DOI 10.1007/s10958-017-3240-8
Journal of Mathematical Sciences, Vol. 221, No. 2, February, 2017

SOME PROBLEMS OF QUALITATIVE ANALYSIS IN THE MODELING OF THE MOTION OF RIGID BODIES IN RESISTIVE MEDIA

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UDC 517; 531.01

Abstract. In this paper, we present a qualitative analysis of plane-parallel and spatial problems on the motion of realistic rigid bodies in a resistive medium and construct a nonlinear model of the influence of the medium on the rigid body.

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In the first chapter, we present a qualitative analysis of plane-parallel and spatial problems on the motion of realistic rigid bodies in a resistive medium. We construct a nonlinear model of the influence of the medium on the rigid body, in which the dependence of the arm of force on the reduced angular velocity of the body is taken into account; in this case, the moment of force is also a function of the angle of attack. Experiments on the motion of homogeneous circular cylinders in water show that these circumstances must be taken into account in modeling (see [9–11, 14–16]). In the study of plane and spatial models of interaction of a rigid body with a medium (both in the presence or absence of an additional tracking force), we find sufficient conditions of stability of one of the key regimes of motion, rectilinear translational motion. We show that under certain conditions, stable or unstable autooscillation regimes in the system can appear.

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 98, Geometry and Mechanics, 2015.

Similar conditions are obtained for the key regime of the motion of a four-dimensional rigid body in a nonconservative force field. In this case, we note mechanical and topological analogies between the motion of low-dimensional bodies in a resistive medium and higher-dimensional bodies in the corresponding nonconservative field.

Chapter 2 is devoted to the study of the problem of the plane-parallel motion of a rigid body that interacts with a medium only on a frontal flat domain of its outer surface. In the construction of the influence function of the medium we use information on properties of the jet flow under quasi-stationarity conditions (for example, when a homogeneous circular cylinder is immersed in water). We do not examine the motion of the medium, but we consider the problem of the rigid-body dynamics in which the characteristic time of motion of the body relative to its center of mass is comparable with the characteristic time of motion of the center itself. In the first chapter we deduce conditions of asymptotic stability of the rectilinear translational deceleration, in [28, 32–34, 36, 40, 62] a new multiparameter family of phase portraits in the space of quasi-velocities is obtained, and in the second chapter we prepare quantitative material for further full-scale experiments on the motion of hollow circular cylinders in a medium.

Acknowledgment. This work was partially supported by the Russian Foundation for Basic Research (project No. 12-01-00020-a).

Chapter 1

ON THE STABILITY OF SOME KEY REGIMES OF MOTION OF A RIGID BODY IN A NONCONSERVATIVE FORCE FIELD

In this paper, we perform a qualitative analysis of plane-parallel and spatial problems on the motion of realistic rigid bodies in resistive media.

We construct a nonlinear model of the influence of a medium on a rigid body in which the dependence of the arm of force on the reduced angular velocity is taken into account and the moment of the force is also a function of the angle of attack. Results of experiments on the motion of homogeneous circular cylinders in water show that these circumstances must be taken into account in modeling (see [8, 12, 13, 18, 23, 25, 29]).

In the study of plane and spatial models of interaction of a rigid body with medium (both in the presence or absence of an additional tracking force), we find sufficient conditions of stability of one of the key regimes of motion, rectilinear translational motion. We show that under certain conditions, stable or unstable autooscillation regimes in the system can appear.

Similar conditions are also obtained for the key regime of the motion of a four-dimensional rigid body in a nonconservative force field; we also note mechanical and topological analogies between the motion of low-dimensional bodies in a resistive meduim and higher-dimensional bodies in the corresponding nonconservative field.

1. Introduction

We study the problem on the motion of a rigid body that interacts with a medium only on a frontal flat domain of its outer surface. In the construction of the influence function of the medium we use information on properties of the jet flow under quasi-stationarity conditions (see [18–20, 30, 60, 61]). We do not examine the motion of the medium, but we consider the problem of the rigid-body dynamics in which the characteristic time of motion of the body relative to its center of mass is comparable with the characteristic time of motion of the center itself.

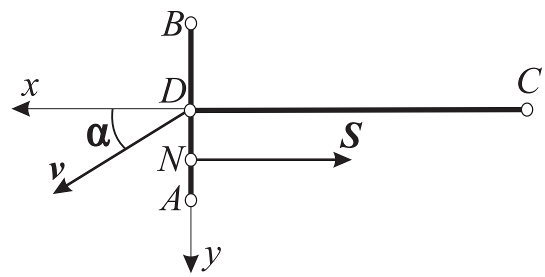

Fig. 1. Plane-parallel motion of a symmetric rigid body is a resistive medium

Since the nonlinear analysis is quite difficult, we neglect the dependence of the moment of force on the angular velocity of the body but take into account the dependence on the angle of attack (see also [24, 26, 27, 31, 41])).

From the practical point of view, the problem on the stability of the nonperturbed (rectilinear translational) motion is of interest; in this motion, velocities of points of the body are perpendicular to the lamina (cavitator).

Results obtained under this simplifying assumption allow one to conclude that there is no conditions under which the systems considered have solutions corresponding to angular oscillations of limited amplitude.

Experiments on the motion of homogeneous circular cylinders in water (see [29]) confirm that in the modeling of the influence of a medium on a rigid body one must take into account the dependence of the moment of force on the angular velocity of the body. In this case, equations of motions contain additional dissipative terms.

In the study of the motion of a body with finite angles of attack, the main problem of the nonlinear analysis is to obtain conditions under which there exist bounded-amplitude oscillations about the nonperturbed motion, which confirms the necessity of the complete nonlinear study.

2. Plane-parallel Motion of a Symmetric Rigid Body in a Resistive Medium

Assume that a *homogeneous* rigid body of mass m performs a plane-parallel motion in a homogeneous medium and some domain of the outer surface of the body is a plane lamina AB under the conditions of jet flow. This means that in the case where tangential forces are absent, the influence of the medium on the lamina is described by a normal force $\mathbf{S}$ applied at a point N (see Fig. 1). The remaining part of the surface of the body is located inside the volume bounded by the jet surface; it is not affected by the medium. Similar conditions appear, for example, after the immersion of a body in water (see [29]). We also assume that the gravity force acting on a body is negligible compared with the resistance force (see also [44]).

Consider a right-hand coordinate system $Dxyz$ attached to the lamina (the axis z is perpendicular to the plane of the figure); for simplicity, we assume that Dzx is the plane of geometric symmetry of the body. Then among possible motions, there exist a regime of rectilinear translational deceleration (nonperturbed motion) that is perpendicular to the lamina AB. The perpendicular bisector Dx passing through the center of gravity C of the body belongs to the line of action of the force $\mathbf{S}$. Under a perturbation of this regime, the velocity vector $\mathbf{v}$ of the point D with respect to the medium deviates from the axis DC of geometric symmetry by an angle α (angle of attack).

To construct a dynamical model, we introduce the first three phase coordinates: the speed v of the point D with respect to the flow of the medium (see Fig. 1), the angle α, and the algebraic value Ω of the projection of the absolute angular velocity of the body on the axis z, $AB = \Delta$.

Assume that the magnitude of the force **S** is a quadratic function of v:

$$S = s_1 v^2, \tag{2.1}$$

where s_1 is a coefficient called the Newton resistance. We represent it in the form

$$s_1 = \frac{\rho P c_x}{2}, \tag{2.2}$$

where c_x is the dimensionless coefficient of the front resistance (here ρ is the density of the medium and P is the area of the lamina). This coefficient depends on the angle of attack, the Strouhal number, and other parameters. We introduce the "Strouhal-type" dimensionless phase variable

$$\omega \cong \frac{\Omega \Delta}{v} \tag{2.3}$$

and the auxiliary function

$$s = s_1 \operatorname{sgn} \cos \alpha; \tag{2.4}$$

the influence of the medium on the body is described by the pair of functions (y_N, s).

We restrict ourselves to the case of the dependence of the coefficient c_x on the angle of attack, i.e., we assume that s is a function of α and $y_N = DN$ is a function of two dimensionless variables (α, ω).

As was noted above, previous works were devoted to the study of a plane interaction in which only the dependence of the pair (y_N, s) on the angle of attack is takes into account. Here we examine plane-parallel and spatial motions of bodies in the nonlinear case where s depends on the angle of attack and, in addition, the function y_N depends on the reduced angular velocity ω.

The free deceleration of a body (i.e., the motion of a body under the action of a single resistance force; see case I below) *with small angles of attack* is an example of nonlinear dynamical systems that describe the interaction of a medium with a body with account of rotational derivatives of the moment of force with respect to the angular velocity. The term "retational derivative" is often used in hydrodynamics in the case where differentiation of dynamical functions is performed in a noninertial coordinate system; if the moment of force depends on the angular velocity, then it is linearly involved in the equations of motion.

The nonperturbed motion is determined by the equations

$$\alpha(t) \equiv 0, \quad \omega(t) \equiv 0. \tag{2.5}$$

Therefore, the function $y_N(\alpha, \omega)$ for small (α, ω) has the form

$$y_N = \Delta(k\alpha - h\omega), \tag{2.6}$$

where k and h are some constants. Since the function s is even (due to the geometric symmetry of the body), we neglect the dependence of s on α.

Linearized models of the force action of the medium contain three parameters $s = s_1$, k, and h, which depend on the shape of the lamina. The first of these parameters, the coefficient s, is dimensional, whereas the parameters k and h are dimensionless. The parameters s and k can be found experimentally by weight measuring in wind tunnels. In [22], one can find information on theoretic methods of finding these values for some shapes of laminas; this information allows one to assume that $k > 0$. As for the parameter h (which provides the dependence of the moment of force on the angular velocity), the necessity of introduction of this parameter to the model theory is not a priori obvious.

The study of properties of motions of some classes of bodies in the Institute of Mechanics of the M. V. Lomonosov Moscow State University (see [29]) began from experiments with homogeneous circular cylinders. The experiments allow one to state the following conclusions.

1. The nonperturbed regime of the motion of a body in water is unstable, at least with respect to the orientation angle of the body. These experiments give a possibility of finding the dimensionless parameters k and h of the influence of a medium on a rigid body.

2. In the modeling of the influence of a medium on a body, one must take into account an additional parameter equivalent to the rotational derivative of the moment of hydroaerodynamical forces with respect to the angular velocity of the body, which leads to an additional dissipation in the system.

The magnitude of the coefficient of damping moment was estimated in [9] for some cases of motion of bodies in water. The estimates obtained indicates the instability of the unperturbed motion of a rigid body in water with respect to the angle of attack and the angular velocity. Formally, increasing this coefficient, one can achieve a motion, which in some media (for example, in clay) is stable in the above sense, as experiments show (see [1–3]). However, this stability is perhaps achieved owing to the damping in the system caused by forces tangential to the lamina.

Under the same assumptions about the character of interaction of a body with a medium, we distinguish a class of problems in which the body is affected by a tracking force (thrust) $\mathbf{T}$ along a straight line CD (see Fig. 1). For one of such problems considered in [12] under the assumption that the thrust is constant, the instability of the nonperturbed regime was proved.

We distinguish the cases of motion that were analyzed in detail.

I. A (free) deceleration of a body, i.e., a motion under the action of a single resistance force (tracking forces ar absent).

II. A motion of a body in which the speed of the center of the lamina is constant (The system contains a nonintegrable constraint):

$$v \equiv \text{const}. \tag{2.7}$$

III. A motion of a body in which the velocity of the center of mass is constant:

$$\mathbf{V}_C \equiv \mathbf{const}. \tag{2.8}$$

Note that in the case I, the nonperturbed regime is also called the *rectilinear translational deceleration.*

The position of a body on the plane is determined by the coordinates (x_0, y_0) of the point D and the angle of deviation φ. The polar coordinates (v, α) of the endpoint of the velocity vector of the point D and the algebraic value of the projection of the angular velocity Ω are related to the variables $(\dot{x_0}, \dot{y_0}, \dot{\varphi}, \varphi)$ by the (nonintegrable) kinematic relations:

$$\dot{\varphi} = \Omega, \quad \dot{x_0} = v\cos(\alpha + \varphi), \quad \dot{y_0} = v\sin(\alpha + \varphi). \tag{2.9}$$

Thus, the phase state of the system is determined by the functions $(v, \alpha, \Omega, x_0, y_0, \varphi)$, and the first three values are considered as quasi-velocities.

Since the kinetic energy of the body and the generalized forces are independent of the position of the body on the plane, the coordinates (x_0, y_0, φ) are cyclic; this leads to the decreasing of the order of the general system of equations of motion.

The equations of motion of the center of mass (in the projections to the axes Dxy) and the equations for the kinetic moment in the König axes for a closed system of differential equations are considered in the three-dimensional phase space of quasi-velocities (here $\sigma = DC$, I is the central moment of inertia; differentiation is performed with respect to time):

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha - \Omega v\sin\alpha + \sigma\Omega^2 = -\frac{s(\alpha)v^2}{m}, \tag{2.10}$$

$$\dot{v}\sin\alpha + \dot{\alpha}v\cos\alpha + \Omega v\cos\alpha - \sigma\dot{\Omega} = 0, \tag{2.11}$$

$$I\dot{\Omega} = y_N(\alpha, \omega)s(\alpha)v^2, \quad \omega \cong \frac{\Delta\Omega}{v}. \tag{2.12}$$

Equations (2.9), (2.10)–(2.12) form a complete system describing the plane-parallel motion of a rigid body in a resistive medium under quasi-stationarity conditions. If we consider problem II or III on the motion of a body in the presence of a tracking force, then the right-hand side of Eq. (2.10) contains the function

$$\frac{T - s(\alpha)v^2}{m}. \tag{2.13}$$

In particular, to provide the condition (2.7), the magnitude T of the tracking force must be chosen as follows:

$$T = T(v, \alpha, \Omega) = m\sigma\Omega^2 + s(\alpha)v^2\left[1 - \frac{m\sigma}{I}y_N(\alpha, \omega)\frac{\sin\alpha}{\cos\alpha}\right]; \tag{2.14}$$

the first equation in (2.10) holds identically. Note that the cases II and III have only methodological significance since they allow one to reduce the order of the system of equations of motion and lead to important mechanical analogies (see also [6, 7, 35, 37, 38, 64, 65]).

3. Influence Functions of Medium Depending on the Angular Velocity of the Body

The dynamical system (2.10)–(2.12) contains the functions $y_N(\alpha, \omega)$ and $s(\alpha)$ that determine the influence of the medium on the body. The function y_N (cf. (2.6)) depends on the angle of attack α and the reduced angular velocity ω. In particular, if we neglect the dependence on ω (this was done in a series of earlier papers: the so-called *simplest assumption* about influence functions of medium), then the function y_N depends only on the angle of attack, $y_N = y(\alpha)$, and the dependence on its single argument can be determined from experimental information on properties of jet circumfluence (see [29, 46, 48, 51]). In this case, one can apply the method of "immersion" of the problem into a more general class of problems.

The main purpose of this work is the consideration of the influence of rotational derivatives of the moment of the action force with respect to the components of the angular velocity of the body, which requires the introduction of additional arguments in the influence function of the medium, which is a nontrivial problem of modeling. As was noted above, in this paper we consider the case where the angular velocity is involved as an argument *only* in the function y_N and neglect the dependence of the coefficient s on the angular velocity.

Similarly to (2.6), we represent y_N as follows:

$$y_N(\alpha, \omega) \cong y_N(\alpha, \Omega/v) = y(\alpha) - \frac{H\Omega}{v}; \tag{3.1}$$

experiments show (see [29]) that $H > 0$.

Then Eq. (2.12) becomes

$$I\dot{\Omega} = F(\alpha)v^2 - Hs(\alpha)\Omega v, \quad F(\alpha) = y(\alpha)s(\alpha). \tag{3.2}$$

The system (2.10), (2.11), (3.2) contains the functions $F(\alpha)$ and $s(\alpha)$, whose explicit analytical form is quite difficult to find even for laminas of simple geometric shape. For this reason, we use the method of "immersion" of the problem considered into a wider class of problems in which only qualitative properties of functions $F(\alpha)$ and $s(\alpha)$ are taken into account.

We use the following representations of the functions $y(\alpha)$ and $s(\alpha)$ analytically obtained by Chaplygin (see [6, 7]) for the plane-parallel jet circumfluence of an infinite lamina:

$$y(\alpha) = A\sin\alpha \in \{y\}, \quad A = y'(0) > 0, \tag{3.3}$$

$$s(\alpha) = B\cos\alpha \in \{s\}, \quad B = s(0) > 0. \tag{3.4}$$

This result allows one to construct functional classes $\{y\}$ and $\{s\}$. Combining (3.3) and (3.4) with experimental information on the properties of jet circumfluence, we formally describe these classes consisting of sufficiently smooth, 2π-periodic functions ($y(\alpha)$ is odd whereas $s(\alpha)$ is even) that satisfy the following conditions: $y(\alpha) > 0$ for $\alpha \in (0, \pi)$ and

$$y'(0) > 0, \quad y'(\pi) < 0 \tag{3.5}$$

(the functional class $\{y\} = Y$); $s(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $s(\alpha) < 0$ for $\alpha \in (\pi/2, \pi)$, and

$$s(0) > 0, \quad s'(\pi/2) < 0 \tag{3.6}$$

(the functional class $\{s\} = \Sigma$). Both functions y and s change their sign under the replacement of α by $\alpha + \pi$. Thus,

$$y \in Y, \ s \in \Sigma. \tag{3.7}$$

The conditions listed above imply that the function F introduced in (3.2) is a sufficiently smooth, odd, π-periodic function satisfying the following conditions: $F(\alpha) > 0$ for $\alpha \in (0, \pi/2)$,

$$F'(0) > 0, \quad F'(\pi/2) < 0 \tag{3.8}$$

(the functional class $\{F\} = \Phi$).

In particular, the analytic function (see [6, 7])

$$F = F_0(\alpha) = AB \sin\alpha \cos\alpha \in \Phi, \quad AB = y'(0)s(0), \tag{3.9}$$

is a typical representative of the functional class Φ.

In connection with the instability of the nonperturbed motion indicated in [29, 45, 47, 51, 53–56, 59], we raise the following question: Do finite-amplitude angular oscillations of the symmetry axis exist?

We state this question in a more general form: Does a pair of influence functions y and s exist such that for some solution of the dynamical part of the equations of motion the inequality $0 < \alpha(t) < \alpha^* < \pi/2$ holds starting from some time instant $t = t_1$?

Under the simplest assumption about the functions y_N and s, it was proved earlier that in the quasi-stationary model of the interaction of a medium with a symmetric body (when y_N and s depend only on the angle of attack), for any admissible pair of functions y and s in the full range ($0 < \alpha < \pi/2$) of finite angles of attack, there is no finite-amplitude oscillation solutions in the system.

Thus, for a possible positive answer to the question raised above, we will take into account the dependence of the moment of the influence force of the medium on the angular velocity; we will use formula (3.1) for $H > 0$. It turns out that under some assumptions, one can expect a positive answer.

From the practical point of vies, the analysis of dynamical equations only in a neighborhood of the nonperturbed motion is interesting, since for some values of critical angles of attack, the lateral surface is blurred and the model of the influence of a medium on the body becomes invalid. However, for bodies with lateral surfaces of various shapes the critical values of angles are different and unknown, which leads to the necessity to study the whole range of angles.

Thus, for the study of the plane-parallel circumfluence of a lamina we use several classes of dynamical systems defined by pairs of influence functions, which considerably complicates the qualitative analysis (see also [57, 58]).

4. Motion of a Body in a Resistive Medium under a Tracking Force

4.1. Case II. Consider the motion of a body in a medium under a tracking force that provides the condition (2.7) during the motion. As was noted above, it suffices to choose the magnitude of this force so that the first equation (2.10) holds identically. Then, in addition to the parameters of the system introduced above, the positive parameter v appears, and the dynamical part of the equations of motion of the body in the case (2.7) takes the form of the following second-order system:

$$\begin{aligned} &\dot{\alpha} v \cos\alpha + \Omega v \cos\alpha - \sigma\dot{\Omega} = 0, \\ &I\dot{\Omega} = F(\alpha)v^2 - Hs(\alpha)\Omega v, \quad H > 0. \end{aligned} \tag{4.1}$$

Outside the union of the straight lines

$$O = \left\{(\alpha, \Omega) \in \mathbf{R}^2 : \ \alpha = \frac{\pi}{2} + \pi k, \ k \in \mathbf{Z}\right\} \tag{4.2}$$

this system is equivalent to the following system of the normal form:

$$\begin{aligned} &\dot{\alpha} = -\Omega + \frac{\sigma v}{I}\frac{F(\alpha)}{\cos\alpha} - \frac{\sigma}{I}H\frac{s(\alpha)}{\cos\alpha}\Omega, \\ &\dot{\Omega} = \frac{v^2}{I}F(\alpha) - H\frac{v}{I}\Omega s(\alpha). \end{aligned} \tag{4.3}$$

First, we examine the stability of its trivial solution corresponding to the *nonperturbed motion.* For this, we write the corresponding characteristic equation near the origin:

$$\lambda^2 + \lambda v\left[\frac{BH}{I} - \sigma n_0^2\right] + n_0^2 v^2 = 0, \tag{4.4}$$

where

$$A = y'_N(0), \quad B = s(0), \quad n_0^2 = \frac{F'(0)}{I} = \frac{y'_N(0)s(0)}{I} = \frac{AB}{I}. \tag{4.5}$$

Introduce the following three positive dimensionless parameters:

$$\mu_1 = 2\frac{B}{mn_0}, \quad \mu_2 = \sigma n_0, \quad \mu_3 = \frac{BH}{In_0}, \tag{4.6}$$

and the dimensionless differentiation and the substitution

$$\langle\ \cdot\ \rangle = n_0 v \langle\ '\ \rangle, \quad \Omega \mapsto n_0 v\Omega. \tag{4.7}$$

Then the system (4.3) takes the following form:

$$\begin{aligned} &\alpha' = -\Omega + \frac{\sigma}{In_0}\frac{F(\alpha)}{\cos\alpha} - \frac{\sigma H}{I}\Omega\frac{s(\alpha)}{\cos\alpha}, \\ &\Omega' = \frac{F(\alpha)}{In_0^2} - \frac{H}{In_0}\Omega s(\alpha). \end{aligned} \tag{4.8}$$

The following proposition is obvious.

Proposition 1. *For* $\mu_3 > \mu_2$ (*respectively,* $\mu_3 < \mu_2$) *the trivial solution of the system* (4.3) *is asymptotically stable* (*is repulsive*).

For the possible birth of a limit cycle near the origin, we examine the stability of the trivial solution of the system (4.3) under the *critical* relation of the parameters:

$$\mu_3 = \mu_2. \tag{4.9}$$

For this, we perform the following change of phase variables $(\alpha, \Omega) \mapsto (a, w)$ in the system (4.8):

$$\alpha = a, \quad \Omega = \frac{|\omega_0|}{1 + \mu_2^2}(\mu_2 a - w), \quad \omega_0 = 1; \tag{4.10}$$

then the system becomes

$$\begin{aligned} &a' = |\omega_0|w + A_3a^3 + A_4a^2w + \bar{o}_1((a^2 + w^2)^{3/2}), \\ &w' = -|\omega_0|a + A_1a^3 + A_2a^2w + \bar{o}_2((a^2 + w^2)^{3/2}), \end{aligned} \tag{4.11}$$

where

$$
\begin{aligned}
A_1 &= -\frac{f_3}{6In_0^2} + \frac{Hs_2}{2In_0}\frac{\mu_2}{1+\mu_2^2} + \frac{\mu_2^2}{2(1+\mu_2^2)},\\
A_2 &= -\frac{Hs_2}{2In_0}\frac{1}{1+\mu_2^2} + \frac{\mu_2^3}{2(1+\mu_2^2)},\\
A_3 &= \frac{\mu_2 f_3}{6In_0^2} - \frac{\mu_2 Hs_2}{2In_0}\frac{\mu_2}{1+\mu_2^2} + \frac{\mu_2}{2(1+\mu_2^2)},\\
A_4 &= \frac{Hs_2}{2In_0}\frac{\mu_2}{1+\mu_2^2} + \frac{\mu_2^2}{2(1+\mu_2^2)}, \quad s_2 = s''(0), \quad f_3 = F'''(0).
\end{aligned} \tag{4.12}
$$

Introduce the following auxiliary index (see [49, 50]):

$$
\begin{aligned}
\mathrm{In} &= |\omega_0|\left\{Y_{111}^1 + Y_{122}^1 + Y_{112}^2 + Y_{222}^2\right\} + \\
&+ \left(Y_{11}^1 Y_{11}^2 - Y_{11}^1 Y_{12}^1 + Y_{11}^2 Y_{12}^2 + Y_{22}^2 Y_{12}^2 - Y_{22}^1 Y_{12}^1 - Y_{22}^1 Y_{22}^2\right),\\
Y_{jkl}^i &= \frac{\partial^3 Y_i}{\partial y_j \partial y_k \partial y_l}(0,0), \quad Y_{jk}^i = \frac{\partial^2 Y_i}{\partial y_j \partial y_k}(0,0),
\end{aligned} \tag{4.13}
$$

where

$$
\begin{pmatrix} Y_1(a,w) \\ Y_2(a,w) \end{pmatrix} \tag{4.14}
$$

is the right-hand side of the system (4.11).

More specifically, for the system (4.11) the index just constructed has the form

$$
\mathrm{In} = 6A_3 + 2A_2 = \frac{\mu_2 f_3}{In_0^2} - \frac{Hs_2}{In_0(1+\mu_2^2)}(1+3\mu_2^2) + \frac{\mu_2}{1+\mu_2^2}(3+\mu_2^2). \tag{4.15}
$$

Since for this system

$$
Y_{jk}^i = \frac{\partial^2 Y_i}{\partial y_j \partial y_k}(0,0) = 0 \tag{4.16}
$$

(owing to the oddness of its right-hand side with respect to phase variables) for all indices i, j, and k, the following proposition provides necessary and sufficient conditions of asymptotic stability (and instability) of the origin for $\mathrm{In} \neq 0$.

Proposition 2. *If* $\mathrm{In} < 0$ (*respectively,* $\mathrm{In} > 0$) *and the inequality*

$$
|\mu_3 - \mu_2| < 2 \tag{4.17}
$$

holds, then the origin of the phase plane $\mathbf{R}^2\{a,w\}$ *of the system* (4.11) (*respectively,* (4.3)) *for the critical relation of the parameters* $\mu_3 = \mu_2$ *is a weak stable* (*respectively, unstable*) *focus.*

In this case, the condition (4.17) is necessary since only under this condition is the origin of the plane $\mathbf{R}^2\{a,w\}$ a (stable or unstable, straight or weak) focus.

The following theorem is a consequence of the well-known Poincaré–Andronov–Hopf theorem (see [21]).

Theorem 1. *Assume that for the system* (4.3) *the inequality* (4.17) *holds. Then the following assertions are valid.*

(1) If $\mathrm{In} < 0$, *then for each fixed* μ_2, *there exist* $\delta_1, \delta_2 > 0$ *such that for* $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$, *the origin is a straight stable focus, whereas for* $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$ *the origin is a straight unstable focus encircled by a stable limit cycle whose size increases as* $\sqrt{|\mu_2 - \mu_3|}$ *when* μ_3 *decreases from* μ_2 *to* $\mu_2 - \delta_2$.

(2) *If* In > 0, *then for each fixed* μ_2, *there exist* $\delta_1, \delta_2 > 0$ *such that for* $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$, *the origin is a straight unstable focus, whereas for* $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$ *the origin is a straight stable focus encircled by an unstable limit cycle whose size increases as* $\sqrt{|\mu_2 - \mu_3|}$ *when* μ_3 *increases from* μ_2 *to* $\mu_2 + \delta_1$.

It is easy to verify the condition $\mu_3 > \mu_2$ (or $\mu_3 < \mu_2$) since in each case these parameters depend either on only the first derivatives of the influence functions y_N and s or on their values. Conversely, the condition In < 0 (or In > 0) is difficult to verify since for each specific body, the explicit form and higher derivatives of the influence functions y_N and s are unknown.

4.2. Case III. Consider the motion of a body in a medium under a tracking force that provides the condition (2.8) during the motion.

Then the right-hand side of Eq. (2.10) contains zero instead of $-s(\alpha)v^2/m$ since the body is influenced by a nonconservative force couple:

$$T - s(\alpha)v^2 \equiv 0. \tag{4.18}$$

Obviously, in this case the magnitude of the tracking force T is equal to

$$T = T(v, \alpha, \Omega) = s(\alpha)v^2, \quad \mathbf{T} \equiv -\mathbf{S}. \tag{4.19}$$

Similarly to the choice of the influence function, we take the dynamical functions s and y_N of the system (2.10)–(2.12) in the form (3.1), (3.7). Moreover, the system considered also contains an additional damping moment of a nonconservative force (note that in some domains of the phase space this moment can become accelerating).

Introduce the new dimensionless phase variable and the dimensionless differentiation by the formulas

$$\Omega = n_0 v \omega, \quad \langle \cdot \rangle = n_0 v \langle \, ' \, \rangle; \tag{4.20}$$

then the system (2.10)–(2.12) can be rewritten in the following form:

$$v' = v\Psi(\alpha, \omega), \tag{4.21}$$

$$\begin{aligned} \alpha' &= -\omega + \mu_2\omega^2 \sin\alpha + \frac{\mu_2}{In_0^2}F(\alpha)\cos\alpha - \frac{\mu_2}{In_0}H\omega s(\alpha)\cos\alpha, \\ \omega' &= \frac{F(\alpha)}{In_0^2} + \mu_2\omega^3\cos\alpha - \frac{\mu_2}{In_0^2}\omega F(\alpha)\sin\alpha - \frac{H}{In_0}\omega s(\alpha) + \frac{\mu_2}{In_0}H\omega^2 s(\alpha)\sin\alpha, \end{aligned} \tag{4.22}$$

$$\Psi(\alpha, \omega) = -\mu_2\omega^2\cos\alpha + \frac{\mu_2}{In_0^2}F(\alpha)\sin\alpha - \frac{\mu_2 H}{In_0}\omega s(\alpha)\sin\alpha;$$

the dimensionless parameters $b = \mu_2$ and $H_1 = \mu_3$ are as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{I}, \quad H_1 = \frac{BH}{In_0}. \tag{4.23}$$

The last two equations (4.22) of the system (4.21), (4.22) form an independent second-order subsystem on the phase cylinder $\mathbf{S}^1\{\alpha \mod 2\pi\} \times \mathbf{R}^1\{\omega\}$.

First, we examine the stability of its trivial solution corresponding the *nonperturbed motion*; for this, we write the corresponding characteristic equation near the origin:

$$\lambda^2 + \lambda[\mu_3 - \mu_2] + 1 = 0. \tag{4.24}$$

The following assertion is obvious.

Proposition 3. *For* $\mu_3 > \mu_2$ (*respectively,* $\mu_3 < \mu_2$) *the trivial solution of the system* (4.22) *is asymptotically stable* (*is repulsive*).

For the possible birth of a limit cycle near the origin, we examine the stability of the trivial solution of the system (4.22) under the *critical* relation of the parameters:

$$\mu_3 = \mu_2. \tag{4.25}$$

For this, we perform the follow change of phase variables $(\alpha, \Omega) \mapsto (a, w)$ in the system (4.22):

$$\alpha = a, \quad \omega = \frac{|\omega_0|}{1+\mu_2^2}(\mu_2 a - w), \quad \omega_0 = 1, \tag{4.26}$$

which transforms it to the following system:

$$\begin{aligned} a' &= |\omega_0| w + B_1 a^3 + B_2 a^2 w + B_3 a w^2 + \bar{o}_1((a^2+w^2)^{3/2}), \\ w' &= -|\omega_0| a + B_4 a^3 + B_5 a^2 w + B_6 a w^2 + B_7 w^3 + \bar{o}_2((a^2+w^2)^{3/2}), \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} B_1 &= \frac{\mu_2 f_3}{6In_0^2} - \frac{Hs_2}{2In_0}\frac{\mu_2^2}{1+\mu_2^2} + \frac{\mu_2^3}{(1+\mu_2^2)^2} - \frac{\mu_2}{2(1+\mu_2^2)}, \\ B_2 &= \frac{Hs_2}{2In_0}\frac{\mu_2}{1+\mu_2^2} - \frac{2\mu_2^2}{(1+\mu_2^2)^2} - \frac{\mu_2^2}{2(1+\mu_2^2)}, \\ B_3 &= \frac{\mu_2}{(1+\mu_2^2)^2}, \\ B_4 &= -\frac{f_3}{6In_0^2} + \frac{Hs_2}{2In_0}\frac{\mu_2}{1+\mu_2^2} + \frac{\mu_2^2}{2(1+\mu_2^2)}, \\ B_5 &= -\frac{Hs_2}{2In_0}\frac{1}{1+\mu_2^2} + \frac{\mu_2^3}{2(1+\mu_2^2)} - \frac{\mu_2}{(1+\mu_2^2)^2}, \\ B_6 &= -\frac{\mu_2^2(3+\mu_2^2)}{(1+\mu_2^2)^2}, \\ B_7 &= \frac{\mu_2}{(1+\mu_2^2)^2}, \quad s_2 = s''(0), \quad f_3 = F'''(0). \end{aligned} \tag{4.28}$$

Introduce the following auxiliary index In similarly to (4.13). Namely, for the system (4.27) the index introduced has the form

$$\mathrm{In} = 6B_1 + 2B_3 + 2B_5 + 6B_7 = \frac{\mu_2 f_3}{In_0^2} - \frac{Hs_2}{In_0}\frac{1+3\mu_2^2}{1+\mu_2^2} + \frac{\mu_2}{1+\mu_2^2}(3+\mu_2^2), \tag{4.29}$$

coinciding with the index (4.15) for the system (4.11).

Since for this system

$$Y^i_{jk} = \frac{\partial^2 Y_i}{\partial y_j \partial y_k}(0,0) = 0 \tag{4.30}$$

(owing to the oddness of its right-hand side with respect to phase variables) for all indices i, j, and k, the following proposition provides a necessary and sufficient condition of asymptotic stability (instability) of the origin for $\mathrm{In} \neq 0$.

Proposition 4. *If* $\mathrm{In} < 0$ (*respectively,* $\mathrm{In} > 0$) *and the inequality*

$$|\mu_3 - \mu_2| < 2 \tag{4.31}$$

holds, then the origin of the phase plane $\mathbf{R}^2\{a, w\}$ *of the system* (4.27) (*respectively,* (4.22)) *under the critical relation of the parameters* $\mu_3 = \mu_2$ *is a weak stable* (*unstable*) *focus.*

In this case, the condition (4.31) is necessary since under this condition the origin of the plane $\mathbf{R}^2\{a, w\}$ is a (stable or unstable, straight or weak) focus.

The following theorem is a consequence of the well-known Poincaré–Andronov–Hopf theorem (see [21]).

Theorem 2. *Assume that for the system* (4.22) *the inequality* (4.31) *holds. Then the following assertions hold.*

(1) If $\mathrm{In} < 0$, *then for each fixed* μ_2, *there exist* $\delta_1, \delta_2 > 0$ *such that for* $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$, *the origin is a straight stable focus, whereas for* $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$, *the origin is a straight unstable focus encircled by a stable limit cycle whose size increases as* $\sqrt{|\mu_2 - \mu_3|}$ *when* μ_3 *decreases from* μ_2 *to* $\mu_2 - \delta_2$.

(2) If $\mathrm{In} > 0$, *then for each fixed* μ_2, *there exist* $\delta_1, \delta_2 > 0$ *such that for* $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$, *the origin is a straight unstable focus, whereas for* $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$, *the origin is a straight stable focus encircled by an unstable limit cycle whose size increases as* $\sqrt{|\mu_2 - \mu_3|}$ *when* μ_3 *increases from* μ_2 *to* $\mu_2 + \delta_1$.

It is easy to verify the condition $\mu_3 > \mu_2$ (or $\mu_3 < \mu_2$) since in each case these parameters depend either only on first derivatives of the influence functions y_N and s or only on their values. Conversely, the condition $\mathrm{In} < 0$ (or $\mathrm{In} > 0$) is difficult to verify since for each specific body, the explicit form and higher derivatives of the influence functions y_N and s are unknown.

5. Free Deceleration of a Rigid Body in a Resistive Medium (Case I)

Further, we consider the case of the motion of a body where the thrust control is turned off and the body freely moves (with deceleration) in a resistive medium (case I).

The right-hand side of Eq. (2.10) contains the function $-s(\alpha)v^2/m$ and the relation $\mathbf{T} \equiv \mathbf{0}$ holds.

Similarly to the choice of the influence function, we represent the dynamical functions s and y_N of the system (2.10)–(2.12) in the form (3.1), (3.7). As above, the system considered also contains an additional damping moment of a nonconservative force (in some domains of the phase space this moment can be accelerating).

Introduce the new dimensionless phase variable and the dimensionless differentiation by the formulas

$$\Omega = n_0 v \omega, \quad \langle \,\cdot\, \rangle = n_0 v \left\langle \,'\, \right\rangle; \tag{5.1}$$

then the system (2.10)–(2.12) can be transformed to the following form:

$$v' = v\Psi(\alpha, \omega), \tag{5.2}$$

$$\begin{aligned} \alpha' &= -\omega + \mu_2\omega^2 \sin\alpha + \frac{\mu_2}{In_0^2}F(\alpha)\cos\alpha - \frac{\mu_2}{In_0}H\omega s(\alpha)\cos\alpha + \frac{s(\alpha)}{mn_0}\sin\alpha, \\ \omega' &= \frac{F(\alpha)}{In_0^2} + \mu_2\omega^3\cos\alpha - \frac{\mu_2}{In_0^2}\omega F(\alpha)\sin\alpha - \frac{H}{In_0}\omega s(\alpha) + \frac{\mu_2}{In_0}H\omega^2 s(\alpha)\sin\alpha + \frac{s(\alpha)}{mn_0}\omega\cos\alpha, \end{aligned} \tag{5.3}$$

$$\Psi(\alpha, \omega) = -\mu_2\omega^2\cos\alpha + \frac{\mu_2}{In_0^2}F(\alpha)\sin\alpha - \frac{\mu_2 H}{In_0}\omega s(\alpha)\sin\alpha - \frac{s(\alpha)}{mn_0}\cos\alpha,$$

where the dimensionless parameters μ_1, $b = \mu_2$, and $H_1 = \mu_3$ are as follows:

$$\mu_1 = 2\frac{B}{mn_0}, \quad b = \sigma n_0, \quad n_0^2 = \frac{AB}{I}, \quad H_1 = \frac{BH}{In_0}. \tag{5.4}$$

The last two equations (5.3) of the system (5.2), (5.3) form an independent second-order subsystem on the phase cylinder $\mathbf{S}^1\{\alpha \mod 2\pi\} \times \mathbf{R}^1\{\omega\}$.

As above, we examine the stability of the trivial solution of the system (5.3), which, obviously, corresponds to the rectilinear translational deceleration (*nonperturbed motion*).

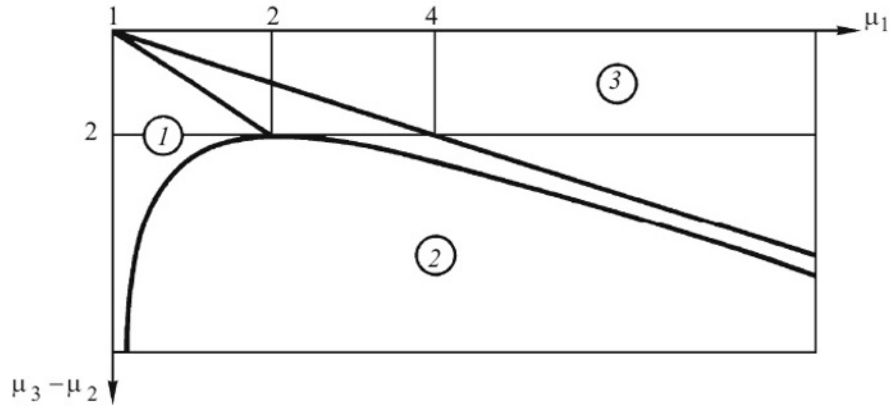

Fig. 2. General structure of rearrangements of trajectories of the vector field of the system (5.3)

We write the corresponding characteristic equation near the origin:

$$\lambda^2 - \lambda\left[\mu_1 + \mu_2 - \mu_3\right] + \frac{\mu_1}{2}\left(\frac{\mu_1}{2} + \mu_2 - \mu_3\right) + 1 = 0. \tag{5.5}$$

Proposition 5. *Assume that the inequality* (4.17) (*or* (4.31)) *holds. Then for* $\mu_3 > \mu_1 + \mu_2$ (*respectively,* $\mu_3 < \mu_1 + \mu_2$) *the trivial solution of the system* (5.3) *is asymptotic stable* (*respectively, is repulsive*)*.*

The general structure of rearrangements of trajectories of the vector field of the system (5.3) near the origin is presented in Fig. 2; the domains 1, 2, and 3 correspond to the attracting, saddle, and repulsive points, respectively.

To study the possibility of birth of a limit cycle near the origin, we examine the stability of the trivial solution of the system (5.3) under the *critical* combination of the parameters:

$$\mu_3 = \mu_1 + \mu_2. \tag{5.6}$$

For this, we perform the following change of phase variables $(\alpha, \omega) \mapsto (a, w)$ in the system (5.3):

$$\alpha = a, \quad \omega = \frac{(\mu_2 + \mu_1/2)\,a - \omega_0 w}{1 + \mu_1\mu_2 + \mu_2^2}, \quad \omega_0 = \sqrt{1 - \frac{\mu_1^2}{4}}. \tag{5.7}$$

Then we obtain the system

$$\begin{aligned} a' &= |\omega_0| w + C_1 a^3 + C_2 a^2 w + C_3 a w^2 + \bar{o}_1((a^2 + w^2)^{3/2}), \\ w' &= -|\omega_0| a + C_4 a^3 + C_5 a^2 w + C_6 a w^2 + C_7 w^3 + \bar{o}_2((a^2 + w^2)^{3/2}), \end{aligned} \tag{5.8}$$

where

$$
\begin{aligned}
C_1 &= \frac{\mu_2 f_3}{6In_0^2} - \frac{Hs_2}{2In_0}\frac{\mu_2(\mu_2+\mu_1/2)}{1+\mu_1\mu_2+\mu_2^2} + \frac{s_2}{2mn_0} - \frac{\mu_2}{2} - \frac{\mu_1}{12} \\
&+ \frac{\mu_2(\mu_1+\mu_2)(\mu_2+\mu_1/2)}{2(1+\mu_1\mu_2+\mu_2^2)} + \frac{\mu_2(\mu_2+\mu_1/2)^2}{(1+\mu_1\mu_2+\mu_2^2)^2}, \\
C_2 &= \frac{Hs_2}{2In_0}\frac{\mu_2\omega_0}{1+\mu_1\mu_2+\mu_2^2} - \frac{2\mu_2(\mu_2+\mu_1/2)\omega_0}{(1+\mu_1\mu_2+\mu_2^2)^2} - \frac{\mu_2(\mu_2+\mu_1/2)\omega_0}{2(1+\mu_1\mu_2+\mu_2^2)}, \\
C_3 &= \frac{\mu_2\omega_0^2}{(1+\mu_1\mu_2+\mu_2^2)^2}, \\
C_4 &= -\left(1+\frac{\mu_1\mu_2}{2}\right)\frac{f_3}{6In_0^2\omega_0} + \frac{Hs_2}{2In_0}\frac{(\mu_2+\mu_1/2)(1+\mu_1\mu_2/2)}{(1+\mu_1\mu_2+\mu_2^2)\omega_0} + \\
&+ \frac{\mu_2+\mu_1/2}{2(1+\mu_1\mu_2+\mu_2^2)\omega_0}\cdot\left[\left(\mu_2+\frac{\mu_1}{3}\right) - \frac{\mu_1\mu_2}{6}(\mu_1+\mu_2)\right], \\
C_5 &= -\frac{Hs_2}{2In_0}\frac{1+\mu_1\mu_2-\mu_1^2/2}{1+\mu_1\mu_2+\mu_2^2} + \frac{s_2}{2mn_0} + \frac{\mu_2(\mu_2+\mu_1/2)^2}{(1+\mu_1\mu_2+\mu_2^2)^2} + \\
&+ \frac{2\mu_2(\mu_1+\mu_2)^2-4\mu_2-\mu_1}{4(1+\mu_1\mu_2+\mu_2^2)}, \\
C_6 &= -\frac{2\mu_2(\mu_2+\mu_1/2)\omega_0}{(1+\mu_1\mu_2+\mu_2^2)^2} - \frac{\mu_2(\mu_1+\mu_2)\omega_0}{1+\mu_1\mu_2+\mu_2^2}, \\
C_7 &= \frac{\mu_2\omega_0^2}{(1+\mu_1\mu_2+\mu_2^2)^2}, \quad s_2 = s''(0), \quad f_3 = F'''(0).
\end{aligned} \tag{5.9}
$$

We introduce the following auxiliary index In similar to (4.13). Namely, for the system (5.8) this index has the form

$$
\begin{aligned}
\mathrm{In} &= 6B_1 + 2B_3 + 2B_5 + 6B_7 \\
&= \frac{\mu_2 f_3}{In_0^2} - \frac{Hs_2}{In_0}\frac{1+3\mu_2^2+5\mu_1\mu_2/2-\mu_1^2/2}{1+\mu_1\mu_2+\mu_2^2} + 4\frac{s_2}{mn_0} + \frac{\mu_2(\mu_1+\mu_2)(\mu_2+2\mu_1)+3\mu_2-\mu_1}{1+\mu_1\mu_2+\mu_2^2}.
\end{aligned} \tag{5.10}
$$

Since for the system considered

$$
Y^i_{jk} = \frac{\partial^2 Y_i}{\partial y_j \partial y_k}(0,0) = 0 \tag{5.11}
$$

(owing to the oddness of its right-hand side with respect to the phase variables) for all indices i, j, and k, the following assertion states a necessary and sufficiently condition of asymptotic stability (or instability) of the origin for $\mathrm{In} \neq 0$.

Proposition 6. *If* $\mathrm{In} < 0$ (*respectively,* $\mathrm{In} > 0$) *and the inequality* (4.31) *holds, then the origin of the phase plane* $\mathbf{R}^2\{a, w\}$ *of the system* (5.8) (*respectively,* (5.3)) *under the critical relation for the parameter* $\mu_3 = \mu_1 + \mu_2$ *is a weak stable* (*respectively, unstable*) *focus.*

In the case considered, the condition (4.31) is necessary since only under this condition is the origin of the plane $\mathbf{R}^2\{a, w\}$ a (stable or unstable, straight or weak) focus.

The following theorem is a consequence of the well-known Poincaré–Andronov–Hopf theorem (see [21]).

Theorem 3. *Assume that for the system* (5.3) *the inequality* (4.31) *holds. Then the following assertions hold*:

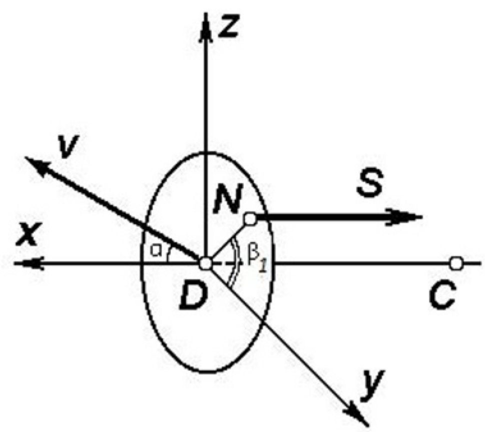

Fig. 3. Spatial motion of an axisymmetric rigid body in a resistive medium

(1) *If* In < 0, *then for all fixed* μ_1 *and* μ_2, *there exist* $\delta_1, \delta_2 > 0$ *such that for* $\mu_3 \in (\mu_1+\mu_2, \mu_1+\mu_2+\delta_1)$, *the origin is a straight stable focus, whereas for* $\mu_3 \in (\mu_1+\mu_2-\delta_2, \mu_1+\mu_2)$ *the origin is a straight unstable focus encircled by a stable limit cycle whose size increases as* $\sqrt{|\mu_1+\mu_2-\mu_3|}$ *when* μ_3 *decreases from* $\mu_1+\mu_2$ *to* $\mu_1+\mu_2-\delta_2$.

(2) *If* In > 0, *then for all fixed* μ_1 *and* μ_2, *there exist* $\delta_1, \delta_2 > 0$ *such that for* $\mu_3 \in (\mu_1+\mu_2-\delta_2, \mu_1+\mu_2)$, *the origin is a straight unstable focus, whereas for* $\mu_3 \in (\mu_1+\mu_2, \mu_1+\mu_2+\delta_1)$, *the origin is a straight stable focus encircled by an unstable limit cycle whose size increases as* $\sqrt{|\mu_1+\mu_2-\mu_3|}$ *when* μ_3 *increases from* $\mu_1+\mu_2$ *to* $\mu_1+\mu_2+\delta_1$.

It is easy to verify the condition $\mu_3 > \mu_1+\mu_2$ (or $\mu_3 < \mu_1+\mu_2$) since in each case these parameters depend either on only first derivatives of the influence functions y_N and s or on their values. Conversely, the condition In < 0 (or In > 0) is difficult to verify since for each specific body, the explicit form and higher derivatives of the influence functions y_N and s are unknown.

6. Spatial Motion of an Axisymmetric Rigid Body in a Resistive Medium

Consider the problem on the spatial motion of a *homogeneous* axisymmetric rigid body of mass m whose surface has a part in the shape of a flat circular disk. Assume that this body interacts with a medium according to the laws of jet circumfluence. The remaining part of the surface of the body is located inside the volume bounded by the jet surface; it is not affected by the medium. Similar conditions appear, for example, after the immersion of a body in water (see [29]).

Assume that tangential forces acting on the disk vanish. Then the influence force **S** applied to the body at a point N preserves its orientation with respect to the body (it is directed along the normal to the disk) and is quadratic with respect to the speed of its center D (Newton's resistance, see Fig. 3). We also assume that the gravity force acting on a body is negligible compared with the resistance force.

Under all the conditions listed above, among all motions of the body, there exists the regime of *rectilinear translational deceleration*, which is similar to the case of a plane-parallel (nonperturbed) motion: the body can perform translational motion in the direction of its axis of symmetry, i.e., perpendicularly to the disk.

We attach to the body a right coordinate system $Dxyz$ (see Fig. 3) whose applicate axis Dx is directed along the axis of geometric symmetry of the body. The axes Dy and Dz are rigidly attached to the circular disk and form a right coordinate system. We denote the components of the vector of angular velocity Ω in the system $Dxyz$ by $\{\Omega_x, \Omega_y, \Omega_z\}$. The tensor of inertia of a dynamically symmetric body in th coordinate system $Dxyz$ has a diagonal form: $\operatorname{diag}\{I_1, I_2, I_2\}$.

Now we use the quasi-stationarity hypothesis and assume, for simplicity, that the function $R_1 = DN$ is determined by the angle of attack α, i.e., the angle between the velocity vector $\mathbf{v}$ of the center D of the disk and the straight line Dx. Thus, $DN = R_1(\alpha, \dots)$.

Moreover, we assume that the magnitude of the resistance force is $S = |\mathbf{S}| = s_1(\alpha)v^2$, $v = |\mathbf{v}|$. For simplicity (as in the case of a plane-parallel motion), instead of the resistance coefficient $s_1(\alpha)$, we introduce the auxiliary alternating function $s(\alpha)$: $s_1 = s_1(\alpha) = s(\alpha)\operatorname{sgn}\cos\alpha > 0$. Thus, the pair of functions $R_1(\alpha, \dots)$ and $s(\alpha)$ determines the force and moment characteristics of the influence of the medium on the disk under the model conditions posed above.

6.1. Dynamical part of the equations of spatial motion. Consider the spherical coordinates (v, α, β_1) of the endpoint of the velocity vector $\mathbf{v} = \mathbf{v}_D$ of the point D with respect to the flow, where the angle β_1 is measured in the plane of the disk (see Fig. 3). The coordinates (v, α, β_1) are expressed by nonintegrable relations through the cyclic kinematic variables and their derivatives (see [52]). Therefore, we take the triple (v, α, β_1) as quasi-velocities and, in addition, consider the components $(\Omega_x, \Omega_y, \Omega_z)$ of the angular velocity in the coordinate system attached to the body. Obviously, in this coordinate system

$$\mathbf{v}_D = \{v\cos\alpha,\ v\sin\alpha\cos\beta_1,\ v\sin\alpha\sin\beta_1\}. \tag{6.1}$$

By the theorems on the motion of the center of mass (in the projections onto the axes of the coordinate system $Dxyz$) and the kinetic moment with respect to these axes, we obtain the dynamical part of the differential equations of motion in the six-dimensional phase space of quasi-velocities (here σ is the distance DC). The first group of equations corresponds to the motion of the center of mass itself, and the second group to rotation about the center of mass:

$$\begin{aligned}
&\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha + \Omega_y v\sin\alpha\sin\beta_1 - \Omega_z v\sin\alpha\cos\beta_1 + \sigma(\Omega_y^2 + \Omega_z^2) = -\frac{s(\alpha)v^2}{m},\\
&\dot{v}\sin\alpha\cos\beta_1 + \dot{\alpha}v\cos\alpha\cos\beta_1 - \dot{\beta}_1 v\sin\alpha\sin\beta_1 + \Omega_z v\cos\alpha\\
&\Omega_x v\sin\alpha\sin\beta_1 - \sigma\Omega_x\Omega_y - \sigma\dot{\Omega}_z = 0,\\
&\dot{v}\sin\alpha\sin\beta_1 + \dot{\alpha}v\cos\alpha\sin\beta_1 + \dot{\beta}_1 v\sin\alpha\cos\beta_1 + \Omega_x v\sin\alpha\cos\beta_1\\
&\Omega_y v\cos\alpha - \sigma\Omega_x\Omega_z + \sigma\dot{\Omega}_y = 0,\\
&I_1\dot{\Omega}_x = 0,\\
&I_2\dot{\Omega}_y + (I_1 - I_2)\Omega_x\Omega_z = -z_N s(\alpha)v^2,\\
&I_2\dot{\Omega}_z + (I_2 - I_1)\Omega_x\Omega_y = y_N s(\alpha)v^2,
\end{aligned} \tag{6.2}$$

where $(0, y_N, z_N)$ are the coordinates of the point N in the system $Dxyz$.

6.2. Motion of a symmetric body under the action of a resistance force and a tracking force in the case II. We distinguish a class of problems on the influence of a medium on a body in which a tracking force acts along the axis of the geometric symmetry of the body (cf. the case of a plane-parallel motion) under some conditions that provide the realization of required classes of motion. In this case, the tracking force itself is the reaction of constraints imposed on the body. If the tracking force is absent, then the motion of the body is the spatial free deceleration in a resistive medium. In the case considered, the tracking force provides the condition (2.7) during the motion (case II), namely, $v \equiv \text{const}$. Similarly to the plane-parallel motion, one can consider the case where the tracking force provides the condition (2.8) (the case III, see below).

By Eqs. (6.2), we have the cyclic invariant relation

$$\Omega_x \equiv \Omega_{x0} = \text{const.} \tag{6.3}$$

6.3. Dynamical equations in the case of zero twist of a rigid body about the longitudinal axis. In the sequel, we analyze the case of zero twist of a rigid body about its longitudinal axis, i.e., when the condition $\Omega_{x0} = 0$ holds.

Then the independent dynamical part of the equations of motion in the four-dimensional phase spaces has the following form:

$$\dot\alpha \cos\alpha \cos\beta_1 - \dot\beta_1 v \sin\alpha \sin\beta_1 + \Omega_z v \cos\alpha - \sigma\dot\Omega_z = 0, \tag{6.4}$$

$$\dot\alpha \cos\alpha \sin\beta_1 + \dot\beta_1 v \sin\alpha \cos\beta_1 - \Omega_y v \cos\alpha + \sigma\dot\Omega_y = 0, \tag{6.5}$$

$$I_2\dot\Omega_y = -z_N s(\alpha) v^2, \tag{6.6}$$

$$I_2\dot\Omega_z = y_N s(\alpha) v^2. \tag{6.7}$$

Here y_N and z_N are the Cartesian coordinates in the plane of the disk containing the application point N of the resistance force.

The system (6.4)–(6.7) contains the influence functions y_N, z_N, and s that can be qualitatively described (similarly to the case of a plane-parallel motion) by using experimental information on properties of jet circumfluence.

First, we analyze the system (6.4)–(6.7) for the following influence functions (Chaplygin functions):

$$\begin{gathered} y_N = A\sin\alpha\cos\beta_1 - h\frac{\Omega_z}{v}, \quad z_N = A\sin\alpha\sin\beta_1 + h\frac{\Omega_y}{v}, \\ s(\alpha) = B\cos\alpha, \quad A = \left.\frac{\partial y_N}{\partial\alpha}\right|_{\alpha=0,\,\beta_1=0} = \left.\frac{\partial z_N}{\partial\alpha}\right|_{\alpha=0,\,\beta_1=\pi/2}, \quad B = s(0),\ h > 0. \end{gathered} \tag{6.8}$$

This systems is said to be a *reference system.* (Note the a similar analysis can be also performed for each pair of influence functions y_N, z_N, and s; see below.)

In Eqs. (6.8), h is the coefficient of terms that are proportional to rotational derivatives of the moment of hydroaerodynamical forces (in the case considered, influence forces) with respect to the components of the angular velocity of the rigid body (see also [4, 5]).

The system (6.4)–(6.7) is a dynamical system with variable dissipation with zero mean (in the case considered, the mean with respect to the angle of attack; see [42, 43]). This means that the integral (over the period of the angle of attack) of the divergence of its right-hand side responsible for the change of the phase volume (after the corresponding reduction of the system) vanishes. The system is "semi-conservative" in some sense.

We project the angular velocity on the movable axes attached to the body so that

$$z_1 = \Omega_y\cos\beta_1 + \Omega_z\sin\beta_1, \quad z_2 = -\Omega_y\sin\beta_1 + \Omega_z\cos\beta_1 \tag{6.9}$$

and introduce the dimensionless variables w_k, $k = 1, 2$, and the dimensionless parameters by the formulas

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{I_2}, \quad H_1 = \frac{Bh}{I_2 n_0}, \quad z_k = n_0 v w_k,\ k = 1, 2, \quad \langle\,\cdot\,\rangle = n_0 v \langle\,'\,\rangle. \tag{6.10}$$

Then we obtain the following fourth-order analytic dynamical system, called the *reference system*:

$$\alpha' = -(1 + bH_1)w_2 + b\sin\alpha, \tag{6.11}$$

$$w_2' = \sin\alpha\cos\alpha - (1 + bH_1)w_1^2\frac{\cos\alpha}{\sin\alpha} - H_1 w_2\cos\alpha, \tag{6.12}$$

$$w_1' = (1 + bH_1)w_1 w_2\frac{\cos\alpha}{\sin\alpha} - H_1 w_1\cos\alpha, \tag{6.13}$$

$$\beta_1' = (1 + bH_1)w_1 \frac{\cos\alpha}{\sin\alpha}. \tag{6.14}$$

This system contains the independent third-order subsystem (6.11)–(6.13).

For $b = H_1$, the divergence of the right-hand side of the systems (6.11)–(6.13) and (6.11)–(6.14) after the change of variables $w^* = \ln|w_1|$ identically vanishes; this allows one to assume that these systems are conservative.

6.4. On the stability of the rectilinear translational motion. We examine the stability of the key regime, i.e., the nonperturbed motion, with respect to perturbations of the angle of attack and angular velocity (i.e., with respect to the variables α, w_1, and w_2). In other words, we examine the stability of the trivial solution of the independent third-order system (6.11)–(6.13) (after the redefinition of the system at the origin).

Consider the following positive-definite function on the phase space of the third-order system (6.11)–(6.13):

$$V(\alpha, w_1, w_2) = (1 + b^2)(w_2^2 + w_1^2) - 2bw_2 \sin\alpha + \sin^2\alpha. \tag{6.15}$$

Theorem 4. *The function* (6.15) *is a Lyapunov* (*Chetaev*) *function for the system* (6.11)–(6.13), *i.e., its derivative with respect to the system* (6.11)–(6.13) *is negative definite for* $b < H_1$ *and positive definite for* $b > H_1$.

Corollary 1. *After redefinition of the right-hand sides at the origin, the system* (6.11)–(6.13) *has an attractive singularity at the origin for* $b < H_1$ *and a repulsive singilarity for* $b > H_1$.

Proof of Theorem 4. Indeed, the derivative of the function (6.15) with respect to the system (6.11)–(6.13) is equal to

$$2(b - H_1)\cos\alpha[w_1^2 + w_2^2]. \tag{6.16}$$

□

We note that a similar theorem is valid for systems of the general form for all admissible influence functions y_N, z_N, and s. The condition of asymptotic stability of the origin for the system of reduced dynamical equations with respect to the variables (α, w_1, w_2) is $b < H_1$.

Indeed, in the general case where the admissible influence functions y_N and z_N can be represented in the form

$$y_N = R(\alpha)\cos\beta_1 - h_1\frac{\Omega_z}{v}, \quad z_N = R(\alpha)\sin\beta_1 + h_1\frac{\Omega_y}{v}, \tag{6.17}$$

whereas the functions R and s satisfy the conditions (3.7) (in this case, the function R corresponds to the function y), the dynamical equations of motion take the following form:

$$\begin{aligned}
\alpha' &= -w_2 + \frac{\sigma}{I_2 n_0}\frac{F(\alpha)}{\cos\alpha} - \frac{\sigma h_1}{I_2} w_2 \frac{s(\alpha)}{\cos\alpha}, \\
w_2' &= \frac{F(\alpha)}{I_2 n_0^2} - w_1^2\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma h_1}{I_2} w_1^2 \frac{s(\alpha)}{\sin\alpha} - \frac{h_1}{I_2 n_0} w_2 s(\alpha), \\
w_1' &= w_1 w_2 \frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2} w_1 w_2 \frac{s(\alpha)}{\sin\alpha} - \frac{h_1}{I_2 n_0} w_1 s(\alpha), \\
\beta_1' &= w_1 \frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2} w_1 \frac{s(\alpha)}{\sin\alpha},
\end{aligned} \tag{6.18}$$

where $F(\alpha) = R(\alpha)s(\alpha)$.

Consider the following function similar to (6.15):

$$V(\alpha, w_1, w_2) = w_2^2 + (1 + b^2)w_1^2 + [bw_2 - \sin\alpha]^2; \tag{6.19}$$

it is positive definite in some neighborhood of the origin.

Theorem 5. *The function* (6.19) *is a Lyapunov* (*Chetaev*) *function for the system* (6.18)*, i.e., its derivative with respect to the system* (6.18) *in a neighborhood of the origin is negative definite for* $\sigma R'(0) < h_1$ *and positive definite for* $\sigma R'(0) > h_1$*.*

Corollary 2. *For* $\sigma R'(0) < h_1$*, the system* (6.18) *has an attracting singularity at the origin and a repulsive singularity for* $\sigma R'(0) > h_1$*.*

Proof of Theorem 5. Indeed, the derivative of the function (6.19) with respect to the system (6.18) is equal to

$$2\left(b\cos\alpha - \frac{h_1}{I_2 n_0}s(\alpha)\right)[w_1^2 + w_2^2] + 2w_2\left\{\frac{F(\alpha)}{I_2 n_0} - \sin\alpha\cos\alpha\right\} \tag{6.20}$$

and in a neighborhood of the origin can be represented in the form

$$2\left(b - \frac{h_1 B}{I_2 n_0}\right)[w_1^2 + w_2^2] + \bar{o}(\alpha^2 + z_1^2 + z_2^2). \tag{6.21}$$

□

Turning to the problem on the motion of homogeneous circular cylinders, we can conclude that the asymptotic stability occurs if the inequality $\sigma k < hD$ holds, where D is the diameter of the cylinder, σ is the distance DC, and k and h are dimensionless influence parameters.

6.5. Motion of a symmetric body under the action of a resistance force and a tracking force in the case III. In this case, the tracking force during the motion provides the fulfillment of the condition (2.8) (the case III), namely, $\mathbf{V}_C \equiv \mathbf{const}$. By Eqs. (6.2), the following cyclic invariant relations holds:

$$\Omega_x \equiv \Omega_{x0} = \text{const}. \tag{6.22}$$

6.6. Dynamical equations in the case of zero twist of a rigid body about the longitudinal axis. We examine the case of zero twist of a rigid body about its longitudinal axis, i.e., if $\Omega_{x0} = 0$. Then on the right-hand side of the first equation of the system (6.2) instead of $-s(\alpha)v^2/m$ we obtain identical zero since a nonconservative force couple acts on the body:

$$T - s(\alpha)v^2 \equiv 0. \tag{6.23}$$

Obviously, the tracking force T has the form

$$T = T(v, \alpha, \Omega) = s(\alpha)v^2, \quad \mathbf{T} \equiv -\mathbf{S}. \tag{6.24}$$

Similarly to the choice of the influence functions, we take the dynamical functions s, y_N, and z_N in the form (3.7) and (6.17) (the function R corresponds to the function y). Moreover, the system considered contains also an additional damping moment of a nonconservative force (note that in some domains of the phase space this moment can be accelerating).

We project the angular velocity on the movable axes so that

$$z_1 = \Omega_y\cos\beta_1 + \Omega_z\sin\beta_1, \quad z_2 = -\Omega_y\sin\beta_1 + \Omega_z\cos\beta_1 \tag{6.25}$$

and introduce the new dimensionless phase variables and differentiation by the formulas

$$z_k = n_0 v Z_k, \; k = 1, 2, \quad \langle\,\cdot\,\rangle = n_0 v\,\langle\,'\,\rangle. \tag{6.26}$$

Then the system (6.2) takes the following form:

$$v' = v\Psi_1(\alpha, Z_1, Z_2), \tag{6.27}$$

$$\alpha' = -Z_2 + \mu_2(Z_1^2 + Z_2^2)\sin\alpha + \frac{\sigma}{I_2 n_0}F(\alpha)\cos\alpha - \frac{\sigma h_1}{I_2}Z_2 s(\alpha)\cos\alpha, \tag{6.28}$$

$$Z_2' = \frac{F(\alpha)}{I_2 n_0^2} - Z_2\Psi_1(\alpha, Z_1, Z_2) - Z_1^2\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma h_1}{I_2}Z_1^2\frac{s(\alpha)}{\sin\alpha} - \frac{h_1}{I_2 n_0}Z_2 s(\alpha), \tag{6.29}$$

$$Z_1' = -Z_1\Psi_1(\alpha, Z_1, Z_2) + Z_1 Z_2\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2}Z_1 Z_2\frac{s(\alpha)}{\sin\alpha} - \frac{h_1}{I_2 n_0}Z_1 s(\alpha), \tag{6.30}$$

$$\beta_1' = Z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2}Z_1\frac{s(\alpha)}{\sin\alpha}, \tag{6.31}$$

where

$$\Psi_1(\alpha, Z_1, Z_2) = -\mu_2(Z_1^2 + Z_2^2)\cos\alpha + \frac{\sigma}{I_2 n_0}F(\alpha)\sin\alpha - \frac{\sigma h_1}{I_2}Z_2 s(\alpha)\sin\alpha,$$

and in the case of Chaplygin influence functions (6.8), it takes the form of the following analytic system:

$$v' = v\Psi_1(\alpha, Z_1, Z_2), \tag{6.32}$$

$$\alpha' = -Z_2 + \mu_2(Z_1^2 + Z_2^2)\sin\alpha + \mu_2\sin\alpha\cos^2\alpha - \mu_2\mu_3 Z_2\cos^2\alpha, \tag{6.33}$$

$$Z_2' = \sin\alpha\cos\alpha - Z_2\Psi_1(\alpha, Z_1, Z_2) - (1 + \mu_2\mu_3)Z_1^2\frac{\cos\alpha}{\sin\alpha} - \mu_3 Z_2\cos\alpha, \tag{6.34}$$

$$Z_1' = -Z_1\Psi_1(\alpha, Z_1, Z_2) + (1 + \mu_2\mu_3)Z_1 Z_2\frac{\cos\alpha}{\sin\alpha} - \mu_3 Z_1\cos\alpha, \tag{6.35}$$

$$\beta_1' = (1 + \mu_2\mu_3)Z_1\frac{\cos\alpha}{\sin\alpha}, \tag{6.36}$$

where

$$\Psi_1(\alpha, Z_1, Z_2) = -\mu_2(Z_1^2 + Z_2^2)\cos\alpha + \mu_2\sin^2\alpha\cos\alpha - \mu_2\mu_3 Z_2\sin\alpha\cos\alpha.$$

We introduce the dimensionless parameters $b = \mu_2$ and $H_1 = \mu_3$ as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{I_2}, \quad H_1 = \frac{Bh_1}{I_2 n_0}. \tag{6.37}$$

Equations (6.28)–(6.31) of the system (6.27)–(6.31) form an independent fourth-order subsystem, whereas Eqs. (6.28)–(6.30) form an independent third-order subsystem.

6.7. On the stability of the rectilinear translational motion. We examine the stability of the key regime (nonperturbed motion) with respect to perturbations of the angle of attack and the angular velocity, i.e., with respect to the variables α, Z_1, and Z_2. In other words, we examine the stability of the trivial solution of the independent third-order system (6.28)–(6.30) (after the redefinition of the system to the origin).

The follow important assertion is valid.

Proposition 7. *The plane*

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R}^3 : \ Z_1 = 0\} \tag{6.38}$$

is an integral manifold for the system (6.28)–(6.30)*.*

Moreover, after the formal substitution $Z_1 = 0$ in the system (6.28)–(6.30), the remaining two equations for α and Z_2 form a system that describes the dynamics of the plane-parallel motion of the body, and the system obtained coincides with (4.22).

Thus, the plane (6.38) contains the phase portrait of the flat dynamics. Moreover, the plane (6.38) divides the three-dimensional phase space into two parts:

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R}^3 : \ 0 < \alpha < \pi, \ Z_1 > 0\} \tag{6.39}$$

and

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R}^3 : \ 0 < \alpha < \pi, \ Z_1 < 0\}; \tag{6.40}$$

motions in each of these parts are independent. However, the system possesses the following symmetry:

(*i*) the α- and Z_2-components of the vector field of the system (6.28)–(6.30) *preserves their signs* under the symmetry

$$\begin{pmatrix} \alpha \\ Z_1 \\ Z_2 \end{pmatrix} \to \begin{pmatrix} \alpha \\ -Z_1 \\ Z_2 \end{pmatrix} \tag{6.41}$$

with respect to the plane (6.38);

(*ii*) the Z_1-component of the vector fields of the system (6.28)–(6.30) *changes its sign* under the symmetry (6.41) with respect to the plane (6.38).

These facts show that it suffices to study the system (6.28)–(6.30) in the semi-bounded layer (6.39), which, however, cannot be treated as a phase space.

We note the following important consequence: it is possible to take the function

$$V_1(\alpha, Z_1) = Z_1 \sin \alpha \tag{6.42}$$

as a Lyapunov (Chetaev) function in the semi-bounded layer (6.39) since it is positive definite here.

Theorem 6. *The function* (6.42) *is a Lyapunov* (*Chetaev*) *function for the system* (6.28)–(6.30), *i.e., its derivative with respect to the system* (6.28)–(6.30) *is negative definite for* $\mu_2 < \mu_3$ *and positive definite for* $\mu_2 > \mu_3$.

Corollary 3. *After redefinition of the right-hand sides at the origin, the system* (6.28)–(6.30) *has an attractive singularity at the origin for* $\mu_2 < \mu_3$ *and a repulsive singularity for* $\mu_2 > \mu_3$.

Proof of Theorem 6. Indeed, the derivative of the function (6.42) with respect to the system (6.28)–(6.30) has the form

$$(\mu_2 - \mu_3) Z_1 \alpha + \overline{o}(\alpha^2 + Z_1^2 + Z_2^2). \tag{6.43}$$

□

In particular, a similar theorem is valid for systems of the form (6.33)–(6.35) with the Chaplygin influence functions (6.8).

We also consider the function (it is similar to (6.15))

$$V(\alpha, Z_1, Z_2) = Z_2^2 + (1 + b^2) Z_1^2 + [bZ_2 - \sin \alpha]^2, \tag{6.44}$$

which is positive definite in some neighborhood of the origin.

Theorem 7. *The function* (6.44) *is a Lyapunov (Chetaev) function for the system* (6.28)–(6.30), *i.e., its derivative with respect to the system* (6.28)–(6.30) *is negative definite for* $\mu_2 < \mu_3$ *and positive definite for* $\mu_2 > \mu_3$.

Corollary 4. *After redefinition of the right-hand sides at the origin, the system* (6.28)–(6.30) *has at the origin an attractive singularity for* $\mu_2 < \mu_3$ *and a repulsive singularity for* $\mu_2 > \mu_3$.

Proof of Theorem 7. Indeed, the derivative of the function (6.44) with respect to the system (6.28)–(6.30) has the form

$$2(\mu_2 - \mu_3)(Z_1^2 + Z_2^2) + \overline{o}(\alpha^2 + Z_1^2 + Z_2^2). \tag{6.45}$$

□

Passing to the problem on the motion of homogeneous circular cylinders, we can conclude that the asymptotic stability occurs if the inequality $\sigma k < hD$ holds, where D is the diameter of the cylinder, σ is the distance DC, and k and h are dimensionless influence parameters.

7. Spatial Free Deceleration of a Rigid Body in a Resistive Medium (Case I)

Further, we consider the case of the motion of a body when the thrust is turned off and the body performs a spatial free motion (deceleration) in a resistive medium (case I).

Then the right-hand side of the first equation of the system (6.2) contains the function $-s(\alpha)v^2/m$ since the equality $\mathbf{T} \equiv \mathbf{0}$ holds.

Similarly to the choice of influence function, we can take the dynamical functions s, y_N, and z_N in the system (6.2) in the form (3.7), (6.17) (the function R corresponds to the function y). As above, the system considered also contains an additional damping moment of a nonconservative force (in some domains of the phase space, this moment can be accelerating).

7.1. Dynamical equations of motion of a symmetric body under the action of a resistance force in the absence of the proper rotation (problem on the spatial free deceleration). By Eqs. (6.2), during the motion, the following cyclic invariant relation holds:

$$\Omega_x \equiv \Omega_{x0} = \text{const}. \tag{7.1}$$

In the sequel, we examine the case of zero twist of a rigid body about its longitudinal axis, i.e., the case where the following condition holds:

$$\Omega_{x0} = 0. \tag{7.2}$$

Projecting the angular velocity on the movable axes so that

$$z_1 = \Omega_y \cos\beta_1 + \Omega_z \sin\beta_1, \quad z_2 = -\Omega_y \sin\beta_1 + \Omega_z \cos\beta_1 \tag{7.3}$$

and introducing, as above, the new dimensionless phase variables and the new differentiation by the formulas

$$z_k = n_0 v Z_k,\ k = 1, 2, \quad \langle\, \cdot \,\rangle = n_0 v \,\langle\, ' \,\rangle, \tag{7.4}$$

we transform the system (6.2) to the following form:

$$v' = v\Psi_1(\alpha, Z_1, Z_2), \tag{7.5}$$

$$\alpha' = -Z_2 + \mu_2(Z_1^2 + Z_2^2)\sin\alpha + \frac{\sigma}{I_2 n_0}F(\alpha)\cos\alpha - \frac{\sigma h_1}{I_2}Z_2 s(\alpha)\cos\alpha + \frac{s(\alpha)}{m n_0}\cos\alpha, \tag{7.6}$$

$$Z_2' = \frac{F(\alpha)}{I_2 n_0^2} - Z_2\Psi_1(\alpha, Z_1, Z_2) - Z_1^2\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma h_1}{I_2}Z_1^2\frac{s(\alpha)}{\sin\alpha} - \frac{h_1}{I_2 n_0}Z_2 s(\alpha), \tag{7.7}$$

$$Z_1' = -Z_1\Psi_1(\alpha, Z_1, Z_2) + Z_1 Z_2\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2}Z_1 Z_2\frac{s(\alpha)}{\sin\alpha} - \frac{h_1}{I_2 n_0}Z_1 s(\alpha), \tag{7.8}$$

$$\beta_1' = Z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2}Z_1\frac{s(\alpha)}{\sin\alpha}, \tag{7.9}$$

where

$$\Psi_1(\alpha, Z_1, Z_2) = -\mu_2(Z_1^2 + Z_2^2)\cos\alpha + \frac{\sigma}{I_2 n_0}F(\alpha)\sin\alpha - \frac{s(\alpha)}{m n_0}\cos\alpha - \frac{\sigma h_1}{I_2}Z_2 s(\alpha)\sin\alpha,$$

and in the case of Chaplygin influence function it takes the form of the analytical system of equations

$$v' = v\Psi_1(\alpha, Z_1, Z_2), \tag{7.10}$$

$$\alpha' = -Z_2 + \mu_2(Z_1^2 + Z_2^2)\sin\alpha + \mu_2\sin\alpha\cos^2\alpha - \mu_2\mu_3 Z_2\cos^2\alpha + \frac{\mu_1}{2}\sin\alpha\cos\alpha, \tag{7.11}$$

$$Z_2' = \sin\alpha\cos\alpha - Z_2\Psi_1(\alpha, Z_1, Z_2) - (1 + \mu_2\mu_3)Z_1^2\frac{\cos\alpha}{\sin\alpha} - \mu_3 Z_2\cos\alpha, \tag{7.12}$$

$$Z_1' = -Z_1\Psi_1(\alpha, Z_1, Z_2) + (1 + \mu_2\mu_3)Z_1 Z_2\frac{\cos\alpha}{\sin\alpha} - \mu_3 Z_1\cos\alpha, \tag{7.13}$$

$$\beta_1' = (1 + \mu_2\mu_3)Z_1\frac{\cos\alpha}{\sin\alpha}, \tag{7.14}$$

where

$$\Psi_1(\alpha, Z_1, Z_2) = -\mu_2(Z_1^2 + Z_2^2)\cos\alpha + \mu_2\sin^2\alpha\cos\alpha - \frac{\mu_1}{2}\cos^2\alpha - \mu_2\mu_3 Z_2\sin\alpha\cos\alpha.$$

In the sequel, as above, we choose the dimensionless parameters μ_1, $b = \mu_2$, and $H_1 = \mu_3$ as follows:

$$\mu_1 = 2\frac{B}{mn_0}, \quad b = \sigma n_0, \quad n_0^2 = \frac{AB}{I_2}, \quad H_1 = \frac{Bh_1}{I_2 n_0}. \tag{7.15}$$

Equations (7.6)–(7.9) of the system (7.5)–(7.9) form an independent fourth-order subsystem and Eqs. (7.6)–(7.8) form an independent third-order subsystem.

7.2. On the stability of the rectilinear translational deceleration. We examine the stability of the key regime, i.e., the nonperturbed motion, with respect to the perturbations of the angle of attack and the angular velocity, i.e., with respect to the variables α, Z_1, and Z_2. In other words, we examine the stability of the trivial solution of the independent third-order system (7.6)–(7.8) (after the redefinition of the system at the origin).

The following important assertion is valid.

Proposition 8. *The plane*

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R}^3: \ Z_1 = 0\} \tag{7.16}$$

Is an integral manifold for the system (7.6)–(7.8).

Moreover, after the formal substitution $Z_1 = 0$ in the system (7.6)–(7.8), the remaining two equations for α and Z_2 form a system that describes the dynamics of the plane-parallel motion of the body; the system obtained coincides with (5.3).

Thus, the plane (7.16) contains the phase portrait from the flat dynamics. Moreover, the plane (7.16) divides the three-dimensional phase space into two parts:

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R}^3: \ 0 < \alpha < \pi, \ Z_1 > 0\} \tag{7.17}$$

and

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R}^3: \ 0 < \alpha < \pi, \ Z_1 < 0\}; \tag{7.18}$$

motions in each of these parts are independent, but not arbitrarily, since the system possesses the symmetry (6.41).

This facts show that it suffices to study the system (7.6)–(7.8) in the semi-bounded layer (7.17), which is not a phase space.

An important consequence of these facts is the possibility of using the function

$$V_1(\alpha, Z_1) = Z_1\sin\alpha \tag{7.19}$$

as a Lyapunov (Chetaev) function in the semi-bounded layer (7.17) since the function considered is positive definite in it.

Theorem 8. *For the system* (7.6)–(7.8), *the function* (7.19) *is a Lyapunov* (*Chetaev*) *function, i.e., its derivative with respect to the system* (7.6)–(7.8) *is negative definite for* $\mu_3 > \mu_1 + \mu_2$ *and positive definite for* $\mu_3 < \mu_1 + \mu_2$.

Corollary 5. *After redefinition of the right-hand sides at the origin, the system* (7.6)–(7.8) *has an attractive singularity at the origin for* $\mu_3 > \mu_1 + \mu_2$ *and a repulsive singularity for* $\mu_3 < \mu_1 + \mu_2$.

Proof of Theorem 8. Indeed, the derivative of the function (7.19) with respect to the system (7.6)–(7.8) can be represented in the form

$$(\mu_1 + \mu_2 - \mu_3)Z_1\alpha + \overline{o}(\alpha^2 + Z_1^2 + Z_2^2). \tag{7.20}$$

□

In particular, a similar theorem is also valid for systems of the form (7.11)–(7.13) with the Chaplygin influence function (6.8).

For the problem on the motion of homogeneous circular cylinders, we can conclude that the asymptotic stability occurs if the following inequality holds:

$$\sigma k + \frac{2I_2}{mD} < hD, \tag{7.21}$$

where D is the diameter of the cylinder, σ is the distance DC, and k and h are dimensionless influence parameters, or

$$\sigma Dk + 2r_1^2 < hD^2, \tag{7.22}$$

where r_1 is the radius if inertia of the cylinder (for details, see the following chapter).

We see that Theorem 8 yields the same conditions of the asymptotic stability with respect to a part of the variables (α, Z_1, Z_2) as Proposition 5, in which dynamical systems from the dynamics of the plane-parallel motion is involved.

In the case of a spatial motion, the systems obtained have singularities at the origin, which is caused by the degeneracy of the spherical coordinates of the velocity vector $\mathbf{v}$ of the frontal disk (cavitator); this can be overcome by a redefinition of the right-hand sides of dynamical systems.

8. Conclusion for Two-Dimensional and Three-Dimensional Problems

Thus, the instability of the simplest motion of a body, i.e., the rectilinear translational deceleration, is used methodologically, namely, for the definition of unknown influence parameters under the quasi-stationarity conditions.

Experiments on the motion of homogeneous circular cylinders in water conducted in the Institute of Mechanics of the Moscow State University confirmed that in the modeling of the influence of a medium on a rigid body, one must take into account an additional parameter that describes dissipation in the system.

In the study of the class of decelerating motions of a body with finite angles of attack, the main question is the search for conditions under which autooscillations in a finite neighborhood of the rectilinear translational deceleration occur. Thus, the necessity of the complete nonlinear study becomes obvious.

In the initial stage of this study, we neglected the damping influence of the medium. In the functional language, this means that the pair of dynamical functions describing the influence of the medium depends only on a single parameter, namely, the angle of attack. Dynamical systems that appear in this way are systems with variable dissipation. Hence we must develop methods of examining such systems.

In the dynamics of a rigid body interacting with a medium, we often obtain either systems with variable dissipation with nonzero mean (the problem of the free deceleration of a rigid body) or systems in which the loss of energy during a period can vanish (the problem on the motion of a rigid body in a resistive medium under the action of a tracking force). In this paper, we use methods that allow one to perform the analytical study of some model problems on the plane-parallel and spatial motion of the rigid body.

In the qualitative description of the interaction of a body with a medium, due to the use of experimental information on the properties of the jet circumfluence, a spread in the modeling of force and moment characteristics appears. This explains the necessity of the notion of relative roughness (relative structure stability) and the proof of roughness for the systems considered. Note that many of systems considered are simply (absolutely) rough in the sense of Andronov–Pontryagin.

Results obtained under the assumption of the absence of damping influence of a medium on a rigid body allow one to conclude that there are no conditions under which autooscillations in a finite neighborhood of the rectilinear translational deceleration exist.

We systematize the study of the motion of a rigid body in a medium in the case where the damping moment is taken into account. This moment causes an additional dissipation of the system and leads to the stability of the rectilinear translational deceleration.

Thus, account of the damping influence of the medium under certain conditions leads to a positive answer to the main question of the existence of stable autooscillations during the motion of a body in a medium with finite angles of attack.

9. On the Stability of the Trivial Solution with respect to a Part of the Variables for the Four-Dimensional Problem

Now we examine some dynamical equations of motion of the four-dimensional rigid body considered in [42].

9.1. System of dynamical equations of motion with a nonintegrable constraint. Consider the system of dynamical equation on the tangent bundle $T_*\mathbf{S}^3\{z_3, z_2, z_1; \alpha, \beta_1, \beta_2\}$ of the three-dimensional sphere $\mathbf{S}^3\{\alpha, \beta_1, \beta_2\}$, which describes the motion of a dynamically symmetric four-dimensional rigid body with a nonintegrable constraint in a nonconservative field (see [47]):

$$\dot{\alpha} = -(1+\mu_2\mu_3)\, z_3 + \mu_2 \sin\alpha, \tag{9.1}$$

$$\dot{z}_3 = \sin\alpha\cos\alpha - (1+\mu_2\mu_3)\,(z_1^2+z_2^2)\frac{\cos\alpha}{\sin\alpha} - \mu_3 z_3 \cos\alpha, \tag{9.2}$$

$$\dot{z}_2 = (1+\mu_2\mu_3)\, z_2 z_3 \frac{\cos\alpha}{\sin\alpha} + (1+\mu_2\mu_3)\, z_1^2 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \mu_3 z_2\cos\alpha, \tag{9.3}$$

$$\dot{z}_1 = (1+\mu_2\mu_3)\, z_1 z_3 \frac{\cos\alpha}{\sin\alpha} - (1+\mu_2\mu_3)\, z_1 z_2 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \mu_3 z_1\cos\alpha, \tag{9.4}$$

$$\dot{\beta}_1 = (1+\mu_2\mu_3)\, z_2 \frac{\cos\alpha}{\sin\alpha}, \tag{9.5}$$

$$\dot{\beta}_2 = -(1+\mu_2\mu_3)\, z_1 \frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{9.6}$$

We examine the stability of its trivial solution with respect to perturbations of the variables α, z_1, z_2, and z_3 (note that this system can be continuously redefined at the origin).

Consider the function

$$V(\alpha, z_1, z_2, z_3) = (1+\mu_2^2)(z_3^2+z_2^2+z_1^2) - 2\mu_2 z_3\sin\alpha + \sin^2\alpha, \tag{9.7}$$

which is positive definite in a neighborhood of the origin.

Theorem 9. *For the system* (9.1)–(9.6)*, the Function* (9.7) *is a Lyapunov* (*Chetaev*) *function, i.e., its derivative with respect to the system* (9.1)–(9.6) *is negative definite for* $\mu_2 < \mu_3$ *and positive definite for* $\mu_2 > \mu_3$.

Corollary 6. *After redefinition of the right-hand sides at the origin, the system* (9.1)–(9.6) *has an attractive singularity at the origin for* $\mu_2 < \mu_3$ *and a repulsive singularity for* $\mu_2 > \mu_3$.

Proof of Theorem 9. Indeed, the derivative of the function (9.7) with respect to the system (9.1)–(9.6) can be represented in the form

$$2(\mu_2-\mu_3)(z_1^2+z_2^2+z_3^2)\cos\alpha. \tag{9.8}$$

□

9.2. System of dynamical equations of motion under the action of a nonconservative force couple. Consider the system of dynamical equation on the tangent bundle $T_*\mathbf{S}^3\{Z_3, Z_2, Z_1; \alpha, \beta_1, \beta_2\}$ of the three-dimensional sphere $\mathbf{S}^3\{\alpha, \beta_1, \beta_2\}$, which describes the motion of a dynamically symmetric four-dimensional rigid body in a nonconservative field under the action of a force couple [42, 47]:

$$v' = v\Psi(\alpha, \beta_1, \beta_2, Z), \tag{9.9}$$

$$\alpha' = -Z_3 + \mu_2(Z_1^2 + Z_2^2 + Z_3^2)\sin\alpha + \mu_2 \sin\alpha\cos^2\alpha - \mu_2\mu_3 Z_3\cos^2\alpha, \tag{9.10}$$

$$Z_3' = \sin\alpha\cos\alpha - (1+\mu_2\mu_3)(Z_1^2 + Z_2^2)\frac{\cos\alpha}{\sin\alpha} + \mu_2 Z_3(Z_1^2 + Z_2^2 + Z_3^2)\cos\alpha \\ - \mu_2 Z_3 \sin^2\alpha\cos\alpha + \mu_2\mu_3 Z_3^2\sin\alpha\cos\alpha - \mu_3 Z_3\cos\alpha, \tag{9.11}$$

$$Z_2' = (1+\mu_2\mu_3) Z_2 Z_3\frac{\cos\alpha}{\sin\alpha} + (1+\mu_2\mu_3) Z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + \mu_2 Z_2(Z_1^2 + Z_2^2 + Z_3^2)\cos\alpha \\ - \mu_2 Z_2\sin^2\alpha\cos\alpha + \mu_2\mu_3 Z_2 Z_3\sin\alpha\cos\alpha - \mu_3 Z_2\cos\alpha, \tag{9.12}$$

$$Z_1' = (1+\mu_2\mu_3) Z_1 Z_3\frac{\cos\alpha}{\sin\alpha} - (1+\mu_2\mu_3) Z_1 Z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + \mu_2 Z_1(Z_1^2 + Z_2^2 + Z_3^2)\cos\alpha \\ - \mu_2 Z_1\sin^2\alpha\cos\alpha + \mu_2\mu_3 Z_1 Z_3\sin\alpha\cos\alpha - \mu_3 Z_1\cos\alpha, \tag{9.13}$$

$$\beta_1' = (1+\mu_2\mu_3) Z_2\frac{\cos\alpha}{\sin\alpha}, \tag{9.14}$$

$$\beta_2' = -(1+\mu_2\mu_3) Z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{9.15}$$

where

$$\Psi(\alpha, \beta_1, \beta_2, Z) = -\mu_2(Z_1^2 + Z_2^2 + Z_3^2)\cos\alpha + \mu_2\sin^2\alpha\cos\alpha - \mu_2\mu_3 Z_3\sin\alpha\cos\alpha.$$

We examine the stability of its trivial solution with respect to perturbations of the variables α, Z_1, Z_2, and Z_3 (after the continuous redefinition of the system at the origin).

Consider the function

$$V(\alpha, Z_1, Z_2, Z_3) = (1+\mu_2^2)(Z_3^2 + Z_2^2 + Z_1^2) - 2\mu_2 Z_3\sin\alpha + \sin^2\alpha, \tag{9.16}$$

which is positive definite in a neighborhood of the origin.

Theorem 10. *For the system* (9.9)–(9.15), *the function* (9.16) *is a Lyapunov* (*Chetaev*) *function, i.e., its derivative with respect to the system* (9.9)–(9.15) *is negative definite for* $\mu_2 < \mu_3$ *and positive definite for* $\mu_2 > \mu_3$.

Corollary 7. *After redefinition of the right-hand sides at the origin, the system* (9.9)–(9.15) *has an attractive singularity at the origin for* $\mu_2 < \mu_3$ *and a repulsive singularity* $\mu_2 > \mu_3$.

Proof of Theorem 10. Indeed, the derivative of the function (9.16) with respect to the system (9.9)–(9.15) has the form

$$2(\mu_2 - \mu_3)(Z_1^2 + Z_2^2 + Z_3^2) + \overline{o}(\alpha^2 + Z_1^2 + Z_2^2 + Z_3^2). \tag{9.17}$$

□

Chapter 2

ANALYSIS OF DATA FOR EXPERIMENTS ON THE MOTION OF BODIES IN A MEDIUM

In this chapter, we present the next stage of examination of the problem on plane-parallel motion of a rigid body interacting with a medium only at the frontal flat part of its surface.

In the construction of the influence function of the medium, we use information on properties of jet circumfluence under the quasi-stationarity condition (for example, in the problem on the motion of homogeneous circular cylinders in water). We do not study the motion of the medium; instead, we examine the problem of the rigid-body dynamics in which the characteristic time of motion of the body with respect to its center of mass is comparable with the characteristic time of motion of the center itself. In [51], the problem was examined under the conditions of the asymptotic stability of the rectilinear translational deceleration; in [50], a new multi-parameter family of phase portraits in the space of quasi-velocities was obtained. In the present chapter, we present data used for performing nature experiments on the motion of hollow circular cylinders in a medium.

10. Preliminaries

First, we give a brief summary of the previous stages of research. Due to the complexity of the nonlinear analysis of the motion, in the initial stage of the study we neglected the dependence of the moment of the influence force on the angular velocity of the body and considered only the dependence on the angle of attack.

From the practical point of view, it is important to study the stability of the nonperturbed (rectilinear translational) motion where the velocities of points of the body are perpendicular to the lamina (cavitator).

Results obtained under this simplest assumption allow one to conclude that there is no conditions under which the systems considered possess solutions corresponding to angular oscillations of the body with limited amplitude.

Experiments on the motion of homogeneous circular cylinders in water justify the fact that in the modeling of the influence of a medium on a rigid body, it is necessary to take into account the dependence of the moment of the influence force on the angular velocity of the body. In this case, the equations of motion contain additional dissipative terms.

In the study of the motion of a body with finite angles of attack, the main problem of the nonlinear analysis is the search for conditions under which oscillations with limited amplitude near the nonperturbed motion appear. This justifies the necessity of a complete nonlinear study.

In previous papers, we used the instability of the rectilinear translational deceleration to find unknown parameters of the influence function under the quasi-stationarity conditions.

The accounting of the damping influence of the medium under certain conditions leads to a positive answer to the principal question of the nonlinear analysis: During the motion of a body in a medium with finite angles of attack, can stable autooscillations appear that are caused by the additional dependence of the influence function of the medium on the angular velocity of the body, which leads to the appearance of an additional dissipation in the system?

Moreover, in the study of dissipative dynamical systems of certain types, we obtained a new multi-parameter family of phase portraits on the two-dimensional cylinder that consists of an infinite number of topologically nonequivalent phase portraits that change their topological types when parameters of the system vary.

11. Data Preparation for Nature Experiments

11.1. Problem on the immersion of a homogeneous circular cylinder in water. We consider the problem on the immersion of a homogeneous circular cylinder in water. The relation between the values of physical parameters for which the rectilinear translational deceleration can be stable is as follows:

$$\mu_3 > \mu_1 + \mu_2 \tag{11.1}$$

or

$$h\frac{mD^2}{I} - 2 - k\frac{m\sigma D}{I} > 0. \tag{11.2}$$

Moreover, if the left-hand side of the inequality (11.2) vanishes, then we speak of the *critical case*.

Recall that D is the diameter of the circular cylinder, σ is the distance from its center of mass to the frontal end, I and m are the inertia and mass characteristics of the cylinder, and the constants k and h are dimensionless parameters that describe the influence of the medium on the cylinder.

For the parameters k and h in the case of bodies with circular frontal end, we have obtained an estimate, namely, $k = h = 0.1$. Thus, the condition (11.2) allows one to construct a rigid body (a circular cylinder) for which the rectilinear translational deceleration can be stable. For this, we must choose the parameters σ, D, I, and m of the cylinder based on the condition (11.2).

Analyzing the inequality (11.2), we arrive at the following conclusion. The inertia and mass parameters of homogeneous cylinders are such that the inequality (11.2) cannot hold for $h = 0.1$. Indeed, for $h = 0.1$ the left-hand side of (11.2) has the form

$$F_1(k, h, m, I, \sigma, D)|_{k=h=0.1} = h\frac{mD^2}{I} - 2 - k\frac{m\sigma D}{I}\bigg|_{k=h=0.1} = F_2(\sigma, D), \tag{11.3}$$

whose right-hand side, up to a positive factor, is always negative and is equal to

$$-3D^2 - 12\sigma D - 80\sigma^2; \tag{11.4}$$

this corresponds to the exponential instability of the rectilinear translational deceleration. Here we have taken into account the fact that the central moment of inertia of the cylinder has the form

$$I = m\left(\frac{\sigma^2}{3} + \frac{D^2}{16}\right). \tag{11.5}$$

Moreover, from an analysis of the left-hand side of (11.2) with respect to change of h, we see that it can vanish only for the minimum critical value of h_* satisfying the equation

$$\left(10h_* - \frac{5}{4}\right) - \bar{\sigma} - \frac{20}{3}\bar{\sigma}^2 = 0, \quad \bar{\sigma} = \frac{\sigma}{D}; \tag{11.6}$$

its root

$$h_* = 0.125 \tag{11.7}$$

is greater than the value $h = 0.1$.

The conditions (11.6) and (11.7) lead us to the following intermediate conclusion. The rectilinear translational deceleration of a homogeneous circular cylinder in water *cannot be stable* with respect to perturbations of the angle of attack and the angular velocity.

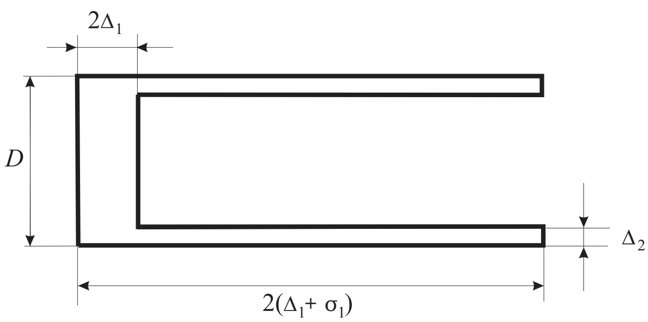

Fig. 4. Hollow cylinder

11.2. Problem on the immersion of a hollow circular cylinder in water. Now we consider the problem on the search for geometric, inertial, and mass parameters of a hollow cylinder that provide the stability. Namely, consider a hollow cylinder (see Fig. 4) whose geometric, inertial, and mass characteristics must satisfy the required inequality for the fixed value $h = 0.1$.

The composite rigid body consists of the cylindrical frontal homogeneous part of diameter D and height $2\Delta_1$ and lateral surface of length $2\sigma_1$ and width Δ_2 (see Fig. 4).

We calculate the following parameters of the composite body that are involved in the inequality (11.2): the distance σ from the center of mass to the frontal circular end and the (central) radius of inertia ρ of the body:

$$\sigma = \frac{\Delta_1^2 D^2 + 4\sigma_1\Delta_2(D-\Delta_2)(\sigma_1 + 2\Delta_1)}{\Delta_1^2 D^2 + 4\sigma_1\Delta_2(D-\Delta_2)}, \tag{11.8}$$

$$\rho^2 = \frac{\Delta_1^2 D^2 +}{\Delta_1^2 D^2 + 4\sigma_1\Delta_2(D-\Delta_2)} \left\{ \frac{4}{3}\Delta_1^2 + \frac{D^2}{16} - 2\Delta_1\sigma + \sigma^2 \right\} + \frac{4\sigma_1\Delta_2(D-\Delta_2)}{\Delta_1^2 D^2 + 4\sigma_1\Delta_2(D-\Delta_2)} \times \left\{ \frac{\sigma_1^2}{3} + \frac{D^2}{8} - \frac{\Delta_2(D-\Delta_2)}{4} + \frac{D^4\Delta_1^2(\sigma_1+\Delta_1)^2}{\Delta_1^2 D^2 + 4\sigma_1\Delta_2(D-\Delta_2)} \right\}. \tag{11.9}$$

We can use the complete equalities (11.8) and (11.9), but this is not decisive since it suffices to accept the following assumptions:

$$\Delta_1^2 \approx \Delta_2^2 \approx \Delta_1\Delta_2 \approx 0. \tag{11.10}$$

All geometric parameters are dimensionless:

$$\overline{\Delta_1} = \frac{\Delta_1}{D}, \quad \overline{\Delta_2} = \frac{\Delta_2}{D}, \quad \overline{\sigma_1} = \frac{\sigma_1}{D}; \tag{11.11}$$

for brevity, we omit the bar in the sequel.

Then the left-hand side of (11.2) under the assumptions (11.10) for $h = 0.1$ in the critical case leads to the equality

$$\Delta_1\left(-\frac{1}{4}\right) + \sigma_1\Delta_2\left(\frac{7}{2}\right) - 4\sigma_1^2\Delta_2 = 0. \tag{11.12}$$

We find a critical value σ_1^* of the dimensionless length of the lateral walls of the composite body:

$$\sigma_1^* = \frac{7}{16} + \frac{1}{8}\sqrt{\frac{49}{4} - 4\frac{\Delta_1}{\Delta_2}}. \tag{11.13}$$

From Eq. (11.13) we see that the ratio Δ_1/Δ_2 takes values in the following interval:

$$0 < \frac{\Delta_1}{\Delta_2} < \frac{49}{16} = 3.0625. \tag{11.14}$$

Formally, as $\Delta_1 \to 0$ (the frontal end becomes infinitesimally thin), the critical value tends to

$$\sigma_1^* = 0.875. \tag{11.15}$$

In the particular case where $\Delta_1 = \Delta_2$, we have

$$\sigma_1^* = \frac{1}{16}(7 + \sqrt{33}) \approx 0.797, \tag{11.16}$$

and as $\Delta_1/\Delta_2 \to 49/16$, we have

$$\sigma_1^* = 0.4375. \tag{11.17}$$

Thus, we can choose σ_1^* as follows:

$$0.4375 \le \sigma_1^* \le 0.875, \tag{11.18}$$

despite the fact that the expressions (11.15)–(11.17) cover only convenient specific cases.

For example, if we take $\Delta_1 = \Delta_2 = 0.1$ (i.e., if $D = 30$ mm, then $\Delta_1 = \Delta_2 = 3$ mm), then the length of the lateral walls is $2\sigma_1 \approx 1.6D \approx 47.8$ mm and the total "critical" length of the composite body is $47.8 + 6 \approx 54$ mm.

Finally, we note that we can "correct" the constant h that describes the influence of the medium on the body by representing σ_1^* as follows. The linearized critical equality (11.12) takes the form

$$\Delta_1 \left(10h - \frac{5}{4}\right) + \sigma_1\Delta_2 \left(40h - \frac{1}{2}\right) - 4\sigma_1^2\Delta_2 = 0, \tag{11.19}$$

and the required value of σ_1^* can be obtained from the equation

$$\sigma_1^* = \frac{1}{8}\left\{\left(40h - \frac{1}{2}\right) + \sqrt{\left(40h - \frac{1}{2}\right)^2 + 16\left(10h - \frac{5}{4}\right)\frac{\Delta_1}{\Delta_2}}\right\}. \tag{11.20}$$

11.3. Possible motions of a rigid body in a resistive medium with bounded angle of attack. As was noted above, if the parameters of the problem admit a critical case (the left-hand side of Eq. (11.2) vanishes), then, depending on the higher derivatives of the influence functions y_N and s, the rectilinear translational deceleration of the body can be stable or unstable with respect to perturbations of the angle of attack and the angular velocity.

Above we have found sufficient conditions of such stability or instability that contain inequalities for higher derivatives of the influence functions. The main difficulty is the impossibility of direct measurement of these derivatives in experiments.

Now we show how one can examine the behavior of the body near the rectilinear translational deceleration (i.e., stable or unstable angular oscillations) by using experimental information for estimating higher derivatives of the influence functions.

First, we note that the inequality

$$\frac{DI\rho_0}{m^2} < \frac{8k}{c_x\pi} \tag{11.21}$$

guarantees the *oscillation* stability (one can change the mass of the body taking various metals). In addition to the known parameters, we also consider the density ρ_0 of the medium (water in the case considered) and the dimensionless coefficient of frontal resistance $c_x = 0.82$.

Indeed, in the CGS system, the inequality (11.21) is equivalent to

$$\frac{D\rho^2}{m} < 0.31,$$

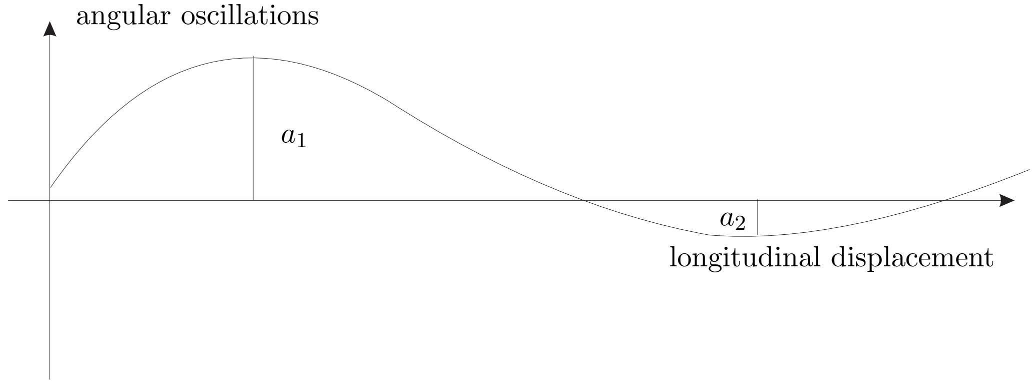

Fig. 5.

where $[m] =$, $[D] = [\rho] =$, and ρ is the (central) radius of inertia expressed by the formula (11.9).

Further, in the case of oscillation motion, we need to obtain experimental information on three or more semi-oscillations with amplitudes a_1, a_2, and a_3 (i.e., one and a half periods). Examining values of parameters close to critical values, we obtain from the theorem on the birth of limit cycles *two conclusions* on the stability of the key regime (the rectilinear translational deceleration) and the character of angular oscillations of the body (see **I** and **II** below).

First, we make an important remark on serial measurements of the amplitudes a_1, a_2, and a_3.

Remark 1. The sequence of ratios

$$\frac{a_2}{a_1}, \frac{a_3}{a_2}, \dots \tag{11.22}$$

of the amplitudes $a_1, a_2, a_3, \dots$ (and further if we can measure more than three semi-oscillations) determines the character of oscillations. For example, if the amplitudes

$$a_1, a_2, a_3, \dots \tag{11.23}$$

are similar to an increasing (decreasing) geometric progression (in particular, the ratios a_2/a_1 and a_3/a_2 are approximately equal), then we can suggest that angular oscillations increase (descrease) sufficiently rapidly. If the values (11.23) increase whereas their ratios a_2/a_1, $a_3/a_2, \dots$ (see (11.22)) decrease, then we observe the transition to angular oscillations of bounded amplitude.

I. Assume that in the nature experiment with parameters corresponding to the *critical* case we observe stable oscillations of the deflection angle. Then a small decrease of the length of the body (see Example 1 below) may cause the damping of angular oscillations (see Fig. 5).

In contrast, a small increase of the length of the body (see Example 2 below) may cause the growth of angular oscillations and subsequent stable angular auto-oscillations of the body (see Fig. 6); in this case, one must pay attention to the rate of change of the amplitude (see Remark 1).

Moreover, repeating the experiments for the body from Example 2 for sufficiently large perturbations of the initial angle of attack and (or) angular velocity, we can observe the transition to stable angular auto-oscillations with finite amplitudes (see Fig. 7) that are similar to the previous case (see Fig. 6).

Example 1. The total length of the body is equal to $50 < 54\,\text{mm}$.

Example 2. The total length of the body is equal to $60 > 54\,\text{mm}$.

II. Assume that in the nature experiment with parameters corresponding to the *critical* case we observe the growth of angular oscillations. Then a small decrease of the length of the body (see

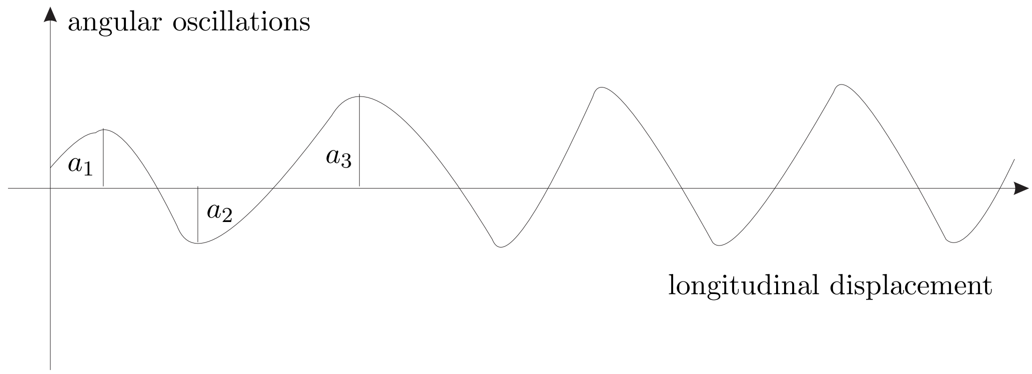

Fig. 6.

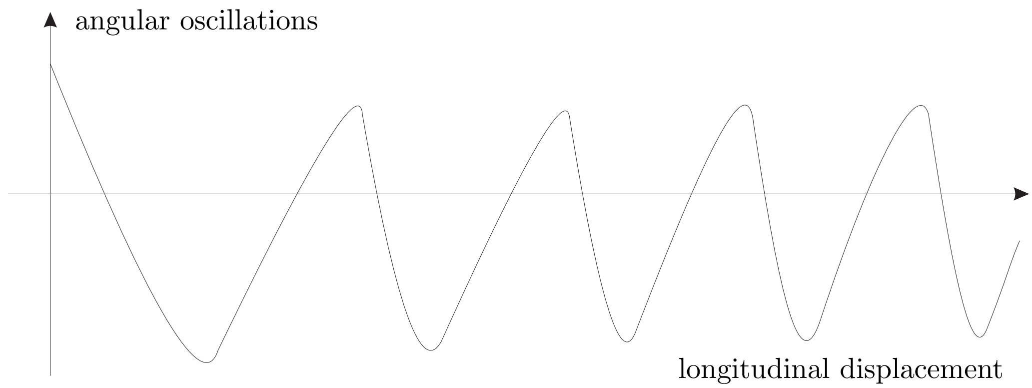

Fig. 7.

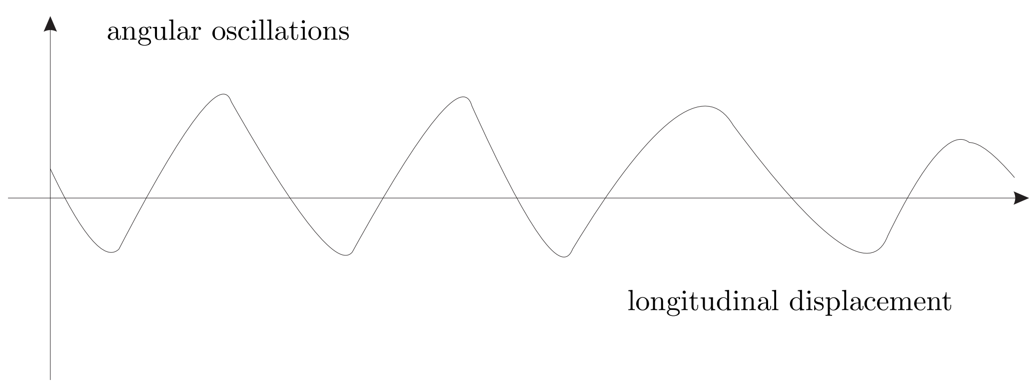

Fig. 8.

Example 1 above) may cause stable oscillations of bounded amplitude (see Fig. 8). Moreover, one must pay attention to the rate of change of the amplitude of oscillations (see Remark 1).

Moreover, in experiments with bodies from Example 1 for finite perturbations of the initial angle of attack and (or) the angular velocity, the transition from unstable auto-oscillations to their growth (see Fig. 9) is possible. A small increase of the length of the body (see Example 2) perhaps leads to the growth of angular oscillations (see Fig. 10).

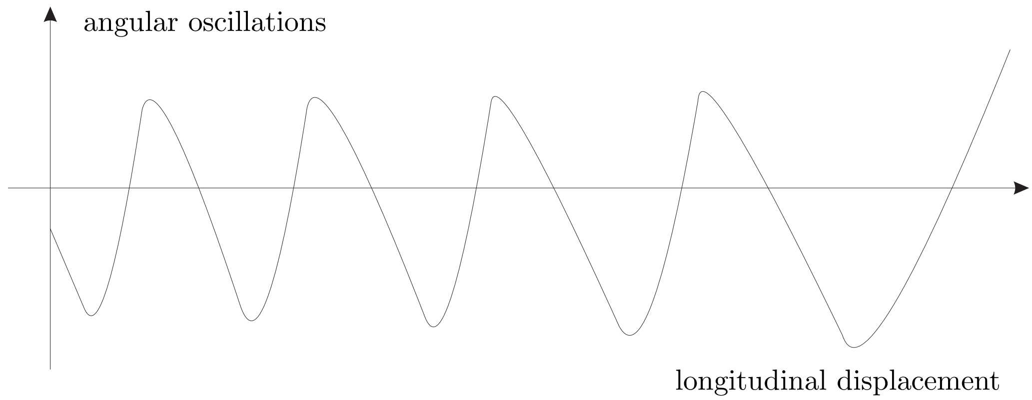

Fig. 9.

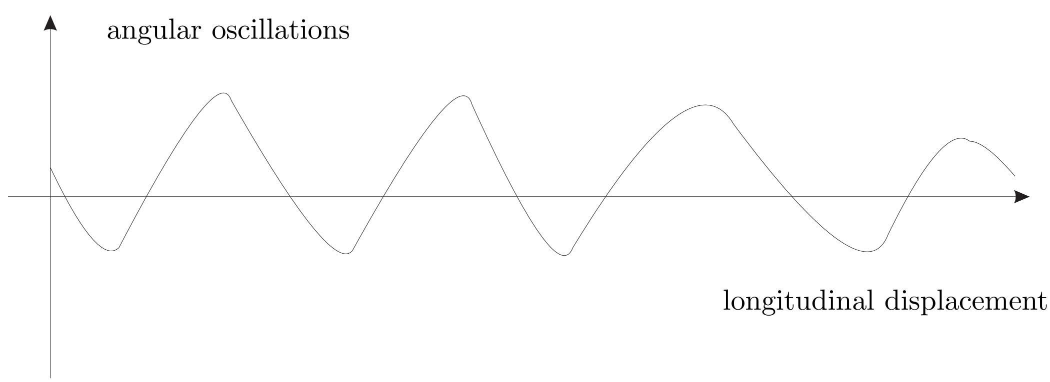

Fig. 10.

12. Conclusion

In the study of the model described above, we have found sufficient conditions of asymptotic stability of one of the key regimes, namely, the rectilinear translational deceleration. For the case of homogeneous circular cylinders, we obtained specific estimates of their inertia and mass characteristic based on results of experiments, including experiments for determining dimensionless influence parameters.

We also showed that under certain conditions for higher derivatives of the influence functions (the arm of the influence force and the resistance coefficient), stable or unstable auto-oscillation regimes of the motion of the system can appear. In this case, the measurement of higher derivatives of the influence functions is the main difficulty since for an arbitrary body, we know neither the explicit form nor signs of higher derivatives at some points.

Applying methods of study of dissipative dynamical systems that appear in the problem on free deceleration, we have obtained a new multi-parameter family of phase portraits on the two-dimensional cylinder of quasi-velocities consisting of an infinite number of topologically nonequivalent portraits whose topological type changes in a degenerate way when parameters of the system change. The family obtained possesses either stable or unstable auto-oscillation regimes in a finite range of values of the angle of attack. In this case, the domain of physical parameters is a set of finite measure in the infinite-dimensional space of parameters of the system, so that the results obtained are typical.

The results obtained allow one to construct hollow circular cylinders that provide the necessary stability in nature experiments.

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