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ON THE ROBUST STABILIZATION OF ONE CLASS OF NONLINEAR DISCRETE SYSTEMS

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We study the problem of robust linear stabilization of a family of nonlinear discrete control systems with uncertainties and nonlinearly dependent control. We establish sufficient conditions for the robust stabilization and synthesize linear regulators of state engaged in the robust stabilization. The obtained necessary conditions for the robust stabilization are close to sufficient.

1. Introduction

The problem of stabilization of controlled systems is one of the most complex problems in the modern control theory, which is extensively investigated by numerous authors [1–4, 13–22]. The problem of robust stabilization of systems [2, 4, 13, 18–21, 24–28, 31, 32] caused by the presence of uncertainties in the mathematical description of control systems occupies a significant place in the theory of stabilization. The Lyapunov theorem on stability in the first approximation can be regarded as the first result in this direction. As one of the most efficient methods for the investigation of the problem of stabilization of nonlinear systems, one can mention the method of Lyapunov functions, which is a powerful tool for the analysis and synthesis of control systems that enables one to obtain many important results [1, 2, 13, 15, 17, 18, 20, 22, 29].

In the present paper, we consider the problem of one-dimensional stabilization of a family of nonlinear discrete systems based on the Lyapunov functions and algebraic Lyapunov equations. We use an approach based on the method of quadratic stabilization [4] guaranteeing the existence of the general Lyapunov function for a given family of systems.

In the Euclidean space, we consider a family of nonlinear discrete control systems

$$x(k+1) = (A + A_0(k, x(k)))x(k) + (B + B_0(k, x(k)))u(k) + \varphi_0(k, x(k), u(k)),$$
$$(k, x(k), u(k)) \in N_0 \times R^n \times R^r, \tag{1}$$

where $N_0 = \{0, 1, 2, \ldots\}$, $x(k) \in R^n$ is the vector of state of the system and $u(k) \in R^r$ is the vector of control. Assume that A and B are given constant real $n \times n$ and $n \times r$ matrices, respectively. It is known that the real matrices $A_0(k, x)$ and $B_0(k, x)$ are defined in $N_0 \times R^n$ and satisfy the conditions

$$\|A_0(k, x)\| \le l_0\|x\|^\omega + d, \quad \|B_0(k, x)\| \le l_0\|x\|^\omega + d, \tag{2}$$

where $\omega > 0$, $l_0 \ge 0$, and $0 \le d < d_0$. It is also known that the real functions $\varphi_0(k, x, u)$ satisfy the following condition in $N_0 \times R^n \times R^r$:

$$\|\varphi_0(k, x, u)\| \le l_1(\|x\| + \|u\|)^{1+\omega}, \quad l_1 \ge 0. \tag{3}$$

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The *robust linear stabilization* is understood as the law of control $u = Px$ common for the entire family (1) and guaranteeing the asymptotic stability of the trivial solutions of systems (1). In other words, the trivial solutions of systems (1) must be stable and there should exist a domain of attraction of the origin of coordinates invariant under the set of incompletely defined functional parameters $A_0(k, x)$, $B_0(k, x)$, and $\varphi_0(k, x, u)$.

We say that the expression

$$\Delta V(x) = V(f(k, x)) - V(x)$$

is the *first difference of a function $V(x)$ by the system*

$$x(k + 1) = f(k, x(k)). \tag{4}$$

It is clear that if $x(k)$ is a solution of system (4), then

$$\Delta V(x(k)) = V(f(k, x(k)) - V(x(k)) = V(x(k + 1)) - V(x(k)).$$

Assume that

(i) $C^- = \{\lambda \in C : |\lambda| < 1\}$ and $C^+ = \{\lambda \in C : |\lambda| \geq 1\}$;

(ii) I is the identity matrix of the corresponding order;

(iii) the symbol $*$ denotes the operation of transposition;

(iv) $L = \mathrm{Lin}\,(B, AB, \ldots, A^{n-1}B)$ is the linear span of the column vectors of the matrices B, AB, $\ldots$, $A^{n-1}B$;

(v) $\sigma(\cdot)$ is the spectrum of a matrix in parentheses;

(vi) e_i is the vector that coincides with the ith column of the identity matrix of the corresponding order;

(vii) $R^{n\times m}$ is the set of $(n \times m)$ matrices.

A polynomial is called *stable* if its roots belong to the ball C^-.

2. Auxiliary Result

Let $A_1 \in R^{m\times m}$, let $B_1 \in R^{m\times r}$, let rank $B_1 = r$, and let $b_1, \ldots, b_r$ be columns of the matrix B_1. It is known (see [5–7]) that, for each polynomial

$$\varphi(\lambda) = \lambda^m + \gamma_1\lambda^{m-1} + \ldots + \gamma_m$$

with real coefficients γ_i to have a matrix $K \in R^{r\times m}$ such that the characteristic polynomial $\chi_{A_1+B_1K}(\lambda)$ of the matrix $A_1 + B_1K$ coincides with the polynomial $\varphi(\lambda)$, it is necessary and sufficient that rank $Q = m$, where $Q = (B_1, A_1B_1, \ldots, A_1^{m-1}B_1)$.

The algorithms of determination of the matrix K are known (see [5, 11, 12]). In the present paper, we use the algorithm proposed in [5]. This algorithm can be described as follows:

1. Formation of a nonsingular matrix

$$W = (b_1, A_1b_1, \ldots, A_1^{m_1-1}b_1, \ldots, b_k, A_1b_k, \ldots, A^{m_k-1}b_k) \in R^{m\times m}$$

of linearly independent column vectors of the matrix Q, where $m_1 + \ldots + m_k = m$, m_i is the least natural number such that the vector $A_1^{m_i} b_i$ linearly depends on the previous vectors of the matrix W;

2. Formation of the $(r \times m)$ matrix

$$S = (0, \ldots, 0, e_2, 0, \ldots, 0, e_3, \ldots, 0, \ldots, 0, e_k, 0, \ldots, 0) \subset R^{r \times m},$$

where e_2 is the m_1th column, e_3 is the $(m_1 + m_2)$th column, ..., e_k is the $(m_1 + \ldots + m_{k-1})$th column of the matrix S; $e_i \in R^r$;

3. Determination of the matrix $K_1 = SW^{-1} \in R^{r \times m}$;

4. Determination of the characteristic polynomial

$$\chi_{\widehat{A}_1}(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \ldots + a_m$$

of the matrix $\widehat{A}_1 = A_1 + B_1 K_1$;

5. Formation of the matrices

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \ldots & -a_1 \end{pmatrix},$$

$$b_0 = (0, \ldots, 0, 1)^* \in R^{m \times 1},$$

$$T_0 = (A_0^{m-1} b_0, A_0^{m-2} b_0, \ldots, b_0) \in R^{m \times m},$$

$$T_1 = (\widehat{A}_1^{m-1} b_1, \widehat{A}_1^{m-2} b_1, \ldots, b_1) \in R^{m \times m},$$

$$T = T_0 T_1^{-1};$$

6. Determination of the vector $f = (\gamma_m - a_m, \ldots, \gamma_1 - a_1)$;

7. Determination of the matrix

$$K = K_1 - e_1 f T. \tag{5}$$

Let $\dim L = m$, where $L = \mathrm{Lin}\,(B, AB, \ldots, A^{n-1}B)$, $A \in R^{n \times n}$, $B \in R^{n \times r}$, $\mathrm{rank}\, B = r$, let $v_1, \ldots, v_m$ be a basis of the subspace L, and let $v_{m+1}, \ldots, v_n$ be a basis of $L^{\perp}$. We form a matrix

$$F = (v_1, \ldots, v_m, v_{m+1}, \ldots, v_n). \tag{6}$$

Theorem 1. *If there is no eigenvector x_0 of the matrix A^* associated with the corresponding eigenvalue from C^+ and satisfying the relation $x_0^* B = 0$, then the inclusion $\sigma(A + BP) \subset C^-$ takes place for each $(r \times n)$ matrix of the form*

$$P = KH^* F^{-1}, \tag{7}$$

where F is taken from (6), $H^* = (e_1, \dots, e_m)^*$, e_i is the ith column of the $(n \times n)$ identity matrix,

$$K = SW^{-1} - e_1 f T^{-1}, \tag{8}$$

$$W = (b_1, A_1 b_1, \dots, A_1^{m_1 - 1} b_1, \dots, b_k, A_1 b_k, \dots, A_1^{m_k - 1} b_k) \in R^{m \times m},$$

$b_i \in R^m$ are columns of the matrix $B_1 = H^* F^{-1} B \in R^{m \times r}$, $A_1 = H^* F^{-1} AFH \in R^{m \times m}$, $m_1 + \ldots + m_k = m$, m_i is the least natural number such that the vector $A_1^{m_i} b_i$ linearly depends on the previous vectors of the matrix W;

$$S = (0, \dots, 0, e_2, 0, \dots, 0, e_3, \dots, 0, \dots, 0, e_k, 0, \dots, 0) \in R^{r \times m},$$

e_2 is the m_1th column of the matrix S; e_3 is its $(m_1 + m_2)$th column, $\dots$, and e_k is its $(m_1 + \ldots + m_{k-1})$th column;

$$f = (\gamma_m - a_m, \dots, \gamma_1 - a_1) \in R^{1 \times m},$$

$a_1, \dots, a_m$ are the coefficients of the characteristic polynomial

$$\chi_{A_1 + B_1 S W^{-1}}(\lambda) = \lambda^m + a\lambda^{m-1} + \ldots + a_m$$

of the matrix $\widehat{A}_1 = A_1 + B_1 S W^{-1}$; $\gamma_1, \dots, \gamma_m$ are the coefficients of an arbitrary stable polynomial $\varphi(\lambda) = \lambda^m + \gamma_1 \lambda^{m-1} + \ldots + \gamma_m$ with real coefficients;

$$T = T_1 T_0^{-1},$$

$$T_1 = (\widehat{A}_1^{m-1} b_1, \quad \widehat{A}_1^{m-2} b_1, \dots, b_1),$$

$$T_0 = (A_0^{m-1} b_0, A_0^{m-2} b_0, \dots, b_0),$$

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \dots & -a_1 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in R^m.$$

Proof. We first prove the following relation for the eigenvalues λ of the matrix A satisfying the condition $\lambda \in C^+$:

$$\operatorname{rank}(F^{-1}AF - \lambda I \quad F^{-1}B) = n. \tag{9}$$

Assume the contrary. Then

$$\operatorname{rank}(F^{-1}AF - \lambda I \quad F^{-1}B) < n$$

for some eigenvalue $\lambda \in C^+$. In this case, there exists a nonzero n-dimensional vector x_0 for which

$$x_0^*(F^{-1}AF - \lambda I \ F^{-1}B) = 0$$

or, in the equivalent form, $x_0^*(F^{-1}AF - \lambda I) = 0$ and $x_0^* F^{-1}B = 0$. The last equalities mean that $y_0 = (F^{-1})^* x_0$ is the eigenvector of the matrix A^* associated with the eigenvalue $\overline{\lambda} \in C^+$ and satisfying the relation $y_0^* B = 0$, which contradicts the condition of the theorem.

We now show that the matrices

$$\widehat{A} = F^{-1}AF, \quad \widehat{B} = F^{-1}B \tag{10}$$

have the structures

$$\widehat{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where $A_1 \in R^{m \times m}$ and $A_3 \in R^{(n-m)\times(n-m)}$.

Indeed, the columns of the matrix B belong to the subspace L. Hence, each of them is a linear combination of the vectors $v_1, \ldots, v_m$. Therefore, by virtue of (10), the matrix $\widehat{B}$ must have the form $\begin{pmatrix} B_1 \\ 0 \end{pmatrix}$, where B_1 is an $(m \times r)$ matrix. Since the subspace L is A-invariant, $Av_i \in L$, $i = \overline{1, m}$. In this case, it follows from (10) that

$$F\widehat{A} = AF = \left(\sum_{i=1}^{m} \alpha_{1i} v_i \ldots \sum_{i=1}^{m} \alpha_{m\,i} v_i \quad \sum_{i=1}^{n} \alpha_{m+1,\,i} v_i \ldots \sum_{i=1}^{n} \alpha_{n\,i}\, v_i \right).$$

This implies that the matrix $\widehat{A}$ has the form

$$\widehat{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where A_3 is a square matrix of order $n - m$. Equality (9) now takes the form

$$\operatorname{rank} \begin{pmatrix} A_1 - \lambda I & A_2 & B_1 \\ 0 & A_3 - \lambda I & 0 \end{pmatrix} = n, \quad \lambda \in C^+. \tag{11}$$

It follows from (11) that $\operatorname{rank}(A_3 - \lambda I) = n - m$ for each $\lambda \in C^+$. The last equality means that $\sigma(A_3) \subset C^-$. We now show that

$$\operatorname{rank}(B_1, A_1 B_1, \ldots, A_1^{m-1} B_1) = m.$$

Indeed, in view of the fact that

$$F^{-1}A^kB = F^{-1}A^kFF^{-1}B = (F^{-1}AF)^k \begin{pmatrix} B_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^k \begin{pmatrix} B_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_1^k B_1 \\ 0 \end{pmatrix},$$

we arrive at the following chain of equalities for block matrices:

$$\begin{aligned} m &= \operatorname{rank}(B\ AB\ \ldots\ A^{n-1}B) \\ &= \operatorname{rank}\left(F^{-1}B\ F^{-1}AB\ \ldots\ F^{-1}A^{n-1}B\right) \\ &= \operatorname{rank}\begin{pmatrix} B_1 & A_1B_1 & \ldots & A_1^{n-1}B_1 \\ 0 & 0 & \ldots & 0 \end{pmatrix} \\ &= \operatorname{rank}\left(B_1\ A_1B_1\ \ldots\ A_1^{n-1}B_1\right) \\ &= \operatorname{rank}\left(B_1\ A_1B_1\ \ldots\ A_1^{m-1}B_1\right). \end{aligned}$$

By using the algorithm described above under the condition that the roots of the polynomial $\varphi(\lambda)$ belong to the ball C^-, we arrive at the inclusion

$$\sigma(A_1 + B_1K) \subset C^-,$$

where the matrix K is the same as in (5).

The spectrum of the matrix $\begin{pmatrix} A_1 + B_1K & A_2 \\ 0 & A_3 \end{pmatrix}$ belongs to C^- because the inclusions $\sigma(A_1 + B_1K) \subset C^-$ and $\sigma(A_3) \subset C^-$ are true. By using the relations

$$\begin{aligned} F\begin{pmatrix} A_1 + B_1K & A_2 \\ 0 & A_3 \end{pmatrix}F^{-1} &= A + F\begin{pmatrix} B_1 \\ 0 \end{pmatrix}(K\ 0)F^{-1} = A + B(K\ 0)\begin{pmatrix} H^*F^{-1} \\ * \end{pmatrix} \\ &= A + BKH^*F^{-1} = A + BP, \end{aligned}$$

where $P = KH^*F^{-1}$, we obtain the inclusion $\sigma(A + BP) \subset C^-$.

Theorem 1 is proved.

3. Main Results

Thus, Theorem 1 enables us to conclude that if there is no eigenvector x_0 of the matrix A^* corresponding to an eigenvalue from C^+ and satisfying the relation $x_0^*B = 0$, then there exists an $(r \times n)$ matrix P such that

$$\sigma(A + BP) \subset C^-. \tag{12}$$

By using this matrix P, we show that the discrete Lyapunov equation

$$(A + BP)^*Q(A + BP) - Q = -I \tag{13}$$

possesses (see [4, 6, 7]) a unique positive-definite solution $Q = Q(P)$.

Theorem 2 presented below gives sufficient conditions for the robust stabilization and indicates the regulators performing the robust stabilization.

Theorem 2. *Assume that conditions (2) and (3) are satisfied for family (1) and that there is no eigenvector x_0 of the matrix A^* corresponding to an eigenvalue from C^+ and satisfying the relation $x_0^* B = 0$. Then, for*

$$d_0 = \frac{1}{1+\|P\|}\left(\sqrt{\|A+BP\|^2+\frac{1}{\|Q\|}} - \|A+BP\|\right),$$

family (1) admits the robust linear stabilization and, as a stabilizing control, one can use $u = Px$, where P is taken from (7) and Q is taken from (13).

Proof. Substituting the control $u = Px$ in (1), we obtain

$$x(k+1) = (A+BP)x(k) + A_0(k, x(k))x(k) + B_0(k, x(k))Px(k) + \varphi_0(k, x(k), Px(k)). \tag{14}$$

As a Lyapunov function for the obtained system, we take the quadratic form

$$V(x) = x^* Q x,$$

where Q is taken from (13).

We now determine the first difference of the function $V(x)$ by the analyzed system:

$$\Delta V(x) = ((A+BP)x + A_0 x + B_0 P x + \varphi_0)^* \; Q\,((A+BP)x + A_0 x + B_0 P x + \varphi_0) - x^* Q x.$$

In view of (13), we have

$$x^*(A+BP)^* Q (A+BP)x - x^* Q x = -\|x\|^2.$$

Therefore,

$$\Delta V(x) = -\|x\|^2 + 2\psi^* Q (A+BP)x + \psi^* Q \psi, \tag{15}$$

where

$$\psi(k, x) = A_0(k, x)x + B_0(k, x)Px + \varphi_0(k, x, Px).$$

By using the estimate

$$|2\psi^* Q(A+BP)x + \psi^* Q\psi| \le \|x\|^2 \left(c_1\|x\|^{2\omega} + c_2\|x\|^{\omega} + c_3\right),$$

where

$$c_1 = \|Q\|(1+\|P\|)^2(l_0 + l_1(1+\|P\|)^{\omega})^2,$$

$$c_2 = \|Q\|(l_0 + l_1(1 + \|P\|)^{\omega})(2a_1(1 + \|P\|) + 2d(1 + \|P\|)^2),$$
$$c_3 = \|Q\|(2a_1 d(1 + \|P\|) + d^2(1 + \|P\|)^2), \tag{16}$$
$$a_1 = \|A + BP\|,$$

we arrive at the inequality

$$2\psi^* Q(A + BP)x + \psi^* Q\psi \le \|x\|^2(c_1\|x\|^{2\omega} + c_2\|x\|^{\omega} + c_3).$$

Substituting this inequality in (15), we get

$$\Delta V(x) \le \|x\|^2(-1 + c_1\|x\|^{2\omega} + c_2\|x\|^{\omega} + c_3).$$

This implies that, for

$$0 \le d < \frac{1}{1 + \|P\|}\left(\sqrt{\|A + BP\|^2 + \frac{1}{\|Q\|}} - \|A + BP\|\right) = \frac{1}{a_2}\left(\sqrt{a_1^2 + \frac{1}{\|Q\|}} - a_1\right),$$

the first difference $\Delta V(x)$ satisfies, in the ball

$$\|x\| \le \left(\frac{-c_2 + \sqrt{c_2^2 - 4c_1(c_3 - 1)}}{2c_1}\right)^{1/\omega},$$

the inequality $\Delta V(x) \le W(x)$, where

$$W(x) = \|x\|^2(-1 + c_1\|x\|^{2\omega} + c_2\|x\|^{\omega} + c_3)$$

is a function negative-definite in this ball. Hence, the conditions of the theorem on asymptotic stability of the trivial solution of Eq. (14) (see [33], Proposition 2) are satisfied in this ball.

Theorem 2 is proved.

Example 1. Consider a family of systems of the second order

$$x_1(k+1) = 2x_1(k) + (1 + (d - 2x_2^2(k))\sin^2 \alpha x_1(k))u(k) + (x_1^2(k) + u^2(k))\,2^{-\alpha^2 k^2},$$

$$x_2(k+1) = \frac{1}{2}x_2(k) + (d + x_1^2(k))(\cos \alpha k)x_2(k) + \frac{x_1^2(k) + u^2(k)}{1 + \alpha^2 k^2},$$

$$\alpha \in R.$$

Here,

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & \left(d + x_1^2\right)\cos k\alpha \end{pmatrix},$$

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} \left(d - 2x_2^2\right)\sin^2\alpha\, x_1 \\ 0 \end{pmatrix},$$

$$\varphi_0 = \begin{pmatrix} (x_1^2 + u^2)2^{-\alpha^2 k^2} \\ \dfrac{x_1^2 + u^2}{1 + \alpha^2 k^2} \end{pmatrix}, \quad \|A_0\| \le (d + x_1^2)|\cos\alpha k| \le 2\|x\|^2 + d,$$

$$\|B_0\| \le (d + 2x_2^2)\sin^2\alpha x_1 \le 2\|x\|^2 + d, \quad \|\varphi_0\| \le (x_1^2 + u^2)\sqrt{2} \le \sqrt{2}(\|x\| + |u|)^2.$$

The matrix A possesses the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$. The vector

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is the eigenvector of the matrix A^* associated with the eigenvalue $\lambda_1 = 2$ and, furthermore, $x_0^* B = 1$. The subspace L has the form $L = \text{Lin}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and, moreover, $m = 1$. It is obvious that $L^\perp = \text{Lin}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Hence,

$$F = F^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We determine the matrix P by relation (7): $P = KH^*F^{-1}$. Since $m = 1$, we have $H^* = (1\ 0)$. The matrix K is given by relation (8):

$$K = SW^{-1} - e_1 f T^{-1}.$$

We have $e_1 = 1$, $B_1 = b_1 = 1$, $W = 1$, $S = 0$, $A_1 = 2$, $\widehat{A}_1 = 2$, $\chi_{\widehat{A}_1} = \lambda - 2$, and $a_1 = -2$.

As a stable polynomial, we take $\varphi(\lambda) = \lambda$. Then $\gamma_1 = 0$, $f = \gamma_1 - a_1 = 2$, $T_1 = 1$, $A_0 = 2$, $b_0 = 1$, $T_0 = 1$, $T = T_1 T_0^{-1} = 1$, $K = -2$, $P = (-2\ 0)$,

$$A + BP = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}, \quad \|Q\| = \frac{4}{3}, \quad d_0 = \frac{1}{6},$$

whence it follows that

$$\sigma(A + BP) = \left\{0, \frac{1}{2}\right\}.$$

Thus, for

$$0 \le d < \frac{1}{6},$$

this family admits the robust linear stabilization by the control $u = -2x_1$.

The theorem presented below gives necessary conditions for the robust stabilization of Eqs. (1).

Theorem 3. *Let conditions (2) and (3) be satisfied for family (1). In order that the family of systems (1) admit a robust linear stabilization, it is necessary that the eigenvector x_0 of the matrix A^* corresponding to the eigenvalue λ_j with $|\lambda_j| > 1$ and satisfying the relation $x_0^* B = 0$ do not exist.*

Proof. Assume that family (1) admits a robust stabilization by the control $u = Px$. Substituting the control $u = Px$ in (1), we get (14).

Assume the contrary, i.e., that there exists an eigenvector x_0 of the matrix A^* corresponding to the eigenvalue λ_j with $|\lambda_j| > 1$ and satisfying the relation $x_0^* B = 0$.

By using the equalities $A^* x_0 - \lambda_j x_0 = 0$ and $x_0^* B = 0$, we arrive at the equality

$$x_0^*(A + BP - \lambda_j I) = 0.$$

Hence, λ_j is an eigenvalue of the matrix $A + BP$.

Thus, the matrix $A+BP$ has at least one eigenvalue λ satisfying the condition $|\lambda| > 1$. There are two possible cases: either the matrix $A + BP$ has eigenvalues λ satisfying the condition $|\lambda| \le 1$ or all eigenvalues of the matrix $A + BP$ satisfy the condition $|\lambda| > 1$.

Consider the first case. Let T be a nonsingular real $(n\times n)$ matrix that reduces the matrix $A+BP$ to the block diagonal form

$$T^{-1}(A + BP)T = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where

$$\sigma(A_1) = \{\lambda \in \sigma(A + BP) : |\lambda| > 1\}, \quad \sigma(A_2) = \{\lambda \in \sigma(A + BP) : |\lambda| \le 1\},$$

A_1 is an $m \times m$ matrix, and A_2 is an $(n-m)\times(n-m)$ matrix.

Substituting $x = Ty$ in (14), we obtain

$$y(k + 1) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} y(k) + \psi(k, y(k)), \tag{17}$$

where

$$\psi(k, y) = T^{-1}(A_0(k, Ty)Ty + B_0(k, Ty)PTy + \varphi_0(k, Ty, PTy)), \tag{18}$$

and the following estimates are true:

$$\|A_0(k, Ty)\| \le l_0\|T\|^{\omega}\|y\|^{\omega} + d, \quad \|B_0(k, Ty)\| \le l_0\|T\|^{\omega}\|y\|^{\omega} + d,$$

$$\|\varphi_0(k, Ty, PTy)\| \le l_1(\|T\| + \|P\|\|T\|)^{1+\omega}\|y\|^{1+\omega}.$$

In view of the linearity of the substitution $x = Ty$, the trivial solutions of system (17) are asymptotically stable.

System (17) can be represented in the form

$$y_1(k+1) = A_1 y_1(k) + \psi_1(k, y(k)), \quad y_1 \in R^m,$$
$$y_2(k+1) = A_2 y_2(k) + \psi_2(k, y(k)), \quad y_2 \in R^{n-m}. \tag{19}$$

Since

$$\|\psi_i(k, y)\| \le \|\psi(k, y)\|, \quad i = 1, 2,$$

in view of (18), we obtain the following estimate for the functions $\psi_i(k, y)$:

$$\begin{aligned}\|\psi_i(k, y)\| &\le \|\psi(k, y)\| \\ &\le \|T^{-1}(A(k, Ty)Ty + B(k, Ty)PTy + \varphi(k, Ty, PTy))\| \\ &\le M_1\|y\|^{1+\omega} + dM_2\|y\|,\end{aligned}$$

where

$$M_1 = \|T^{-1}\|(l_0\|T\|^{1+\omega}(1 + \|P\|) + l_1\|T\|^{1+\omega}(1 + \|P\|)^{1+\omega})$$

and

$$M_2 = \|T^{-1}\|\,\|T\|(1 + \|P\|).$$

We now denote $\min_{\lambda_j \in \sigma(A_1)} |\lambda_j| = 1 + \gamma, \gamma > 0$ and choose a number $q > 0$ such that $1 < q < 1 + \gamma$. Then we get the inclusions

$$\sigma\left(\frac{1}{q} A_1\right) \subset C^+ \quad \text{and} \quad \sigma\left(\frac{1}{q} A_2\right) \subset C^-.$$

In this case, the algebraic matrix Lyapunov equations

$$\left(\frac{1}{q} A_1\right)^* Q_1 \left(\frac{1}{q} A_1\right) - Q_1 = I, \tag{20}$$

$$\left(\frac{1}{q} A_2\right)^* Q_1 \left(\frac{1}{q} A_2\right) - Q_2 = -I \tag{21}$$

have solutions Q_1 and Q_2, which are positive-definite matrices of the corresponding orders.

As a Lyapunov function for system (19), we take the quadratic form

$$V(y) = y_1^* Q_1 y_1 - y_2^* Q_2 y_2.$$

It is obvious that $V(y) > 0$ at the points $y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$, $y_1 \neq 0$. We now determine the first difference of the function $V(y)$. By virtue of system (19), we find

$$\Delta V(y) = (A_1 y_1 + \psi_1)^* Q_1 (A_1 y_1 + \psi_1) - (A_2 y_2 + \psi_2)^* Q_2 (A_2 y_2 + \psi_2) - y_1^* Q_1 y_1 + y_2^* Q_2 y_2.$$

In view of (20) and (21), the equalities

$$y_1^* A_1^* Q_1 A_1\, y_1 = q^2(\|y_1\|^2 + y_1^* Q_1\, y_1),$$

$$y_2^* A_2^* Q_2 A_2\, y_2 = q^2(-\|y_2\|^2 + y_2^* Q_2\, y_2)$$

are true. Hence,

$$\Delta V(y) = q^2\|y\|^2 + (q^2-1)V(y) + 2\psi_1^* Q_1 A_1\, y_1 - 2\psi_2^* Q_2 A_2\, y_2 + \psi_1^* Q_1 \psi_1 - \psi_2^* Q_2 \psi_2\,. \tag{22}$$

By using the estimate

$$|2\psi_1^* Q_1 A_1\, y_1 - 2\psi_2^* Q_2 A_2\, y_2 + \psi_1^* Q_1 \psi_1 - \psi_2^* Q_2 \psi_2| \le \|y\|^2 \left(c_1\|y\|^{2\omega} + c_2\|y\|^{\omega} + c_3\right),$$

where

$$c_1 = (\|Q_1\| + \|Q_2\|)M_1^2,$$

$$c_2 = 2(\|Q_1 A_1\| + \|Q_2 A_2\|)M_1 + 2(\|Q_1\| + \|Q_2\|)dM_1M_2,$$

$$c_3 = 2(\|Q_1 A_1\| + \|Q_2 A_2\|)dM_2 + (\|Q_1\| + \|Q_2\|)d^2M_2^2,$$

we arrive at the inequality

$$2\psi_1^* Q_1 A_1\, y_1 - 2\psi_2^* Q_2 A_2\, y_2 + \psi_1^* Q_1 \psi_1 - \psi_2^* Q_2 \psi_2 \ge -\|y\|^2 \left(c_1\|y\|^{2\omega} + c_2\|y\|^{\omega} + c_3\right).$$

Substituting this inequality in (22), we obtain

$$\Delta V(y) \ge (q^2-1)V(y) + \left(q^2 - \left(c_1\|y\|^{2\omega} + c_2\|y\|^{\omega} + c_3\right)\right)\|y\|^2.$$

It follows from the last inequality that, for

$$d < \frac{-(\|Q_1A_1\| + \|Q_2A_2\|) + \sqrt{(\|Q_1A_1\| + \|Q_2A_2\|)^2 + (\|Q_1\| + \|Q_2\|)q^2}}{(\|Q_1\| + \|Q_2\|)\|T^{-1}\|\,\|T\|(1+\|P\|)},$$

in the ball

$$\|y\| \le \left(\frac{-c_2 + \sqrt{c_2^2 - 4c_1(c_3 - q^2)}}{2c_1}\right)^{1/\omega},$$

the following inequality is true:

$$\Delta V(y) \ge (q^2-1)V(y). \tag{23}$$

By $y(k, k_0, y_0)$ we denote a solution of Eq. (17) satisfying the initial condition $y(k_0, k_0, y_0) = y_0$. It follows from (23) that

$$V(y(k+1, k_0, y_0)) \geq (q^2)^{k-k_0+1} V(y_0),$$

which implies that the trivial solution of Eq. (17) is unstable. We arrive at a contradiction.

We now consider the second case. Since all eigenvalues of the matrix $A + BP$ belong to the set $\{\lambda : |\lambda| > 1\}$, the matrix Lyapunov equation

$$(A + BP)^* Q(A + BP) - Q = I \tag{24}$$

has a unique positive-definite solution Q. As a Lyapunov function for system (14), we take the quadratic form $V(x) = x^* Qx$.

We determine the first difference of the function $V(x)$ for this system. We have

$$\Delta V(x) = ((A + BP)x + A_0 x + B_0 Px + \varphi_0)^* Q((A + BP)x + A_0 x + B_0 Px + \varphi_0) - x^* Qx.$$

Since Eq. (24) yields the equality

$$x^*(A + BP)^* Q(A + BP)x - x^* Qx = \|x\|^2,$$

we get

$$\Delta V(x) = \|x\|^2 + 2\psi^* Q(A + BP)x + \psi^* Q\psi, \tag{25}$$

where

$$\psi(k, x) = A_0(k, x)x + B_0(k, x)Px + \varphi_0(k, x, Px).$$

By using the estimate

$$|2\psi^* Q(A + BP)x + \psi^* Q\psi| \leq \|x\|^2(c_1\|x\|^{2\omega} + c_2\|x\|^{\omega} + c_3),$$

where c_i is taken from (16), we deduce the inequality

$$2\psi^* Q(A + BP)x + \psi^* Q\psi \geq -\|x\|^2(c_1\|x\|^{2\omega} + c_2\|x\|^{\omega} + c_3).$$

Substituting this inequality in (25), we obtain

$$\Delta V(x) \geq \|x\|^2(1 - c_1\|x\|^{2\omega} - c_2\|x\|^{\omega} - c_3).$$

The last inequality implies that, for

$$0 \leq d < \frac{1}{1 + \|P\|}\left(\sqrt{\|A + BP\|^2 + \frac{1}{\|Q\|}} - \|A + BP\|\right) = \frac{1}{a_2}\left(\sqrt{a_1^2 + \frac{1}{\|Q\|}} - a_1\right),$$

the first difference $\Delta V(x)$ satisfies, in the ball

$$\|x\| \le \left(\frac{-c_2 + \sqrt{c_2^2 - 4c_1(c_3 - 1)}}{2c_1} \right)^{1/\omega},$$

the inequality $\Delta V(x) \ge W(x)$, where

$$W(x) = \|x\|^2(1 - c_1\|x\|^{2\omega} - c_2\|x\|^{\omega} - c_3)$$

is a positive-definite function in this ball. Hence, in this ball, all conditions of the theorem on instability of the trivial solution of Eqs. (14) are satisfied (see [33], Proposition 4).

Hence, the control $u = Px$ does not stabilize family (1). We arrive at a contradiction.

Theorem 3 is proved.

Example 2. In the family of systems from Example 1, we replace the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

by the matrix

$$A = \begin{pmatrix} 2 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$$

and the vector $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by the vector $B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. It is easy to see that the obtained new family of systems satisfies the conditions of Theorem 3 and, hence, does not admit robust linear stabilization.

4. Conclusions

In the present paper, for the family of objects

$$x(k+1) = (A + A_0(k, x(k)))x(k) + (B + B_0(k, x(k)))u(k) + \varphi_0(k, x(k), u(k)),$$

nonlinear in control and with functional uncertainties, we synthesize a set of general regulators linear with respect to the state $u = Px$ and guaranteeing the possibility of robust linear stabilization of this family. We estimate the admissible value of the parameter d for each stabilizing regulator. The procedure of synthesis is based on the method of square stabilization.

REFERENCES

1. V. I. Korobov, *Method of Control Functions* [in Russian], Institute of Computer Research, Izhevsk (2007).
2. V. I. Korobov and A. V. Lutsenko, "Robust stabilization of one class of nonlinear systems," *Avtomat. Telemekh.,* Issue 8, 99–112 (2014).
3. V. I. Korobov and A. V. Lutsenko, "Stabilization of linear autonomous discrete systems," *Visn. Kharkiv. Nats. Univ., Ser. Mat. Prykl. Mat. Mekh.,* No. 711, 28–36 (2005).
4. B. T. Polyak and P. S. Shcherbakov, *Robust Stability and Control* [in Russian], Nauka, Moscow (2002).
5. Yu. I. Andreev, *Control by Finite-Dimensional Linear Objects* [in Russian], Nauka, Moscow (1976).
6. A. Halanay and D. Wexler, *Teoria Calitativă a Sistemelor cu Impulsuri,* Editura Academiei Republicii Socialiste România, Bucureşti (1968).
7. V. I. Zubov, *Theory of Equations of Controlled Motion* [in Russian], Leningrad University, Leningrad (1980).
8. C. E. Langenhop, "On the stabilization of linear systems," *Proc. Amer. Math. Soc.,* **15**, No. 5, 735–742 (1964).
9. V. M. Popov, "Hyperstability and optimality of automatic systems with several control functions," *Rev. Roum. Sci. Techn. Sér. Élecfrotechn. Énerg.,* **9**, 629–690 (1964).
10. W. M. Wonham, "On pole assignment in multi-input controllable linear systems," *IEEE Trans. Automat. Contr.,* **12**, No. 6, 660–665 (1967).
11. W. M. Wonham, *Linear Multivariable Control: a Geometric Approach,* Springer, New York (1979).
12. J. W. Polderman and J. C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach,* Springer, New York (1998).
13. Z. Qu, "Global stabilization of nonlinear systems with a class of unmatched uncertainties," *Syst. Contr. Lett.,* **18**, 301–307 (1992).
14. S. L. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory,* SIAM, Philadelphia (1994).
15. P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *LMI Control Toolbox,* The MathWorks, Natick (1995).
16. Y. Cao, Y. Sun, and W. Mao, "Output feedback decentralized stabilization: ILMI approach," *Syst. Contr. Lett.,* **35**, 183–194 (1998).
17. S. Nguang and M. Fu, "Global quadratic stabilization of a class of nonlinear systems," *Int. J. Robust Nonlinear Contr.,* **8**, 483–497 (1998).
18. S. Battilotti, "Robust stabilization of nonlinear systems with pointwise norm-bounded uncertainties: a control Lyapunov function approach," *IEEE Trans. Automat. Contr.,* **44**, No. 1, 3–17 (1999).
19. G. Dullerud and F. Paganini, *A Course in Robust Control Theory — a Convex Approach,* Springer, New York (2000).
20. D. Siljak and D. Stipanovic, "Robust stabilization of nonlinear systems," *Math. Probl. Eng.,* **6**, 461–493 (2000).
21. D. Stipanovic and D. Siljak, "Robust stability and stabilization of discrete-time nonlinear systems: The LMI approach," *Int. J. Contr.,* **74**, 873–879 (2001).
22. N. K. Khalil, *Nonlinear Systems,* Prentice Hall, New York (2002).
23. S. H. Zak, *Systems and Control,* Oxford University Press, Oxford (2002).
24. D. Ho and G. Lu, "Robust stabilization for a class of discrete-time nonlinear systems via output feedback: the unified LMI approach," *Int. J. Contr.,* **76**, 105–115 (2003).
25. Z. Zuo, J. Wang, and L. Huang, "Robust stabilization for non-linear discrete-time systems," *Int. J. Contr.,* **77**, 384–388 (2004).
26. M. Yu, L. Wang, and T. Chu, "Robust stabilization of nonlinear sampled-data systems," *Amer. Contr. Conf.,* 3421–3426 (2005).
27. F. Amato, *Robust Control of Linear Systems Subject to Uncertain Time-Varying Parameters,* Springer, Berlin (2006).
28. I. E. Zuber and A. Kh. Gelig, "Synthesis of robust stabilizing control for nonlinear systems," in: *The 6th European Nonlinear Dynamics Conf. (ENOC 2008) (June 30–July 4, 2008, St. Petersburg)*, St. Petersburg, **4** (2008).
29. Y. G. Sun and L. Wang, "Stabilization of planar discrete-time switched systems: switched Lyapunov functional approach," *Nonlin. Anal. Hybrid Syst.,* **2**, 1062–1068 (2008).
30. W. Zhang, A. Abate, J. Hu, and M. P. Vitus, "Exponential stabilization of discrete-time switched linear systems," *Automatica,* **45**, 2526–2536 (2009).
31. D. Jabri, K. Guelton, N. Manamanni, A. Jaadari, and C. Chinh, "Robust stabilization of nonlinear systems based on a switched fuzzy control law," *CEAI,* **14**, No. 2, 40–49 (2012).
32. G. Rajchakit, T. Rojsiraphisal, and M. Rajchakit, "Robust stability and stabilization of uncertain switched discrete-time systems," *Adv. Difference Equat.,* **134**, 1–15 (2012).
33. W. Hahn, "Uber die Anwendung der Methode von Ljapunov auf Differenzengleichungen," *Math. Ann.,* **136**, 430–441 (1958).