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1-D Schrödinger operators with local interactions on a discrete set with unbounded potential

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Abstract. We study spectral properties of the one-dimensional Schrödinger operators $H_{X,\alpha,q} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) + \sum_{x_n \in X} \alpha_n \delta(x - x_n)$ with local interactions, $d_* = 0$, and an unbounded potential q being a piecewise constant function, by using the technique of boundary triplets and the corresponding Weyl functions. Under various sufficient conditions for the self-adjointness and discreteness of Jacobi matrices, we obtain the condition of self-adjointness and discreteness for the operator $H_{X,\alpha,q}$.

Keywords. Schrödinger operator, unbounded potential, self-adjoint, local point interaction, discreteness.

1. Introduction

Let $\mathbb{R}_+ = [0, +\infty)$, and let $X = \{x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ be a strictly increasing sequence ($x_{n+1} > x_n$ for all $n \in \mathbb{N}$) such that $x_n \to +\infty$. We set $x_0 = 0$ and $d_n := x_n - x_{n-1}$ for all $n \in \mathbb{N}$. Let also

$$d_* := \inf_{n\in\mathbb{N}} d_n = \inf_{n\in\mathbb{N}} (x_n - x_{n-1}), \qquad d^* := \sup_{n\in\mathbb{N}} d_n = \sup_{n\in\mathbb{N}} (x_n - x_{n-1}).$$

Let $H_{X,\alpha,q}$ be the minimal symmetric operator associated in $L^2(\mathbb{R}_+)$ with the differential expression

$$\ell_{X,\alpha,q} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) + \sum_{x_n \in X} \alpha_n \delta(x - x_n), \qquad x \geq 0. \tag{1.1}$$

Namely, assuming that $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $q : \mathbb{R}_+ \to \mathbb{R}$ is locally square integrable function on $\mathbb{R}_+$, $q \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, define the pre-minimal operator $H^0_{X,\alpha,q}$ in $L^2(\mathbb{R}_+)$ by the differential expression

$$\tau_q := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) \tag{1.2}$$

on the domain

$$\mathrm{dom}(H^0_{X,\alpha,q}) = \left\{ f \in W^{2,2}_{\mathrm{comp}}(\mathbb{R}_+ \setminus X) : f'(0) = 0, \quad \begin{array}{c} f(x_n+) = f(x_n-) \\ f'(x_n+) - f'(x_n-) = \alpha_n f(x_n) \end{array}, \ n \in \mathbb{N} \right\}. \tag{1.3}$$

Clearly, $H^0_{X,\alpha,q}$ is symmetric, and we denote its closure by $H_{X,\alpha,q}$. Note that if all $\alpha_n = 0$, the operator $H_{X,0,q} =: H^N_q$ is the Neumann realization of the expression (1.2).

The operator $H_{X,\alpha,q}$ describes δ-interactions on a discrete set $X = \{x_n\}_{n\in\mathbb{N}}$, and the coefficient α_n is called the strength of the interaction at $x = x_n$. Let us stress that the operator $H_{X,\alpha,q}$ is symmetric, but not automatically self-adjoint even in the case $q \equiv 0$ (see [21, 22, 34]).

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Schrödinger operators with point interaction on a finite or discrete set arise in various physical applications (see [3]). In recent years, spectral properties of the operator $H_{X,\alpha,q}$ have been studied in numerous papers (see, e.g., [4, 7, 8, 15, 16, 18, 20–25, 27, 30, 31, 33, 34], and [22] for a comprehensive review).

Here, we will study spectral properties of the Hamiltonian $H_{X,\alpha,q}$ with $d_* = 0$ and an unbounded potential q being a piecewise constant function. Namely, later on in this paper we make the following assumption:

Hypothesis 1. *Assume that*

$$q(x) \equiv q_n > 0, \quad x \in (x_{n-1}, x_n), \tag{1.4}$$

for all $n \in \mathbb{N}$, *and the sequence* $\{q_n\}_{n\in\mathbb{N}}$ *satisfies the condition*

$$\sup_{n\in\mathbb{N}} d_n\sqrt{q_n} =: c < \infty. \tag{1.5}$$

We mention that relation (1.5) covers the very important case in our considerations:

$$d_n\sqrt{q_n} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.6}$$

Our main tool is a powerful approach developed recently in [21]. Namely, applying the technique of boundary triplets and the corresponding Weyl function (see [12, 13, 17]), it was shown in [21] that spectral properties of the operator $H_{X,\alpha,q}$ with a bounded potential $q \in L^\infty(\mathbb{R}_+)$ closely correlate with the corresponding properties of a certain class of Jacobi matrices. Similar results were obtained later for Schrödinger operators with a matrix-valued potential [24], as well as for Dirac operators [9]. Our main aim is to extend the results of [21] to the case of unbounded potentials satisfying Hypothesis 1. Namely, let us consider the Jacobi (three-diagonal) matrix

$$B_{X,\alpha,q} = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{1.7}$$

where

$$a_n = -\frac{\sqrt{q_{n+1}}}{r_n r_{n+1}\sinh(d_{n+1}\sqrt{q_{n+1}})}, \qquad r_n := \sqrt{d_n + d_{n+1}}, \tag{1.8}$$

$$b_n = \frac{\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})}{d_n + d_{n+1}}. \tag{1.9}$$

Our main result reads as follows.

Theorem 1.1. *Assume that Hypothesis 1 holds, and $H_{X,\alpha,q}$ is the minimal symmetric operator associated with* (1.1). *Let also $B_{X,\alpha,q}$ be the minimal operator associated with the Jacobi matrix* (1.7). *Then*

(i) The deficiency indices of $H_{X,\alpha,q}$ and $B_{X,\alpha,q}$ are equal, and

$$\mathrm{n}_\pm(H_{X,\alpha,q}) = \mathrm{n}_\pm(B_{X,\alpha,q}) \le 1.$$

In particular, $H_{X,\alpha,q}$ is self-adjoint, iff $B_{X,\alpha,q}$ is self-adjoint.

(ii) The operator $H_{X,\alpha,q}$ is lower semibounded, iff so is the operator $B_{X,\alpha,q}$.

In addition, we assume that $H_{X,\alpha,q}$ (and, hence, $B_{X,\alpha,q}$) is self-adjoint. Then:

(iii) The operator $H_{X,\alpha,q}$ is nonnegative, iff so is $B_{X,\alpha,q}$.

(iv) The total multiplicities of the negative spectra of $H_{X,\alpha,q}$ and $B_{X,\alpha,q}$ coincide:

$$\kappa_-(H_{X,\alpha,q}) = \kappa_-(B_{X,\alpha,q}). \tag{1.10}$$

(v) For any $p \in (0,\infty]$, the following equivalence holds:

$$E_{H_{X,\alpha,q}}(\mathbb{R}_-)H_{X,\alpha,q} \in \mathfrak{S}_p \iff E_{B_{X,\alpha,q}}(\mathbb{R}_-)B_{X,\alpha,q} \in \mathfrak{S}_p.$$

In particular, the negative part of the spectrum $H_{X,\alpha,q}$ is discrete, iff the same holds for the negative spectrum of $B_{X,\alpha,q}$.

(vi) $\sigma_c(H_{X,\alpha,q}) \subseteq [0,\infty)$, iff $\sigma_c(B_{X,\alpha,q}) \subseteq [0,\infty)$.

(vii) $\sigma_c(H_{X,\alpha,q}) \subset (0,\infty)$, iff $\sigma_c(B_{X,\alpha,q}) \subset (0,\infty)$.

(viii) The operator $H_{X,\alpha,q}$ has a purely discrete spectrum, iff $\lim\limits_{n\to\infty} d_n = 0$, and $B_{X,\alpha,q}$ has a purely discrete spectrum.

(ix) Let $\widetilde{\alpha} = \{\widetilde{\alpha}_k\}_{k=1}^\infty \subset \mathbb{R}$, and let $B_{X,\widetilde{\alpha},q}$ be the minimal operator associated with matrix (1.7) and constructed by the sequence $\widetilde{\alpha}$ instead of α. If $H_{X,\widetilde{\alpha},q} = H^*_{X,\widetilde{\alpha},q}$, then $B_{X,\widetilde{\alpha},q} = B^*_{X,\widetilde{\alpha},q}$, and, for any $p \in (0,+\infty]$, the following equivalence holds:

$$(H_{X,\alpha,q} - \mathrm{i})^{-1} - (H_{X,\widetilde{\alpha},q} - \mathrm{i})^{-1} \in \mathfrak{S}_p \iff (B_{X,\alpha,q} - \mathrm{i})^{-1} - (B_{X,\widetilde{\alpha},q} - \mathrm{i})^{-1} \in \mathfrak{S}_p.$$

Combining Theorem 1.1(i) with the Carleman test (see, e.g., [1, Chapter II]), we obtain the following result.

Proposition 1.2. *Assume that Hypothesis 1 holds. Then the Hamiltonian $H_{X,\alpha,q}$ is self-adjoint for any $\alpha = \{\alpha_n\}_{n=1}^\infty \subset \mathbb{R}$ provided that*

$$\sum_{n=1}^\infty d_n^2 = \infty. \tag{1.11}$$

Note that this result is sharp. Namely, if $\{d_n\}_{n=1}^\infty \in l^2$ and the coefficients $X = \{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\alpha_n \in \mathbb{R}$ satisfy certain concavity assumptions, then the operator $H_{X,\alpha,q}$ is symmetric with $\mathrm{n}_\pm(H_{X,\alpha,q}) = 1$ (see Proposition 6.8). Note that Proposition 1.2 in the case where $q \in L^\infty(\mathbb{R}_+)$ was first proved in [21]. A more general result was proved later in [30].

Investigating the discreteness and absolute continuity of spectra of the operator $B_{X,\alpha,q}$, we arrive at the following sufficient condition (see Propositions 6.17 and 6.23).

Proposition 1.3. *Assume that Hypothesis 1 holds and $\lim_{n\to\infty} d_n\sqrt{q_n} = 0$. Assume also that $\lim_{n\to\infty} d_n = 0$ and the operator $B_{X,\alpha,q}$ is self-adjoint. If*

$$\lim_{n\to\infty}\left|\frac{\alpha_{n-1}}{d_n} + q_n\right| = \infty,\ \lim_{k\to\infty}\frac{1}{d_k(\alpha_k + q_{k+1}d_{k+1})} > -\frac{1}{4}$$
$$\text{and}\qquad \lim_{n\to\infty}\frac{1}{d_n\alpha_{n-1}} > -\frac{1}{4}, \tag{1.12}$$

then the operator $H_{X,\alpha,q}$ has a discrete spectrum.

This result is of interest only in the case where the operator $H_{X,\alpha,q}$ is not semibounded, since the lower-bounded below Hamiltonian $H_{X,\alpha,q}$ is always self-adjoint (see [4]).

Proposition 1.4. *Assume that Hypothesis 1 holds, and assume that*

$$\sum_{n=1}^{\infty} \frac{|\alpha_n|}{d_{n+1}} < \infty. \tag{1.13}$$

Then the absolutely continuous part $H^{\mathrm{ac}}_{X,\alpha,q}$ of the Hamiltonian $H_{X,\alpha,q}$ is unitarily equivalent to the operator $H_q^N := H_{X,0,q}$ that is the Neumann realization of (1.2) *in $L^2(\mathbb{R}_+)$. In particular,*

$$\sigma_{ac}(H_{X,\alpha,q}) = \sigma_{ac}(H_q^N), \tag{1.14}$$

where $\mathrm{dom}(H_q^N) = \mathrm{dom}(H_{X,0,q}) \subset \{W^{2,2}(\mathbb{R}_+) : f'(0) = 0\}$.

If, in addition, $q \in L^1(\mathbb{R}_+)$, then $\sigma_{ac}(H_{X,\alpha,q}) = \mathbb{R}_+$.

The main results were announced in [5].

Notation. Let $\mathfrak{H}$ and $\mathcal{H}$ stand for the separable Hilbert spaces. Further, $[\mathfrak{H}, \mathcal{H}]$ denotes the set of bounded operators from $\mathfrak{H}$ to $\mathcal{H}$; $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$; $\mathfrak{S}_p(\mathcal{H})$, $(p \in (0,\infty))$, denotes the Neumann–Schatten ideal in $\mathcal{H}$. In particular, $\mathfrak{S}_\infty(\mathcal{H})$ is a set of compact operators in $\mathcal{H}$, $\mathfrak{S}_1(\mathcal{H})$ is a trace class of operators in $\mathcal{H}$, $\mathcal{C}(\mathfrak{H})$ and $\widetilde{\mathcal{C}}(\mathfrak{H})$ are the sets of closed operators and linear relations in $\mathfrak{H}$, respectively. Let T be a linear operator in the Hilbert space $\mathfrak{H}$. In what follows, $\mathrm{dom}(T)$, $\ker(T)$, and $\mathrm{ran}(T)$ denote the domain, kernel, and range of T, respectively; $\sigma(T)$, $\rho(T)$, and $\widehat{\rho}(T)$ denote the spectrum, resolvent set, and set of regular-type points of T, respectively; $R_T(\lambda) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$, is the resolvent of T.

By $W^{2,2}(\mathbb{R}_+ \setminus X)$, $W_0^{2,2}(\mathbb{R}_+ \setminus X)$, and $W^{2,2}_{\mathrm{loc}}(\mathbb{R}_+ \setminus X)$, we denote the Sobolev spaces

$$\begin{aligned} W^{2,2}(\mathbb{R}_+ \setminus X) &:= \{f \in L^2(\mathbb{R}_+) : f, f' \in AC_{loc}(\mathbb{R}_+ \setminus X), f'' \in L^2(\mathbb{R}_+)\}, \\ W_0^{2,2}(\mathbb{R}_+ \setminus X) &:= \{f \in W^{2,2}(\mathbb{R}_+) : f(x_k) = f'(x_k) = 0 \,\text{for all}\, x_k \in X\}, \\ W^{2,2}_{\mathrm{comp}}(\mathbb{R}_+ \setminus X) &:= \{f \in W^{2,2}(\mathbb{R}_+ \setminus X) : \mathrm{supp}\, f \text{ is compact in } \mathbb{R}_+\}. \end{aligned}$$

Let I be a subset of $\mathbb{Z}$, $I \subseteq \mathbb{Z}$. We denote, by $l^2(I, \mathcal{H})$, the Hilbert space of $\mathcal{H}$-valued sequences such that $\|f\|^2 = \sum_{n\in I} \|f_n\|^2_{\mathcal{H}} < \infty$; $l^2_0(I, \mathcal{H})$ is a set of sequences with a finite number of nonzero components; we also abbreviate $l^2 := l^2(\mathbb{N}, \mathbb{C})$, $l^2_0 := l^2_0(\mathbb{N}, \mathbb{C})$.

2. Preliminaries

2.1. Boundary triplets and Weyl functions

In this section, we briefly recall the notion of abstract boundary triplets and associated Weyl functions in the extension theory of symmetric operators (for a detailed study of boundary triplets, we refer the reader to [12, 13, 17]).

Linear relations, boundary triplets, and self-adjoint extensions

1. The set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$ is the a of closed linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. Recall that $\mathrm{dom}(\Theta) = \big\{f : \{f, f'\} \in \Theta\big\}$, $\mathrm{ran}(\Theta) = \big\{f' : \{f, f'\} \in \Theta\big\}$ and $\mathrm{mul}\,(\Theta) = \big\{f' : \{0, f'\} \in \Theta\big\}$ are the domain, range, and multivalued part of Θ, respectively. A closed linear operator A in $\mathcal{H}$ is identified with its graph $\mathrm{gr}(A)$, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators in $\mathcal{H}$ is viewed as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$. In particular, a linear relation Θ is an operator, iff $\mathrm{mul}\,(\Theta)$ is trivial. For the definition of the inverse linear relation, resolvent set, and spectrum of linear relations, we refer to [14]. We recall that the adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} : (f', h)_{\mathcal{H}} = (f, h')_{\mathcal{H}} \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is said to be *symmetric*, if $\Theta \subset \Theta^*$, and self-adjoint, if $\Theta = \Theta^*$.

For a symmetric linear relation $\Theta \subseteq \Theta^*$ in $\mathcal{H}$, the multivalued part $\mathrm{mul}\,(\Theta)$ is the orthogonal complement of $\mathrm{dom}(\Theta)$ in $\mathcal{H}$. Setting $\mathcal{H}_{\mathrm{op}} := \overline{\mathrm{dom}(\Theta)}$ and $\mathcal{H}_\infty = \mathrm{mul}\,(\Theta)$, one arrives at the orthogonal decomposition $\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty$, where Θ_{op} is a symmetric operator in $\mathcal{H}_{\mathrm{op}}$, the operator part of Θ, and $\Theta_\infty = \left\{ \left(\begin{smallmatrix} 0 \\ f' \end{smallmatrix} \right) : f' \in \mathrm{mul}\,(\Theta) \right\}$ is a "pure" linear relation in $\mathcal{H}_\infty$.

2. Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $\mathrm{n}_\pm(A) = \dim \mathfrak{N}_{\pm \mathrm{i}} \leq \infty$, $\mathfrak{N}_z := \ker(A^* - z)$.

Definition 2.1 ([17]). *A triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is called a* boundary triplet *for the adjoint operator* A^*, *if* $\mathcal{H}$ *is a Hilbert space and* $\Gamma_0, \Gamma_1 : \ \mathrm{dom}(A^*) \to \mathcal{H}$ *are bounded linear mappings such that the abstract Green identity*

$$(A^* f, g)_{\mathfrak{H}} - (f, A^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(A^*), \tag{2.1}$$

holds, and the mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \mathrm{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ *is surjective.*

First, we note that a boundary triplet for A^* exists, since the deficiency indices of A are assumed to be equal. Moreover, $\mathrm{n}_\pm(A) = \dim(\mathcal{H})$ and $A = A^* \upharpoonright (\ker(\Gamma_0) \cap \ker(\Gamma_1))$ hold. Note also that a boundary triplet for A^* is not unique.

A closed extension $\widetilde{A}$ of A is called *proper*, if $A \subseteq \widetilde{A} \subseteq A^*$. Two proper extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ of A are called *disjoint*, if $\mathrm{dom}(\widetilde{A}_1) \cap \mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A)$, and *transversal*, if, in addition, $\mathrm{dom}(\widetilde{A}_1) \dotplus \mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A^*)$. The set of all proper extensions of A is denoted by $\mathrm{Ext}A$. Fixing a boundary triplet Π, one can parametrize the set $\mathrm{Ext}A$ in the following way.

Proposition 2.2 ([13]). *Let* A *be as above, and let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*. *Then the mapping* $\Gamma = \{\Gamma_0, \Gamma_1\} : \mathrm{dom}(A^*) \to \mathcal{H} \times \mathcal{H}$ *establishes a bijective correspondence between the sets* Ext_A *and* $\widetilde{\mathcal{C}}(\mathcal{H})$ *as follows:*

$$\Theta \mapsto A_\Theta := A^* \upharpoonright \Gamma^{-1}\Theta = A^* \upharpoonright \left\{ f \in \mathrm{dom}(A^*) : \ \{\Gamma_0 f, \Gamma_1 f\} \in \Theta \right\}. \tag{2.2}$$

At the same time, the following relations hold:

(i) $A_\Theta^* = A_{\Theta^*}$.

(ii) The extensions A_Θ *and* A_0 *are disjoint (transversal), iff* $\Theta \in \mathcal{C}(\mathcal{H})$ $\big(\Theta \in [\mathcal{H}]\big)$. *In this case,* A_Θ *admits a representation* $A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$.

(iii) $A_\Theta \in \mathcal{C}(\mathfrak{H})$, *iff* $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$.

(iv) $A_{\Theta_1} \subseteq A_{\Theta_2}$, *iff* $\Theta_1 \subseteq \Theta_2$.

(v) A_Θ *is symmetric (self–adjoint), iff the same is true for* Θ, *and* $\mathrm{n}_\pm(A_\Theta) = \mathrm{n}_\pm(\Theta)$ *holds.*

(vi) *Let $A_\Theta = A^*_\Theta$ and $A_{\widetilde{\Theta}} = A^*_{\widetilde{\Theta}}$. Then, for any $p \in (0, +\infty]$, the following equivalence holds:*

$$(A_\Theta - \mathrm{i})^{-1} - (A_{\widetilde{\Theta}} - \mathrm{i})^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \iff (\Theta - \mathrm{i})^{-1} - (\widetilde{\Theta} - \mathrm{i})^{-1} \in \mathfrak{S}_p(\mathcal{H}).$$

Moreover, if $\mathrm{dom}(\Theta) = \mathrm{dom}(\widetilde{\Theta})$, *then the following implication is valid:*

$$\overline{\Theta - \widetilde{\Theta}} \in \mathfrak{S}_p(\mathcal{H}) \implies (A_\Theta - \mathrm{i})^{-1} - (A_{\widetilde{\Theta}} - \mathrm{i})^{-1} \in \mathfrak{S}_p(\mathfrak{H}).$$

Proposition 2.2 immediately implies that the extensions $A_0 := A^* \restriction \ker(\Gamma_0)$ and $A_1 := A^* \restriction \ker(\Gamma_1)$ are self-adjoint. Clearly, $A_j = A_{\Theta_j}$, $j \in \{0, 1\}$, where the subspaces $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\}$ are self-adjoint relations in $\mathcal{H}$. Note that Θ_0 is a "pure" linear relation.

Weyl functions, γ-fields, and Krein-type formula for resolvents

1. In [12,13], the concept of the classical Weyl–Titchmarsh m-function from the theory of Sturm–Liouville operators was generalized to the case of symmetric operators with equal deficiency indices. The role of abstract Weyl functions in the extension theory is similar to that of the classical Weyl–Titchmarsh m-function in the spectral theory of singular Sturm–Liouville operators.

Definition 2.3 ([12]). *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator-valued functions $\gamma : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M : \rho(A_0) \to [\mathcal{H}]$ defined by*

$$\gamma(z) := \big(\Gamma_0 \restriction \mathfrak{N}_z\big)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1 \gamma(z), \qquad z \in \rho(A_0), \tag{2.3}$$

are called the γ-field *and the* Weyl function, *respectively, corresponding to the boundary triplet* Π.

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.3) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$, and the following relations hold (see [12]):

$$\gamma(z) = \big(I + (z - \zeta)(A_0 - z)^{-1}\big)\gamma(\zeta), \tag{2.4}$$

$$M(z) - M(\zeta)^* = (z - \overline{\zeta})\gamma(\zeta)^*\gamma(z), \tag{2.5}$$

$$\gamma^*(\overline{z}) = \Gamma_1 (A_0 - z)^{-1}, \qquad z,\ \zeta \in \rho(A_0). \tag{2.6}$$

Identity (2.5) implies that $M(\cdot)$ is an $R_{\mathcal{H}}$-function (or *Nevanlinna function*). In other words, $M(\cdot)$ is an ($[\mathcal{H}]$-valued) holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ and

$$\operatorname{Im} z \cdot \operatorname{Im} M(z) \geq 0, \qquad M(z^*) = M(\overline{z}), \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.7}$$

In addition, it follows from (2.5) that $M(\cdot)$ satisfies $0 \in \rho(\operatorname{Im} M(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Since A is densely defined, $M(\cdot)$ admits an integral representation (see, e.g., [13]):

$$M(z) = C_0 + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma_M(t), \qquad z \in \rho(A_0), \tag{2.8}$$

where $\Sigma_M(\cdot)$ is an operator-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma_M(t) \in [\mathcal{H}]$ and $C_0 = C_0^* \in [\mathcal{H}]$. The integral in (2.8) is understood in the strong sense.

In contrast to the spectral measures of self-adjoint operators, the measure $\Sigma_M(\cdot)$ is not necessarily orthogonal. However, the measure Σ_M is uniquely determined by the Nevanlinna function $M(\cdot)$. The operator-valued measure Σ_M is called *the spectral measure* of $M(\cdot)$. If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ up to unitary equivalence (see [13,26]). Due to this fact, the spectral properties of A_0 can be expressed in terms of $M(\cdot)$.

2. The following result provides a description of resolvents and spectra of proper extensions of the operator A in terms of the Weyl function $M(\cdot)$ and the corresponding boundary parameters.

Proposition 2.4 ([12])**.** *For any $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, the following Krein-type formula holds:*

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\overline{z}), \quad z \in \rho(A_0) \cap \rho(A_\Theta). \tag{2.9}$$

Moreover, if $z \in \rho(A_0)$, then

$$z \in \sigma_i(A_\Theta) \quad \Leftrightarrow \quad 0 \in \sigma_i(\Theta - M(z)), \qquad i \in \{\mathrm{p},\ \mathrm{c},\ \mathrm{r}\}.$$

Formula (2.9) is a generalization of the well-known Krein formula for canonical resolvents (cf. [2]). We note also that all the objects in (2.9) are expressed in terms of the boundary triplet Π.

The following result follows from (2.9).

Proposition 2.5 ([12])**.** *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $\Theta_1, \Theta_2 \in \widetilde{\mathcal{C}}(\mathcal{H})$, and let $\mathfrak{S}_p$, $p \in (0,\infty)$, be the Neumann–Schatten ideal. Then,*

(i) for any $z \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})$, $\zeta \in \rho(\Theta_1) \cap \rho(\Theta_2)$, the following equivalence holds:

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \iff (\Theta_1 - \zeta)^{-1} - (\Theta_2 - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H}). \tag{2.10}$$

(ii) If, in addition, $\Theta_1, \Theta_2 \in \mathcal{C}(\mathcal{H})$ and $\mathrm{dom}(\Theta_1) = \mathrm{dom}(\Theta_2)$, then

$$\overline{\Theta_1 - \Theta_2} \in \mathfrak{S}_p(\mathcal{H}) \Longrightarrow (A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}). \tag{2.11}$$

(iii) Moreover, if $\Theta_1, \Theta_2 \in [\mathcal{H}]$, then implication (2.11) *becomes the equivalence.*

Extensions of a nonnegative operator

Assume that a symmetric operator $A \in \mathcal{C}(\mathfrak{H})$ is nonnegative. Then the set $\mathrm{Ext}_A(0,\infty)$ of its nonnegative self-adjoint extensions is nonempty (see [2, 19]). Moreover, there is a maximal nonnegative extension A_F (also called the *Friedrichs* or *hard* extension), and there is a minimal nonnegative extension A_K (*Krein* or *soft* extension) satisfying the relation

$$(A_F + x)^{-1} \le (\widetilde{A} + x)^{-1} \le (A_K + x)^{-1}, \qquad x \in (0,\infty), \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty)$$

(for details we refer the reader to [2, 17]).

Proposition 2.6 ([12])**.** *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 = A_0^* \ge 0$. Let $M(\cdot)$ be the corresponding Weyl function. Then $A_0 = A_\mathrm{F}$ $(A_0 = A_\mathrm{K})$ iff*

$$\lim_{x \downarrow -\infty}(M(x)f, f) = -\infty, \quad \left(\lim_{x\uparrow 0}(M(x)f, f) = +\infty\right), \quad f \in \mathcal{H} \setminus \{0\}. \tag{2.12}$$

Proposition 2.7 ([12])**.** *Let A be a nonnegative symmetric operator in $\mathfrak{H}$. Assume that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, and $M(\cdot)$ is the corresponding Weyl function. Let also $A_0 = A_\mathrm{F}$ be the Friedrichs extension. Then the following assertions hold:*

(i) the linear relation $\Theta \in \widetilde{\mathcal{C}}_\mathrm{self}(\mathcal{H})$ is semibounded below;

(ii) a self-adjoint extension A_Θ is semibounded below;

the equivalence holds, iff $M(\cdot)$ uniformly tends to $-\infty$ as $x \to -\infty$, i.e., for any $a > 0$, there exists $x_a < 0$ such that $M(x_a) < -a \cdot I_\mathcal{H}$.

In this case, we will write $M(x) \rightrightarrows -\infty$ as $x \to -\infty$.

3. Direct sums of boundary triplets

Let S_n be a densely defined symmetric operator in a Hilbert space $\mathfrak{H}_n$ with $\mathrm{n}_+(S_n) = \mathrm{n}_-(S_n) \leq \infty$, $n \in \mathbb{N}$. Consider the operator $A := \bigoplus_{n=1}^{\infty} S_n$ acting in $\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$, the Hilbert direct sum of Hilbert spaces $\mathfrak{H}_n$. By definition, $\mathfrak{H} = \{f = \oplus_{n=1}^{\infty} f_n : \ f_n \in \mathfrak{H}_n, \ \sum_{n=1}^{\infty} \|f_n\|^2 < \infty\}$. Clearly,

$$A^* = \bigoplus_{n=1}^{\infty} S_n^*,$$

$$\mathrm{dom}(A^*) = \{f = \oplus_{n=1}^{\infty} f_n \in \mathfrak{H} : \ f_n \in \mathrm{dom}(S_n^*), \ \sum_{n \in \mathbb{N}} \|S_n^* f_n\|^2 < \infty\}. \tag{3.1}$$

We equip the domains $\mathrm{dom}(S_n^*) =: \mathfrak{H}_{n+}$ and $\mathrm{dom}(A^*) =: \mathfrak{H}_+$ with the graph norms $\|f_n\|^2_{\mathfrak{H}_{n+}} := \|f_n\|^2 + \|S_n^* f_n\|^2$ and $\|f\|^2_{\mathfrak{H}_+} := \|f\|^2 + \|A^* f\|^2 = \sum_n \|f_n\|^2_{\mathfrak{H}_{n+}}$, respectively.

Further, let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. By $\|\Gamma_j^{(n)}\|$, we denote the norm of the linear mapping $\Gamma_j^{(n)} \in [\mathfrak{H}_{n+}, \mathcal{H}_n]$, $j \in \{0, 1\}$, $n \in \mathbb{N}$.

Let $\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ be a Hilbert direct sum of $\mathcal{H}_n$. Define mappings Γ_0 and Γ_1 by setting

$$\Gamma_j := \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)},$$

$$\mathrm{dom}(\Gamma_j) = \big\{f = \oplus_{n=1}^{\infty} f_n \in \mathrm{dom}(A^*) : \ \sum_{n \in \mathbb{N}} \|\Gamma_j^{(n)} f_n\|^2_{\mathcal{H}_n} < \infty\big\}. \tag{3.2}$$

Clearly, $\mathrm{dom}(\Gamma) := \mathrm{dom}(\Gamma_1) \cap \mathrm{dom}(\Gamma_0)$ is dense in $\mathfrak{H}_+$. We define the operators $S_{nj} := S_n^* \upharpoonright \ker \Gamma_j^{(n)}$ and $A_j := \bigoplus_{n=1}^{\infty} S_{nj}$, $j \in \{0, 1\}$. Then A_0 and A_1 are self-adjoint extensions of A. Note that A_0 and A_1 are disjoint but not necessarily transversal.

Definition 3.1. *Let Γ_j be defined by (3.2) and $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$. A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ will be called a direct sum of boundary triplets and will be assigned as $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n$.*

The following criteria were obtained in [9, 21].

Theorem 3.2. *Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^*, and let $M_n(\cdot)$ be the corresponding Weyl function, $n \in \mathbb{N}$. The direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for the operator $A^* = \bigoplus_{n=1}^{\infty} S_n^*$, iff*

$$C_1 = \sup_n \|M_n(i)\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_2 = \sup_n \|(\mathrm{Im}\, M_n(i))^{-1}\|_{\mathcal{H}_n} < \infty. \tag{3.3}$$

Theorem 3.2 makes it possible to construct a boundary triplet by regularizing an arbitrary direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets.

Theorem 3.3 ([28, 29])**.** *Let S_n be a symmetric operator in $\mathfrak{H}_n$ with deficiency indices $\mathrm{n}_\pm(S_k) = \mathrm{n}_n \leq \infty$ and $S_{n0} = S_{n0}^* \in \mathrm{Ext} S_n$, $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for S_n^* such that $\ker \Gamma_0^{(n)} = \mathrm{dom}(S_{n0})$, and $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ satisfying $\ker \Gamma_0 = \mathrm{dom}(\widetilde{A}_0) := \bigoplus_{n=1}^{\infty} \mathrm{dom}(S_{n0})$.*

Next, we assume that the operator $A = \bigoplus_{n=1}^{\infty} S_n$ has a regular real point, i.e., there exists $a = \overline{a} \in \hat{\rho}(A)$. The latter is equivalent to the existence of $\varepsilon > 0$ such that

$$(a-\varepsilon, a+\varepsilon) \subset \cap_{n=1}^{\infty} \widehat{\rho}(S_n). \tag{3.4}$$

Emphasize that the condition $a \in \cap_{n=1}^{\infty} \widehat{\rho}(S_n)$ is not sufficient for the inclusion $a \in \widehat{\rho}(A)$ to hold.

It is known that, under condition (3.4) for every $k \in \mathbb{N}$, there exists a self-adjoint extension $\widetilde{S}_k = \widetilde{S}_k^*$ of S_k preserving the gap $(a-\varepsilon, a+\varepsilon)$. Moreover, the Weyl function of the pair $\{S_k, \widetilde{S}_k\}$ is regular within the gap $(a-\varepsilon, a+\varepsilon)$.

Theorem 3.4 ([9, Theorem 2.12]). *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying* (3.4). *Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a-\varepsilon, a+\varepsilon) \subset \rho(S_{n0})$, and let $M_n(\cdot)$ be the corresponding Weyl function. Then*

(i) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms a B-generalized boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$, iff

$$C_3 := \sup_{n\in\mathbb{N}} \|M_n(a)\|_{\mathcal{H}_n} < \infty \qquad \text{and} \qquad C_4 := \sup_{n\in\mathbb{N}} \|M_n'(a)\|_{\mathcal{H}_n} < \infty, \tag{3.5}$$

where $M_n'(a) := (dM_n(z)/dz)|_{z=a}$.

(ii) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$, iff, in addition to (3.5)*, the following condition is satisfied:*

$$C_5 := \sup_{n\in\mathbb{N}} \|\big(M_n'(a)\big)^{-1}\|_{\mathcal{H}_n} < \infty. \tag{3.6}$$

Corollary 3.5 ([9, Corollary 2.13]). *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying* (3.4)*. Let also $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a-\varepsilon, a+\varepsilon) \subset \rho(S_{n0})$, $S_{n0} = S_n^* \restriction \ker(\widetilde{\Gamma}_0^{(n)})$, and let $\widetilde{M}_n(\cdot)$ be the corresponding Weyl function. Assume also that, for some operators R_n such that $R_n, R_n^{-1} \in [\mathcal{H}_n]$, the following conditions are satisfied:*

$$\sup_n \|R_n^{-1}(\widetilde{M}_n'(a))(R_n^{-1})^*\|_{\mathcal{H}_n} < \infty \quad \text{and}$$

$$\sup_n \|R_n^*(\widetilde{M}_n'(a))^{-1}R_n\|_{\mathcal{H}_n} < \infty, \quad n \in \mathbb{N}. \tag{3.7}$$

Then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets

$$\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\} \quad \text{with} \quad \Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)},$$

$$\Gamma_1^{(n)} := (R_n^{-1})^*\big(\widetilde{\Gamma}_1^{(n)} - \widetilde{M}_n(a)\widetilde{\Gamma}_0^{(n)}\big), \tag{3.8}$$

forms a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$.

4. First boundary triplet for the operator H_n

In what follows, $\mathbb{R}_+ = [0, +\infty)$, and $X = \{x_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$ is a strictly increasing sequence.

Consider the following symmetric operator in $L^2(x_{n-1}, x_n)$:

$$H_n = -\frac{d^2}{dx^2} + q_n, \qquad \operatorname{dom}(H_n) = W_0^{2,2}[x_{n-1}, x_n], \tag{4.1}$$

where q_n satisfies (1.5).

Lemma 4.1. *Assume that Hypothesis 1 holds. Then the operator H_n is a symmetric one with deficiency indices $n_\pm(H_n) = 2$. Its adjoint H_n^* is given by*

$$H_n^* = H_n, \qquad \mathrm{dom}(H_n^*) = W^{2,2}[x_{n-1}, x_n].$$

Moreover, the following assertions hold:

(i) A boundary triplet for the operator H_n^* can be chosen as follows:

$$\mathcal{H} = \mathbb{C}^2, \qquad \widetilde{\Gamma}_0^{(n)} = \begin{pmatrix} f(x_{n-1}) \\ f'(x_n) \end{pmatrix}, \qquad \widetilde{\Gamma}_1^{(n)} = \begin{pmatrix} f'(x_{n-1}) \\ f(x_n) \end{pmatrix}. \tag{4.2}$$

(ii) The corresponding Weyl function $\widetilde{M}_n(\cdot)$ is

$$\widetilde{M}_n(z) = \begin{pmatrix} \sqrt{z-q_n}\tan(d_n\sqrt{z-q_n}) & \frac{1}{\cos(d_n\sqrt{z-q_n})} \\ \frac{1}{\cos(d_n\sqrt{z-q_n})} & \frac{\tan(d_n\sqrt{z-q_n})}{\sqrt{z-q_n}} \end{pmatrix}. \tag{4.3}$$

Proof. It is straightforward. □

Clearly, $H_{\min}$ is a closed operator with $n_\pm(H_{\min}) = \infty$, and

$$H_{\max} := H_{\min}^* = \bigoplus_{n=1}^{\infty} H_n^*,$$

$$\mathrm{dom}(H_{\max}) \subseteq W^{2,2}(\mathbb{R}_+\backslash X) = \bigoplus_{n=1}^{\infty} W^{2,2}[x_{n-1}, x_n].$$

Proposition 4.2. *Assume that Hypothesis 1 holds. Let $X = \{x_n\}_{n=0}^\infty$ be as above and $d^* < +\infty$. Define the mappings $\Gamma_j^{(n)} : W^{2,2}[x_{n-1}, x_n] \to \mathbb{C}^2$, $n \in \mathbb{N}$, $j \in \{0,1\}$, by setting*

$$\Gamma_0^{(n)} = \begin{pmatrix} d_n^{1/2} f(x_{n-1}) \\ d_n^{3/2} f'(x_n) \end{pmatrix}, \tag{4.4}$$

$$\Gamma_1^{(n)} = \begin{pmatrix} d_n^{-1/2} f'(x_{n-1}) + \sqrt{\frac{q_n}{d_n}}\tanh(d_n\sqrt{q_n}) f(x_{n-1}) - \frac{d_n^{-1/2} f'(x_n)}{\cosh(d_n\sqrt{q_n})} \\ d_n^{-3/2} f(x_n) - \frac{d_n^{-3/2} f(x_{n-1})}{\cosh(d_n\sqrt{q_n})} - \frac{\tanh(d_n\sqrt{q_n}) f'(x_n)}{\sqrt{q_n d_n^3}} \end{pmatrix}. \tag{4.5}$$

Define the function $M_n(z)$ given by

$$M_n(z) = \begin{pmatrix} \frac{1}{d_n}(\sqrt{z-q_n}\tan(d_n\sqrt{z-q_n}) + \sqrt{q_n}\tanh(d_n\sqrt{q_n})) & \frac{1}{d_n^2}\left(\frac{1}{\cos(d_n\sqrt{z-q_n})} - \frac{1}{\cosh(d_n\sqrt{z-q_n})}\right) \\ \frac{1}{d_n^2}\left(\frac{1}{\cos(d_n\sqrt{z-q_n})} - \frac{1}{\cosh(d_n\sqrt{z-q_n})}\right) & \frac{1}{d_n^3}\left(\frac{\tan(d_n\sqrt{z-q_n})}{\sqrt{z-q_n}} - \frac{\tanh(d_n\sqrt{q_n})}{\sqrt{q_n}}\right) \end{pmatrix}. \tag{4.6}$$

Then:

(i) For any $n \in \mathbb{N}$, the triplet $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is the boundary triplet for the operator H_n^*.

(ii) The Weyl function $M_n(z)$ corresponding to the triplet Π_n takes the form (4.6).

(iii) The direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *with* $\mathcal{H} = \mathbb{C}^2$ *and* $\Gamma_j = \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}$, $j \in \{0,1\}$, *is a boundary triplet for the operator* $H_{\min}^* = \bigoplus_{n=1}^{\infty} H_n^*$.

Proof. (i) The proof is straightforward. Note, however, that it follows from Lemma 4.1, since

$$\Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)}, \qquad \Gamma_1^{(n)} := R_n^{-1}(\widetilde{\Gamma}_1^{(n)} - Q_n \widetilde{\Gamma}_0^{(n)}), \qquad n \in \mathbb{N}, \tag{4.7}$$

where

$$R_n := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{3/2} \end{pmatrix},$$

$$Q_n := \widetilde{M}_n(0) = \begin{pmatrix} -\sqrt{q_n}\tanh(d_n\sqrt{q_n}) & \frac{1}{\cosh(d_n\sqrt{q_n})} \\ \frac{1}{\cosh(d_n\sqrt{q_n})} & \frac{\tanh(d_n\sqrt{q_n})}{\sqrt{q_n}} \end{pmatrix}, \quad n \in \mathbb{N}. \tag{4.8}$$

(ii) It easily follows from (4.3) and (4.7) that

$$M_n(z) = R_n^{-1}(\widetilde{M}_n(z) - Q_n)R_n^{-1}, \qquad n \in \mathbb{N}. \tag{4.9}$$

(iii) We set $v_n := d_n\sqrt{q_n}$. Then

$$\begin{aligned} M_n'(0) &= R_n^{-1}\widetilde{M}_n'(0)\, R_n^{-1} \\ &= \begin{pmatrix} \frac{\sinh(v_n)\cosh(v_n)+v_n}{2v_n\cosh^2(v_n)} & \frac{\sinh(v_n)}{2v_n\cosh^2(v_n)} \\ \frac{\sinh(v_n)}{2v_n\cosh^2(v_n)} & \frac{\sinh(v_n)\cosh(v_n)-v_n}{2v_n^3\cosh^2(v_n)} \end{pmatrix}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.10}$$

Clearly, (1.5) yields

$$\sinh(v_n) < 2^{-1}\exp(v_n).$$

Since, in addition, $\lim\limits_{x\to 0}\frac{\sinh(x)}{x} = 1$, the matrices $M_n'(0)$ are uniformly bounded:

$$\sup_{n\in\mathbb{N}} \|M_n'(0)\| =: c_1 < \infty. \tag{4.11}$$

Further,

$$\begin{aligned} (M_n'(0))^{-1} &= R_n(\widetilde{M}_n'(0))^{-1}R_n \\ &= \begin{pmatrix} \frac{2v_n(\sinh(v_n)\cosh(v_n)-v_n)}{\sinh^2(v_n)-v_n^2} & \frac{-2v_n^3\sinh(v_n)}{\sinh^2(v_n)-v_n^2} \\ \frac{-2v_n^3\sinh(v_n)}{\sinh^2(v_n)-v_n^2} & \frac{2v_n^3(\sinh(v_n)\cosh(v_n)+v_n)}{\sinh^2(v_n)-v_n^2} \end{pmatrix}, \ n \in \mathbb{N}. \end{aligned} \tag{4.12}$$

Similarly, (1.5) yields the uniform boundedness of matrices $(M_n'(0))^{-1}$, i.e.,

$$\sup_{n\in\mathbb{N}} \|(M_n'(0))^{-1}\| =: c_2 < \infty. \tag{4.13}$$

We complete the proof by applying Theorem 3.4. □

Remark 4.3. Assume that condition (1.6) holds. Then we have

$$\lim_{n\to\infty} M_n'(0) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}, \qquad n \in \mathbb{N}, \tag{4.14}$$

$$\lim_{n\to\infty} (M_n'(0))^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}, \qquad n \in \mathbb{N}. \tag{4.15}$$

5. Second boundary triplets for the operator H_n

In what follows, $\mathbb{R}_+ = [0,\infty) \subseteq \mathbb{R}$ denotes a bounded interval or positive semiaxis, $X = \{x_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$ is a strictly increasing sequence.

Consider the following symmetric operator in $L^2(x_{n-1}, x_n)$:

$$H_n = -\frac{d^2}{dx^2} + q_n, \qquad \mathrm{dom}(H_n) = W_0^{2,2}[x_{n-1}, x_n], \tag{5.1}$$

where q_n satisfies (1.5).

Lemma 5.1. *Assume that Hypothesis 1 holds. Then the operator H_n is a symmetric one with deficiency indices $n_\pm(H_n) = 2$.*
Its adjoint H_n^* is given by

$$H_n^* = H_n, \qquad \mathrm{dom}(H_n^*) = W^{2,2}[x_{n-1}, x_n].$$

Moreover, the following assertions hold:

(i) A boundary triplet for the operator H_n^* can be chosen as follows:

$$\mathcal{H} = \mathbb{C}^2, \qquad \widetilde{\Gamma}_0^{(n)} = \begin{pmatrix} f(x_{n-1}) \\ -f(x_n) \end{pmatrix}, \qquad \widetilde{\Gamma}_1^{(n)} = \begin{pmatrix} f'(x_{n-1}) \\ f'(x_n) \end{pmatrix}; \tag{5.2}$$

(ii) The corresponding Weyl function $\widetilde{M}_n(\cdot)$ is

$$\widetilde{M}_n(z) = \begin{pmatrix} -\sqrt{z-q_n}\cot(d_n\sqrt{z-q_n}) & -\frac{\sqrt{z-q_n}}{\sin(d_n\sqrt{z-q_n})} \\ -\frac{\sqrt{z-q_n}}{\sin(d_n\sqrt{z-q_n})} & -\sqrt{z-q_n}\cot(d_n\sqrt{z-q_n}) \end{pmatrix}. \tag{5.3}$$

Proof. It is straightforward. □

Proposition 5.2. *Assume that Hypothesis 1 holds. Let also $X = \{x_n\}_{n=0}^{\infty}$ be as above, and let $d^* < +\infty$. For any $n \in \mathbb{N}$, define the boundary triplet $\Pi^{(n)} = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for H_n^* by setting*

$$\Gamma_j^{(n)} : W^{2,2}[x_{n-1}, x_n] \to \mathbb{C}^2, \quad n \in \mathbb{N}, \quad j \in \{0,1\},$$

$$\Gamma_0^{(n)} = \sqrt{d_n}\begin{pmatrix} f(x_{n-1}) \\ -f(x_n) \end{pmatrix}, \tag{5.4}$$

$$\Gamma_1^{(n)} = \frac{1}{\sqrt{d_n}}\begin{pmatrix} f'(x_{n-1}) + \sqrt{q_n} f(x_{n-1})\coth(d_n\sqrt{q_n}) - \frac{\sqrt{q_n} f(x_n)}{\sinh(d_n\sqrt{q_n})} \\ f'(x_n) + \frac{\sqrt{q_n} f(x_{n-1})}{\sinh(d_n\sqrt{q_n})} - \sqrt{q_n} f(x_n)\coth(d_n\sqrt{q_n}) \end{pmatrix}. \tag{5.5}$$

Define the function $M_n(z)$ by

$$M_n(z) = \begin{pmatrix} a_n(z) & b_n(z) \\ b_n(z) & a_n(z) \end{pmatrix}, \tag{5.6}$$

where

$$a_n(z) := \frac{1}{d_n}(-\sqrt{z-q_n}\cot(d_n\sqrt{z-q_n}) + \sqrt{q_n}\coth(d_n\sqrt{q_n})),$$

$$b_n(z) := \frac{1}{d_n}\left(-\frac{\sqrt{z-q_n}}{\sin(d_n\sqrt{z-q_n})} + \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\right).$$

Then:

(i) For any $n \in \mathbb{N}$ the triplet $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is the boundary triplet for the operator H_n^*.

(ii) The Weyl function $M_n(z)$ corresponding to the triplet Π_n takes the form (5.6).

(iii) The direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with $\mathcal{H} = \mathbb{C}^2$ and $\Gamma_j = \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}$, $j \in \{0,1\}$, is a boundary triplet for the operator $H_{\min}^* = \bigoplus_{n=1}^{\infty} H_n^*$.

Proof. (i) The proof is straightforward. Note, however, that it follows from Lemma 5.1, since

$$\Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)}, \quad \Gamma_1^{(n)} := R_n^{-1}(\widetilde{\Gamma}_1^{(n)} - Q_n \widetilde{\Gamma}_0^{(n)}), \qquad n \in \mathbb{N}, \tag{5.7}$$

where

$$R_n := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{1/2} \end{pmatrix},$$

$$Q_n := \widetilde{M}_n(0) = \begin{pmatrix} -\sqrt{q_n}\coth(d_n\sqrt{q_n}) & -\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \\ -\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} & -\sqrt{q_n}\coth(d_n\sqrt{q_n}) \end{pmatrix}, \quad n \in \mathbb{N}. \tag{5.8}$$

(ii) It easily follows from (5.3) and (5.7) that

$$M_n(z) = R_n^{-1}(\widetilde{M}_n(z) - Q_n)R_n^{-1}, \qquad n \in \mathbb{N}. \tag{5.9}$$

(iii) We set $v_n := d_n\sqrt{q_n}$. Then

$$\begin{aligned} M_n'(0) &= R_n^{-1}\widetilde{M}_n'(0)\,R_n^{-1} \\ &= \begin{pmatrix} \frac{\cosh(v_n)\sinh(v_n)-v_n}{2v_n\sinh^2(v_n)} & \frac{\sinh(v_n)-v_n\cosh(v_n)}{2v_n\sinh^2(v_n)} \\ \frac{\sinh(v_n)-v_n\cosh(v_n)}{2v_n\sinh^2(v_n)} & \frac{\cosh(v_n)\sinh(v_n)-v_n}{2v_n\sinh^2(v_n)} \end{pmatrix}, \qquad n \in \mathbb{N}. \end{aligned} \tag{5.10}$$

Clearly, (1.5) yields

$$\cosh(v_n) < 2^{-1}\exp(v_n).$$

Since, in addition, $\lim_{x\to 0} \frac{\sinh(x)}{x} = 1$, the matrices $M_n'(0)$ are uniformly bounded

$$\sup_{n\in\mathbb{N}} \|M_n'(0)\| =: c_3 < \infty. \tag{5.11}$$

Further,

$$\begin{aligned} (M_n'(0))^{-1} &= R_n(\widetilde{M}_n'(0))^{-1}R_n \\ &= \frac{1}{\sinh^2(v_n) - v_n^2} \cdot \begin{pmatrix} 2(\cosh(v_n)\sinh(v_n) - v_n) & 2(v_n\cosh(v_n) - \sinh(v_n)) \\ 2(v_n\cosh(v_n) - \sinh(v_n)) & 2(\cosh(v_n)\sinh(v_n) - v_n) \end{pmatrix}, \ n \in \mathbb{N}. \end{aligned} \tag{5.12}$$

Similarly, (1.5) yields the uniform boundedness of matrices $(M_n'(0))^{-1}$, i.e.,

$$\sup_{n\in\mathbb{N}} \|(M_n'(0))^{-1}\| =: c_4 < \infty. \tag{5.13}$$

We complete the proof by applying Theorem 3.4. □

Remark 5.3. Assume that condition (1.6) is satisfied. Then we get

$$M'_n(0) = R_n^{-1}\widetilde{M}'_n(0)\,R_n^{-1} \longrightarrow \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}, \qquad n \to \infty, \tag{5.14}$$

$$(M'_n(0))^{-1} = R_n(\widetilde{M}'_n(0))^{-1}R_n \longrightarrow \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \qquad n \to \infty. \tag{5.15}$$

Proposition 5.4. *Assume that Hypothesis 1 holds. Let also* Π *be the boundary triplet for the operator* $H^*_{\min}$ *defined in Proposition 5.2, and let* $M(\cdot)$ *be the corresponding Weyl function. If*

$$d^* = \sup_{n\in\mathbb{N}} d_n < +\infty, \tag{5.16}$$

then

$$M(-a^2) \rightrightarrows -\infty \quad \text{as} \quad a \to +\infty. \tag{5.17}$$

Proof. The Weyl function $M(\cdot)$ has the form $M(z) = \oplus_{n=1}^{\infty} M_n(z)$, where $M_n(\cdot)$ is given by (5.6). We now introduce the matrix-valued function

$$M(-a^2; d_n, q_n) := \begin{pmatrix} F_a(d_n, q_n) & G_a(d_n, q_n) \\ G_a(d_n, q_n) & F_a(d_n, q_n) \end{pmatrix}, \tag{5.18}$$

where

$$\begin{aligned} F_a(d_n, q_n) &:= \frac{1}{d_n}\left[-\sqrt{a^2+q_n}\coth(d_n\sqrt{a^2+q_n}) + \sqrt{q_n}\coth(d_n\sqrt{q_n})\right] \\ &= \frac{1}{d_n^2}\left[-\sqrt{d_n^2a^2+d_n^2q_n}\coth(\sqrt{d_n^2a^2+d_n^2q_n}) + d_n\sqrt{q_n}\coth(d_n\sqrt{q_n})\right], \end{aligned} \tag{5.19}$$

$$\begin{aligned} G_a(d_n, q_n) &:= \frac{1}{d_n}\left[-\frac{\sqrt{a^2+q_n}}{\sinh(d_n\sqrt{a^2+q_n})} + \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\right] \\ &= \frac{1}{d_n^2}\left[-\frac{\sqrt{d_n^2a^2+d_n^2q_n}}{\sinh(\sqrt{d_n^2a^2+d_n^2q_n})} + \frac{d_n\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\right]. \end{aligned} \tag{5.20}$$

Let us check that

$$G_a(d_n, q_n) > 0 \quad \text{and} \quad F_a(d_n, q_n) < 0 \quad \text{for} \quad a^2 > 1.$$

Consider the function $f_1(x) := \frac{\sinh(\sqrt{x})}{\sqrt{x}}$. Since

$$\begin{aligned} f_1'(x) &= \left(\frac{\sinh(\sqrt{x})}{\sqrt{x}}\right)' = \frac{\sqrt{x}\cosh(\sqrt{x}) - \sinh(\sqrt{x})}{2x\sqrt{x}} \\ &= \frac{e^{\sqrt{x}}(\sqrt{x}-1) + e^{-\sqrt{x}}(\sqrt{x}+1)}{4x\sqrt{x}} > 0 \quad \text{for} \quad x > 1, \end{aligned}$$

we have that $f_1(x)$ grows, if $x > 1$. Hence, the function $f_1^{-1}(x) = \frac{\sqrt{x}}{\sinh(\sqrt{x})}$ decreases for $x > 1$. This implies that $G_a(d_n, q_n) > 0$, whenever $a^2 > 1$.

Further, consider the function $f_2(x) := \sqrt{x}\coth(\sqrt{x})$. Since

$$f_2'(x) = \left(\sqrt{x}\coth(\sqrt{x})\right)' = \frac{\cosh(\sqrt{x})\sinh(\sqrt{x}) - \sqrt{x}}{2\sqrt{x}\sinh^2(\sqrt{x})} = \frac{\sinh(2\sqrt{x}) - 2\sqrt{x}}{2\sqrt{x}\sinh^2(\sqrt{x})} > 0 \quad \text{for} \quad x > 1,$$

we have that $f_2(x)$ grows, if $x > 1$. Hence, $F_a(d_n, q_n) < 0$ for $a^2 > 1$.

According to Hypothesis 1, we have $d_n\sqrt{q_n} < c$. Since

$$\begin{pmatrix} F_a(d_n,q_n) & G_a(d_n,q_n) \\ G_a(d_n,q_n) & F_a(d_n,q_n) \end{pmatrix} - (F_a(d_n,q_n) + G_a(d_n,q_n))I_2 = G_a(d_n,q_n)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.21}$$

and $G_a(d_n, q_n) > 0$, we get the following inequality:

$$M(-a^2; d_n, q_n) \le (F_a(d_n, q_n) + G_a(d_n, q_n))I_2.$$

Further, consider the functions

$$\begin{aligned} F_a(d_n,q_n) + G_a(d_n,q_n) = \frac{1}{d_n^2}\Bigg[& \frac{d_n\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\{\cosh(d_n\sqrt{q_n}) + 1\} \\ & - \frac{\sqrt{a^2d_n^2 + d_n^2q_n}}{\sinh(\sqrt{a^2d_n^2 + d_n^2q_n})}\left\{\cosh(\sqrt{a^2d_n^2 + d_n^2q_n}) + 1\right\}\Bigg] \end{aligned} \tag{5.22}$$

and $g(x) := \frac{\sqrt{x}}{\sinh(\sqrt{x})}(1 + \cosh(\sqrt{x}))$. Since

$$g'(x) = \left(\frac{\sqrt{x}}{\sinh(\sqrt{x})}(1 + \cosh(\sqrt{x}))\right)' = \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sqrt{x}\sinh^2(\frac{\sqrt{x}}{2})} > 0, \tag{5.23}$$

the function $g(\cdot)$ grows. Applying the Lagrange theorem to the right-hand side of (5.22), we get

$$F_a(d_n,q_n) + G_a(d_n,q_n) = -\frac{1}{d_n^2}\left(g(a^2d_n^2 + d_n^2q_n) - g(d_n^2q_n)\right) = -a^2g'(\xi_n), \tag{5.24}$$

where $\xi_n \in (d_n^2q_n, a^2d_n^2 + d_n^2q_n)$. Further, since $\lim_{x\to 0} g'(x) = \frac{1}{6} > 0$, there exists $\varepsilon > 0$ such that

$$g'(x) > \frac{1}{12}, \qquad x \in [\varepsilon, \infty). \tag{5.25}$$

On the other hand,

$$\lim_{x\to\infty} \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sinh^2(\frac{\sqrt{x}}{2})} = \frac{1}{2}. \tag{5.26}$$

Combining this relation with the obvious inequality $\sinh(\sqrt{x}) > \sqrt{x}$, $x > 0$, we arrive at the two-sided estimate

$$C_1 < \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sinh^2(\frac{\sqrt{x}}{2})} < C_2, \qquad x \in [\varepsilon, \infty). \tag{5.27}$$

In view of (5.23), we have

$$\frac{C_1}{\sqrt{x}} < g'(x) = \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sqrt{x}\sinh^2(\frac{\sqrt{x}}{2})} < \frac{C_2}{\sqrt{x}}, \qquad x \in [\varepsilon, \infty). \tag{5.28}$$

Using $d_n^2q_n < c^2$ (see Hypothesis 1) and (5.16), we obtain

$$\frac{C_1}{\sqrt{a^2d_n^2 + d_n^2q_n}} > \frac{C_1}{\sqrt{a^2d_n^2 + c^2}} > \frac{C_1}{\sqrt{a^2(d^*)^2 + c^2}}. \tag{5.29}$$

Combining the latter with (5.25) and (5.28), we have

$$\inf_{x\in(d_n^2 q_n, a^2 d_n^2 + d_n^2 q_n)} g'(x) > \beta(a),$$
$$\text{where}\quad \beta(a) = \min\left\{\frac{1}{12}, \frac{C_1}{\sqrt{a^2(d^*)^2 + c^2}}\right\}. \tag{5.30}$$

Choosing $a > \frac{\sqrt{3}c}{d^*}$, we continue this inequality as

$$\inf_{x\in(d_n^2 q_n, a^2 d_n^2 + d_n^2 q_n)} g'(x) > \frac{C_1}{2ad^*}, \qquad a > \frac{\sqrt{3}c}{d^*}. \tag{5.31}$$

Combining this estimate with (5.24) yields

$$\sup_n (F_a(d_n, q_n) + G_a(d_n, q_n)) \le -a^2 \cdot \frac{C_1}{2ad^*} = -a\frac{C_1}{2d^*}. \tag{5.32}$$

Since $M_n(-a^2) = M(-a^2, d_n)$, the preceding inequality yields

$$M(-a^2) = \oplus_{n=1}^\infty M_n(-a^2) \le -a\frac{C_1}{2d^*}, \qquad a > \frac{\sqrt{3}c}{d^*}. \tag{5.33}$$

Relation (5.17) is obviously satisfied. □

Combining Proposition 5.2 with Proposition 2.2, we arrive at the following parametrization of the set $\mathrm{Ext}H_{\min}$ of closed proper extensions of the operator $H_{\min}$:

$$\widetilde{H} = H_\Theta := H_{\min}^*\lceil \mathrm{dom}(H_\Theta),$$
$$\mathrm{dom}(H_\Theta) = \{f \in \mathrm{dom}(H_{\min}^*):\ \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}, \tag{5.34}$$

where $\Theta \in \widetilde{\mathcal{C}}(l^2)$ and $\Gamma_0,\ \Gamma_1$ are defined by (5.4)-(5.5).

Theorem 5.5. *Let $\Pi = \oplus_{n=1}^\infty \Pi_n$ be a boundary triplet for $H_{\min}^*$ defined in Proposition 5.2, $\Theta, \widetilde{\Theta} \in \widetilde{\mathcal{C}}(\mathcal{H})$, and let $H_\Theta, H_{\widetilde{\Theta}} \in \mathrm{Ext}H_{\min}$ be proper extensions of $H_{\min}$ defined by (5.34). Then*

(i) The operator H_Θ is symmetric (self-adjoint), iff so is Θ, and $\mathrm{n}_\pm(H_{\min}) = \mathrm{n}_\pm(\Theta)$.

(ii) The self-adjoint (symmetric) operator H_Θ is lower semibounded, iff so is Θ.

(iii) Let $\Theta = \Theta^*$. Then $\kappa_-(H_\Theta) = \kappa_-(\Theta)$. In particular, $H_\Theta \ge 0$ iff $\Theta \ge 0$.

(iv) For any $p \in (0, \infty]$, $z \in \rho(H_\Theta) \cap \rho(H_{\widetilde{\Theta}})$, and $\zeta \in \rho(\Theta) \cap \rho(\widetilde{\Theta})$, the following equivalence holds

$$(H_\Theta - z)^{-1} - (H_{\widetilde{\Theta}} - z)^{-1} \in \mathfrak{S}_p \quad \Longleftrightarrow \quad (\Theta - \zeta)^{-1} - (\widetilde{\Theta} - \zeta)^{-1} \in \mathfrak{S}_p.$$

(v) The operator $H_\Theta = H_{\Theta}^*$ has a discrete spectrum, iff $d_n \searrow 0$, and Θ has a discrete spectrum.

Proof. (i) follows from Proposition 2.2.

$(ii), (iii)$ Combining Proposition 2.7 with Proposition 5.4 yields the first statement.

(iv) is implied by Proposition 2.5.

(v) First, we show that conditions are sufficient. Indeed, the operator

$$H_0 := H_{\min}^*\lceil \ker(\Gamma_0) = \oplus_{n\in\mathbb{N}} H_{n0}, \qquad H_{n0} := H_n^*\lceil \ker(\Gamma_0^{(n)}), \tag{5.35}$$

has a discrete spectrum, if $\lim_{n\to\infty} d_n = 0$. Moreover, the Krein resolvent formula and the discreteness of $\sigma(\Theta)$ yield $\mathcal{R}_{H_\Theta}(z) - \mathcal{R}_{H_0}(z) \in \mathfrak{S}_\infty,\ z \in \mathbb{C}_+$, and, hence, $\mathcal{R}_{H_\Theta}(z) \in \mathfrak{S}_\infty$.

Let us show that the condition $d_n \searrow 0$ is necessary for the discreteness of $\sigma(H_\Theta)$. Without loss of generality, assume that $0 \in \rho(H_\Theta)$. Assume also that $\limsup_{n\to\infty} d_n > 0$ and H_Θ has a discrete spectrum. Then there exists a sequence $\{d_{n_k}\}_{k=1}^\infty$ such that $d_{n_k} \geq d_*/2 > 0$. For $\varepsilon \in (0, d_*/2)$, define the function

$$\varphi_\varepsilon(\cdot) \in W_2^2(\mathbb{R}), \qquad \varphi_\varepsilon(x) = \begin{cases} 1, & \varepsilon \leq x \leq d_* - \varepsilon, \\ 0, & x \notin [0, d_*]. \end{cases}$$

Note that $\varphi_k(x) := P_\mathcal{I}\varphi_\varepsilon(x + x_{n_k}) \in \mathrm{dom}(H_\Theta)$, where $P_\mathcal{I}$ is the orthoprojection in $L^2(\mathbb{R})$ onto $L^2(\mathcal{I})$. Moreover, $\|\varphi_k\|_{L^2} \equiv \text{const}$ and $\|H_\Theta\varphi_k\|_{L^2} \equiv \text{const}$. Since the functions $\varphi_k(\cdot)$ have disjoint supports, the operator $(H_\Theta)^{-1}$ is not compact. We arrive at the contradiction.

□

Remark 5.6. Clearly, all statements of Theorem 5.5 with exception for *(ii)*–*(iii)* remain valid for the boundary triplet $\Pi = \oplus_1^\infty \Pi_n$ with Π_n defined by (4.4)–(4.5) in place of (5.4)–(5.5).

Corollary 5.7. *If a is large enough, then $H_\Theta \geq -a^2$, whenever $\Theta \geq -\frac{a}{2d^*} I_{l_2}$.*

6. Schrödinger operators with δ-interactions

Now, we return to the symmetric differential operator $H^0_{X,\alpha,q}$ in $L^2(\mathbb{R}_+)$

$$\begin{aligned} H^0_{X,\alpha,q} &:= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_n, \\ \mathrm{dom}(H^0_{X,\alpha,q}) &= \left\{ f \in W^{2,2}_{\mathrm{comp}}(\mathcal{I}\setminus X) : \begin{array}{c} f'(0) = 0,\ f(x_n+) = f(x_n-) \\ f'(x_n+) - f'(x_n-) = \alpha_n f(x_n) \end{array} \right\}. \end{aligned} \tag{6.1}$$

As above, we denote, by $H_{X,\alpha,q}$, the closure of $H^0_{X,\alpha,q}$, $H_{X,\alpha,q} = \overline{H^0_{X,\alpha,q}}$.

6.1. Parametrization of the operator $H_{X,\alpha,q}$

Let $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$, and let $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ be the boundary triplets defined in Propositions 5.2 and 4.2, respectively. According to Proposition 2.2, the extension $H_{X,\alpha} (\in \mathrm{Ext} H_{\min})$ admits two representations

$$\begin{aligned} H_{X,\alpha,q} &= H_{\Theta_j} := H^*_{\min}\lceil \mathrm{dom}(H_{\Theta_j}), \\ \mathrm{dom}(H_{\Theta_j}) &= \{f \in \mathrm{dom}(H^*_{\min}) :\ \{\Gamma_0^j f, \Gamma_1^j f\} \in \Theta_j\}, \quad j \in \{1, 2\}, \end{aligned} \tag{6.2}$$

where $\Theta_j \in \widetilde{\mathcal{C}}(\mathcal{H})$ $(j \in \{1,2\})$ are closed symmetric linear relations. In this section, we show that Θ_2, as well as the operator part Θ_1' of Θ_1, is a Jacobi matrix.

1. The first parametrization. At first, we consider the triplet $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ constructed in Proposition 4.2. For any α, the operators $H_{X,\alpha}$ and $H_0^{(1)} := H^*_{\min}\lceil \ker(\Gamma_0^1)$ are disjoint. Hence, Θ_1 in (6.2) is a (closed) operator in $\mathcal{H} = l^2(\mathbb{N})$, $\Theta_1 \in \mathcal{C}(l^2)$. More precisely, consider the Jacobi matrix

$$B_{X,\alpha,q} := \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ 0 & 0 & a_3 & b_4 & a_4 & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{6.3}$$

where

$$b_{2k-1} = d_k^{-1}(\alpha_{k-1} + \sqrt{q_k}\tanh(d_k\sqrt{q_k})), \qquad b_{2k} = -\frac{\tanh(d_k\sqrt{q_k})}{d_k^3\sqrt{q_k}},$$

$$a_{2k-1} = -\frac{1}{d_k^2\cosh(d_k\sqrt{q_k})}, \qquad a_{2k} = d_k^{-3/2}d_{k+1}^{-1/2}.$$

Let $\tau_{X,\alpha,q}$ be a second-order difference expression associated with (6.3). We defines the corresponding minimal symmetric operator in l^2 by (see [1,6])

$$B^0_{X,\alpha,q}f := \tau_{X,\alpha,q}f, \qquad f \in \mathrm{dom}(B^0_{X,\alpha,q}) := l^2_0 \quad \text{and} \quad B_{X,\alpha,q} = \overline{B^0_{X,\alpha,q}}. \tag{6.4}$$

Recall that $B_{X,\alpha,q}$ has equal deficiency indices, and $\mathrm{n}_+(B_{X,\alpha,q}) = \mathrm{n}_-(B_{X,\alpha,q}) \leq 1$.

In addition, we note that $B_{X,\alpha,q}$ admits the representation

$$B_{X,\alpha,q} = R_X^{-1}(\widetilde{B}_\alpha - Q_X)R_X^{-1},$$

$$\text{where} \quad \widetilde{B}_\alpha := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.5}$$

and $R_X = \oplus_{n=1}^\infty R_n,\ Q_X = \oplus_{n=1}^\infty Q_n$ are given by (4.8).

Proposition 6.1. *Let $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ be the boundary triplet for $H^*_{\min}$ constructed in Proposition 4.2, and let $B_{X,\alpha,q}$ be the minimal Jacobi operator defined by* (6.3)–(6.4). *Then $\Theta_1 = B_{X,\alpha,q}$, i.e.,*

$$H_{X,\alpha,q} = H_{B_{X,\alpha,q}} = H^*_{\min}\lceil\mathrm{dom}(H_{B_{X,\alpha,q}}),$$

$$\mathrm{dom}(H_{B_{X,\alpha,q}}) = \{f \in \mathrm{dom}(H^*_{\min}) : \Gamma_1^1 f = B_{X,\alpha,q}\Gamma_0^1 f\}.$$

Proof. Let $f \in W^{2,2}_{\mathrm{comp}}(\mathbb{R}_+ \setminus X)$. Then $f \in \mathrm{dom}(H_{X,\alpha,q})$, iff $\widetilde{\Gamma}_1^1 f = \widetilde{B}_\alpha\widetilde{\Gamma}_0^1 f$. Here, $\widetilde{\Gamma}_j^1 := \oplus_{n\in\mathbb{N}}\widetilde{\Gamma}_j^{(n)}$ where $\widetilde{\Gamma}_j^{(n)},\ j \in \{0,1\}$, are defined by (4.5), and $\widetilde{B}_\alpha$ is defined by (6.5). Combining (5.7) and (5.8) with (6.5), we rewrite the equality $\widetilde{\Gamma}_1^1 f = \widetilde{B}_\alpha\widetilde{\Gamma}_0^1 f$ as $\Gamma_1^1 f = B_{X,\alpha}\Gamma_0^1 f$.

Taking the closures, we complete the proof. □

Remark 6.2. Note that matrix (6.3) has negative off-diagonal entries, although the off-diagonal entries in the classical theory of Jacobi operators are assumed to be positive. But it is known (see, e.g., [35]) that the (minimal) operator $B_{X,\alpha,q}$ is unitarily equivalent to the minimal Jacobi operator associated with the matrix

$$B'_{X,\alpha,q} := \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ 0 & 0 & a_3 & b_4 & a_4 & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{6.6}$$

where

$$b_{2k-1} = d_k^{-1}(\alpha_{k-1} + \sqrt{q_k}\tanh(d_k\sqrt{q_k})), \qquad b_{2k} = -\frac{\tanh(d_k\sqrt{q_k})}{d_k^3\sqrt{q_k}},$$

$$a_{2k-1} = \frac{1}{d_k^2 \cosh(d_k\sqrt{q_k})}, \qquad a_{2k} = d_k^{-3/2} d_{k+1}^{-1/2}.$$

In the sequel, we will identify the operators $B_{X,\alpha,q}$ and $B'_{X,\alpha,q}$, while investigating those spectral properties of the operator $H_{X,\alpha,q}$, which are invariant under unitary transformations.

2. The second parametrization. Let us consider the boundary triplet $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ constructed in Proposition 5.2. Now the operators $H_{X,\alpha,q}$ and $H_0^{(2)} := H_{\min}^* \lceil \ker(\Gamma_0^2)$ are not disjoint. Hence, by Proposition 2.2(ii), the corresponding linear relation Θ_2 in (6.2) is not an operator, i.e., it has a nontrivial multivalued part, $\operatorname{mul}\Theta_2 := \{f \in \mathcal{H} :\ \{0, f\} \in \Theta_2\} \neq \{0\}$.

Let $f \in W^{2,2}_{\text{comp}}(\mathbb{R}_+ \setminus X)$. Then $\Gamma_0^2 f, \Gamma_1^2 f \in l_0^2$ and $f \in \operatorname{dom}(H_{X,\alpha,q})$, iff $C_{X,\alpha,q}\Gamma_1 f = D_{X,\alpha,q}\Gamma_0 f$, where

$$C_{X,\alpha,q} := CR_X, \qquad D_{X,\alpha,q} := (D_\alpha - CQ_X)R_X^{-1}, \tag{6.7}$$

$$C := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$D_\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.8}$$

and $R_X = \oplus_{n=1}^\infty R_n,\ Q_X = \oplus_{n=1}^\infty Q_n$ are defined by (5.8);

$$C_{X,\alpha,q} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -d_1^{1/2} & d_2^{1/2} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -d_2^{1/2} & d_3^{1/2} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.9}$$

$$D_{X,\alpha,q} := (a_{i,j})_{i,j=1}^\infty, \tag{6.10}$$

where

$$a_{1,1} = d_1^{-1/2},$$

$$a_{2k,2k} = d_k^{-1/2}, \quad a_{2k,2k+1} = d_{k+1}^{-1/2},$$

$$a_{2k+1,2k-1} = -\frac{d_k^{-1/2}\sqrt{q_k}}{\sinh(d_k\sqrt{q_k})}, \quad a_{2k+1,2k} = -d_k^{-1/2}\sqrt{q_k}\coth(d_k\sqrt{q_k}),$$

$$a_{2k+1,2k+1} = d_{k+1}^{-1/2}(\alpha_k + \sqrt{q_{k+1}}\coth(d_k\sqrt{q_k})),$$

$$a_{2k+1,2k+2} = -\frac{d_{k+1}^{-1/2}\sqrt{q_{k+1}}}{\sinh(d_{k+1}\sqrt{q_{k+1}})},$$

$$a_{i,j} = 0 \quad \text{otherwise}.$$

We now define a linear relation Θ_2^0 by

$$\Theta_2^0 = \{\{f,g\} \in l_0^2 \oplus l_0^2 : \ D_{X,\alpha,q} f = C_{X,\alpha,q} g\}. \tag{6.11}$$

Hence, we get obviously

$$H_{X,\alpha,q}^0 = H_{\min}^* \lceil \operatorname{dom}(H_{X,\alpha,q}^0),$$
$$\operatorname{dom}(H_{X,\alpha,q}^0) = \{f \in W_{\text{comp}}^{2,2}(\mathbb{R}_+ \setminus X) : \ \{\Gamma_0^2 f, \Gamma_1^2 f\} \in \Theta_2^0\}. \tag{6.12}$$

Direct calculations show that Θ_2^0 is symmetric. Moreover, (6.12) implies that the closure of Θ_2^0 is Θ_2. Hence, Θ_2 is a closed symmetric linear relation. Therefore (see Subsection 2.1), Θ_2 admits the representation

$$\Theta_2 = \Theta_2^{\text{op}} \oplus \Theta_2^{\infty}, \quad \mathcal{H} = \mathcal{H}_{\text{op}} \oplus \mathcal{H}_{\infty},$$
$$\mathcal{H}_{\text{op}} = \overline{\operatorname{dom}(\Theta_2)} = \overline{\operatorname{dom}(\Theta_2^{\text{op}})}, \quad \mathcal{H}_{\infty} := \operatorname{mul}\Theta_2, \tag{6.13}$$

where $\Theta_2^{\text{op}} (\in \mathcal{C}(\mathcal{H}_{\text{op}}))$ is the operator part of Θ_2. Moreover, it follows from (6.7) that

$$\operatorname{mul}\Theta_2 = \ker(C_{X,\alpha}) = \overline{R_X^{-1}(\ker C)}, \qquad \Theta_2^{\infty} = \{\{0,f\} : \ f \in \operatorname{mul}\Theta_2\}. \tag{6.14}$$

Since $\mathcal{H}_{\text{op}} = \overline{\operatorname{ran}(R_X C^*)}$, the system $\{\mathbf{f}_n\}_{n=1}^{\infty}$, $\mathbf{f}_n := \frac{\sqrt{d_n}\mathrm{e}_{2n} - \sqrt{d_{n+1}}\mathrm{e}_{2n+1}}{\sqrt{d_n + d_{n+1}}}$, forms the orthonormal basis in $\mathcal{H}_{\text{op}}$. Next, we show that the operator part Θ_2^{op} of Θ_2 is unitarily equivalent to the minimal Jacobi operator

$$B_{X,\alpha,q} = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.15}$$

where

$$b_n = r_n^{-2}(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})),$$
$$a_n = -\frac{\sqrt{q_{n+1}}}{r_n r_{n+1}\sinh(d_{n+1}\sqrt{q_{n+1}})},$$

and $r_n := \sqrt{d_n + d_{n+1}}$, $n \in \mathbb{N}$. We show that $\{\mathbf{f}_n\}_{n=1}^{\infty} \subset \operatorname{dom}(\Theta_2^{\text{op}})$. Assume that there exists $\mathbf{g}_n$ such that $\{\mathbf{f}_n, \mathbf{g}_n\} \in \Theta_2^{\text{op}}$, i.e., $\mathbf{g}_n = \Theta_2^{\text{op}}\mathbf{f}_n$. The latter yields $\mathbf{g}_n \in \mathcal{H}_{\text{op}}$ and, hence, $\mathbf{g}_n = \sum_{k=1}^{\infty} g_{n,k}\mathbf{f}_k$. Moreover, after direct calculations, we obtain

$$D_{X,\alpha,q}\mathbf{f}_1 = r_1^{-1}\big(-(\alpha_1 + \sqrt{q_1}\coth(d_1\sqrt{q_1}) + \sqrt{q_2}\coth(d_2\sqrt{q_2}))\mathrm{e}_3$$
$$+\sqrt{q_2}\sinh^{-1}(d_2\sqrt{q_2})\mathrm{e}_5\big),$$
$$D_{X,\alpha,q}\mathbf{f}_n = r_n^{-1}\left(\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\mathrm{e}_{2n-1} - (\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n})\right.$$
$$\left.+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\mathrm{e}_{2n+1} + \frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})}\mathrm{e}_{2n+3}\right), \quad n \geq 2,$$
$$C_{X,\alpha,q}\mathbf{g}_n = -\sum_{k=1}^{\infty} g_{n,k} r_k \mathrm{e}_{2k+1}, \quad n \geq 1.$$

Hence, $\{\mathbf{f}_n, \mathbf{g}_n\} \in \Theta$, i.e., the equality $D_{X,\alpha,q}\mathbf{f}_n = C_{X,\alpha,q}\mathbf{g}_n$ holds, iff

$$g_{n,n-1} = -\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n},$$

$$g_{n,n} = \frac{1}{r_n^2}\big(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}\big),$$

$$g_{n,n+1} = -\frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})r_n r_{n+1}}, \qquad n \geq 2,$$

and $g_{n,k} = 0$ for all $k \notin \{n-1, n, n+1\}$. Hence, $\mathbf{f}_n \in \operatorname{dom}(\Theta_2^{\mathrm{op}})$, and the matrix representation of the operator Θ_2^{op} in the basis $\{\mathbf{f}_n\}_{n=1}^{\infty}$ coincides with the matrix $B_{X,\alpha,q}$ defined by (6.15).

Since the operator $B_{X,\alpha,q}$ of the form (6.3) and (6.15) is closed, we conclude that Θ_1^{op} and $B_{X,\alpha,q}$ are unitarily equivalent.

Proposition 6.3. *Let $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ be the boundary triplet constructed in Proposition 5.2, and let the linear relation Θ_2 be defined by* (6.2)*. Then Θ_2 admits representation* (6.13)*, where the "pure" relation Θ_2^{∞} is determined by* (6.14) *and* (6.8)*, and the operator part Θ_2^{op} is unitarily equivalent to the minimal Jacobi operator $B_{X,\alpha,q}$ of the form* (6.4) *and* (6.15)*.*

6.2. Self–adjontness

Theorem 6.4. *The operator $H_{X,\alpha,q}$ has equal deficiency indices $\mathrm{n}_+(H_{X,\alpha,q}) = \mathrm{n}_-(H_{X,\alpha,q}) \leq 1$. Moreover, $\mathrm{n}_\pm(H_{X,\alpha,q}) = \mathrm{n}_\pm(B_{X,\alpha,q})$, where $B_{X,\alpha,q}$ is the minimal operator associated with the Jacobi matrix either* (6.3) *or* (6.15)*. In particular, $H_{X,\alpha,q}$ is self-adjoint, iff $B_{X,\alpha,q}$ is.*

Proof. Combining Theorem 5.5 (i) with Propositions 6.1 and 6.3, we arrive at the equality $\mathrm{n}_\pm(H_{X,\alpha,q}) = \mathrm{n}_\pm(B_{X,\alpha,q})$. It remains to note that, for Jacobi matrices, $\mathrm{n}_\pm(B_{X,\alpha,q}) \leq 1$ (see [1,6]). □

Corollary 6.5. *Let $B_{X,\alpha,q}^{(1)}$ and $B_{X,\alpha,q}^{(2)}$ be the minimal Jacobi operators associated with* (6.3) *and* (6.15)*, respectively. Then $\mathrm{n}_\pm(B_{X,\alpha,q}^{(1)}) = \mathrm{n}_\pm(B_{X,\alpha,q}^{(2)})$. In particular, $B_{X,\alpha,q}^{(1)}$ is self-adjoint, iff so is $B_{X,\alpha,q}^{(2)}$.*

Proof. The assertion immediately follows from Theorem 6.4. □

Proposition 6.6. *Assume that Hypothesis 1 is valid. Then the Hamiltonian $H_{X,\alpha,q}$ is self-adjoint for any $\alpha = \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$ provided that*

$$\sum_{n=1}^{\infty} d_n^2 = \infty. \tag{6.16}$$

Proof. Consider the Jacobi matrix $B_{X,\alpha,q}$ (6.6). By Carleman's theorem [1], [6, Chapter VII.1.2], $B_{X,\alpha,q}$ is self-adjoint, whenever

$$\sum_{n=1}^{\infty}(d_n^2\cosh(d_n\sqrt{q_n}) + d_n^{3/2}d_{n+1}^{1/2}) = \infty. \tag{6.17}$$

Obviously,

$$d_n^2\cosh(d_n\sqrt{q_n}) \sim d_n^2, \text{ and } d_n^2 < d_n^2 + d_n^{3/2}d_{n+1}^{1/2} \leq \frac{7}{4}d_n^2 + \frac{1}{4}d_{n+1}^2 \text{ as } n \to \infty,$$

and, hence, relations (6.16) and (6.17) are equivalent.

Now, the result follows from Theorem 6.4. □

Corollary 6.7 ([16]). *If* $\limsup_n d_n > 0$ *(in particular,* $d_* = \liminf_n d_n > 0$*), then* $H_{X,\alpha}$ *is self-adjoint.*

Let us present the sufficient conditions of self-adjointness in the case where (6.16) does not hold.

Proposition 6.8. *Let* $\{d_n\}_{n=1}^{\infty} \in l^2$,

$$c_1 \le d_n\sqrt{q_n} \le c_2, \quad c_1, c_2 > 0, \tag{6.18}$$

and let

$$d_{n-1} \cdot d_{n+1} \ge d_n^2, \quad n \in \mathbb{N}. \tag{6.19}$$

If, in addition, the strengths α_n *of* δ*-interactions satisfy the relation*

$$\sum_{n=1}^{\infty} d_{n+1}\,|\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}| < \infty, \tag{6.20}$$

then the operator $H_{X,\alpha,q}$ *is symmetric with* $\mathrm{n}_{\pm}(H_{X,\alpha,q}) = 1$.

Proof. Consider the Jacobi matrix (6.15). To apply [25, Theorem 1], we denote $a_n := r_n^{-2}|\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}|$ and $b_n := \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n}$, $n \in \mathbb{N}$, and define a sequence $\{c_n\}_{n=1}^{\infty}$ as follows:

$$c_1 := b_1, \qquad c_2 := 1, \qquad c_{n+1} := -\frac{b_{n-1}}{b_n}c_{n-1}, \qquad n \in \mathbb{N}.$$

It is easily seen that

$$c_{n+1} = (-1)^{n+1} r_{n+1}\frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})} \cdot \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \cdot \frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}}$$

$$\times \frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \ldots \cdot \widetilde{c}, \quad n \in \mathbb{N};$$

$$\widetilde{c} := \begin{cases} c_1 r_1^{-1}, & n = 2k, \\ c_2 r_2^{-1}, & n = 2k+1. \end{cases}$$

Using both conditions (6.24)–(6.19) and the obvious inequality $\sinh(x) > x$, $x > 0$, we obtain

$$\begin{aligned}
&\frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})} \cdot \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \cdot \frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}} \cdot \frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \ldots \\
&= \frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}} \cdot \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \cdot \frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})} \cdot \ldots \\
&= \sqrt{\frac{\sinh(d_{n+2}\sqrt{q_{n+2}})}{\sqrt{q_{n+2}}}}\left(\frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \sqrt{\frac{\sqrt{q_{n+2}}}{\sinh(d_{n+2}\sqrt{q_{n+2}})} \cdot \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}}\right) \\
&\quad \times \left(\frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}} \cdot \sqrt{\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \cdot \frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})}}\right) \\
&\le C\sqrt{d_{n+2}}, \qquad n \in \mathbb{N}.
\end{aligned} \tag{6.21}$$

Therefore,

$$|c_{n+1}| \le C\widetilde{c}r_{n+1}\sqrt{d_{n+2}} = \sqrt{2}C\widetilde{c}(d_{n+2} + \sqrt{d_{n+1}d_{n+2}}) \le \sqrt{2}C\widetilde{c}\left(\frac{3}{2}d_{n+2} + \frac{1}{2}d_{n+1}\right),$$

and, hence, $\{c_n\}_{n=1}^{\infty} \in l^2$. On the other hand, it follows from (6.20) and (6.21) that $\sum_{n=1}^{\infty}|a_n|c_n^2 < \infty$, i.e.,

$$\sum_{n=1}^{\infty}\frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}}\,|\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}| < \infty.$$

Since $\sinh(x) > x,\ x > 0$, we easily get conditions (6.20). By [25, Theorem 1], this inequality and the inclusion $\{c_n\}_{n=1}^{\infty} \in l^2$ yield $\mathrm{n}_{\pm}(B_{X,\alpha,q}) = 1$. It remains to apply Theorem 6.4. □

Corollary 6.9. *Let the assumptions of Proposition 6.8 be satisfied. If*

$$d_n(q_n)^{\frac{3}{2}} \le c, \qquad c > 0, \tag{6.22}$$

then condition (6.20) *is equivalent to*

$$\sum_{n=1}^{\infty} d_{n+1}\left|\alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}} + \frac{1}{3}\left(d_n\sqrt{q_n} + d_{n+1}\sqrt{q_{n+1}}\right)\right| < \infty. \tag{6.23}$$

Proof. Using

$$\coth(x) = \frac{1}{x} + \frac{x}{3} - O(x^3),$$

$\{d_n\}_{n=1}^{\infty} \in l^2$, and (6.22), we prove the assertion. □

Remark 6.10. Let the assumptions of Proposition 6.8 be satisfied. Note that condition (6.20) is automatically satisfied, whenever

$$\alpha_n = -(\sqrt{q_n} + \sqrt{q_{n+1}}).$$

Proposition 6.11. *Assume that Hypothesis 1 is valid, and assume that* (6.16) *does not hold. Let also* $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ *and* $X = \{x_n\}_{n=1}^{\infty}$ *satisfy one of the following conditions:*

(i)

$$c_1 \le d_n\sqrt{q_n} \le c_2, \quad c_1, c_2 > 0, \tag{6.24}$$

and

$$\sum_{n=1}^{\infty}|\alpha_n|d_n d_{n+1}r_{n-1}r_{n+1} = \infty, \tag{6.25}$$

where $r_n = \sqrt{d_n + d_{n+1}}$.

(ii) There exists a positive constant $C_1 > 0$ *such that*

$$\alpha_n + \sqrt{q_n}\left(M_1 + \frac{r_n}{Mr_{n-1}}\right) + \sqrt{q_{n+1}}\left(M_1 + \frac{r_n}{Mr_{n+1}}\right) \le C_1(d_n + d_{n+1}), \qquad n \in \mathbb{N}, \tag{6.26}$$

where

$$M = \liminf_{n\to\infty}\sinh(d_n\sqrt{q_n}), \qquad M_1 = \limsup_{n\to\infty}\coth(d_n\sqrt{q_n}). \tag{6.27}$$

(*iii*) *There exists a positive constant* $C_2 > 0$ *such that*

$$\alpha_n + \sqrt{q_n}\left(M_2 - \frac{r_n}{Mr_{n-1}}\right) + \sqrt{q_{n+1}}\left(M_2 - \frac{r_n}{Mr_{n+1}}\right) \geq -C_2(d_n + d_{n+1}), \qquad n \in \mathbb{N}, \tag{6.28}$$

where

$$M = \liminf_{n\to\infty} \sinh(d_n\sqrt{q_n}), \qquad M_2 = \liminf_{n\to\infty} \coth(d_n\sqrt{q_n}). \tag{6.29}$$

Then the operator $H_{X,\alpha,q}$ *is self-adjoint in* $L^2(\mathbb{R}_+)$.

Proof. (*i*) Applying the Dennis–Wall test ([1, p. 25, Problem 2]) to matrix (6.15), we obtain that the condition

$$\sum_{n=1}^{\infty} |\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}| \times \frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}} r_{n-1}r_{n+1} = \infty \tag{6.30}$$

yields the self-adjointness of the minimal operator $B_{X,\alpha,q}$ associated with (6.15).

Obviously,

$$\begin{aligned} &|\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})| \frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}} \\ &\geq (|\alpha_n| - |\sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})|) \frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}}. \end{aligned} \tag{6.31}$$

Since $\sinh(x) > x,\ x > 0$, we get

$$|\alpha_n|\frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}} \geq |\alpha_n| d_n d_{n+1}. \tag{6.32}$$

Condition (6.24) implies that

$$|\sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})| \times \frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}} \leq \frac{\sinh(2c_2)}{2c_1}\left(d_n + d_{n+1}\right). \tag{6.33}$$

Since $\{d_n\}_{n=1}^{\infty} \in l^2$, we get

$$\sum_{n=1}^{\infty} |\sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})| \times \frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}} < \infty. \tag{6.34}$$

Combining (6.31)–(6.32) with (6.34), we get that relations (6.25) and (6.30) are equivalent. By Theorem 6.4, $H_{X,\alpha,q} = H^*_{X,\alpha,q}$.

$(ii)-(iii)$ Applying results in [6, Theorem VII.1.4] (see also [1, Problem 3, p. 37]) to the Jacobi matrix (6.15), we obtain that the conditions

$$\begin{aligned} &-\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n}+\frac{1}{r_n^2}\big(\alpha_n+\sqrt{q_n}\coth(d_n\sqrt{q_n}) \\ &+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\big)-\frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})r_nr_{n+1}}\le C_1 \end{aligned} \tag{6.35}$$

and

$$\begin{aligned} &-\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n}-\frac{1}{r_n^2}\big(\alpha_n+\sqrt{q_n}\coth(d_n\sqrt{q_n}) \\ &+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\big)-\frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})r_nr_{n+1}}\le C_2 \end{aligned} \tag{6.36}$$

guarantee the self-adjointness of $B_{X,\alpha,q}$. Since $d_n\sqrt{q_n}$ is bounded, we get easily conditions (6.26) and (6.38), by using conditions (6.27) and (6.29). Theorem 6.4 completes the proof. □

Corollary 6.12. *Let the assumptions of Proposition 6.11 be satisfied. If, in addition,* $\lim_{n\to\infty} d_n\sqrt{q_n}=0$, *then conditions* (6.26)–(6.38) *are equivalent to*

$$\alpha_n+\frac{1}{d_n}\left(1+\frac{r_n}{r_{n-1}}\right)+\frac{1}{d_{n+1}}\left(1+\frac{r_n}{r_{n+1}}\right)\le C_1(d_n+d_{n+1}),\qquad n\in\mathbb{N} \tag{6.37}$$

and

$$\alpha_n+\frac{1}{d_n}\left(1-\frac{r_n}{r_{n-1}}\right)+\frac{1}{d_{n+1}}\left(1-\frac{r_n}{r_{n+1}}\right)\ge -C_2(d_n+d_{n+1}),\qquad n\in\mathbb{N}, \tag{6.38}$$

respectively.

Example 6.13. Let $d_n:=\frac{1}{n}$, $n\in\mathbb{N}$. Consider the operator

$$H_A:=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q(x)+\sum_{n=1}^{\infty}\alpha_n\delta(x-x_n). \tag{6.39}$$

Clearly, $\{d_n\}_{n=1}^{\infty}\in l^2$, i.e., condition (6.16) is violated. Applying Propositions 6.8 and 6.11 and executing some direct calculations, we obtain

(i) If $\sum_{n=1}^{\infty}\frac{|\alpha_n|}{n^3}=\infty$, then the operator H_A is self-adjoint (cf. Proposition 6.11 (i)).

(ii) If $\alpha_n\le -2(c_2M_1+\frac{c_2}{M})n-(c_2M_1+\frac{c_2}{M})+O(n^{-1})$, then H_A is self-adjoint (cf. Proposition 6.11 (ii)).

(iii) If $\alpha_n\ge-\frac{K}{n}$, $n\in\mathbb{N}$, $K\equiv\text{const}>0$, then H_A is self-adjoint (cf. Proposition 6.11 (iii)).

(iv) If $\alpha_n=-(\sqrt{q_n}+\sqrt{q_{n+1}})+O(n^{-\varepsilon})$, then $\mathrm{n}_{\pm}(H_A)=1$ (cf. Proposition 6.8).

6.3. Operators with discrete spectrum

Theorem 6.14. *Assume that Hypothesis 1 is valid. Let* $B_{X,\alpha,q}$ *be the minimal Jacobi operator defined either by* (6.3) *or* (6.15).

(i) *If* $\mathrm{n}_{\pm}(B_{X,\alpha,q})=1$, *then any self-adjoint extension of* $H_{X,\alpha,q}$ *has a discrete spectrum.*

(ii) *If* $B_{X,\alpha,q}=B^*_{X,\alpha,q}$, *then the Hamiltonian* $H_{X,\alpha,q}(=H^*_{X,\alpha,q})$ *has a discrete spectrum, iff*

- $\lim_{n\to\infty} d_n = 0$, *and*
- $B_{X,\alpha,q}$ *has a discrete spectrum.*

Proof. 1) To be precise, let $B_{X,\alpha,q}$ be defined by (6.3). Since $\mathrm{n}_{\pm}(B_{X,\alpha,q}) = 1$, any self-adjoint extension of $B_{X,\alpha,q}$ has a discrete spectrum (see [1,6]). Moreover, by Corollary 6.7, $\lim_{n\to\infty} d_n = 0$. Hence the operator H_0 defined by (5.35) has a discrete spectrum as well. The Krein resolvent formula (2.9) implies that any self-adjoint extension of $H_{X,\alpha,q}$ is discrete.

2) This follows from Theorem 5.5 (*iv*) and Remark 5.6. □

Proposition 6.15. *Assume that Hypothesis 1 is valid. Let the operator $B_{X,\alpha,q}$ defined by* (6.15) *be self-adjoint, and let* $\lim_{n\to\infty} d_n = 0$. *Assume also that* $\alpha_n < 0$, *and there exist*

$$\liminf_{n\to\infty} \sinh(d_n\sqrt{q_n}) = C, \qquad \limsup_{n\to\infty} \coth(d_n\sqrt{q_n}) = C_2 > 0, \tag{6.40}$$

and also

$$\begin{gathered}\lim_{n\to\infty} \frac{\left|\alpha_n + C_2(\sqrt{q_n} + \sqrt{q_{n+1}})\right|}{\left(d_n + d_{n+1}\right)} = \infty, \\ \lim_{n\to\infty} q_{n+1}C^{-2}\left(\alpha_n + C_2(\sqrt{q_n} + c\sqrt{q_{n+1}})\right)^{-1}\left(\alpha_{n+1} + C_2(\sqrt{q_{n+1}} + c\sqrt{q_{n+2}})\right)^{-1} < \frac{1}{4}.\end{gathered} \tag{6.41}$$

Then the operator $H_{X,\alpha,q}$ has a purely discrete spectrum.

Proof. Applying [10, Theorem 8] to the Jacobi matrix $B_{X,\alpha,q}$ of the form (6.15), we get the sufficient conditions of discreteness of the spectrum:

$$\lim_{n\to\infty} \frac{1}{r_n^2}\left(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}\right) = \infty \tag{6.42}$$

and

$$\begin{aligned}\lim_{n\to\infty} q_{n+1}\sinh^{-2}(d_{n+1}\sqrt{q_{n+1}})\left(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\right)^{-1} \\ \times\left(\alpha_{n+1} + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}) + \sqrt{q_{n+2}}\coth(d_{n+2}\sqrt{q_{n+2}})\right)^{-1} < \frac{1}{4}.\end{aligned} \tag{6.43}$$

Since $\alpha_n < 0$, we get regard for the boundedness of $d_n\sqrt{q_n}$ that conditions (6.42) and (6.43) are equivalent to (6.40) and (6.41), respectively. Since $\lim_{n\to\infty} d_n = 0$, the same is true for $H_{X,\alpha,q}$ by Theorem 6.14. □

Remark 6.16. If $H_{X,\alpha,q}$ is a semibounded operator, in particular, if $\alpha_n > 0$, then the assertion of Proposition 6.15 follows immediately from the analogous classical Molchanov discreteness criterion (see [4]).

Proposition 6.17. *Assume that Hypothesis 1 is valid,* $\lim_{n\to\infty} d_n = 0$, *and* $d_n\sqrt{q_n} \to 0$ *as* $n \to \infty$. *Let also the operator $B_{X,\alpha,q}$ defined by* (6.3)–(6.4) *be self-adjoint. Let the following conditions be satisfied:*

$$\lim_{n\to\infty}\left|\frac{\alpha_{n-1}}{d_n} + q_n\right| = \infty, \quad \lim_{n\to\infty} \frac{1}{d_n(\alpha_n + q_{n+1}d_{n+1})} > -\frac{1}{4}$$

$$\text{and} \qquad \lim_{n\to\infty} \frac{1}{d_n\alpha_{n-1}} > -\frac{1}{4}. \tag{6.44}$$

Then the operator $H_{X,\alpha,q}$ has discrete spectrum.

Proof. We apply [10, Theorem 8] to the operator $B'_{X,\alpha,q}$ of the form (6.6) and prove the statement in at least two steps.

At first, we consider the case $b_n = b_{2k-1}$ and $a_n = a_{2k-1}$. We obtain the following sufficient conditions of discreteness of the spectrum of $B'_{X,\alpha}$:

$$\lim_{k\to\infty}\left|\frac{\alpha_{k-1}}{d_k} + q_k\right| = \infty \qquad \text{and} \qquad \lim_{k\to\infty}\frac{1}{d_k\alpha_{k-1}} > -\frac{1}{4}. \tag{6.45}$$

Similarly, if $b_n = b_{2k}$ and $a_n = a_{2k}$, we obtain

$$\lim_{k\to\infty}\left|\frac{1}{d_k^2}\right| = \infty \qquad \text{and} \qquad \lim_{k\to\infty}\frac{1}{d_k(\alpha_k + q_{k+1}d_{k+1})} > -\frac{1}{4}. \tag{6.46}$$

Since $\lim_{n\to\infty} d_n = 0$ and q_n is unbounded, conditions (6.45)–(6.46) are equivalent to (6.44). Theorem 6.14 completes the proof. □

Remark 6.18. In the case of $q \in L^\infty(\mathbb{R}_+)$, Proposition 6.17 was obtained in [21].

Corollary 6.19. *Let the assumptions of Proposition 6.17 be satisfied,* $\alpha_n + q_{n+1}d_{n+1} < 0$, *and let the following conditions be met:*

$$\lim_{n\to\infty}\frac{1}{d_n(\alpha_n + q_{n+1}d_{n+1})} > -\frac{1}{4} \qquad \textit{and} \qquad \lim_{n\to\infty}\frac{1}{d_n\alpha_{n-1}} > -\frac{1}{4}. \tag{6.47}$$

Then the operator $H_{X,\alpha,q}$ *has a discrete spectrum.*

Proof. If $\alpha_n + q_{n+1}d_{n+1} < 0$, then the condition

$$\lim_{n\to\infty}\frac{1}{d_n(\alpha_n + q_{n+1}d_{n+1})} > -\frac{1}{4}$$

yields the relation

$$|\alpha_n + q_{n+1}d_{n+1}| > \frac{4}{d_n}.$$

Combining the latter with the condition $\lim_{n\to\infty} d_n = 0$, we get

$$\lim_{n\to\infty}\left|\frac{\alpha_{n-1}}{d_n} + q_n\right| = \infty.$$

In this case, conditions (6.44) are reduced to (6.47). □

Remark 6.20. Note that if $\alpha_n + q_{n+1}d_{n+1} > 0$, then the condition $\lim_{n\to\infty}\frac{1}{d_n(\alpha_n+q_{n+1}d_{n+1})} > -\frac{1}{4}$ in (6.44) is automatically satisfied and can be omitted.

6.4. Resolvent comparability

Proposition 6.21. *Assume that Hypothesis 1 is valid. Suppose also that* $H_{X,\alpha,q}$ *and* $H_{X,\widetilde{\alpha},q}$ *are self-adjoint, and* $B_{X,\alpha,q}$ *and* $B_{X,\widetilde{\alpha},q}$ *are the corresponding (self-adjoint) Jacobi operators defined either by (6.3) or (6.6). Then, for any* $p \in (0,\infty]$, *the inclusion*

$$(H_{X,\alpha,q} - z)^{-1} - (H_{X,\widetilde{\alpha},q} - z)^{-1} \in \mathfrak{S}_p \tag{6.48}$$

is equivalent to the inclusion

$$(B_{X,\alpha,q} - \mathrm{i})^{-1} - (B_{X,\widetilde{\alpha},q} - \mathrm{i})^{-1} \in \mathfrak{S}_p. \tag{6.49}$$

Proof. From Theorem 2.5, we get the result with $B_{X,\alpha,q}$ defined by (6.6). The result with the matrices defined by (6.3) follows from Proposition 6.1. □

Corollary 6.22. *Assume that Hypothesis 1 is valid. If* $\left\{\frac{\alpha_n-\widetilde{\alpha_n}}{d_{n+1}}\right\}_{n=1}^{\infty} \in l^p,\ p \in (0,\infty)$ *(*$\in c_0,\ p=\infty$*), then inclusion* (6.48) *holds.*

Proof. Note that the condition $B_{X,\widetilde{\alpha},q} - B_{X,\alpha,q} \in \mathfrak{S}_p$ is sufficient for inclusion (6.48) to hold. Clearly, $l_0^2 \subset \operatorname{dom}(B_{X,\alpha,q}) \cap \operatorname{dom}(B_{X,\widetilde{\alpha},q})$. On the other hand, for any $f \in l^{2,0}$, (6.5) yields the inclusion $(B_{X,\widetilde{\alpha},q} - B_{X,\alpha,q}) \in \mathfrak{S}_p$, i.e.,

$$B_{X,\widetilde{\alpha},q}f - B_{X,\alpha,q}f = R_X^{-1}\big(\widetilde{B}_{\widetilde{\alpha}} - \widetilde{B}_{\alpha}\big)R_X^{-1}f = \oplus_{n=1}^{\infty}\begin{pmatrix} \frac{\alpha_n-\widetilde{\alpha_n}}{d_{n+1}} & 0 \\ 0 & 0 \end{pmatrix} f$$

for all finite sequences $f \in l^2(\mathbb{N})$. Hence, due to the assumption of Corollary 6.22, we get

$$\overline{B_{X,\widetilde{\alpha},q} - B_{X,\alpha,q}} \in \mathfrak{S}_p \subset [\mathcal{H}]$$

and $\operatorname{dom}(B_{X,\alpha,q}) = \operatorname{dom}(B_{X,\widetilde{\alpha},q})$. It remains to apply Proposition 2.5. Finally, Proposition 6.21 completes the proof. □

Proposition 6.23. *Assume that Hypothesis 1 is valid. Let also $d^* < \infty$. If*

$$\sum_{n=1}^{\infty} \frac{|\alpha_n|}{d_{n+1}} < \infty, \tag{6.50}$$

then

$$\sigma_{ac}(H_{X,\alpha,q}) = \sigma_{ac}(H_{X,0,q}). \tag{6.51}$$

If, in addition, $q(\cdot) \in L^1(\mathbb{R}_+)$, then $\sigma_{ac}(H_{X,\alpha,q}) = \mathbb{R}_+$.

Proof. Applying Corollary 6.22 for $p = 1$ to the Hamiltonians $H_{X,\alpha,q}$ and $H_{X,0,q}$ and using (6.50), we get that inclusion (6.48) holds. Now, the result follows from the Kato–Rozenblum theorem (cf. [32, Theorem XI.9]). The assertion is proved.

If $q(\cdot) \in L^1(\mathbb{R}_+)$, then $\sigma_{ac}(H_{X,0,q}) = \mathbb{R}_+$. Hence,

$$\sigma_{ac}(H_{X,\alpha,q}) = \sigma_{ac}(H_{X,0,q}) = \mathbb{R}_+.$$

□

Remark 6.24. In the case of $q \in L^\infty(\mathbb{R}_+)$, Proposition 6.23 was established in [4].

Example 6.25. Let $x_0 = 0$,

$$x_n := \begin{cases} k, & n = 2k-1, \\ k + \frac{1}{k^3}, & n = 2k, \end{cases} \qquad k \in \mathbb{N}, \tag{6.52}$$

and let

$$d_n := \begin{cases} 1 - \frac{1}{(k-1)^3}, & n = 2k-1, \\ \frac{1}{k^3}, & n = 2k, \end{cases} \qquad k \in \mathbb{N}. \tag{6.53}$$

We set

$$q(x) := \begin{cases} k, & x \in [x_{2k-1}, x_{2k}], \\ 0, & \text{otherwise}, \end{cases} \qquad k \in \mathbb{N}. \tag{6.54}$$

Consider the minimal symmetric operator $H_{X,\alpha,q}$ associated with (1.1) in $L^2(\mathbb{R}_+)$.

Define

$$q_n(x) := \begin{cases} k, & n = 2k, \\ 0, & n = 2k-1. \end{cases} \quad k \in \mathbb{N}. \tag{6.55}$$

In addition, we suppose that

$$\sum_{k=1}^{\infty} \left(k^3 \alpha_{2k-1} + \alpha_{2k} \right) < \infty.$$

Since d_n and $q_n(\cdot)$ satisfy Hypothesis 1 and $q(\cdot) \in L^1(\mathbb{R}_+)$, Proposition 6.23 immediately yields

$$\sigma_{ac}(H_{X,\alpha,q}) = \mathbb{R}_+. \tag{6.56}$$

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