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ON CONVERGENCE RATE ESTIMATES FOR SOME BIRTH AND DEATH PROCESSES

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Homogeneous birth and death processes with a finite number of states are studied. We analyze the slowest and fastest rates of convergence to the limit mode. Estimates of these bounds for some classes of mean-field models are obtained. The asymptotics of the convergence rate for some models of chemical kinetics is studied in the case where the number of system states tends to infinity.

1. Introduction

The problem of analyzing the rate of convergence to the limit mode for birth and death processes (BDP) has been studied for the last two decades in many papers (see, e.g., [1–3]). The relation between spectrum bounds of BDP's intensity matrix and the slowest and fastest rates of convergence is well known. The common approach (also applicable to the inhomogeneous BDP with a countable state space) was proposed in [11, 12]. The method is based on two main components: the logarithmic operator norm and the related estimates, and special transforms of the reduced intensity matrix. The first applications of mean-field models were considered in [5–7].

Let $X(t),\ t \geqslant 0$, be the BDP with a phase space $E = \{0, 1, \dots, N\}$, and $a_k > 0$, $0 \leqslant k \leqslant N-1$, and $b_k > 0$, $1 \leqslant k \leqslant N$, $\quad b_0 = a_N = 0$ be its birth and death intensities respectively.

Denote by Σ the spectrum of the corresponding intensity matrix. Let $(-\xi)$ and $(-\beta)$ be the minimum and maximum points of $\Sigma \setminus \{0\}$ respectively. It is well known that all points of $\Sigma \setminus \{0\}$ are real, distinct and negative.

In the present paper we give the estimates of ξ and β for some mean-field models and in some cases study their asymptotics when $N \to \infty$.

Let us consider a direct Kolmogorov system for the BDP $X(t)$:

$$\begin{pmatrix} \dfrac{dp_0}{dt} \\ \dfrac{dp_1}{dt} \\ \vdots \\ \dfrac{dp_N}{dt} \end{pmatrix} = \begin{pmatrix} -a_0 & b_1 & 0 & 0 & \cdots \\ a_0 & -(a_1+b_1) & b_2 & 0 & \cdots \\ 0 & a_1 & -(a_2+b_2) & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & \cdots & \cdots & a_{N-1} & -b_N \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_N \end{pmatrix}, \tag{1}$$

where $p_0 = p_0(t),\ p_1 = p_1(t), \dots, p_N = p_N(t)$ are the state probabilities for $X(t)$.

Setting $p_0(t) = 1 - \sum\limits_{i=1}^{N} p_i(t)$, we come to the system of differential equations of the form

$$\frac{d\mathbf{z}(t)}{dt} = B\mathbf{z}(t) + \mathbf{f}. \tag{2}$$

Further analysis is based on studying the convergence rate of solutions of the system

$$\frac{d\mathbf{x}(t)}{dt} = B\mathbf{x}(t) \tag{3}$$

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to zero. To study this system, special renormalizations and the logarithmic norm are used (see the cited papers). Here we formulate only the main results, on which further estimates are based.

To formulate the main results, let us consider auxiliary sequences $\{\delta_i\}, \{\sigma_i\}$ of positive numbers and the values

$$\alpha_i = a_i + b_{i+1} - a_{i+1}\delta_{i+1} - \delta_i^{-1} b_i, \tag{4}$$

$$\chi_i = a_i + b_{i+1} + a_{i+1}\sigma_{i+1} + \sigma_i^{-1} b_i. \tag{5}$$

The following double-sided estimates for the convergence rate are valid.

Theorem 1. *Let the birth and death intensities be positive, and the number β satisfy the condition*

$$\min_{0 \leqslant i \leqslant N-1} \alpha_i \leqslant \beta \leqslant \max_{0 \leqslant i \leqslant N-1} \alpha_i.$$

Then for some appropriate norm

$$\| x^* (t) - x^{**} (t) \| \leqslant \| x^* (0) - x^{**} (0) \| e^{-\beta t}$$

for all $t \geqslant 0$, and there is a unique positive sequence $\{\delta_i^*\}$ such that $\beta^* = \alpha_i$ for all i; hence, the corresponding convergence rate estimate is precise.

Theorem 2. *Let the birth and death intensities be positive, and the number ξ satisfy the condition*

$$\min_{0 \leqslant i \leqslant N-1} \chi_i \leqslant \xi \leqslant \max_{0 \leqslant i \leqslant N-1} \chi_i.$$

Then for some appropriate norm

$$\| x^* (t) - x^{**} (t) \| \geqslant \| x^* (0) - x^{**} (0) \| e^{-\xi t}$$

for all $t \geqslant 0$, and there is a unique positive sequence $\{\sigma_i^*\}$ such that $\xi^* = \chi_i$ for all i; hence, the corresponding convergence rate estimate is precise.

Note that in practice δ_k^* and σ_k^*, are usually intractable, and the main interest is in finding the sequences giving the closest estimates for β and χ.

2. Mean-field models

Consider an ergodic system with local interaction of particles (see [8]) φ_t, $t \geqslant 0$, on the complete graph with N nodes (cells). Such systems are well known in statistical physics as mean-field models (see the references in [4]). They are defined by $2N$ jump intensities λ_i, μ_i, $i = 0, \dots, N-1$, where λ_i and μ_i are birth (death) intensities in a free (occupied) graph cell when i adjacent cells are occupied. The relation between such process and the BDP is well known: $X_t^{(N)} = |\varphi_t|$, $t \geqslant 0$, where $|\varphi_t|$ is the number of occupied cells at the moment t, and $X_t^{(N)}$, $t \geqslant 0$, is a DBP with intensities

$$a_k = (N-k)\lambda_k, \quad b_{k+1} = (k+1)\mu_k, \quad k = 0, \dots, N-1. \tag{6}$$

The behavior of $\beta = \beta_N$ and $\chi = \chi_N$ for some classes of mean-field models was studied in [5–7].

In the present paper we consider other classes of such models describing stochastic models of chemical kinetics.

Let us set

$$A_k = \lambda_k + \mu_k,$$

$$B_k = \lambda_k + \mu_k + k\frac{\lambda_{k-1}\mu_k - \lambda_k\mu_{k-1}}{\lambda_{k-1}},$$

$$C_k = \lambda_k + \mu_k + (N-k-1)\frac{\lambda_k\mu_{k+1} - \lambda_{k+1}\mu_k}{\mu_{k+1}},$$

$$D_k = 2\mu_k + (N-2k)\left(\lambda_k - \mu_k\right),$$

$$E_k = (N-k)\lambda_k + (k+1)\mu_k.$$

Theorem 3. *Let λ_k descend and μ_k grow in k. Then the following estimate is valid:*

$$\mathfrak{A} \leqslant \beta_N \leqslant \mathfrak{B}, \tag{7}$$

where

$$\mathfrak{A} = \max\left(\min\left(A_k\right), \min\left(B_k\right), \min\left(C_k\right), \min\left(D_k\right)\right), \tag{8}$$

$$\mathfrak{B} = \min\left(\min\left(E_k\right), \max\left(B_k\right), \max\left(C_k\right), \max\left(D_k\right)\right).$$

Proof. Let us prove the validity of each estimate. First, let us note that for any choice of δ_k we have $\alpha_i < a_i + b_{i+1} = E_i$ for any i. Hence, $\beta_N < \min\left(E_k\right)$.

Then, setting all $\delta_k = 1$, we get $\alpha_k = (N-k-1)\left(\lambda_k - \lambda_{k+1}\right) + k\left(\mu_k - \mu_{k-1}\right) + \lambda_k + \mu_k \geqslant A_k$ for any k. From here we obtain the estimate $\beta_N \geqslant \min\left(A_k\right)$.

Setting $\delta_k = \frac{\lambda_{k-1}}{\lambda_k}$, we get $\alpha_k = B_k$, and thus

$$\min\left(B_k\right) \leqslant \beta_N \leqslant \max\left(B_k\right).$$

The remaining two estimates are obtained similarly, the first one by choosing $\delta_k = \frac{\mu_{k-1}}{\mu_k}$, and the second one by choosing $\delta_k = \frac{\mu_{k-1}}{\lambda_k}$.

Let us introduce the following values:

$$F_k = (2N-2k-1)\lambda_k + \mu_k + k\frac{\lambda_{k-1}\mu_k + \lambda_k\mu_{k-1}}{\lambda_{k-1}},$$

$$G_k = \lambda_k + (2k+1)\mu_k + (N-k-1)\frac{\lambda_k\mu_{k+1} + \lambda_{k+1}\mu_k}{\mu_{k+1}},$$

$$H_k = N\left(\lambda_k + \mu_k\right),$$

$$L_k = 2N\lambda_k + (2k+1)\left(\mu_k - \lambda_k\right).$$

Theorem 4. *Let λ_k descend and μ_k grow in k. Then the following estimate is valid:*

$$\mathfrak{C} \leqslant \xi_N \leqslant \mathfrak{D}, \tag{9}$$

where

$$\mathfrak{C} = \max\left(\max\left(E_k\right), \min\left(F_k\right), \min\left(G_k\right), \min\left(H_k\right)\right), \tag{10}$$

$$\mathfrak{D} = \min\left(\max\left(F_k\right), \max\left(G_k\right), \max\left(H_k\right), \max\left(L_k\right)\right).$$

Proof. First, let us note that for any choice of σ_k we have $\chi_i > a_i + b_{i+1} = E_i$ for any i. Hence, $\xi_N > \max(E_k)$.

Other inequalities are proved as in the precious theorem. That is, setting all $\sigma_k = 1$, we get

$$\chi_k = (N-k-1)(\lambda_k + \lambda_{k+1}) + k(\mu_k + \mu_{k-1}) + \lambda_k + \mu_k \leqslant L_k$$

for any k. From here the estimate $\xi_N \leqslant \max(L_k)$ follows.

Setting $\sigma_k = \frac{\lambda_{k-1}}{\lambda_k}$, we get $\alpha_k = F_k$, and thus $\min(F_k) \leqslant \xi_N \leqslant \max(F_k)$. The remaining two estimates are obtained similarly, the first one by choosing $\sigma_k = \frac{\mu_{k-1}}{\mu_k}$, and the second one by choosing $\sigma_k = \frac{\mu_{k-1}}{\lambda_k}$.

3. Examples

Let us consider some chemical kinetics models, described in [10]. The first results of estimating β are given in [9].

Example 1. Consider the reaction $A + B \leftrightarrow C$. It takes place in a closed vessel of volume V. The number of molecules of the substance C is described by the mean-field model with intensities of the form $\lambda_k = \frac{a}{V}(N-k)$, $\mu_k = b$. Let us analyze the behavior of β_N in the case where $V = N^r$ as $N \to \infty$. First, note that in this example $\lambda_k \mu_{k+1} - \lambda_{k+1}\mu_k = \frac{ab}{N^r}$.

The most precise results are obtained for $r > 1$. In this case we sequentially have

$$C_k = b + \frac{a}{N^r}(2N - 2k + 1), \tag{11}$$

$$\lim_{N\to\infty} \beta_N = b.$$

For $r = 1$ from (11) we get $\beta_N = O(1)$.

Let now $0 \leqslant r < 1$. Consider

$$B_k = \frac{a}{N^r}(N-k) + b + \frac{bk}{N-k+1}. \tag{12}$$

Setting $x = N - k$ and analyzing the corresponding function, we get $2\sqrt{abN^{1-r}} \leqslant B_k \leqslant bN + o(1)$, and hence for large N the following estimate is valid:

$$O\left(N^{\frac{1-r}{2}}\right) \leqslant \beta_N \leqslant O(N).$$

Let us now analyze the behavior of ξ_N. On the one hand, from the formula

$$E_k = (N-k)\lambda_k + (k+1)\mu_k = \frac{a(N-k)^2}{N^r} + (k+1)b$$

and Theorem 4, it follows that $\xi_N > \max\left(aN^{2-r},\ bN\right)$. On the other hand, analyzing the right-hand estimate in (9) and the values H_k, we obtain

$$H_k = N(\lambda_k + \mu_k) = aN^{1-r}(N-k) + bN,$$

and hence $\xi_N < H_0 = aN^{2-r} + bN$.

Now for $r > 1$ and $0 \leqslant r < 1$ we have respectively:

$$\lim_{N\to\infty} \frac{\xi_N}{N} = b \quad \text{and} \quad \lim_{N\to\infty} \frac{\xi_N}{N^{2-r}} = a,$$

and finally, for $r = 1$ and large N

$$\max(a, b) \leqslant \frac{\xi_N}{N} \leqslant a + b.$$

Example 2. Consider the reaction $A \leftrightarrow 2C$. This reaction is described by the mean-field model with intensities of the form $\lambda_k = a,\ \mu_k = \frac{b}{V}k$, assuming that $k \geqslant 1$. Analyzing the behavior of β_N in the case where $V = N^r$ as $N \to \infty$, we get, as in Example 1, interchanging B_k and C_k in our considerations:

$\lim\limits_{N\to\infty} \beta_N = a \quad$ for $r > 1$;

$\beta_N = O(1) \quad$ for $r = 1$;

$O\left(N^{\frac{1-r}{2}}\right) \leqslant \beta_N \leqslant O(N) \quad$ for $0 \leqslant r < 1$ and large N.

The behavior of ξ_N in this situation is studied in the same way as in Example 1.

From the formula

$$E_k = (N-k)\lambda_k + (k+1)\mu_k = a(N-k) + \frac{bk(k+1)}{N^r}$$

and Theorem 4 it follows that $\xi_N > \max\left(aN, bN^{2-r}\right)$. On the other hand, analyzing the right-hand estimate in (9) and the values H_k, we obtain

$$H_k = N(\lambda_k + \mu_k) = aN + bkN^{1-r},$$

hence, $\xi_N < H_1 < aN + bN^{2-r}$.

Now we have, for $r > 1$ and $0 \leqslant r < 1$ respectively,

$$\lim_{N\to\infty} \frac{\xi_N}{N} = a \quad \text{and} \quad \lim_{N\to\infty} \frac{\xi_N}{N^{2-r}} = b,$$

and finally, for $r = 1$ and large N

$$\max(a, b) \leqslant \frac{\xi_N}{N} \leqslant a + b.$$

Example 3. Consider the reaction $A + B \leftrightarrow C + D$. This reaction is described by the mean-field model with intensities $\lambda_k = \frac{a}{V}(N-k),\ \mu_k = \frac{b}{V}k$.

Let us analyze the behavior of β_N again in the situation where $V = N^r$ as $N \to \infty$. In this example, $\lambda_k\mu_{k+1} - \lambda_{k+1}\mu_k = \frac{abN}{N^{2r}}$. For definiteness let us limit ourselves to the case $a \geqslant b$. Then the sequence

$$C_k = N^{-r}\left(a(N-k) + bk + \frac{N-k-1}{k+1}aN^{1-r}\right)$$

monotonically decreases, and hence

$$C_{N-1} = aN^{-r} + b(N-1)N^{-r} \leqslant \beta_N \leqslant C_1 \leqslant aN^{1-r} + a(N-1)N^{1-2r}. \tag{13}$$

First, let $r > 1$. In this case from (13) we get $\beta_N = O\left(N^{1-r}\right)$.

If in addition $a = b$, then we get $\lim\limits_{N\to\infty} \frac{\beta_N}{N^{1-r}} = b$.

For $r = 1$ from (13) we get $\beta_N = O(1)$.

$$\text{If} \quad 0 \leqslant r < 1, \quad \text{then} \quad O\left(N^{1-r}\right) \leqslant \beta_N \leqslant O\left(N^{2(1-r)}\right).$$

Let us analyze the behavior of ξ_N. From the formula

$$E_k = (N-k)\lambda_k + (k+1)\mu_k = \frac{a(N-k)^2}{N^r} + \frac{bk(k+1)}{N^r}$$

and Theorem 4 it follows that $\xi_N \geqslant (N-1)^{2-r} \max(a,b) = a(N-1)^{2-r}$. On the other hand, analyzing the right-hand estimate in (9) and the values H_k, we obtain

$$H_k = N(\lambda_k + \mu_k) = N^{1-r}(a(N-k)+bk),$$

hence $\xi_N \leqslant H_1 \leqslant aN^{2-r}$.

Then for any $r \geqslant 0$

$$\lim_{N\to\infty} \frac{\xi_N}{N^{2-r}} = a.$$

Now let us consider a more general mean-field model, for which the precise asymptotics of the lower spectrum bound can be found.

Theorem 5. *Let $\lambda_k = \alpha(N-k)^a$, and $\mu_k = \beta(k+1)^b$, where a, b, α, β are some positive constants, and $a \neq b$. Then for $a < b$*

$$\lim_{N\to\infty} \frac{\xi_N}{N^{1+b}} = \beta,$$

and for $a > b$

$$\lim_{N\to\infty} \frac{\xi_N}{N^{1+a}} = \alpha.$$

Proof. Let $a < b$. Set $\sigma_k = \dfrac{\beta\left(N^{1+b} - k^{1+b}\right)}{\alpha(N-k)^{1+a}}$. Then

$$\chi_k = N^{1+b}\left(\beta + \frac{\alpha(N-k)^{1+a}}{N^{1+b} - k^{1+b}}\right)$$

monotonically decreases and hence

$$\beta N^{1+b} < \chi_{N-1} \leqslant \chi_k \leqslant \chi_0 = \beta N^{1+b} + \alpha N^{1+a}.$$

From here the first equality follows.

To prove the second one we set

$$\sigma_k = \frac{\beta k^{1+b}}{\alpha\left(N^{1+a} - (N-k)^{1+a}\right)}.$$

Now we have

$$\chi_k = N^{1+a}\left(\alpha + \frac{\beta(1+k)^{1+b}}{N^{1+a} - (N-k-1)^{1+a}}\right).$$

Let us prove that χ_k grows. Consider the function $f(x) = \frac{x^{1+b}}{N^{1+a} - (N-x)^{1+a}}$. It is easy to verify that the sign of $f'(x)$ is the same as the sign of the function

$$g(x) = (1+b)N^{1+a} - (N-x)^a\left((1+b)N + x(a-b)\right).$$

In addition, $g(0) = 0$, and $g'(x) > 0$ for all $x \in (0, N)$. Thus, $f(x)$ grows, and hence χ_k also grows. From here we obtain the estimate

$$\alpha N^{1+a} \leqslant \chi_k \leqslant \alpha N^{1+a} + \beta N^{1+b},$$

and hence the second equality.

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