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BÉZOUT RINGS WITH FINITE KRULL DIMENSION

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Abstract. It is proven that if R is a commutative Bézout ring of Krull dimension 1, with stable range 2, then R is an elementary divisor ring.

Let R be a commutative ring with identity. Recall that the Krull dimension of R is the maximal length n of a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals inside R. By convention, a ring R has Krull dimension -1 if and only if it is trivial (i.e., $1_R = 0_R$) [5]. By a Bézout ring we mean a ring in which all finitely generated ideals are principal. An $(n \times m)$ matrix $A = (a_{ij})$ is said to be diagonal if $a_{ij} = 0$ for all $i \neq j$. We say that a matrix A of dimension $n \times m$ admits a diagonal reduction if there exist invertible matrices $P \in \mathrm{GL}_n(R)$ and $Q \in \mathrm{GL}_m(R)$ such that PAQ is a diagonal matrix. We say that two matrices A and B over a ring R are equivalent if there exist invertible matrices P and Q such that $B = PAQ$. Following [3], we say that if every matrix over R is equivalent to a diagonal matrix (d_{ii}) with the property that every d_{ii} is a divisor of $d_{i+1,i+1}$, then R is an elementary divisor ring. A ring R is said to be a Hermite ring if every (1×2) matrix over R admits diagonal reduction. A row $(a_1; a_2; \ldots; a_n)$ over a ring R is called unimodular if $a_1R + a_2R + \cdots + a_nR = R$. If $(a_1; a_2; \ldots; a_n)$ is a unimodular n-row over a ring R, then we say that $(a_1; a_2; \ldots; a_n)$ is reducible if there exists an $(n-1)$-row $(b_1; b_2; \ldots; b_{n-1})$ such that the $(n-1)$-row $(a_1 + a_nb_1; a_2 + a_nb_2; \ldots; a_{n-1} + a_nb_{n-1})$ is unimodular. A ring R is said to have stable range n if n is the least positive integer such that every unimodular $(n+1)$-row is reducible. A commutative Bézout ring R with identity is said to be adequate if it satisfies such conditions: for every $a, b \in R$, with $a \neq 0$, there exist $a_i, d \in R$ such that

(i) $a = a_id$,
(ii) $(a_i, b) = (1)$,
(iii) for every nonunit divisor d' of d, we have $(d', b) \neq (1)$ [2].

In Theorems 1 and 2, we obtain the generalizations of the results in [1, 7].

Theorem 1. *If R is a commutative Bézout ring of Krull dimension 1, with stable range 2, then R is an elementary divisor ring. In fact, it is adequate.*

Proof. According to the results of [9], we can assume that the ring R is a reduced ring. As a Bézout ring with stable range 2 is Hermite, by [4, p. 232] we must prove that if $a, d \in R$, then there exist elements $b, c \in R$ with $a = bc$ and no nonunit factor of c is relatively prime to d. Consider the following sequence of elements of R: $a_1 = a/(a, d)$, $a_2 = a_1/(a_1, d)$, $a_3 = a_2/(a_2, d)$, $\ldots$. We claim that for some integer n, $(a_n, d) = 1$. Otherwise, look at the following chain of ideals of R:

$$(a, d) \subseteq (a_1, d) \subseteq (a_2, d) \subseteq \ldots .$$

The union is a proper ideal of R and so is contained in a maximal ideal M. Since the ring R is reduced, according to the results of [6, 2.1] R_M is a Bézout domain; moreover, it is a valuation domain. In the valuation domain R_M, if $(a_i, d) = (a_i)$ for some i, then $(a_{i+1}, d) = R_M$, a contradiction. The alternative is that $(a_i, d) = (d)$ for each i, but this implies that $a \in (d^i)R_M$ for each i. I.e. $a \in \bigcap (d^i)R_M$. Then by [6, p. 187] $\bigcap (d^i)R_M \in \operatorname{spec} R_M$ and, therefore, $a = 0$, since R_M is a one-dimensional domain. This justifies the claim. □

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Theorem 2. *If R is a commutative semihereditary Bézout ring of Krull dimension 2, then R is an elementary divisor ring.*

Proof. The following theorem has been proven in [7]: Let R be a commutative semihereditary Bézout ring. Then R is an elementary divisor ring if and only if R/dR is an elementary divisor ring for all nonzero divisors $d \in R$. Let d be a nonzero divisor of R. By [6] d is not contained in any minimal prime ideal. So R/dR is commutative Bézout ring of Krull dimension 1, with stable range 2. Then R/dR is an elementary divisor ring and by Theorem 1 R also is an elementary divisor ring. □

Open problem. Following [8], a ring R is fractionally P provided that the classical quotient ring $Q(R/I)$ of the ring R/I satisfies P for every ideal I of the ring R. In [10], Theorem 7 was proved: A fractionally regular Bézout ring of a stable range 2 is an elementary divisor ring. The author asks the following question: Is every commutative Bézout ring of Krull dimension 1 with stable range 2 fractionally regular?

REFERENCES

1. J. W. Brewer, C. Naude, and G. Naude, "On Bézout domains, elementary divisor rings, and pole assignability," *Commun. Algebra*,**12 (24)**, 2987–3003 (1984).
2. O. Helmer, "The elementary divisor theorem for certain rings without chain condition," *Bull. Amer. Math. Soc.*, **49**, 225–236 (1943).
3. I. Kaplansky, "Elementary divisor rings and modules," *Trans. Amer. Math. Soc.*, **66**, 464–491 (1949).
4. I. Kaplansky, *Infinite Abelian Groups*, Univ. Michigan, Ann Arbor (1969).
5. I. Kaplansky, *Commutative Rings*, Univ. Chicago Press (1974).
6. E. Maltis, "The minimal spectrum of a reduced ring," *Illinois J. Math.*, **27**, No. 3, 353–391 (1983).
7. T. S. Shores, "Modules over semihereditary Bézout rings," *Proc. Amer. Math. Soc.*, **46**, 211–213 (1974).
8. P. Vamos, "The decomposition of finitely generated modules and fractionally self-injective rings," *J. London Math. Soc.*, **16**, 209–220 (1977).
9. B. V. Zabavsky, "Diagonalization of matrices over rings of finite stable rank," *Visnyk Lviv. Univ.*, **61**, 206–211 (2003).
10. B. V. Zabavsky, "Fractionally regular Bézout rings," *Mat. Stud.*, **32**, No. 1, 76–80 (2009).

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