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ON THE LATTICE OF SUBVARIETIES OF THE WREATH PRODUCT OF THE VARIETY OF SEMILATTICES AND THE VARIETY OF SEMIGROUPS WITH ZERO MULTIPLICATION

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Abstract. It is known that the monoid wreath product of any two semigroup varieties that are atoms in the lattice of all semigroup varieties may have a finite as well as an infinite lattice of subvarieties. If this lattice is finite, then as a rule it has at most eleven elements. This was proved in a paper of the author in 2007. The exclusion is the monoid wreath product $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$ of the variety of semilattices and the variety of semigroups with zero multiplication. The number of elements of the lattice $L(\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2)$ of subvarieties of $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$ is still unknown. In our paper, we show that the lattice $L(\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2)$ contains no less than 33 elements. In addition, we give some exponential upper bound of the cardinality of this lattice.

1. Introduction

Going over from the study of products of group varieties [10] to the study of products of semigroup varieties, we have some different variants of the definitions of such a product. One variant of the definition was proposed by A. I. Malcev in [9]. This product has been studied in a number papers of mathematicians of Ekaterinburg (the science school due to L. N. Shevrin) and some other authors (see [14, 16]). Another variant was proposed by J. Rhodes as a wreath product or semidirect product of semigroup varieties and semivarieties (see [1, papers 5–7]. Later this approach has been described in monographs due to S. Eilenberg [3] and G. Lallement [8, Chaps. 4–6]. This approach has been used also by some mathematicians. There have arisen different definitions: general wreath product, monoid wreath product, and standard wreath product of semigroup varieties. In the first and the second cases, we have the associative wreath product. The standard wreath product of semigroup varieties is not associative (see [7, 18, 19]).

We recall that the ordered monoid of semigroup varieties under the operation of monoid wreath product has been studied in [22]. In our paper, we deal with the monoid wreath product of varieties. The computation of the wreath product of atoms of the lattice of semigroup varieties has been realized in [23]. A semigroup variety $\mathbf{U}$ is called a *Cross* if it is finitely based, is generated by a finite semigroup, and has a finite lattice of subvarieties (see, e.g., [12]). The atoms of the lattice of all semigroup varieties are well known (see [4, 16]). These are precisely the varieties $\mathbf{N}_2$ of all semigroups with zero multiplication, $\mathbf{L}$ of all semigroups of left zeroes, $\mathbf{R}$ of all semigroups of right zeroes, $\mathbf{Sl}$ of all semilattices, and $\mathbf{A}_p$ of all Abelian groups of prime exponent p. The lattice $L(\mathbf{U}\,\mathrm{w}\,\mathrm{V})$ is infinite in the following four cases of the wreath product of atoms: $\mathbf{A}_p\,\mathrm{w}\,\mathbf{A}_p$ (p is prime), $\mathbf{Sl}\,\mathrm{w}\,\mathbf{Sl}$, $\mathbf{Sl}\,\mathrm{w}\,\mathbf{R}$, and $\mathbf{Sl}\,\mathrm{w}\,\mathbf{A}_p$ (p is prime).

In other cases, the monoid wreath product $\mathbf{U}\,\mathrm{w}\,\mathbf{V}$ of two atoms $\mathbf{U}$ and $\mathbf{V}$ of the lattice of semigroup varieties has a finite lattice of subvarieties. The cardinality of such a lattice, as a rule, is not greater than 11 [23, Theorem 3.1]. However, in the case of the wreath product of the varieties of all semilattices and all semigroups with zero multiplication we have another situation. It was proved in [23] that the lattice $L(\mathbf{Sl}\,\mathrm{w}\,\mathbf{R})$ is finite. But precise computation of this lattice is not as easy as for other wreath products of atoms if such a lattice of subvarieties is finite. Thus, we can set two connected problems.

Problem 1. Describe the lattice $L(\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2)$ of subvarieties.

Problem 2. To give lower and upper bounds of the cardinality of the lattice $L(\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2)$ of all subvarieties.

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It is clear that the solution of Problem 1 covers Problem 2. But now we do not have the full solution of Problem 1. We consider the full solution of this problem as an important step in the study of the monoid wreath product of semigroup varieties.

Further the variety $\mathbf{Sl} \mathrm{w} \mathbf{N}_2$ will be denoted by $\mathbf{W}$. In this paper, we have some progress in the solution of Problems 1 and 2. We indicate two disjoint sublattices L' and L_0 of the lattice $L = L(\mathbf{W})$ of subvarieties. In addition, we have $|L'| = 13$ and $|L_0| = 20$. So, $|L| \geq 33$. In this paper, we shall get an upper bound for the cardinality of the lattice L. But this bound is not exact, namely, $|L| \leq 2^(127) + 33$.

In Sec. 2, we give some needed definitions and notation and give some results from [23]. So, we describe the approach to the solution of Problems 1 and 2. In Sec. 3, we give a complete description of all subvarieties in L that do not contain the variety $\mathbf{Sl}$ of all semilattices. Such subvarieties form a sublattice L' of L containing 13 elements. In Sec. 4, we give a sublattice L'' of L_0 consisting of all subvarieties in $L = L(\mathbf{Sl} \mathrm{w} \mathbf{N}_2)$ containing the variety $\mathbf{Sl}$) of all semilattices. So, $|L| \geq 33$. We also find an upper bound for the cardinality of the lattice L, but this estimation is not good.

In conclusion, note that the results on two disjoint sublattices of 13 and 20 elements were announced in [21].

2. Preliminaries

We shall use the usual terminology of the theory of semigroups and the theory of varieties (see [2, 15, 16]). We recall some definitions and some notation.

Let X be a countable alphabet. Let us denote the letters of this alphabet by $x, y, z, t, x_1, x_2, \ldots, x_k, \ldots, y_1, y_2, \ldots, y_k, \ldots, z_1, z_2, \ldots, z_k, \ldots$, and so on. If u and v are words over the alphabet X, then $u \approx v$ denotes the identity over X; $|u|$ denotes the length of the word u, $c(u)$ denotes the set of letters that appear in the word u, $h_k(u)$ is the prefix of the word u of length k. An identity $u \approx v$ is called *homotypic* if the equality $c(u) = c(v)$ holds, and *heterotypic* if the inequality $c(u) \neq c(v)$ holds.

An element a of a semigroup S is called *periodic* if it satisfies the equality $a^{m+n} = a^m$ for some natural numbers m, n. If m and n are the least numbers with such properties, then the number $m + n$ is called the *order*, m is called the *index*, and n is called the *period* of the element a. It is easy to see that a monogenic semigroup $\langle a \rangle$, generated by the element a of order $l = m + n$, contains l elements.

A semigroup is called *uniform periodic* if the identity

$$x^m = x^{m+n}$$

is true for some natural numbers m and n. A semigroup is called *nil semigroup of index m* if the identity

$$x^m = 0$$

holds in it for some m. A semigroup S is called *nilpotent of step m* if the identity

$$x_1 x_2 \ldots x_m = 0$$

holds. A nonempty word w is called *linear* if none of its letters occurs in w more than once.

The *wreath product of semigroups S and R by a right R-set A* is the semigroup $T = S^A \mathrm{w} R$ defined on the Cartesian product $S^A \times R$ with S^A being the set of all functions of A into S, and multiplication given by formulas

$$(f, p)(g, q) = (f^q g, pq), \quad (f^q g)(a) = f(a) g(ap)$$

for any $a \in A$ [17, 19, 20]. First of all, we now are interested in the wreath product of semigroup varieties. In the study of the wreath product of semigroup varieties, we have chosen the monoid wreath product of semigroup varieties [18–20] as the most suitable. The monoid wreath product of semigroup varieties is generated by the set of all extended standard products in which the passive semigroup belongs to the first variety and the active semigroup belongs to the second variety [19, 20]. Note that in [18] the monoid wreath product of semigroup varieties is called the wreath product of semigroup varieties. We recall that a wreath product of semigroups $T = S^A \mathrm{w} R$ is called an *extended standard wreath product of semigroups* if the R-set A coincides with the least monoid R^1 containing the semigroup R. The extended standard wreath product of semigroups S and R is denoted by $T = S \,\mathrm{w}_1\, R$. Further in this paper, we consider only the extended standard wreath product of semigroups.

In [23], there have been proved some results concerning the variety $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$, which we need now. We formulate these results.

Lemma 2.1 ([23, Corollary 2.13]). *If* $\operatorname{var} S = \mathbf{Sl}$, *then an identity* $u \approx v$ *belongs to the set* $I(\mathbf{W})$ *if and only if the following conditions hold*:

(1) *either* $u \approx v$ *is a trivial identity or* $|u|, |v| \geq 3$;
(2) $h_2(u) = h_2(v)$;
(3) *replacing the common beginning* $h_2(u) = h_2(v)$ *in the identity* $u \equiv h_2(u)u_1 \approx h_2(v)v_1 \equiv v$ *by the subword* z_1z_2, *where* $z_1, z_2 \notin c(uv)$, *yields the identity* $z_1z_2u_1 = z_1z_2v_1$, *where* $c(u_1) = c(v_1)$.

Lemma 2.2 ([23, Corollary 2.14]). *The variety* $\mathbf{W}$ *has the basis of identities*

$$z_1z_2y \approx z_1z_2y^2, \tag{2.1}$$

$$z_1z_2yx \approx z_1z_2xy. \tag{2.2}$$

Proposition 2.1 ([23, Proposition 3.8]). *The lattice* $L(\mathbf{W})$ *is finite.*

In particular, in the proof of this proposition in [23] there has been established the following fact. If in the subvariety $\mathbf{U} \subset \mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$ some heterotypic identity is true, then the identity

$$z_1z_2y \approx z_1z_2x$$

is true in it. If each identity in $I(\mathbf{U})$ is homotypic, i.e., for every identity $u \approx v$ in $I(\mathbf{U})$ the equality $c(u) = c(v)$ holds, then the variety $\mathbf{U}$ contains the variety $\mathbf{Sl}$ of all semilattices. In this case, we can assume that each of the words u and v is reduced to some $\mathbf{W}$-equivalent *canonical word*, i.e., it has one of the following forms:

$$z,\ z^2x_1 \dots x_k,\ z^3x_1 \dots x_k,\ z_1z_2x_1 \dots x_k,\ z_1z_2z_1x_1 \dots x_k,\ z_1z_2^2x_1 \dots x_k,\ z_1z_2^2z_1x_1 \dots x_k \quad (k \geq 0). \tag{2.3}$$

In solving Problems 1 and 2, it is important to know whether the considered subvariety $\mathbf{V}$ of the variety $\mathbf{W}$ contains the variety $\mathbf{Sl}$ of all semilattices or not. The fact $\mathbf{Sl} \subseteq \mathbf{V}$ is known to be equivalent to the fact that each identity from $I(\mathbf{V})$ is homotypic. It is easy to note that

$$L(\mathbf{W}) = L' \cup L'',$$

where L' is the lattice of all subvarieties in $\mathbf{W}$ in which there is true a heterotypic identity, and L'' is the lattice of all subvarieties in $\mathbf{W}$ that contain the variety $\mathbf{Sl}$ of all semilattices.

We denote by $\operatorname{var} \Sigma$ the variety of all semigroups in which all semigroup identities of the set Σ are true.

3. The Sublattice L' of All Subvarieties of the Variety W That Do Not Contain the Variety of All Semilattices

Proposition 3.1. *The sublattice* L' *of all subvarieties of the variety* $\mathbf{W}$, *in which some heterotypic identity is true, coincides with the lattice of all subvarieties of the variety* $\mathbf{L}_{2,3} = \operatorname{var}\{z_1z_2x = z_1z_2y\}$. *This lattice is represented in Fig.* 1. *It contains* 13 *elements.*

In Fig. 1, we use the following additional notation:

$$\mathbf{L}_{1,3} = \operatorname{var}\{xy_1y_2 \approx xz_1z_2\},$$

$$\mathbf{N}_3 = \mathbf{L}_{1,3} = \operatorname{var}\{y_1y_2y_3 \approx z_1z_2z_3\},$$

$$\mathbf{L}_{2,2} = \operatorname{var}\{x_1x_2 \approx x_1x_2z\},$$

$$\mathbf{L}_{1,2} = \operatorname{var}\{xy \approx xz\},$$

$$\mathbf{N}_2 = \mathbf{L}_{0,2} = \operatorname{var}\{xy \approx zt\},$$

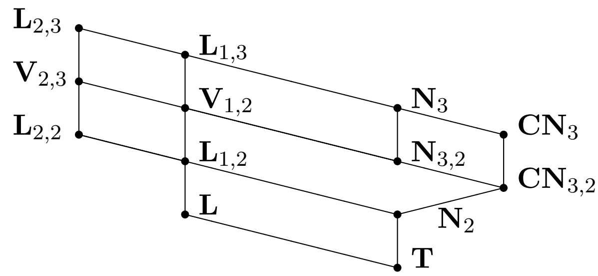

Fig. 1. The sublattice L' of subvarieties of the variety $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$ that do not contain the variety of all semilattices.

$$\begin{aligned}
&\mathbf{L} = \mathbf{L}_{1,1} = \mathrm{var}\{xy \approx x\},\\
&\mathbf{N}_{3,2} = \mathrm{var}\{x^2 \approx y_1y_2y_3\},\\
&\mathbf{CN}_3 = \mathrm{var}\{y_1y_2y_3 \approx z_1z_2z_3,\ xy \approx yx\},\\
&\mathbf{CN}_{3,2} = \mathrm{var}\{x^2 \approx y_1y_2y_3,\ xy \approx yx\},\\
&\mathbf{V}_{2,3} = \mathrm{var}\{x^2 \approx x^3,\ x_1x_2y \approx x_1x_2z\},\\
&\mathbf{V}_{1,3} = \mathrm{var}\{x^2 \approx x^3,\ xy_1y_2 \approx xz_1z_2\}.
\end{aligned}$$

Remark 3.1. The notation $\mathbf{L}_{j,m}$ ($m \geq 1$, $0 \leq j \leq m$) was used in [22] and in [6] for the idempotents under the wreath product of semigroup varieties.

Proof of Proposition 3.1. The proposition is proved by standard arguments using known full descriptions of all identities of the varieties $\mathbf{L}_{2,3}$, $\mathbf{L}_{2,2}$, $\mathbf{L}_{1,3}$, and other varieties indicated in Fig. 1. We do this argument for the case of the variety $\mathbf{L}_{2,3}$. Let $\mathbf{V}$ be a subset of $\mathbf{L}_{2,3}$. It is clear that $u \approx v \in I(\mathbf{L}_{2,3})$ if and only if the following two conditions hold:

(1) $u \approx v$ is trivial or $|u|, |v| \geq 3$;
(2) $h_2(u) = h_2(v)$.

Let $u \approx v \in I(\mathbf{V}) - I(\mathbf{L}_{2,3})$ be true. Then for the identity $u \approx v$, condition (1) or (2) is violated.

Let condition (2) be violated but $h_1(u) = h_1(v)$ be true. In this case, we have some identity of the form $u \equiv xyu' \approx xyv' \equiv v$ with y, z being two different variables belonging to the set $I(\mathbf{V})$. Multiplying this identity on the right by some word we can assume that $|u|, |v| \geq 3$. Therefore, in the variety $\mathbf{V}$ the following sequence of identities is true:

$$xyt \approx xyu' \approx h_3(u) \approx h_3(v) \approx xzv' \approx xzt_1.$$

Thus, in this case we have $\mathbf{V} \subseteq \mathbf{L}_{1,3}$.

Let $h_1(u) \neq h_1(v)$. Then $u \equiv xu' \approx yv' \equiv v$, where x, y are different variables. Setting $\varphi(x) = x_1x_2$, $\varphi(y) = y_1y_2$, and $\varphi(z) = z$ for each $z \in X - \{x, y\}$, we obtain that the identity $x_1x_2u'' \approx y_1y_2v''$ is true in the variety $\mathbf{V}$. Therefore, in the variety $\mathbf{V}$ we have the following chain of identities:

$$x_1x_2t \approx y_1y_2u'' \approx y_1y_2v'' \approx x_1x_2t_1.$$

Thus, in this case, we have $\mathbf{V} \subseteq \mathbf{N}_3$.

Let for the identity $u \approx v$ condition (1) be violated. If $|u| = 1$ and $\mathbf{V} \neq \mathbf{T}$, then we have the identity $x \approx x^2$ in $\mathbf{V}$. Then the identity $xy \approx x^2y \approx x^2z \approx xz$ is true in $\mathbf{V}$. Therefore, in this case, we have $\mathbf{V} \subseteq \mathbf{L}_{1,2}$. If $|u| = 2$ and $|v| \geq 3$, then $x^2 \approx x^3 \in I(\mathbf{V})$ and $\mathbf{V} \subseteq \mathbf{V}_{2,3}$. If $u \equiv xy$, then under $|v| \geq 3$ this identity is equivalent to the identity $xy \approx xyz$. Thus, in this case, we have that $\mathbf{V} \subseteq \mathbf{L}_{2,2}$. If $|u| = |v| = 2$ and the considered identity is nontrivial, then condition (2) is violated. This case has been considered above. The inequality $\mathbf{V}_{2,3} \neq \mathbf{L}_{2,3}$ follows from the fact that the first set $I(\mathbf{V}_{2,3})$ of identities contains an identity $u \equiv v$ with $|u| = 2$ and the second set does not contain such identities. □

4. Subvarieties of the Variety W, Containing the Variety of All Semilattices

Now we shall estimate the number of different possible homotypic identities $u \approx v$ in which the right and left sides are $\mathbf{W}$-canonical words, i.e., words of the form (2.3). It should be noted that the letters z, z_1, and z_2 in these words are used for letters of the beginnings of length 2.

Further, we assume that for an identity $u \approx v$ we have the inequality

$$|u| \leq |v|. \tag{4.1}$$

Now we consider distinct cases of $u \approx v$. Let us introduce a "working" definition.

Definition 4.1. Identities $u \approx v$ and $u_1 \approx v_1$ are called $\mathbf{W}$-*equivalent* for the variety $\mathbf{W}$ if the subvarieties defined in it by these identities coincide.

Definition 4.2. An identity $u \approx v$ is called $\mathbf{W}$-*canonical* in a subvariety $\mathbf{U} \subseteq \mathbf{W}$ if its right and left sides are $\mathbf{W}$-canonical words.

Thus, if $\beta(\mathbf{W})$ is some basis of identities of the variety $\mathbf{W}$, then for $\mathbf{W}$-equivalent identities the subvarieties $\operatorname{var}(\beta(\mathbf{W}),\ u \approx v)$ and $\operatorname{var}(\beta(\mathbf{W}),\ u_1 \approx v_1)$ are equal.

Further we shall consider the subvarieties of $\mathbf{W}$ defined by adding one identity $u \approx v$ to the basis $\beta(\mathbf{W})$, i.e., the subvarieties of the form

$$\mathbf{U} = \operatorname{var}(\beta(\mathbf{W}),\ u \approx v). \tag{4.2}$$

Remark 4.1. A general plan of solving Problems 1 and 2 consists of the following steps. At first, we shall describe all $\mathbf{W}$-canonical non-$\mathbf{W}$-equivalent identities. Then for finding $\mathbf{U}_1 \vee \mathbf{U}_2$ we can use the known equality $I(\mathbf{U}_1 \vee \mathbf{U}_2) = I(\mathbf{U}_1) \cap I(\mathbf{U}_2)$. If the descriptions of all identities of $I(\mathbf{U}_1)$ and $I(\mathbf{U}_2)$ are known, then we can try to find the basis of the variety $\mathbf{U}_1 \vee \mathbf{U}_2$. Moreover, it is known that for the intersection of two semigroup varieties we have the following equality: $I(\mathbf{U}_1 \cap \mathbf{U}_2) = I(\mathbf{U}_1 \cup \mathbf{U}_2)$. In the last case, the basis of the intersection of two semigroup varieties is the union of the bases of the initial varieties (see [16, Sec. 5]). The obtained basis possibly may be simplified. Due to [23, Theorem 1.2], the variety $\mathbf{W}$ is Cross. In particular, this variety is hereditarily finitely based (see [5]). So, any subvariety $\mathbf{U} \subseteq \mathbf{W}$ is the intersection of a finite number of semigroup subvarieties of the form (4.2). Thus, the first problem here is the description of all possible $\mathbf{W}$-canonical non-$\mathbf{W}$-equivalent identities.

Let us introduce the following notation for subvarieties of $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$, containing the variety $\mathbf{Sl}$:

$$\begin{aligned}
&\mathbf{V}_1 = \operatorname{var}\{\beta(\mathbf{W}),\ z^2 \approx z^3\},\\
&\mathbf{V}_2 = \operatorname{var}\{\beta(\mathbf{W}),\ zx \approx zx^2\},\\
&\mathbf{V}_3 = \operatorname{var}\{\beta(\mathbf{W}),\ zx \approx zxz\},\\
&\mathbf{W}_0 = \mathbf{Sl}\,\mathrm{w}\,\mathbf{L} = \operatorname{var}\{zx \approx zx^2,\ zyx \approx zxy\},\\
&\mathbf{V} = \mathbf{Sl} \vee \mathbf{L} \vee \mathbf{N}_2 = \operatorname{var}\{zx \approx z^2x,\ zyx \approx zxy\},\\
&\mathbf{P}' = \operatorname{var} P' = \operatorname{var}\{zx \approx zx^2,\ y^2x \approx x^2y\},\\
&\mathbf{W}_1 = \operatorname{var}\{\beta(\mathbf{W}),\ zyx \approx zxy\},\\
&\mathbf{W}_2 = \operatorname{var}\{\beta(\mathbf{W}),\ zyx \approx zxy,\ yxz \approx xyz\},\\
&\mathbf{W}_3 = \operatorname{var}\{\beta(\mathbf{W}),\ yx \approx xy\},\\
&\mathbf{V}' = \operatorname{var}\{\beta(\mathbf{W}),\ z^2y \approx z^3y\},\\
&\mathbf{U}_0 = \operatorname{var}\{\beta(\mathbf{W}),\ y^2x \approx x^2y\},\\
&\mathbf{W}_{11} = \operatorname{var}\{\beta(\mathbf{W}),\ zyx \approx zxy,\ z^2 \approx z^3\},
\end{aligned}$$

$$\mathbf{W}_{21} = \mathrm{var}\{\beta(\mathbf{W}),\ zyx \approx zxy,\ yxz \approx xyz,\ z^2 \approx z^3\},$$
$$\mathbf{W}_{31} = \mathrm{var}\{\beta(\mathbf{W}),\ yx \approx xy,\ z^2 \approx z^3\},$$
$$\mathbf{U}_1 = \mathrm{var}\{\beta(\mathbf{W}),\ zyx \approx zxy,\ y^2x \approx x^2y\},$$
$$\mathbf{U}_{11} = \mathrm{var}\{\beta(\mathbf{W}),\ zyx \approx zxy,\ y^2x \approx x^2y,\ z^2 \approx z^3\}.$$

Let us add to these subvarieties three more, namely:

$$\mathbf{Sl} = \mathrm{var}\{z \approx z^2,\ yx \approx xy\},$$
$$\mathbf{Sl} \vee \mathbf{L} = \mathrm{var}\{z \approx z^2,\ zyx \approx zxy\},$$
$$\mathbf{Sl} \vee \mathbf{N}_2 = \mathrm{var}\{zy \approx z^2y,\ yx \approx xy\}.$$

Now we have got the sublattice $L_0 \subseteq L''$ of 20 subvarieties. Now let us show that this subset L_0 of 20 subvarieties is a sublattice.

The main aim of the remainder of the present section is a proof of the following proposition.

Proposition 4.1. *The lattice $L(\mathbf{W})$ contains the sublattice $L_0 \subseteq L''$ of* 20 *subvarieties, which are defined only by homotypic identities. The sublattice L_0 is depicted in Fig.* 2. *In particular, the sublattice L_0 contains all subvarieties that are defined in* $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$ *by permutation identities and homotypic identities $u \approx v$ with the condition $|u| \le 2$.*

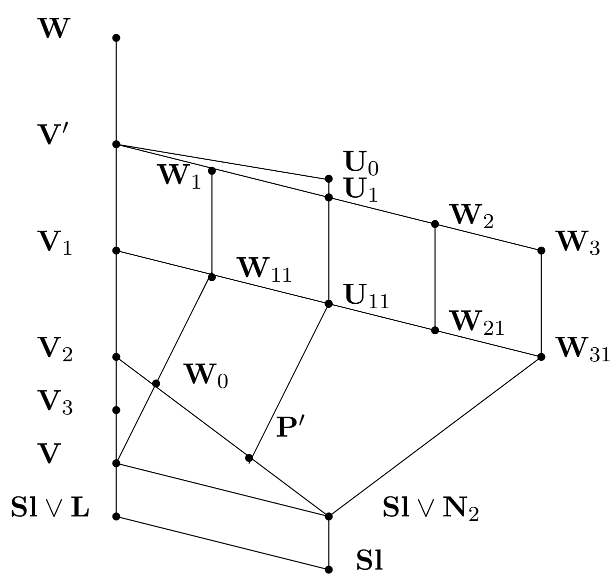

Fig. 2. The sublattice $L_0 \subseteq L''$ of subvarieties of the variety $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$, containing the variety of all semilattices.

Before proving Proposition 4.1, we shall prove a number of lemmas.

Lemma 4.1. *If the identity $u \approx v$ satisfies the condition $|u| = 1$, then the subvariety $\mathbf{U}$ of the form* (4.2) *is contained in the variety $\mathbf{Sl} \vee \mathbf{L}$.*

Proof. If $|v| = 1$, then the assertion is obvious. Let $|v| \ge 1$. Then a homotypic identity is either $z \approx z^2$ or $z \approx z^3$. In any case, the identity $z \approx z^2$ is true in the variety $\mathbf{U}$. Then the identity $zyx \approx zxy$ is also true in the variety $\mathbf{U}$. The lemma is proved. □

Lemma 4.2. *If in the identity we have $u = z^2$, then the variety* $\mathbf{U}$ *of the form* (4.2) *is contained in the variety* $\mathbf{V}_1$.

Proof. It follows from inequality (4.1) and the homotypic identity $u \approx v$ that $|v| = 2$ or $|v| = 3$. In the first case, the identity $u \approx v$ is trivial. In the second case, we have the identity $z^2 \approx z^3$. □

Lemma 4.3. *If in the subvariety* $\mathbf{U}$ *of the form* (4.2) *we have the identity of the form*

$$yx \approx v$$

in which $h_1(v) = x$ and $|v| \geq 2$, then in $\mathbf{U}$ *we have the identity of commutativity*

$$yx \approx xy,$$

i.e., it is contained in the variety $\mathbf{W}_3$.

Proof. It follows from inequality (4.1) and the homotypic identity $z^2 \approx z^3$ that $c(v) = \{x, y\}$, $h_1(v) = x$. Then the word v coincides with one of the following words: (1) xy, (2) xy^2, (3) xyx, (4) $(xy)^2$, (5) x^2y, (6) x^3y. Let us show that in any case the identity of commutativity is true in $\mathbf{U}$. Indeed, (1) is the identity of commutativity.

(3) $\Longrightarrow$ (4). It follows from the identity

$$yx \approx xyx$$

that $xyx \approx yxyx$. Hence $yx \approx yxyx$. Renaming the variables we obtain the identity

$$yx \approx (xy)^2.$$

(4) $\Longrightarrow$ (1). The identity $yx \approx (xy)^2$ implies that the identity $x^2 \approx x^4$ is true in $\mathbf{U}$. Due to the identity (2.2), the identity $x^2 \approx x^3$ is true in $\mathbf{U}$. Now we have the following chain of identities:

$$yx \approx (xy)^2 \approx (yx)^4 \approx (yx)^3 \approx (yx)^2 \approx xy.$$

(5) $\Longrightarrow$ (3). It follows from the identity

$$yx \approx x^2y$$

that the identity $x^2 \approx x^3$ is true in $\mathbf{U}$. So, we also have the following chain of identities:

$$yx \approx x^2y \approx x^3y = x(x^2y) \approx xyx.$$

(2) $\Longrightarrow$ (1). The identity

$$yx \approx xy^2$$

implies that we have the following chain of identities:

$$yx \approx xy^2 \approx y^2x^2 \approx x^2y^4 \approx x^2y^2 \approx xy$$

in the variety $\mathbf{U}$.

(6) $\Longrightarrow$ (3). The identity

$$yx \approx x^3y$$

implies that we have the following chain of identities:

$$yx \approx x^3y \approx x^4y \approx x(yx)$$

in $\mathbf{U}$. Therefore, the identity $yx \approx xy$ is true in it. □

Now let a nontrivial homotypic identity of the form

$$yx \approx yv_1$$

be true in some subvariety $\mathbf{U}$ of the form (4.2). Then the right side of this identity coincides with some word from the following list: yx^2, yxy, y^2x, y^3x, yx^2y.

Lemma 4.4.

(1) *The identities* $yx \approx yx^2y$ *and*

$$yx \approx yxy \tag{4.3}$$

are **W**-*equivalent.*

(2) *The identities* $yx \approx y^3x$ *and*

$$yx \approx y^2x \tag{4.4}$$

are **W**-*equivalent. In addition, the subvariety* $\mathrm{var}\{\beta(\mathbf{W}),\ yx \approx y^2x\}$ *coincides with the variety*

$$\mathbf{V} = \mathbf{Sl} \vee \mathbf{L} \vee \mathbf{N}_2 = \mathrm{var}\{zx \approx z^2x,\ zyx \approx zxy\}.$$

Proof. (1) From (4.3) it follows that

$$yx \approx yxy \approx yxyx \approx yx^2y$$

is true. Conversely, we have the following chain of identities:

$$yx \approx yx^2y \approx yxy^2x \approx yxy.$$

(2) It is clear that we have the following chain of identities:

$$yx \approx y^3x \approx y^4x \approx y^2x.$$

Similarly, the converse implication is also true. The basis of the variety

$$\mathbf{V} = \mathbf{Sl} \vee \mathbf{L} \vee \mathbf{N}_2 = \mathrm{var}\{zx \approx z^2x,\ zyx \approx zxy\}$$

is well known and easily may be calculated. In [4], the basis of it contains the additional identity $zx \approx zx^2$. It is easy to verify that the last identity is a consequence of the first and second identities. Let us identity (4.4) is true in a subvariety **U**. Then the chain of identities

$$zyx \approx z^2yx \approx z^2xy \approx zxy$$

is true in it. Therefore, the variety **U** is contained in **V**. The converse inclusion is clear. □

Remark 4.2. Further it is convenient to assume that the letters z, z_1, z_2, z_3, z_4, and t in **W**-canonical words (2.3) are reserved for letters that occur at the beginnings of length 2 of these words. Moreover, we assume that some of the four letters of the identity $u \approx v$ may coincide in $u \approx v$. In addition, some of these letters may be absent in $u \approx v$. Thus, in fact the identity $u \approx v$ may contain from one to four letters z_i $(i = 1, 2, 3, 4)$. Furthermore, a linear letter of the left side of (4.1) is present if and only if it occurs in the set $c\big(h_2(v)\big) - c\big(h_2(u)\big)$. A similar remark is true for letters z_i $(i = 1, 2, 3, 4)$ of the right side of (4.1).

Lemma 4.5. *If for a homotypic identity*

$$u \equiv u_1yu'x \approx v_1yv'x \equiv v \tag{4.5}$$

there are the inequalities $|u_1|, |v_1| \geq 2$, *and the letter* x *is linear in* u *and in* v, *then identity* (4.5) *is* **W**-*equivalent to a shorter identity, which can be obtained from* (4.5) *by deleting the letter* x, *i.e., the identity*

$$\tilde{u} \equiv u_1yu' \approx v_1yv' \equiv \tilde{v}. \tag{4.6}$$

Proof. Indeed, let the letter y occur in the right and left sides of identity (4.5): $u \equiv u_1yu'x$ and $v \equiv v_1yv'x$. In addition, the lengths of the words satisfy the inequalities $|u_1| \geq 2$ and $|v_1| \geq 2$. Putting in the identity $u \approx v$ $\varphi(x) = y$, we get that the chain of identities

$$u_1yu' = u_1y^2u' = u_1yu'y = v_1yv'y = v_1y^2v' = v_1yv'$$

is true in **V**. Thus, identity (4.6) is true in **V**. Conversely, (4.6) implies (4.5). □

Corollary 4.1. *Any homotypic identity in a subvariety* $\mathbf{U} \subseteq \mathbf{W}$ *is* **W**-*equivalent to a homotypic identity that contains at most one variable* $x \notin \{z_1, z_2, z_3, z_4\}$. *Thus, any* **W**-*canonical identity contains either* $C = \{z_1, z_2, z_3, z_4, x\}$, *or some subset of* C.

Later on, we shall use the following notation associated with the identity $u \approx v$: A_1 is the set $c\big(h_2(u)\big)$ with $A_1 \subseteq \{z_1, z_2\}$ and A_2 is the set $c\big(h_2(v)\big)$ with $A_2 \subseteq \{z_3, z_4\}$. We assume that $z_1 \neq z_2$ and $z_3 \neq z_4$. However, among the letters z_1, z_2, z_3, and z_4 some letters may be equal. Let us assume $A = A_1 \cap A_2$.

Now our main problem is to give an upper and lower bound for the cardinality of the sublattice L''. As a result of learning of this section we shall get such bounds. Recall that in Lemmas 4.1–4.4 we have completely enumerated the set of all possible non-**W**-equivalent homotypic identities of the form $u \approx v$ with the condition $|u| \leq 2$. In the following lemma, we shall get an upper bound for the number of non-**W**-equivalent homotypic identities of the form $u \approx v$ with the condition $|v| \geq |u| \geq 3$.

Lemma 4.6. *The number of possible non-***W***-equivalent homotypic identities of the form $u \approx v$ satisfying the inequalities*

$$|v| \geq |u| \geq 3 \tag{4.7}$$

is at most 142.

Proof. To prove this assertion, consider seven different cases connected with the sets A, A_1, and A_2. Let us denote by D_u the set of all possible beginnings of maximal length of the word u that contain only variables having occurrences in the beginning $h_2(u)$. First of all, note that for counting non-**W**-equivalent homotypic identities we use Lemma 4.5 and its corollary. This fact will not be noted each time.

Case 1. $|A| = 2$. This means that $A_1 = A_2 = \{z_1, z_2\}$. Due to Corollary 4.1 either $c(u) = c(v) = \{z_1, z_2\}$, or $c(u) = c(v) = \{z_1, z_2, x\}$. In this case,

$$D_u = \{z_1z_2,\ z_1z_2z_1,\ z_1z_2^2,\ z_1z_2^2z_1,\ z_2z_1,\ z_2z_1z_2,\ z_2z_1^2,\ z_2z_1^2z_2\}.$$

If $c(u) = \{z_1, z_2\}$, then the number of such identities is 15. If $c(u) = \{z_1, z_2, x\}$, then due to Corollary 4.1 the number of such identities is $C_8^2 - 2 \cdot C_4^2 = 16$. The common number of all non-**W**-equivalent homotypic identities in Case 1 equals 31. From these identities only one is permutable, namely:

$$z_1z_2x \approx z_2z_1x. \tag{4.8}$$

Case 2. $|A| = 1$, $A_1 = A_2 = \{z_1\}$. In this case, we have $c(u) = c(v) = \{z_1, x\}$. There exists only one such identity

$$z_1^2x \approx z_1^3x. \tag{4.9}$$

Case 3. $|A| = 1$, $A_1 = \{z_1\}$, $A_2 = \{z_1, z_3\}$. In this case, we have either $c(u) = c(v) = \{z_1, z_2\}$, or $c(u) = c(v) = \{z_1, z_2, x\}$. Then there are

$$D_{3u} = \{z_1^2,\ z_1^3\} \cdot z_3,$$

$$D_{3v} = \{z_1z_3,\ z_1z_3z_1,\ z_1z_3^2,\ z_1z_3^2z_1;\ z_3z_1,\ z_3z_1z_3,\ z_3z_1^2,\ z_3z_1^2z_3\}.$$

There exist $12 + 8 = 20$ such identities. None of these identities is permutable.

Case 4. $|A| = 1$, $A_1 = \{z_1, z_2\}$, $A_2 = \{z_1, z_3\}$. There are

$$D_{4u} = \{z_1z_2,\ z_1z_2z_1,\ z_1z_2^2,\ z_1z_2^2z_1;\ z_2z_1,\ z_2z_1z_2,\ z_2z_1^2,\ z_2z_1^2z_2\},$$

$$D_{4v} = \{z_1z_3,\ z_1z_3z_1,\ z_1z_3^2,\ z_1z_3^2z_1;\ z_3z_1,\ z_3z_1z_3,\ z_3z_1^2,\ z_3z_1^2z_3\}.$$

In this case, we have either $c(u) = c(v) = \{z_1, z_2, z_3\}$ or $c(u) = c(v) = \{z_1, z_2, z_3, x\}$. It should be taken into account that the subcases $h_2(u) = z_1z_2$, $h_2(v) = z_3z_1$, and $h_2(u) = z_2z_1$, $h_2(v) = z_1z_3$, are identical. Moreover, every identity of the first subcase coincides with some identity of the second after renaming some variables. For example, the identity $z_2z_1z_3 \approx z_1z_3z_1z_2$ coincides with $z_3z_1z_2 \approx z_1z_2z_1z_3$, under the following map of the alphabet: $\varphi(z_1) = z_1$, $\varphi(z_2) = z_3$, $\varphi(z_3) = z_2$.

Taking into account what has been said, there may exist $48+4+2+4+2+1+1 = 62$ non-**W**-equivalent homotypic identities in Case 4. The following identities in Case 4 are permutable:

$$z_1z_2z_3 \approx z_1z_3z_2, \tag{4.10}$$

$$z_1z_2z_3 \approx z_3z_1z_2, \tag{4.11}$$

$$z_2z_1z_3 \approx z_1z_3z_2, \tag{4.12}$$

$$z_2z_1z_3 \approx z_3z_1z_2, \tag{4.13}$$
$$z_1z_2z_3x \approx z_1z_3z_2x, \tag{4.14}$$
$$z_1z_2z_3x \approx z_3z_1z_2x, \tag{4.15}$$
$$z_2z_1z_3x \approx z_1z_3z_2x, \tag{4.16}$$
$$z_2z_1z_3x \approx z_3z_1z_2x. \tag{4.17}$$

It is easy to see that the identity (4.12) coincides with (4.11) up to variable renaming. The same is true for the identities (4.16) and (4.15). Hence, 60 non-**W**-equivalent homotypic identities are possible in Case 4. Moreover, 6 of these identities are permutable.

CASE 5. $|A| = \varnothing$, $A_1 = \{z_1, z_2\}$, $A_2 = \{z_3, z_4\}$. Then we have

$$D_{5u} = \{z_1z_2,\ z_1z_2z_1,\ z_1z_2^2,\ z_1z_2^2z_1;\ z_2z_1,\ z_2z_1z_2,\ z_2z_1^2,\ z_2z_1^2z_2\},$$
$$D_{5v} = \{z_3z_4,\ z_3z_4z_3,\ z_3z_4^2,\ z_3z_4^2z_3;\ z_4z_3,\ z_4z_3z_4,\ z_4z_3^2,\ z_4z_3^2z_4\}.$$

In this case, the positions of z_1 and z_2, as well as z_3 and z_4 are similar. Therefore, we can assume here that $h_2(u) = z_1z_2$ and $h_2(v) = z_3z_4$. Then we have that

$$D_{5u}z_3z_4 = \{z_1z_2z_3z_4,\ z_1z_2z_1z_3z_4,\ z_1z_2^2z_3z_4,\ z_1z_2^2z_1z_3z_4\},$$
$$D_{5v}z_1z_2 = \{z_3z_4z_1z_2,\ z_3z_4z_3z_1z_2,\ z_3z_4^2z_1z_2,\ z_3z_4^2z_3z_1z_2\}.$$

As in previous cases, we have either $c(u) = c(v) = \{z_1, z_2, z_3, z_4\}$ or $c(u) = c(v) = \{z_1, z_2, z_3, z_4, x\}$. Now we have in this case $16 + 1 = 17$ different identities. Among these identities we have only two permutable ones

$$z_1z_2z_3z_4 \approx z_3z_4z_1z_2, \tag{4.18}$$
$$z_1z_2z_3z_4x \approx z_3z_4z_1z_2x. \tag{4.19}$$

CASE 6. $|A| = \varnothing$, $A_1 = \{z_1\}$, $A_2 = \{z_3, z_4\}$. There are

$$D_{6u}z_3z_4 = \{z_1^2z_3z_4,\ z_1^3z_3z_4\},$$
$$D_{6v}z_1 = \{z_3z_4z_1,\ z_3z_4z_3z_1,\ z_3z_4^2z_1,\ z_3z_4^2z_3z_1\}.$$

In this case, the positions of z_3 and z_4 are similar. Therefore, we can assume that $h_2(v) = z_3z_4$. In this case, we have $8 + 1 = 9$ different identities.

CASE 7. $|A| = \varnothing$, $A_1 = \{z_1\}$, $A_2 = \{z_3\}$. Then we have

$$D_{7u}z_3 = \{z_1^2z_3,\ z_1^3z_3\},$$
$$D_{7v}z_1 = \{z_3^2z_1,\ z_3^3z_1\}.$$

In this case, we have the following identities:

$$z_1^2z_3 \approx z_3^2z_1, \quad z_1^3z_3 \approx z_3^2z_1, \quad z_1^2z_3 \approx z_3^3z_1, \quad z_1^3z_3 \approx z_3^3z_1,$$

and also

$$z_1^2z_3x \approx z_3^2z_1x, \quad z_1^3z_3x \approx z_3^2z_1x, \quad z_1^2z_3x \approx z_3^3z_1x, \quad z_1^3z_3x \approx z_3^3z_1x.$$

It should be noted that in this list the second and third identities are the same and the sixth and seventh identities are the same after renaming the variables. So, we have the following four identities in Case 7:

$$x^2y \approx y^2x, \tag{4.20}$$
$$x^3y \approx y^2x, \tag{4.21}$$
$$x^3y \approx y^3x, \tag{4.22}$$
$$x^2yx_1 \approx y^2xx_1. \tag{4.23}$$

Thus, the number of the identities indicated in Lemma 4.6 does not exceed $31+1+20+60+17+9+4 = 142$. □

Lemma 4.7. *The following strict inclusions of varieties hold*:

$$\mathbf{V} \subset \mathbf{V}_3 \subset \mathbf{V}_2 \subset \mathbf{V}_1 \subset \mathbf{W}, \quad \mathbf{V} \subset \mathbf{W}_0 \subset \mathbf{V}_2, \quad \mathbf{P}' \subset \mathbf{W}_0.$$

Moreover,

$$\mathbf{W}_0 \cap \mathbf{V}_3 = \mathbf{V}.$$

Proof. Indeed, let identity (4.4) be true in the variety $\mathbf{U} \subseteq \mathbf{W}$. Then we have the following chain of identities in $\mathbf{U}$:

$$zx \approx z^2x \approx z^3x \approx z^2xz \approx zxz.$$

Conversely, the identity $zx \approx z^2x$ is not deduced from the identities $zx \approx zxz$ and $\beta(\mathbf{W})$, since all initial identities have the property $h_2(u) = h_2(v)$, while the identity $zx \approx z^2x$ does not have this property. Hence, $\mathbf{V} \subset \mathbf{V}_3$.

Further, identity (4.3) implies

$$zx \approx zx^2. \tag{4.24}$$

Indeed,

$$zx \approx zxzx \approx zx^2z \approx zx^2.$$

On the other hand, it should be noted that the identities of the basis of the variety $\mathbf{V}_2$ have the following property: if the letter z has an occurrence only at the leftmost position in the word u, then in any word v connected by a chain of deduction with u the letter z has an occurrence only at the leftmost position in the word v. Identity (4.3) does not have this property. Hence, $\mathbf{V}_3 \subset \mathbf{V}_2$. Clearly, $\mathbf{V}_2 \subset \mathbf{V}_1$. It is easy to see that $\mathbf{W}_0 \cap \mathbf{V}_3 = \mathbf{V}$. Indeed, from identities (4.24), (4.21), and (4.3) we obtain the validity of $zx \approx zxz \approx z^2x$ in $\mathbf{W}_0 \cap \mathbf{V}_3$. This fact means the coincidence of the intersection of these varieties with $\mathbf{V}$. Finally, the lattice $\mathbf{W}_0 = \mathbf{Sl}\,\mathrm{w}\,\mathbf{L}$ has been computed in [23, Proposition 3.6]. □

Remark 4.3. It is easy to see that

$$\mathbf{W}_{11} = \mathbf{W}_1 \cap \mathbf{V}_1, \quad \mathbf{W}_{21} = \mathbf{W}_2 \cap \mathbf{V}_1, \quad \mathbf{W}_{31} = \mathbf{W}_3 \cap \mathbf{V}_1.$$

Moreover, these varieties may be defined by their bases of identities:

$$\begin{aligned}
\mathbf{W}_{11} &= \mathrm{var}\{z_1z_2x \approx z_1z_2x^2,\ zyx \approx zxy,\ z^2 \approx z^3\},\\
\mathbf{W}_{21} &= \mathrm{var}\{z_1z_2x \approx z_1z_2x^2,\ zyx \approx zxy,\ yxz \approx xyz,\ z^2 \approx z^3\},\\
\mathbf{W}_{31} &= \mathrm{var}\{z_1z_2x \approx z_1z_2x^2,\ yx \approx xy,\ z^2 \approx z^3\}.
\end{aligned}$$

Remark 4.4. It is clear from the definitions that there hold the following strict inclusions:

$$\mathbf{W}_{31} \subset \mathbf{W}_3 \subset \mathbf{W}_2 \subset \mathbf{W}_1 \subset \mathbf{V}' \subset \mathbf{W}, \quad \mathbf{Sl} \vee \mathbf{N}_2 \subset \mathbf{W}_{31} \subset \mathbf{W}_{21} \subset \mathbf{W}_{11} \subset \mathbf{V}_1.$$

Lemma 4.8. *Identities* (4.10), (4.14), *and*

$$zxy \approx zy^2x \tag{4.25}$$

are $\mathbf{W}$*-equivalent. Each of them defines in* $\mathbf{W}$ *the subvariety* $\mathbf{W}_1$*.*

Proof. Indeed, let identity (4.14) be true in a subvariety $\mathbf{U} \subseteq \mathbf{W}$. Then in $\mathbf{U}$ we have the following chain of identities:

$$zxy \approx zxy^2 \approx zyxy \approx zy^2x,$$

i.e., identity (4.25) is also true in $\mathbf{U}$. Applying (4.25) to itself, we get that the chain of identities

$$zxy \approx zy^2x \approx zx^2y^2 \approx zx^2y \approx zyx$$

is true in $\mathbf{U}$, i.e., identity (4.10) holds in $\mathbf{U}$. The implication (4.10) $\Longrightarrow$ (4.14) is obvious. □

Remark 4.5. If in some subvariety $\mathbf{U}$ of the variety $\mathbf{W}$ identity (4.8) is true, then identity (4.10) is also true in it.

Proof. Let identity (4.8) hold in a subvariety $\mathbf{U} \subseteq \mathbf{W}$. Then we have the following chain of identities:

$$zyx \approx yzx \approx yzx^2 \approx yxzx \approx xyzx \approx zxyx \approx zx^2y$$

in $\mathbf{U}$. Thus, in $\mathbf{U}$ is true the identity which obtain from (4.25) by renaming variables. By virtue of Lemma 4.8 identity (4.10) is also true in $\mathbf{U}$. □

Remark 4.6. Due to Remark 4.5 we can find a shorter basis of identities of the variety $\mathbf{W}_2$, namely

$$\mathbf{W}_2 = \mathrm{var}\{z_1z_2x \approx z_1z_2x^2, yxz \approx xyz\}.$$

Lemma 4.9. *The following strict inclusions of varieties are valid*:

$$\mathbf{W}_3 \subset \mathbf{W}_2 \subset \mathbf{W}_1 \subset \mathbf{V}' \subset \mathbf{W}.$$

Proof. The indicated nonstrict inclusions are obvious. To prove the strict inclusions, we should analyze some invariant properties of the identities of the mentioned bases that are preserved during a deduction. We enumerate these properties:

(1) $\mathbf{W}_3 \subset \mathbf{W}_2$, since the set of identities $I(\mathbf{W}_3)$ contains the nontrivial identity $yx \approx xy$, for which $|u| = |v| = 2$, and $I(\mathbf{W}_2)$ does not contain this identity;
(2) $\mathbf{W}_2 \subset \mathbf{W}_1$, since the set of identities $I(\mathbf{W}_2)$ contains the nontrivial identity $yxz \approx xyz$, in which $h_1(u) \neq h_1(v)$, and $I(\mathbf{W}_1)$ does not contain this identity;
(3) $\mathbf{W}_1 \subset \mathbf{V}'$, since the set of identities $I(\mathbf{W}_1)$ contains the nontrivial identity $zyx \approx zxy$, in which $h_2(u) \neq h_2(v)$, and the set $I(\mathbf{V}')$ does not contain this identity;
(4) $\mathbf{V}' \subset \mathbf{W}$, since the set of identities $I(\mathbf{V}')$ contains the nontrivial identity $z^2y \approx z^3y$, which is false in $\mathbf{W}$. The last fact is easy to verify by putting $\varphi(z) = (f, a)$, $\varphi(y) = (g, a)$, in the monoid wreath product $T = U_2 \,\mathrm{w}\, N = F(N^1, U_2) \times N$, where $U_2 = \{0, 1\}$ is the two-element semilattice and $N = \{0, a \mid a^2 = 0\}$ is the two-element semigroup with null multiplication. Here g is the identity function, i.e., $g(1) = g(a) = g(0) = 1$, but $f(1) = f(a) = 1$ and $f(0) = 0$. □

Lemma 4.10. *The identities* (4.8), (4.11), (4.13), (4.15), (4.17), (4.18), (4.19), *and also identities*

$$xyz \approx yx^2z, \tag{4.26}$$

$$xyz \approx zyxz \tag{4.27}$$

are $\mathbf{W}$*-equivalent. Each of them defines in* $\mathbf{W}$ *the subvariety* $\mathbf{W}_2$.

Proof. Indeed, the implications (4.11) $\Longrightarrow$ (4.15), (4.13) $\Longrightarrow$ (4.17), and (4.18) $\Longrightarrow$ (4.19) are obvious.

(4.11) $\Longrightarrow$ (4.18). Let $\mathbf{U}$ be a subvariety of $\mathbf{W}$. Applying (4.11) twice, we obtain a chain of identities in $\mathbf{U}$:

$$z_1z_2z_3z_4 \approx z_3z_1z_2z_4 \approx z_3z_4z_1z_2.$$

(4.19) $\Longrightarrow$ (4.8). Let the identity (4.19) be true in a subvariety $\mathbf{U}$ of $\mathbf{W}$. Then the chain of identities

$$yxz \approx yxz^3 \approx z^2yxz \approx z^2xyz \approx xyz^2z \approx xyz$$

holds also in it. Therefore, the identity (4.8) is also true in $\mathbf{U}$.

(4.8) $\Longrightarrow$ (4.13). Let the identity (4.8) be true in a subvariety $\mathbf{U}$. Then we have in it the following chain of identities:

$$xyz \approx xyz^2 \approx xzyz \approx xzy^2z \approx xy^2z^2 \approx xy^2z.$$

Hence, the identity

$$xyz \approx xy^2z \tag{4.28}$$

holds in the variety $\mathbf{U}$. Further, applying (4.28), we have that the chain of identities

$$xyz \approx xy^2z \approx y^2xz \approx y^2zx \approx zy^2x \approx zyx$$

holds in $\mathbf{U}$. Hence, identity (4.13) is true in $\mathbf{U}$.

(4.13) $\Longrightarrow$ (4.26). Let identity (4.13) be true in a subvariety **U** of **W**. Then the following chain of identities holds in it:

$$xyz \approx xyz^2 \approx z^2yx \approx z^2xy \approx z^2x^2y \approx yx^2z^2 \approx yx^2z,$$

i.e., identity (4.26) is true.

(4.26) $\Longrightarrow$ (4.13). Identity (4.26) implies that the following chain of identities is true in the subvariety **U** of **W**:

$$xyz \approx yx^2z \approx yx^2z^2 \approx yxz^2x \approx yxz^3x \approx yxz^2xz \approx yzx^2z \approx yz^2x^2 \approx zyx^2 \approx zyx.$$

(4.26) $\Longrightarrow$ (4.11). Using identity (4.13) obtained in a previous item, we have that the chain of identities

$$xyz \approx yx^2z \approx yx^2z^2 \approx z^2x^2y \approx xz^2y \approx zxy$$

is true in the subvariety **U**, i.e., (4.11) holds in it.

(4.15) $\Longrightarrow$ (4.11). Let identity (4.15) be true in a subvariety **U** of **W**. By renaming the variables, identity (4.15) may be written as

$$xyzt \approx zxyt. \tag{4.29}$$

Setting in (4.29) $\varphi(t) = z$, we get the identity

$$xyz \approx zxyz, \tag{4.30}$$

faithful in **U**. In addition, the chain of identities

$$zxy \approx zxyy \approx xyzy \approx xy^2z$$

is true in it, i.e., the identity

$$zxy \approx xy^2z \tag{4.31}$$

holds in **U**. From here it follows the truth of the chain of identities

$$zxy \approx xy^2z \approx y^2z^2x \approx z^2x^2y^2 \approx z^2xy \approx zxzy \approx zxyz \approx xyz$$

in **U**. Hence, identity (4.11) is true in **U**.

(4.17) $\Longrightarrow$ (4.27). Let identity (4.17) be true in a subvariety **U** of **W**. Setting in (4.17) $\varphi(z_2) = x$, $\varphi(z_1) = y$, $\varphi(z_3) = z$, $\varphi(x) = z$, we get that in **U** identity (4.27) is true.

(4.27) $\Longrightarrow$ (4.13). Identity (4.27) implies that the chain of identities

$$xyz \approx zyxz \approx xyzxz \approx xyzx \approx zyx$$

in a subvariety **U** is true. Thus, the identity (4.13) holds in it. As noted in Remark 4.6 the truth of the identity (4.8) in **U** means the coincidence of the varieties **U** and $\mathbf{W}_2$. □

Lemma 4.11. *The identities* (4.8)*,* (4.30)*,*

$$xyz \approx z^2xy, \tag{4.32}$$

$$xyz \approx z^3xy, \tag{4.33}$$

$$xyx \approx yx^2 \tag{4.34}$$

are **W***-equivalent. Each of them defines in* **W** *the subvariety* $\mathbf{W}_2$*.*

Proof. The implication (4.8) $\Longrightarrow$ (4.34) is obvious.

(4.34) $\Longrightarrow$ (4.30). Let identity (4.34) hold in a subvariety **U** of the variety **W**. Then assuming in (4.34) $\varphi(y) = yz$, $\varphi(x) = x$, we obtain that

$$xyzx \approx yzx^2 \approx yzx$$

holds in **U**, i.e., identity (4.30) is true in it.

(4.30) $\Longrightarrow$ (4.33). Let identity (4.30) hold in a subvariety $\mathbf{U}$ of the variety $\mathbf{W}$. Then the chain of identities

$$xyz \approx zxyz \approx z^2xyz \approx z^3xy$$

holds in $\mathbf{U}$, i.e., identity (4.33) is true in it.

(4.33) $\Longrightarrow$ (4.8). Let identity (4.33) hold in a subvariety $\mathbf{U}$ of the variety $\mathbf{W}$. Then the chain of identities

$$xyz \approx z^3xy \approx z^3yx \approx yxz$$

is true in $\mathbf{U}$. Therefore, (4.8) is true in it.

(4.8) $\Longrightarrow$ (4.32). Let $\mathbf{U}$ be a subvariety of the variety $\mathbf{W}$. According to Remark 4.5 identity (4.10) is also true in $\mathbf{U}$. Then the chain of identities

$$z^2xy \approx xz^2y \approx xyz^2 \approx xyz$$

holds in $\mathbf{U}$, i.e., (4.32) is true in it.

(4.32) $\Longrightarrow$ (4.33). If identity (4.32) holds in a subvariety $\mathbf{U}$ of the variety $\mathbf{W}$, then the chain of identities

$$xyz \approx xyz^2 \approx z^2xyz \approx z^3xy$$

holds in $\mathbf{U}$, i.e., (4.33) is true in it. □

Lemma 4.12. *Identities* (4.20) *and* (4.21) *are* $\mathbf{W}$*-equivalent. Each of them defines in* $\mathbf{W}$ *the subvariety* $\mathbf{U}_0$*, which has the basis of identities of* (2.1), (2.2), *and* (4.20). *The variety* $\mathbf{U}_0$ *contains the variety* $\mathbf{P}'$. *In addition, the variety* $\mathbf{V}'$ *defined by identities* (2.1), (2.2), *and* (4.9) *contains the variety* $\mathbf{U}_0$.

Proof. Let us prove that identities (4.20) and (4.21) are $\mathbf{W}$-equivalent. Indeed, if (4.20) is true in a subvariety $\mathbf{U}$ of the variety $\mathbf{W}$, then the chain of identities

$$x^2y \approx x^2y^2 \approx y^2xy \approx y^3x$$

holds in $\mathbf{U}$, i.e., identity (4.21) is true in it.

Conversely, let identity (4.21) hold in a subvariety $\mathbf{U}$ of the variety $\mathbf{W}$. Then the chain of identities

$$x^2y \approx y^3x \approx y^2xy \approx x^3y^2 \approx x^3y$$

holds in it. From here we have that $x^2y \approx y^3x \approx y^2x$ are true in $\mathbf{U}$, whence identity (4.20) is also true in it. Other assertions of this lemma are obvious or follow from what has been said. □

Lemma 4.13. *The above-defined subvarieties of* $\mathbf{W}$ *satisfy the following conditions*:

$$\mathbf{U}_0 \subset \mathbf{V}', \quad \mathbf{V}_1 \subset \mathbf{V}', \quad \mathbf{W}_1 \subset \mathbf{V}',$$
$$\mathbf{U}_1 = \mathbf{W}_1 \cap \mathbf{U}_0 = \mathrm{var}\{z_1z_2x \approx z_1z_2x^2,\ zyx \approx zxy,\ x^2y \approx y^2x\},$$
$$\mathbf{U}_{11} = \mathbf{V}_1 \cap \mathbf{U}_0 = \mathrm{var}\{z_1z_2x \approx z_1z_2x^2,\ zyx \approx zxy,\ x^2y \approx y^2x,\ z^2 \approx z^3\},$$
$$\mathbf{P}' = \mathbf{V}_2 \cap \mathbf{U}_0.$$

Proof. The mentioned strict inclusions in the lemma are obvious from what was stated above. The basis of intersection of two varieties, as was noted in Remark 4.1, consists of the union of bases of these varieties. It remains to verify that these bases are simplified to bases pointed out for intersections of varieties in the lemma. □

Proof of Proposition 4.1. Most of the inclusions and intersections of varieties pointed out on Fig. 2 have been proved or noted earlier in Lemmas 4.7, 4.12, and 4.13 and in Remark 4.3.

It remains to verify the following strict inclusions:

$$\mathbf{Sl} \vee \mathbf{N}_2 \subset \mathbf{P}', \quad \mathbf{W}_0 \subset \mathbf{W}_{11}, \quad \mathbf{W}_{21} \subset \mathbf{U}_{11} \subset \mathbf{W}_{11}.$$

All these strict inclusions easily follow from the definitions of bases of the indicated subvarieties. Let us verify for example that the inclusion $\mathbf{W}_{21} \subset \mathbf{U}_{11}$ holds. It is sufficient to prove that the identity $x^2y \approx y^2x$ is true in $\mathbf{W}_{21}$. Indeed, the chain of identities

$$x^2y \approx x^2y^2 \approx xy^2x \approx y^2x^2 \approx y^2x$$

holds in it. On the other hand, if $u \approx v \in I(\mathbf{U}_{11})$ and the first letter y of the word u occurs in u only once, then the first letter of the word v coincides with y and occurs in v only once. This property of the set $I(\mathbf{U}_{11})$ is false for the set $I(\mathbf{W}_{21})$. Therefore, the inclusion $\mathbf{W}_{21} \subset \mathbf{U}_{11}$ is strict. □

Corollary 4.2. *The lattice $L(\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2)$ of all subvarieties of $\mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2$ contains the subset $L' \cup L_0$, whence it contains at least 33 elements. In particular, the lattice interval $\mathcal{I}(\mathbf{Sl}, \mathbf{Sl}\,\mathrm{w}\,\mathbf{N}_2)$ contains at least 20 elements.*

Proof. It follows directly from Propositions 3.1 and 4.1. □

5. Concluding Remarks

In Secs. 3 and 4, we established that the lattice $L(\mathbf{W})$ contains the subset $L' \cup L_0$ with 33 elements. On the other hand, it is easy to see that in the lattice L_0 there are only 6 subvarieties $\mathbf{U}$ such that each identity from $I(\mathbf{U})$ has the property 4.7, namely, $\mathbf{W}$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{V}'$, $\mathbf{U}_0$, and $\mathbf{U}_1$. Each of the other 14 subvarieties of L_0 has at least one identity $u \approx v$ with $|u| \leq 2$. Moreover, the subvarieties of type (4.2), i.e., subvarieties defined by a single identity in $\mathbf{W}$, are $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{V}'$, and $\mathbf{U}_0$ and also $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}'$, and $\mathbf{W}_3$. Note that the way of getting an upper bound for the cardinality of the sublattice $L'' \subseteq L(\mathbf{W})$ by means of the description of all non-$\mathbf{W}$-equivalent subvarieties in Remark 4.1 apparently is not efficient. For example, the number of such identities with the property (4.7) due to Lemma 4.6 equals 142. According to Lemmas 4.8, 4.10–4.12, this number reduces to $142 - 15 = 127$. Then the upper bound of the number of all subvarieties equals the cardinality of all subsets of such identities, i.e., 2^{127}. In fact, there are considerably fewer such subvarieties, since different subsets of identities may define the same variety of semigroups. I think that the cardinality of the lattice $L(\mathbf{W})$ is not large and this lattice can be fully described.

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