is easy to see that these functions have singularities for $\zeta = \pm 1$, $\tau = \pm 1$. The matrices $R(\tau, \zeta)$ and $U(\tau, \zeta)$ are formed by functions specifying regular integrals.

Numerical Solution of the System of Integrodifferential Equations

The presence of fixed singularities in the singular components of system (12) affects the behavior of solutions in the vicinities of the points ± 1. Their asymptotics in the vicinities of these points are determined in the same way as in [6, 8]. As a result, we conclude that the derivatives of the unknown functions should be sought in the form

$$\varphi'_\ell(\tau) = (1-\tau)^{-\sigma_{\ell+1}}(1+\tau)^{-\sigma_\ell}\psi_\ell(\tau), \quad \ell = 1, 2, 3, \tag{14}$$

where $\psi_\ell(\tau)$ satisfy the Hölder condition and the exponents are as follows:

$$\sigma_\ell = \frac{\beta_{\ell-1}}{\pi + \beta_{\ell-1}}, \quad 0 \le \beta_{\ell-1} \le \pi, \qquad \sigma_\ell = \frac{2\pi - \beta_{\ell-1}}{3\pi - \beta_{\ell-1}}, \quad \pi \le \beta_{\ell-1} \le 2\pi,$$

$$\beta_{\ell-1} = |\alpha_\ell - \alpha_{\ell-1}|, \quad \ell = 2, 3, \qquad \sigma_1 = \sigma_4 = 0.5\,.$$

Further, we approximate the functions $\psi_\ell(\tau)$ by interpolating polynomials [11]

$$\psi_k(\tau) \approx \psi_{n-1}^{(k)}(\tau), \qquad \psi_{n-1}^{(k)}(\tau) = \sum_{m=1}^{n} \psi_{km} \frac{P_{kn}(\tau)}{(\tau - \tau_{km})P'_{kn}(\tau)},$$
$$\psi_{km} = \psi_k(\tau_{km}), \quad k = 1, 2, 3, \tag{15}$$

where

$$P_{kn}(\tau) = P_n^{-\sigma_{k+1} - \sigma_k}(\tau)$$

are the Jacobi polynomials (orthogonal with the corresponding weight) and τ_{km} are roots of these polynomials. Representing the derivatives of the unknown functions in the form (14) and (15), we use the following quadrature formula [1] for the integrals with the Cauchy kernel:

$$\int_{-1}^{+1} \frac{\varphi'_k(\tau)}{\tau - \zeta_{kj}}\, d\tau = \sum_{m=1}^{n} A_{km} \frac{\psi_{km}}{\tau_{km} - \zeta_{kj}}, \qquad j = 1, 2, \ldots, n_k, \quad n_1 = n_3 = n, \quad n_2 = n + 1. \tag{16}$$

In relation (16), ζ_{kj}, $k = 1, 2, 3$, $j = 1, 2, \ldots, n_k$, are the roots of the Jacobi functions of the second kind

$$J_n^{-\sigma_{k+1} - \sigma_k}(\tau)$$

and A_{km} are the coefficients of the corresponding Gauss–Jacobi quadrature formulas [4].

In what follows, we obtain similar formulas for the integrals with fixed singularities

$$E_{\ell k}^{j} = \int_{-1}^{1} \varphi_k'(\tau) g_{\ell k}(\tau, \zeta_{\ell j})\, d\tau, \qquad \ell, k = 1, 2, 3, \quad \ell \neq k. \tag{17}$$

If $1 \pm \zeta > \varepsilon > 0$, then it follows from relations (13) that the functions $g_{\ell k}(\tau, \zeta)$ are infinitely differentiable. Therefore, we can apply the Gauss–Jacobi quadrature formulas to integrals (17). The main difficulty is connected with the calculation of these integrals as $1 \pm \zeta \to 0$. To this end, we use representation (13)–(15) and perform the transformations

$$\frac{g_{\ell k}(\tau, \zeta)}{\tau - \tau_{km}} = \frac{g_{\ell k}(\tau_{km}, \zeta)}{\tau - \tau_{km}} + \frac{s_{\ell k}\gamma_k^2}{q_{\ell k}(\tau_{km}, \zeta)}\big((1 + s_{\ell k}\tau_{km}) g_{\ell k}(\tau, \zeta)$$

$$- (1 - s_{\ell k}\zeta)\, g_{k\ell}(\zeta, \tau)\big), \qquad \ell, k = 1, 2, 3, \quad \ell \neq k. \tag{18}$$

We now substitute (14), (15), and (18) in (17). As a result, the integral in the first term can be found by using the formula

$$\int_{-1}^{+1} (1-\tau)^{-\sigma_{k+1}} (1+\tau)^{-\sigma_k} \frac{P_{kn}(\tau)}{\tau - \tau_{km}}\, d\tau = A_{km} P_{kn}'(\tau_{km}), \quad k = 1, 2, 3. \tag{19}$$

Analogous integrals of the functions $g_{\ell k}(\tau, \zeta)$ and $g_{k\ell}(\zeta, \tau)$ can be found by the method of taking the integrals with orthogonal polynomials [9] based on the application of the Mellin integral transformation. The final relations for the integrals with fixed singularities take the form

$$E_{\ell k}^{j} = \sum_{m=1}^{n} \psi_{km} \frac{D_{jm}^{\ell k}}{Q_{\ell k}(\tau_{km}, \zeta_{\ell j})}, \qquad \ell, k = 1, 2, 3, \quad \ell \neq k, \tag{20}$$

where

$$D_{jm}^{\ell k} = -s_{\ell k} A_{km} \gamma_k \big(\gamma_k (1 - s_{\ell k}\tau_{km}) \cos\beta + \gamma_\ell (1 + s_{\ell k}\zeta_{\ell j})\big),$$

$$1 + s_{\ell k}\zeta_{\ell j} > \varepsilon > 0,$$

$$D_{jm}^{\ell k} = -s_{\ell k} A_{km} \gamma_k \big(\gamma_k (1 - s_{\ell k}\tau_{km}) \cos\alpha_{\ell k} + \gamma_\ell (1 + s_{\ell k}\zeta_{\ell j})\big)$$

$$- \frac{\gamma_k s_{\ell k} (-1)^{nk}}{P_{kn}'(\tau_{\ell m})} \left(\gamma_k (1 - s_{\ell k}\tau_{km}) B_n^{(1)}\left(\frac{\gamma_\ell (1 + s_{\ell k}\zeta_{\ell j})}{2\gamma_k} \right) \right.$$

$$\left. + \gamma_\ell (1 + s_{\ell k}\zeta_{\ell j}) B_n^{(2)}\left(\frac{\gamma_\ell (1 + s_{\ell k}\zeta_{\ell j})}{2\gamma_k} \right) \right), \qquad 0 \le 1 + s_{\ell k}\zeta_{\ell j} < \varepsilon,$$

$$B_{kn}^{(p)}(y) = \frac{\Gamma(1-\sigma_{k+1}+n)}{n!\,2^{\sigma_{k+1}+\sigma_k}}\left[-\frac{\sin\pi(\sigma_{k+1}+\sigma_k)}{\sin\pi\sigma_{k+1}}\sum_{s=0}^{\infty} c_{sk} y^s \cos\beta(s+2-p)\right.$$

$$\left. + \frac{\sin\pi\sigma_k}{\sin\pi\sigma_{k+1}}\sum_{s=0}^{\infty} d_{sk} y^{-\sigma_{k+1}+s}\cos\beta(\sigma_{k+1}-s-2+p)\right], \quad p = 1,2,$$

$$c_s = \frac{(-1)^s (n+s)!\,\Gamma(\sigma_{k+1}+\sigma_k-n+s)}{s!\,\Gamma(1+s+\sigma_{k+1})},$$

$$d_s = \frac{(-1)^s \Gamma(1-s+n+\sigma_{k+1})\Gamma(\sigma_k-n+s)}{s!\,\Gamma(1+s-\sigma_{k+1})}.$$

Since system (12) also includes the functions $\varphi_\ell(\tau)$, $\ell = 1,2,3$, it is necessary to deduce the quadrature formulas for the corresponding integrals. First, we use obvious formulas

$$\varphi_k(\tau) = \varphi_k(-1) + \int_{-1}^{\tau} \varphi_k'(x)\,dx, \qquad \varphi_k(\tau) = \varphi_k(1) - \int_{\tau}^{1} \varphi_k'(x)\,dx. \tag{21}$$

Then we use relations (14) and (15) and the Darboux–Christoffel identity for the orthogonal polynomials [11]:

$$\frac{P_{kn}(\tau)}{\tau-\tau_{km}} = A_{km} P_{kn}'(\tau_{km}) \sum_{j=0}^{n-1} \frac{P_{kj}(\tau_{km}) P_{kj}(\tau)}{s_{kj}^2}, \tag{22}$$

where

$$s_{kj}^2 = \frac{2^{1-\sigma_{k+1}-\sigma_k}\Gamma(1-\sigma_{k+1}+j)\Gamma(1-\sigma_k+j)}{(1-\sigma_k-\sigma_{k+1}+2j)\Gamma(1-\sigma_k-\sigma_{k+1}+2j)}, \qquad k = 1,2,3.$$

We now substitute (14), (15), and (22) in (21). After integration, we get the following approximate formulas for the unknown functions:

$$\varphi_k(\tau) = \varphi_k(\pm 1) + (1\pm\tau)^{1-\sigma_k^{\mp}} \sum_{m=1}^{n} A_{km} S_{km}^{\mp}(\tau), \qquad \sigma_k^- = \sigma_k, \quad \sigma_k^+ = \sigma_{k+1}, \tag{23}$$

where

$$S_{km}^{\mp} = s_{k0}^{-2} F_k^{\mp}(\tau) - (1\mp\tau)^{1-\sigma_k^{\pm}} \sum_{j=1}^{n-1} \frac{P_{kj-1}^{(1)}(\tau) P_{kj}(\tau_{km})}{2 j s_{kj}^2},$$

$$F_k^{\mp}(\tau) = \frac{F\left(\sigma_k^{\pm}, 1-\sigma_k^{\mp}, 2-\sigma_k^{\mp}, \dfrac{1\pm\tau}{2}\right)}{2^{\sigma_k^{\pm}}(1-\sigma_k^{\mp})}.$$

Representation (23) serves as a basis for the following quadrature formulas with unknown functions:

$$\int_{-1}^{1} \varphi_k(\tau) U_{\ell k}(\tau, \zeta_{\ell j})\, d\tau \;=\; \varphi_k(\pm 1) U^{j}_{\ell k} + \sum_{m=1}^{n} A_{km} \psi_{km} U^{\pm}_{\ell k j}\,, \tag{24}$$

where

$$U^{j}_{\ell k} \;=\; \sum_{p=1}^{n} A_{0p} U_{\ell k}(z_{0p}, \zeta_{\ell j}), \qquad A_{0p} \;=\; \frac{2}{(1 - z^2_{0p})(P'_n(z_{0p}))^2}\,,$$

$P_n(z)$ are Legendre polynomials, z_{0p} are the roots of these polynomials,

$$U^{\pm}_{\ell k j} \;=\; \sum_{p=1}^{n} A^{\pm}_{kp} S^{\pm}_{km}(z^{\pm}_{kp})\, U_{\ell k}(z^{\pm}_{kp}, \zeta_{\ell j}),$$

$$A^{\pm}_{kp} \;=\; \frac{2^{\sigma^{\pm}_k}}{\left(1 - (z^{\pm}_{kp})^2\right)\left(P'^{\pm}_n(z^{\pm}_{kp})\right)^2}\,,$$

$P^-_n(z) = P_n^{0\,1-\sigma^-_k}(z)$ and $P^+_n(z) = P_n^{1-\sigma^+_k\,0}(z)$ are Jacobi polynomials, and $z^{\pm}_{kp}$ are the roots of these polynomials.

We now consider the integrals with logarithmic difference kernels. As a result of integration by parts and application of representations (14) and (15) for the derivatives of the unknown functions, we obtain the following quadrature formulas:

$$\int_{-1}^{1} \varphi_\ell(\tau) \ln\left|\, \tau - \zeta_{\ell j} \,\right| d\tau \;=\; \varphi_\ell(\pm 1) h_{\ell j} + \sum_{m=1}^{n} A_{\ell m} H^{\pm}_{\ell j m} \psi_{\ell j}\,, \tag{25}$$

where

$$h_{\ell j} \;=\; (1 - \zeta_{\ell j})(\ln(1 - \zeta_{\ell j}) - 1) + (1 + \zeta_{\ell j})(\ln(1 + \zeta_{\ell j}) - 1),$$

$$H^{\pm}_{\ell j m} \;=\; \mp(1 - \zeta_{\ell j})(\ln(1 - \zeta_{\ell j}) - 1) - (\tau_{\ell m} - \zeta_{\ell j})(\ln\left|\, \tau_{\ell m} - \zeta_{\ell j} \,\right| - 1).$$

We now apply the quadrature formulas (16), (20), (24), and (25) and the Gauss–Jacobi quadrature formulas and use the roots of the Jacobi polynomials as collocation points. As a result, system (12) is reduced to a system of linear algebraic equations for $\psi_2(-1)$, ψ_{km}, $k = 1, 2, 3$. As a result of the solution of this system, we get the following formulas for the approximate values of the stress intensity factors:

$$\begin{aligned} K^- \;&=\; -G\sqrt{d_1} \cdot 2^{-(1+\sigma_2)} P_{1n}(-1) \sum_{m=1}^{n} \frac{\psi_{1m}}{P'_{1n}(\tau_{1m})(1 + \tau_{1m})}\,, \\ K^+ \;&=\; -G\sqrt{d_3} \cdot 2^{-(1+\sigma_3)} P_{3n}(1) \sum_{m=1}^{n} \frac{\psi_{3m}}{P'_{3n}(\tau_{3m})(1 + \tau_{3m})}\,. \end{aligned} \tag{26}$$

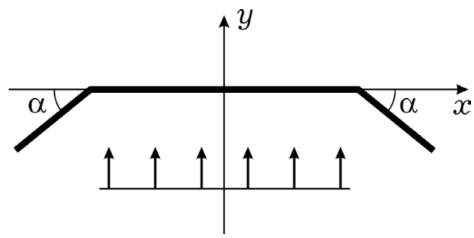

Fig. 2

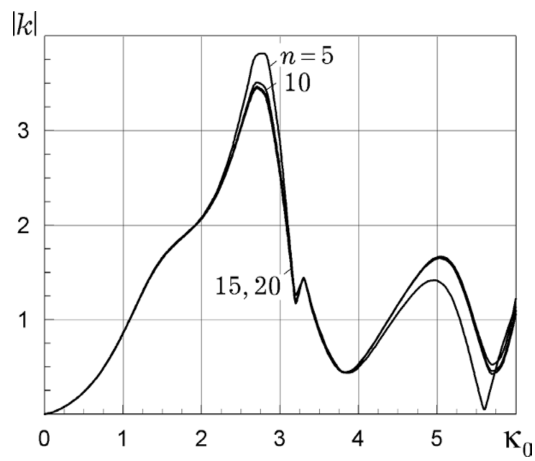

Fig. 3

Results of Numerical Analysis and Conclusions

In our numerical analyses, we made an attempt, first, to study the practical convergence of the method proposed for the numerical solution of the problem. Thus, we considered a crack whose configuration is illustrated in Fig. 2. We accepted the following relation between the lengths of the links: $d_2 = 3d_1 = 3d_3$.

By using relations (26), we determined the dimensionless values of the stress intensity factors

$$k^- = \frac{K^-}{G\sqrt{d_1}} \quad \text{and} \quad k^+ = \frac{K^+}{G\sqrt{d_3}} .$$

Moreover, it follows from the symmetry of the problem that $k^+ = k^- = k$. The calculations were carried out for the angle $\alpha = 45°$.

The results of calculations are presented in the form of plots of the absolute value of the stress intensity factor as a function of the dimensionless wave number $\kappa_0 = \kappa_2 d$. The values $n = 5, 10, 15$, and 20 correspond to the numbers of interpolation nodes in relation (15). We see that it suffices to take at most 20 interpolation nodes in (15) in order to get the values of the stress intensity factors with an error of at most 0.1%. For the waves with small frequencies $\kappa_0 \le 2$, it is sufficient to take five nodes.

We also studied the influence of crack shape on the value of stress intensity factor by using the crack depicted in Fig. 4 as an example. The ratio of the lengths of its links is the same as in the previous case. We performed the numerical analysis of the influence of the angle β on the frequency dependence of the stress intensity factor.

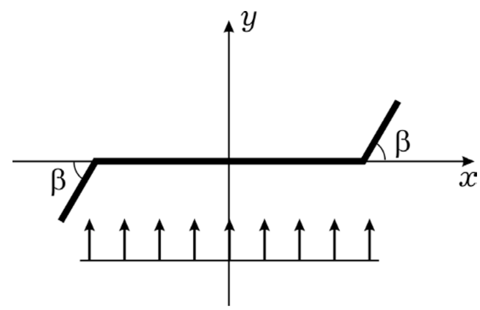

Fig. 4

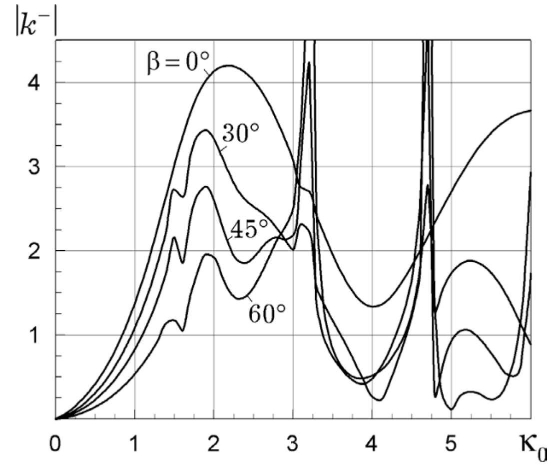

Fig. 5

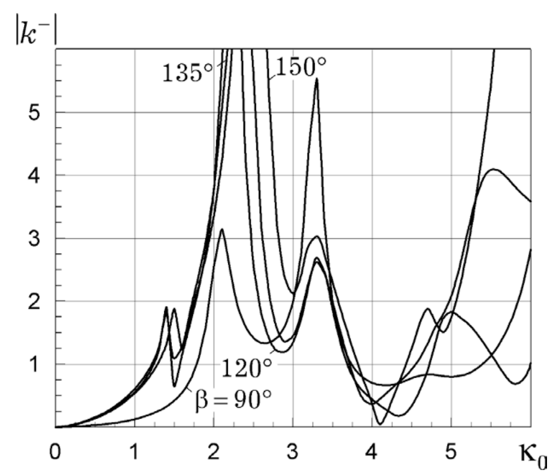

Fig. 6

The results of calculations are presented in Fig. 5 (for $\beta = 0$, 30, 45, and 60°) and in Fig. 6 (for $\beta = 90$, 120, 135, and 150°). It is easy to see that, for acute angles β, the rectilinear crack ($\beta = 0°$) has the maximum values of the stress intensity factor. As this angle increases, the values of stress intensity factor decrease and are minimum in the case where the adjacent links of the crack are perpendicular.

As the angle between the links increases further, the stress intensity factors exhibit a trend to increase and exceed the values corresponding to the rectilinear crack. In general, in view of complexity of the wave field created by the reflection of waves from the crack links, the frequency dependence of the stress intensity factor has well-pronounced maxima whose amplitudes and positions depend on the configuration of the crack.

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