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# ON HIGHER-ORDER GENERALIZED EMDEN-FOWLER DIFFERENTIAL EQUATIONS WITH DELAY ARGUMENT

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We consider a differential equation

$$u^{(n)}(t) + p(t)|u(\tau(t))|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0. \tag{$*$}$$

It is assumed that $n \geq 3$, $p \in L_{\mathrm{loc}}(R_+; R_-)$, $\mu \in C(R_+; (0, +\infty))$, $\tau \in C(R_+; R_+)$, $\tau(t) \leq t$ for $t \in R_+$ and $\lim_{t \to +\infty} \tau(t) = +\infty$. In the case $\mu(t) \equiv \text{const} > 0$, the oscillatory properties of equation $(*)$ are extensively studied, whereas for $\mu(t) \not\equiv \text{const}$, to the best of authors' knowledge, problems of this kind were not investigated at all. We also establish new sufficient conditions for the equation $(*)$ to have Property **B**.

## 1. Introduction

The present work deals with the oscillatory properties of solutions of a functional differential equations of the form

$$u^{(n)}(t) + p(t)|u(\tau(t))|^{\mu(t)} \operatorname{sign} u(\tau(t)) = 0, \tag{1.1}$$

where

$$\begin{gathered} n \geq 3, \quad p \in L_{\mathrm{loc}}(R_+; R_-), \quad \mu \in C(R_+; (0, +\infty)), \quad \tau \in C(R_+; R_+), \\ \tau(t) \leq t \quad \text{for} \quad t \in R_+ \quad \text{and} \quad \lim_{t \to +\infty} \tau(t) = +\infty. \end{gathered} \tag{1.2}$$

It is always assumed that the condition

$$p(t) \leq 0 \quad \text{for} \quad t \in R_+ \tag{1.3}$$

is satisfied.

Let $t_0 \in R_+$. A function $u : [t_0; +\infty) \to R$ is said to be a proper solution of equation (1.1) if it is locally absolutely continuous together with its derivatives up to the order $n-1$, inclusively,

$$\sup\{|u(s)| : s \in [t, +\infty)\} > 0$$

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for $t \geq t_0$, and there exists a function $\overline{u} \in C(R_+; R)$ such that $\overline{u}(t) \equiv u(t)$ on $[t_0, +\infty)$ and the equality

$$\overline{u}^{(n)}(t) + p(t)|\overline{u}(\tau(t))|^{\mu(t)} \operatorname{sign} \overline{u}(\tau(t)) = 0$$

holds almost everywhere for $t \in [t_0, +\infty)$. A proper solution $u : [t_0, +\infty) \to R$ of equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise, the solution $u$ is said to be nonoscillatory.

***Definition 1.1.*** *We say that equation (1.1) has Property* **A** *if any proper solution $u$ is oscillatory for even $n$ and is either oscillatory or satisfies*

$$|u^{(i)}(t)| \downarrow 0 \quad as \quad t \uparrow +\infty, \quad i = 0, \dots, n-1, \tag{1.4}$$

*for odd $n$.*

***Definition 1.2.*** *We say that equation (1.1) has Property* **B** *if any proper solution $u$ is either oscillatory, or satisfies (1.4), or satisfies*

$$|u^{(i)}(t)| \uparrow +\infty \quad as \quad t \uparrow +\infty, \quad i = 0, \dots, n-1, \tag{1.5}$$

*for even $n$ and is either oscillatory or satisfies (1.5) for odd $n$.*

***Definition 1.3.*** *We say that equation (1.1) is almost linear if the condition* $\lim_{t\to+\infty} \mu(t) = 1$ *is satisfied. At the same time, if* $\limsup_{t\to+\infty} \mu(t) \neq 1$ *or* $\liminf_{t\to+\infty} \mu(t) \neq 1$, *then we say that the analyzed equation is an essentially nonlinear differential equation.*

The oscillatory properties of almost linear and essentially nonlinear differential equations with advanced argument were sufficiently well studied in [1–6]. For the Emden–Fowler differential equations with deviating arguments, an essential contribution was made in [7–13]. In the present paper, sufficient conditions are established for the equation (1.1) to have Property **B**. Analogous results for Property **A** are presented in [14].

## 2. Some Auxiliary Lemmas

The following notation is used throughout the work: $\widetilde{C}^{\,n-1}_{\rm loc}([t_0, +\infty))$ denotes the set of all functions $u : [t_0, +\infty) \to R$ absolutely continuous in any finite subinterval of $[t_0, +\infty)$ together with their derivatives of the orders up to and including $n-1$;

$$\alpha = \inf\{\mu(t), t \in R_+\}, \quad \beta = \sup\{\mu(t), t \in R_+\}, \tag{2.1}$$

$$\tau_{(-1)}(t) = \sup\{s \geq 0;\ \tau(s) \leq t\}, \quad \tau_{(-k)} = \tau_{(-1)} \circ \tau_{(-(k-1))}, \quad k = 2, 3, \dots. \tag{2.2}$$

Clearly, $\tau_{(-1)}(t) \geq t$ and $\tau_{(-1)}$ is nondecreasing and coincides with the inverse of $\sigma$ if the latter exists.

**Lemma 2.1** [12]. *Let $u \in \widetilde{C}^{\,n-1}_{\rm loc}([t_0, +\infty))$, $u(t) > 0$, $u^{(n)}(t) \geq 0$ for $t \geq t_0$ and let $u^{(n)}(t) \not\equiv 0$ in any neighborhood of $+\infty$. Then there exist $t_1 \geq t_0$ and $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is even and*

$$\begin{aligned} &u^{(i)}(t) > 0 \quad for \quad t \geq t_1, \quad i = 0, \dots, \ell-1, \\ &(-1)^{i+\ell} u^{(i)}(t) \geq 0 \quad for \quad t \geq t_1, \quad i = \ell, \dots, n. \end{aligned} \tag{2.3$_\ell$}$$

In the case $\ell = 0$, only the second inequality in $(2.3_\ell)$ holds. At the same time, for $\ell = n$, only the first inequality holds and $u^{(n)}(t) \geq 0$.

**Lemma 2.2** [15]**.** *Let $u \in \widetilde{C}_{\mathrm{loc}}^{n-1}([t_0, +\infty))$, let $u^{(n)}(t) \geq 0$, and let $(2.3_\ell)$ be satisfied for some $\ell \in \{1, \dots, n-2\}$, where $\ell + n$ is even. Then*

$$\int_{t_0}^{+\infty} t^{n-\ell-1} u^{(n)}(t)\, dt < +\infty. \tag{2.4}$$

*Moreover, if*

$$\int_{t_0}^{+\infty} t^{n-\ell} u^{(n)}(t)\, dt = +\infty, \tag{$2.5_\ell$}$$

*then there exists $t_1 \geq t_0$ such that*

$$u(t) \geq \frac{t^{\ell-1}}{\ell!} u^{(\ell-1)}(t) \quad \text{for} \quad t \geq t_1, \tag{2.6}$$

$$\frac{u^{(i)}(t)}{t^{\ell-i}} \downarrow, \quad \frac{u^{(i)}(t)}{t^{\ell-i-1}} \uparrow, \quad i = 0, \dots, \ell-1, \tag{$2.7_i$}$$

*and*

$$u^{(\ell-1)}(t) \geq \frac{t}{(n-\ell)!} \int_t^{+\infty} s^{n-\ell-1} u^{(n)}(s)\, ds + \frac{1}{(n-\ell)!} \int_{t_1}^{t} s^{n-\ell} u^{(n)}(s)\, ds. \tag{2.8}$$

***Definition 2.1.*** *Let $t_0 \in R_+$. By $\mathbf{U}_{\ell,\mathbf{t_0}}$ we denote the set of all solutions of equation (1.1) satisfying the condition $(2.3_\ell)$.*

**Lemma 2.3.** *Let conditions (1.2), (1.3) be satisfied, let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$, and let equation (1.1) have a positive proper solution $u : [t_0, +\infty) \to (0, +\infty)$ such that $u \in \mathbf{U}_{\ell,\mathbf{t_0}}$. Moreover, let $\alpha \geq 1$ and let*

$$\int_{t_0}^{+\infty} t^{n-\ell} (c\, \tau^{\ell-1}(t))^{\mu(t)} |p(t)| dt = +\infty \quad \text{for} \quad c \in (0, 1]. \tag{$2.9_{\ell,c}$}$$

*Then, for any $\gamma \in (1, +\infty)$, there exists $t_* > t_0$ such that, for any $k \in N$,*

$$u^{(\ell-1)}(t) \geq \rho^{(\alpha)}_{k,\ell,t_*}(t) \quad \text{for} \quad t \geq \tau_{(-k)}(t_*), \tag{2.10}$$

*where*

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) = \ell\,!\exp\left\{\gamma_\ell(\alpha)\int\limits_{\tau_{(-1)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}|p(\xi)|d\xi\,ds\right\}, \tag{2.11$_\ell$}$$

$$\rho^{(\alpha)}_{i,\ell,t_*}(t) = \ell\,! + \frac{1}{(n-\ell)!}\int\limits_{\tau_{(-1)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}$$

$$\times\left(\frac{1}{\ell\,!}\rho^{(\alpha)}_{i-1,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds,\quad i=2,\dots,k, \tag{2.12$_\ell$}$$

$$\gamma_\ell(\alpha) = \begin{cases}\dfrac{1}{\ell\,!\,(n-\ell)!} & \textit{if}\quad \alpha=1,\\ \gamma & \textit{if}\quad \alpha>1,\end{cases} \tag{2.13$_\ell$}$$

*and $\alpha$ is given by the first equality in (2.1).*

***Proof.*** Let $t_0 \in R_+$, $\ell \in \{1,\dots,n-2\}$ with even $\ell+n$ and let $u \in \mathbf{U}_{\ell,\mathbf{t_0}}$. According to (1.1), (2.3$_\ell$), and (2.9$_{\ell,c}$), it is clear that condition (2.5$_\ell$) is satisfied. Indeed, by (2.3$_\ell$), there exist $t_1 > t_0$ and $c \in (0,1]$ such that

$$u(\tau(t)) \ge c(\tau(t))^{\ell-1}\quad\text{for}\quad t\ge t_1.$$

Thus, it follows from (1.1) that

$$\int\limits_{t_1}^{t} s^{n-\ell}u^{(n)}(s)ds \ge \int\limits_{t_1}^{t} s^{n-\ell}\left(c\,\tau^{\ell-1}(s)\right)^{\mu(s)}|p(s)|ds\quad\text{for}\quad t\ge t_1.$$

Passing to the limit in this inequality, by virtue of (2.9$_{\ell,c}$), we get (2.5$_\ell$).

According to Lemma 2.2, there exists $t_2 > t_1$ such that conditions (2.6)–(2.8) are satisfied for $t \ge t_2$ and

$$u^{(\ell-1)}(t) \ge \frac{t}{(n-\ell)!}\int\limits_{t}^{+\infty} s^{n-\ell-1}(u(\tau(s)))^{\mu(s)}|p(s)|ds$$

$$+\frac{1}{(n-\ell)!}\int\limits_{t_{(-1)}(t_2)}^{t} s^{n-\ell}(u(\tau(s)))^{\mu(s)}|p(s)|ds\quad\text{for}\quad t\ge\tau_{(-1)}(t_2).$$

Therefore, by (2.6), we find

$$u^{(\ell-1)}(t) \ge \frac{1}{(n-\ell)!}\int\limits_{\tau_{(-1)}(t_2)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}$$

$$\times\left(\frac{1}{\ell\,!}u^{(\ell-1)}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds \quad \text{for} \quad t\geq\tau_{(-1)}(t_2). \tag{2.14}$$

According to ($2.7_{\ell-1}$) and ($2.9_{\ell,c}$), we choose $t_*>\tau_{(-1)}(t_2)$ such that

$$\frac{1}{(n-\ell)!}\int\limits_{\tau_{(-1)}(t_2)}^{t_*}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}\left(\frac{1}{\ell\,!}u^{(\ell-1)}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds>\ell\,!. \tag{2.15}$$

By (2.14) and (2.15), we have

$$u^{(\ell-1)}(t)\geq\ell\,!+\frac{1}{(n-\ell)!}\int\limits_{t_*}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}$$

$$\times\left(\frac{1}{\ell\,!}u^{(\ell-1)}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds \quad \text{for} \quad t\geq t_*. \tag{2.16}$$

Let $\alpha=1$. Since $u^{(\ell-1)}(t)/t$ is a nonincreasing function, from (2.16), we obtain

$$u^{(\ell-1)}(t)\geq\ell\,!+\frac{1}{\ell\,!(n-\ell)!}\int\limits_{t_*}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}$$

$$\times\,u^{(\ell-1)}(\xi)|p(\xi)|d\xi\,ds \quad \text{for} \quad t\geq t_*. \tag{2.17}$$

By the second condition in ($2.7_{\ell-1}$), it is obvious that

$$x'(t)\geq\frac{u^{(\ell-1)}(t)}{\ell\,!(n-\ell)!}\int\limits_{t}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}|p(\xi)|d\xi, \tag{2.18}$$

where

$$x(t)=\ell\,!+\frac{1}{\ell\,!(n-\ell)!}\int\limits_{t_*}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}u^{(\ell-1)}(\xi)|p(\xi)|d\xi\,ds. \tag{2.19}$$

Thus, according to (2.17), (2.18), and (2.19), we get

$$x'(t)\geq\frac{x(t)}{\ell\,!(n-\ell)!}\int\limits_{t}^{+\infty}\xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)}|p(\xi)|d\xi \quad \text{for} \quad t\geq t_*.$$

Therefore, since $x(t_*) = \ell\,!$, we conclude that

$$x(t) \geq \ell\,! \exp\left\{ \frac{1}{\ell\,!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi\, ds \right\} \quad \text{for} \quad t \geq t_*.$$

Hence, by (2.16) and (2.19),

$$u^{(\ell-1)}(t) \geq \rho^{(1)}_{1,\ell,t_*}(t) \quad \text{for} \quad t \geq t_*, \tag{2.20}$$

where

$$\rho^{(1)}_{1,\ell,t_*}(t) = \ell\,! \exp\left\{ \frac{1}{\ell\,!(n-\ell)!} \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi\, ds \right\}. \tag{2.21}$$

Therefore, in view of (2.14) and (2.20),

$$u^{(\ell-1)}(t) \geq \rho^{(1)}_{i,\ell,t_*}(t) \quad \text{for} \quad t \geq \tau_{(-i)}(t_*), \quad i = 1, \dots, k, \tag{2.22}$$

where

$$\rho^{(1)}_{i,\ell,t_*}(t) = \ell\,! + \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-i)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)}$$

$$\times \left( \frac{1}{\ell\,!} \rho^{(1)}_{i-1,\ell,t_*}(\tau(\xi)) \right)^{\mu(\xi)} |p(\xi)| d\xi\, ds \quad i = 2, \dots, k. \tag{2.23}$$

We now assume that $\alpha > 1$ and $\gamma \in (1, +\infty)$. Since $u^{(\ell-1)}(t) \uparrow +\infty$ as $t \uparrow +\infty$, without loss of generality, we can assume that

$$\left( \frac{1}{\ell\,!} u^{(\ell-1)}(\tau(t)) \right)^{\alpha-1} \geq \ell\,!(n-\ell)!\,\gamma$$

for $t \geq t_*$. It follows from (2.16) that

$$u^{(\ell-1)}(t) \geq \ell\,! + \gamma \int\limits_{t_*}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)} u^{(\ell-1)}(\xi) |p(\xi)| d\xi\, ds \quad \text{for} \quad t \geq t_*. \tag{2.24}$$

By (2.24), as above, we can show that if $\alpha > 1$, then

$$u^{(\ell-1)}(t) \geq \rho^{(\alpha)}_{k,\ell,t_*}(t) \quad \text{for} \quad t \geq \tau_{(-k)}(t_*), \tag{2.25}$$

where

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) = \ell\,!\,\exp\left\{\gamma \int\limits_{\tau_{(-1)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-2}(\tau(\xi))^{1+(\ell-1)\mu(\xi)} \right.$$

$$\left. \times |p(\xi)| d\xi\, ds \right\} \quad \text{for} \quad t \geq \tau_{(-1)}(t_*), \tag{2.26}$$

$$\rho^{(\alpha)}_{i,\ell,t_*}(t) = \ell\,! + \frac{1}{(n-\ell)!} \int\limits_{\tau_{(-i)}(t_*)}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}$$

$$\times \left(\frac{1}{\ell\,!}\rho^{(\alpha)}_{i-1,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)| d\xi\, ds \quad \text{for} \quad t \geq \tau_{(-i)}(t_*), \quad i = 2, \ldots, k. \tag{2.27}$$

In view of (2.20)–(2.23) and (2.25)–(2.27), it is clear that, for any $\alpha \geq 1$, $k \in N$, and $\gamma \in (1, +\infty)$, there exists $t_* \in R_+$ such that (2.10) holds, where $\gamma_\ell(\alpha)$ is given by (2.13$_\ell$), which proves the validity of the lemma.

***Remark 2.1.*** It is obvious that if $\beta < +\infty$ and (2.9$_{\ell,1}$) holds, then, for any $c \in (0, 1]$, condition (2.9$_{\ell,c}$) is satisfied.

***Remark 2.2.*** Condition (2.9$_{\ell,1}$) is not sufficient for condition (2.5) to be satisfied. Therefore, in this case, it may happen that Lemma 2.3 is incorrect. Indeed, let $\delta \in (0, 1)$. Consider equation (1.1), where $n$ is odd and

$$\tau(t) \equiv t, \quad p(t) = -\frac{n!\, t^{\log_{1/\delta} t}}{t^{n+1}(\delta t - 1)^{\log_{1/\delta} t}}, \quad \mu(t) = \log_{1/\delta} t, \quad t \geq \frac{2}{\delta}.$$

It is clear that the function

$$u(t) = \delta - \frac{1}{t}$$

is a solution of equation (1.1) and satisfies condition (2.3$_1$) for $t \geq \dfrac{2}{\delta}$. On the other hand, condition (2.9$_{1,1}$) holds but condition (2.5$_1$) is not satisfied.

## 3. Necessary Conditions for the Existence of Solutions of Type (2.3$_\ell$)

**Theorem 3.1.** *Let $\ell \in \{1, \ldots, n-2\}$ with even $\ell + n$, let conditions (1.2), (1.3), (2.9$_{\ell,c}$) and*

$$\int\limits_{0}^{+\infty} t^{n-\ell-1}(\tau(t))^{\ell\mu(t)} |p(t)| dt = +\infty \tag{3.1$_\ell$}$$

*be satisfied, and let*

$$\mathbf{U}_{\ell,\mathbf{t_0}} \neq \varnothing$$

*for some $t_0 \in R_+$. Then there exists $t_* > t_0$ such that if $\alpha = 1$, then, for any $k \in N$,*

$$\lim_{t\to+\infty} \frac{1}{t} \int_{\tau_{(-k)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\,\rho^{(1)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds = 0 \tag{3.2}$$

*and if $\alpha > 1$, then, for any $k \in N$, $\gamma \in (1, +\infty)$, and $\delta \in (1, \alpha]$,*

$$\int_{\tau_{(-i)}(t_*)}^{+\infty} \int_{s}^{+\infty} \xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\, ds < +\infty, \tag{3.3}$$

*where $\alpha$ is defined by the first equality in (2.1) and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by (2.11)–(2.13).*

***Proof.*** Let $t_0 \in R_+$, $\ell \in \{1, \dots, n-2\}$, $\mathbf{U}_{\ell,\mathbf{t_0}} \neq \varnothing$, and $\gamma \in (1, +\infty)$. By the definition (see Definition 2.1), equation (1.1) has a proper solution $u \in \mathbf{U}_{\ell,\mathbf{t_0}}$ satisfying condition (2.3$_\ell$) with some $t_1 \geq t_0$. In view of (1.1), (2.3$_\ell$), and (2.9$_{\ell,c}$), it is obvious that condition (2.5$_\ell$) holds. Thus, by Lemma 2.2, there exists $t_1 > t_0$ such that conditions (2.6) and (2.7$_i$) are satisfied. On the other hand, according to Lemma 2.3 (and its proof), there exist $t_2 > t_1$ and $t_* > t_2$ such that

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int_{t_2}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1}(u(\tau(\xi)))^{\mu(\xi)}|p(\xi)|d\xi\, ds \quad \text{for} \quad t \geq t_2 \tag{3.4}$$

and relation (2.10) is true. Without loss of generality, we can assume that $\tau(t) \geq t_2$ for $t \geq t_*$. Therefore, by (2.10), it follows from (3.4) that

$$u^{(\ell-1)}(t) \geq \frac{1}{(n-\ell)!} \int_{\tau_{(-k)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\,u^{(\ell-1)}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds. \tag{3.5}$$

Assume that $\alpha = 1$. Thus, by (2.10) and (3.5), we obtain

$$\begin{aligned} u^{(\ell-1)}(t) \geq &\frac{1}{(n-\ell)!} \int_{\tau_{(-k)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)} \\ &\times \left(\frac{1}{\ell\,!}\,\rho^{(1)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds \quad \text{for} \quad t \geq \tau_{(-k)}(t_*). \end{aligned} \tag{3.6}$$

On the other hand, according to (2.7$_{\ell-1}$) and (3.1$_\ell$), it is clear that

$$u^{(\ell-1)}(t)/t \ \downarrow 0 \quad \text{for} \quad t \uparrow +\infty. \tag{3.7}$$

Therefore, by using (3.7), and (3.6), we get

$$\lim_{t\to+\infty}\frac{1}{t}\int\limits_{\tau_{(-k)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}\left(\frac{1}{\ell\,!}\,\rho_{k,\ell,t_*}^{(1)}(\tau(\xi))\right)^{\mu(\xi)}|p(\xi)|d\xi\,ds=0. \tag{3.8}$$

We now assume that $\alpha>1$ and $\delta\in(1,\alpha]$. Then, by $(2.7_{\ell-1})$, (2.10), and (3.7) we obtain

$$u^{(\ell-1)}(t)\geq\frac{1}{(n-\ell)!}\int\limits_{\tau_{(-k)}(t_*)}^{t}\int\limits_{s}^{+\infty}\xi^{n-\ell-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}$$

$$\times\left(\frac{1}{\ell\,!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\tau(\xi))\right)^{\mu(\xi)-\delta}\left(\frac{1}{\ell\,!}\,u^{(\ell-1)}(\xi)\right)^{\delta}|p(\xi)|d\xi\,ds$$

$$\geq\frac{1}{(n-\ell)!}\int\limits_{\tau_{(-k)}(t_*)}^{t}\left(\frac{1}{\ell\,!}\,u^{(\ell-1)}(\xi)\right)^{\delta}\int\limits_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}$$

$$\times\left(\frac{1}{\ell\,!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\tau(\xi))\right)^{\mu(\xi)-\delta}|p(\xi)|d\xi\,ds.$$

Thus, we get

$$(v(t))^{\delta}\geq\frac{1}{(\ell\,!(n-\ell)!)^{\delta}}\left(\int\limits_{\tau_{(-k)}(t_*)}^{t}v^{\delta}(s)\int\limits_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}\right.$$

$$\left.\times\left(\frac{1}{\ell\,!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\tau(\xi))\right)^{\mu(\xi)-\delta}|p(\xi)|d\xi\,ds\right)^{\delta}, \tag{3.9}$$

where

$$v(t)=\frac{1}{\ell\,!}\,u^{(\ell-1)}(t).$$

In view of $(3.1_\ell)$, it is clear that there exists $t_1>\tau_{(-k)}(t_*)$ such that

$$\int\limits_{\tau_{(-k)}(t_*)}^{t}v^{\delta}(s)\int\limits_{s}^{+\infty}\xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}\left(\frac{1}{\ell\,!}\,\rho_{k,\ell,t_*}^{(\alpha)}(\tau(\xi))\right)^{\mu(\xi)-\delta}$$

$$\times|p(\xi)|d\xi\,ds>0\quad\text{for}\quad t\geq t_1.$$

Therefore, it follows from (3.9) that

$$\int_{t_1}^{t} \frac{\varphi'(s)ds}{(\varphi(s))^{\delta}} \geq \frac{1}{(\ell\,!(n-\ell)!)^{\delta}} \int_{t_1}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}$$

$$\times \left(\frac{1}{\ell\,!}\, \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\, ds \quad \text{for} \quad t \geq t_1, \tag{3.10}$$

where

$$\varphi(t) = \int_{\tau_{(-k)}(t_*)}^{t} (v(s))^{\delta} \int_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\, \rho^{(\alpha)}_{k,\ell,t_*}(\tau(s))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\, ds.$$

By using (3.10), we obtain

$$\int_{t_1}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\, \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\, ds$$

$$\leq \frac{(\ell\,!(n-\ell)!)^{\delta}}{\delta-1} \left(\varphi^{1-\delta}(t_1) - \varphi^{1-\delta}(t)\right) \leq \frac{(\ell\,!(n-\ell)!)^{\delta}}{\delta-1}\, \varphi^{1-\delta}(t_1) \quad \text{for} \quad t \geq t_1.$$

Hence,

$$\int_{t_1}^{+\infty} \int_{s}^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\, \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\, ds \leq +\infty. \tag{3.11}$$

According to (3.8) and (3.11), conditions (3.2) and (3.3) are satisfied, which proves the validity of the theorem.

***Corollary 3.1.*** *Let $\ell \in \{1, \dots, n-1\}$ with even $\ell + n$, let $\beta < +\infty$, let conditions (1.2), (1.3), (2.9$_{\ell,1}$), and (3.1$_\ell$) be satisfied, and let $\mathbf{U}_{\ell,\mathbf{t_0}} \neq \varnothing$ for some $t_0 \in R_+$. Then, for any $\gamma > 1$ there exists $t_* > t_0$ such that if $\alpha = 1$, then relation (3.2) holds for any $k \in N$ and if $\alpha > 1$, then relation (3.3) holds for any $k \in N$ and $\delta \in (1, \alpha]$, where $\alpha$ and $\beta$ are defined by (2.1) and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by (2.11)–(2.13).*

***Proof.*** According to Remark 2.1, it suffices to note that, since $\beta < +\infty$, conditions (2.9$_{\ell,c}$) is satisfied by (2.9$_{\ell,1}$) for any $c \in (0, 1]$.

## 4. Sufficient Conditions for the Nonexistence of Solutions of the Type (2.3$_\ell$)

**Theorem 4.1.** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), (1.3), (2.9$_{\ell,c}$), and (3.1$_\ell$) be satisfied. Moreover, assume that, for $\alpha = 1$,*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_{\tau_{(-k)}(t_*)}^{t} \int_{s}^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\, \rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)} |p(\xi)|d\xi\, ds > 0 \tag{4.1$_\ell$}$$

*for large* $t_* \in R_+$ *and some* $k \in N$ *and, for* $\alpha > 1$,

$$\int\limits_{\tau_{(-k)}(t_*)}^{+\infty} \int\limits_{s}^{+\infty} \xi^{n-\ell-1-\delta}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} \left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{k,\ell,t_*}(\tau(\xi))\right)^{\mu(\xi)-\delta} |p(\xi)|d\xi\, ds = +\infty \tag{4.2$_\ell$}$$

*for some* $k \in N$ *and* $\delta \in (1, \alpha]$. *Then* $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ *for any* $t_0 \in R_+$, *where* $\alpha$ *is defined by the first equality in (2.1), and* $\rho^{(\alpha)}_{k,\ell,t_*}$ *is given by (2.11)–(2.13).*

***Proof.*** Assume the contrary, i.e., that there exists $t_0 \in R_+$ such that $\mathbf{U}_{\ell,\mathbf{t_0}} \neq \varnothing$ (see Definition 2.1). Then equation (1.1) has a proper solution $u : [t_0, +\infty) \to R$ satisfying condition (2.3$_\ell$). Since the conditions of Theorem 3.1 are satisfied, there exists $t_* > t_0$ such that if $\alpha = 1$ ($\alpha > 1$), then condition (3.2) [condition (3.3)] is satisfied, which contradicts (4.1$_\ell$) [(4.2$_\ell$)]. The obtained contradiction proves the validity of the theorem.

**Theorem 4.1′.** *Let* $\ell \in \{1, \dots, n-2\}$ *with even* $\ell + n$, *let conditions (1.2), (1.3), (2.9$_{\ell,1}$) and (3.1$_\ell$) be satisfied, and let* $\beta < +\infty$. *Moreover, if* $\alpha = 1$, $\alpha > 1$, *for any large* $t_* \in R_+$ *and, for some* $k \in N$ *(for some* $k \in N$ *and* $\delta \in (1, \alpha]$), *relation (4.1$_\ell$) [(4.2$_\ell$] holds, then* $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$, *where* $\alpha$ *and* $\beta$ *are given by (2.1).*

***Proof.*** It suffices to note that, since $\beta < +\infty$, condition (2.9$_{\ell,c}$) is satisfied by (2.9$_{\ell,1}$) for any $c \in (0, 1]$. Therefore, all conditions of Theorem 4.1 are satisfied, which proves the validity of the theorem.

***Corollary 4.1.*** *Let* $\ell \in \{1, \dots, n-2\}$ *with even* $\ell + n$, *let* $\alpha = 1$, *let conditions (1.2), (1.3), (2.9$_{\ell,c}$), and (3.1$_\ell$) be satisfied, and let*

$$\limsup_{t\to+\infty} \frac{1}{t} \int\limits_{0}^{t} \int\limits_{s}^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}|p(\xi)|d\xi\, ds > 0. \tag{4.3$_\ell$}$$

*Then, for any* $t_0 \in R_+$, $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$, *where* $\alpha$ *is defined by the first equality in (2.1).*

***Proof.*** Since

$$\rho^{(1)}_{1,\ell,t_*}(\tau(t)) \geq \ell \quad \text{for large} \quad t,$$

it suffices to note that, by (4.3$_\ell$), condition (4.1$_\ell$) is satisfied for $\alpha = 1$ and $k = 1$.

***Corollary 4.1′.*** *Let* $\ell \in \{1, \dots, n-2\}$ *with even* $\ell + n$ *and let conditions (1.2), (1.3), (4.3$_\ell$), and (3.1$_\ell$) be satisfied. In this case, if* $\alpha = 1$ *and* $\beta < +\infty$, *then* $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ *for any* $t_0 \in R_+$, *where* $\alpha$ *and* $\beta$ *are given by (2.1).*

***Proof.*** To prove the corollary, it suffices to note that, since $\beta < +\infty$, condition (2.9$_{\ell,c}$) holds by (4.3$_\ell$).

***Corollary 4.2.*** *Let* $\ell \in \{1, \dots, n-2\}$ *with even* $\ell + n$, *let conditions (1.2), (1.3) and (2.9$_{\ell,c}$) be satisfied, let* $\alpha = 1$, *and let*

$$\liminf_{t\to+\infty} t \int\limits_{t}^{+\infty} s^{n-\ell-2}(\tau(s))^{1+(\ell-1)\mu(s)}|p(\xi)|ds = \gamma > 0. \tag{4.4$_\ell$}$$

*If, in addition, for some $\varepsilon \in (0, \gamma)$,*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-\ell-1} (\tau(\xi))^{\mu(\xi)\left(\ell-1+\frac{\gamma-\varepsilon}{\ell!(n-\ell)!}\right)} |p(\xi)| d\xi\, ds > 0, \tag{4.5$_\ell$}$$

*then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ is given by the first equality in (2.1).*

***Proof.*** Let $\varepsilon \in (0, \gamma)$. According to (4.4$_\ell$), (2.11), and (2.13), it is clear that

$$\rho^{(1)}_{1,\ell,t_*}(\tau(t)) \geq \ell\,!(\tau(t))^{\frac{\gamma-\varepsilon}{\ell!(n-\ell)!}}$$

for large $t$. Therefore, by (4.5$_\ell$), relation (4.1$_\ell$) holds for $k = 1$, which proves the validity of the corollary.

***Corollary 4.2′.*** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), (1.3), (3.1$_\ell$), (4.4$_\ell$), and (4.5$_\ell$) be satisfied. If, in addition, $\alpha = 1$ and $\beta < +\infty$, then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ and $\beta$ are given by (2.1).*

***Proof.*** To prove the corollary, it suffices to note that condition (2.9$_{\ell,c}$) is satisfied because $\beta < +\infty$ by (4.4$_\ell$).

***Corollary 4.3.*** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), (1.3), (2.9$_{\ell,c}$), and (3.1$_\ell$) be satisfied. If, in addition, $\alpha > 1$ and, for some $\delta \in (1, \alpha]$,*

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} |p(\xi)| d\xi\, ds = +\infty, \tag{4.6$_\ell$}$$

*then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$, for any $t_0 \in R_+$, where $\alpha$ is defined by the first condition in (2.1).*

***Proof.*** By virtue of (4.6$_\ell$), condition (4.2$_\ell$) holds for $k = 1$, which proves the validity of the corollary.

***Corollary 4.3′.*** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), (1.3), (3.1$_\ell$), (2.9$_{\ell,1}$), and (4.6$_\ell$) be satisfied. If, in addition, $\alpha > 1$ and $\beta < +\infty$, then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ and $\beta$ are given by (2.1).*

***Proof.*** According to Corollary 4.3, it suffices to note that, since $\beta < +\infty$ by (2.9$_{\ell,1}$), condition (2.9$_{\ell,c}$) holds for any $c \in (0, 1]$.

***Corollary 4.4.*** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), (1.3), (2.9$_{\ell,c}$), (3.1$_\ell$), (4.4$_\ell$), and (4.6$_\ell$) be satisfied. If, in addition, $\alpha > 1$ and there exists $m \in N$ such that*

$$\liminf_{t\to+\infty} \frac{\tau^m(t)}{t} > 0, \tag{4.7}$$

*then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ is given by the first condition in (2.1).*

***Proof.*** By virtue of (4.4$_\ell$), there exist $c > 0$ and $t_1 \in R_+$ such that

$$t \int_t^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi \geq c \quad \text{for} \quad t \geq t_1. \tag{4.8}$$

Let

$$\delta = \frac{1+\alpha}{2} \quad \text{and} \quad m_0 = \frac{\delta(m-1)}{c(\alpha-\delta)}.$$

Thus, by (4.8) and (2.26), there exists $t_* > t_1$ such that

$$\rho^{(\alpha)}_{1,\ell,t_*}(t) \geq t^{m_0 c} \quad \text{for} \quad t \geq t_*.$$

Therefore, for large $t$ we find

$$\left(\frac{\tau(t)}{t}\right)^{\delta} \left(\frac{1}{\ell\,!}\,\rho^{(\alpha)}_{1,\ell,t_*}(\tau(t))\right)^{\mu(t)-\delta} \geq \left(\frac{\tau(t)}{t}\right)^{\delta} \left(\frac{1}{\ell\,!}\,\tau^{m_0 c}(t)\right)^{\alpha-\delta}$$
$$= \frac{1}{(\ell\,!)^{\alpha-\delta}} \left(\frac{(\tau(t))^{1+\frac{m_0 c(\alpha-\delta)}{\delta}}}{t}\right)^{\delta} = (\ell\,!)^{\delta-\alpha} \left(\frac{\tau^m(t)}{t}\right)^{\delta}.$$

Then, by (4.7) and (4.6$_\ell$), it is obvious that (4.2$_\ell$) holds, which proves the corollary.

***Corollary 4.4′.*** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), (1.3), (3.1$_\ell$), (4.6$_\ell$), and (4.7) be satisfied. If, in addition, $\alpha > 1$ and $\beta < +\infty$, then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ and $\beta$ are given by (2.1).*

***Proof.*** Since $\beta < +\infty$, it suffices to note that all conditions of Corollary 4.4 are satisfied.

In a similar way, one can prove the following corollary:

***Corollary 4.5.*** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$, let conditions (1.2), (1.3), (3.1$_\ell$) and (2.9$_{\ell,c}$) be satisfied, and let $\alpha > 1$. If, in addition,*

$$\liminf_{t \to +\infty} t \, \ln t \int_t^{+\infty} \xi^{n-\ell-2} (\tau(\xi))^{1+(\ell-1)\mu(\xi)} |p(\xi)| d\xi > 0 \tag{4.9$_\ell$}$$

*and, for some $\delta \in (1, \alpha]$ and $m \in N$,*

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{n-\ell-1-\delta} (\tau(\xi))^{\delta+(\ell-1)\mu(\xi)} (\ln \tau(\xi))^m |p(\xi)| d\xi \, ds = +\infty, \tag{4.10$_\ell$}$$

*then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ is specified by the first equality in (2.1).*

***Corollary 4.5′.*** *Let $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), (1.3), ($2.9_{\ell,1}$), ($4.9_\ell$), and ($4.10_\ell$) be satisfied. If, in addition, $\alpha > 1$ and $\beta < +\infty$, then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ and $\beta$ are given by (2.1).*

***Corollary 4.6.*** *Let $\alpha > 1$, $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$ and let conditions (1.2), ($3.1_\ell$), and ($2.9_{\ell,c}$) be satisfied. If, in addition, there exist $\gamma \in (0,1)$ and $r \in (0,1)$ such that*

$$\liminf_{t\to+\infty} t^\gamma \int_t^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}|p(\xi)|d\xi > 0, \tag{$4.11_\ell$}$$

$$\liminf_{t\to+\infty} \frac{\tau(t)}{t^r} > 0 \tag{4.12}$$

*and, at least one of the conditions*

$$r\,\alpha \ge 1 \tag{4.13}$$

*or $r\,\alpha < 1$ is satisfied and, for some $\varepsilon > 0$ and $\delta \in (1, \alpha)$,*

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-\ell-1-\delta-\varepsilon+\frac{r(1-\gamma)(\alpha-\delta)}{1-\alpha r}}(\tau(\xi))^{\delta+(\ell-1)\mu(\xi)}|p(\xi)|d\xi\, ds = +\infty, \tag{$4.14_\ell$}$$

*then $\mathbf{U}_{\ell,\mathbf{t_0}} = \varnothing$ for any $t_0 \in R_+$, where $\alpha$ is specified by the first equality in (2.1).*

***Proof.*** It suffices to show that condition ($4.2_\ell$) is satisfied for some $k \in N$. Indeed, according to ($4.11_\ell$) and (4.12), there exist $\gamma \in (0,1)$, $r \in (0,1)$, $c > 0$, and $t_1 \in R_+$ such that

$$t^\gamma \int_t^{+\infty} \xi^{n-\ell-1}(\tau(\xi))^{(\ell-1)\mu(\xi)}|p(\xi)|d\xi \ge c \quad \text{for} \quad t \ge t_1 \tag{4.15}$$

and

$$\tau(t) \ge c\,t^r \quad \text{for} \quad t \ge t_1. \tag{4.16}$$

By ($2.12_\ell$), ($2.11_\ell$), and (4.15), we obtain

$$\rho^{(\alpha)}_{2,\ell,t_*}(t) \ge \frac{c}{(n-\ell)!}\int_{\tau_{(-1)}(t_*)}^{t} s^{-\gamma}ds = \frac{c\left(t^{1-\gamma} - \tau^{1-\gamma}_{(-1)}(t_*)\right)}{(n-\ell)!(1-\gamma)} \quad \text{for} \quad t \ge \tau_{(-1)}(t_*).$$

We now choose $t_2 > \tau_{(-1)}(t_*)$ and $c_1 \in (0, c)$ such that

$$\rho^{(\alpha)}_{2,\ell,t_*}(t) \ge c_1\,t^{1-\gamma} \quad \text{for} \quad t \ge t_2.$$

Therefore, in view of (4.15) and (4.16), we can find $t_3 > t_2$ and $c_2 \in (0, c_1)$ such that, according to (2.12), we get

$$\rho^{(\alpha)}_{3,\ell,t_*}(t) \geq c_2\, t^{(1-\gamma)(1+\alpha r)} \quad \text{for} \quad t \geq t_3.$$

Hence, for any $k_0 \in N$, there exist $t_{k_0}$ and $c_{k_0-1} > 0$ such that

$$\rho^{(\alpha)}_{k_0,\ell,t_*}(t) \geq c_{k_0-1}\, t^{(1-\gamma)(1+\alpha r+\ldots+(\alpha r)^{k_0-2})} \quad \text{for} \quad t \geq t_{k_0}. \tag{4.17}$$

Assume that (4.13) is satisfied. We choose $k_0 \in N$ such that

$$k_0 - 1 \geq \frac{\delta}{r(\alpha-\delta)(1-\gamma)}.$$

Thus, by (4.16), (4.17), and ($2.9_{\ell,1}$), condition ($4.2_\ell$) holds for $k = k_0$.

In this case, the validity of the corollary has already been proved.

Further, assume that $\alpha r < 1$ and, for some $\varepsilon \in (0, (1-\gamma(\alpha-\delta)r)$, relation ($4.14_\ell$) is satisfied. We choose $k_0 \in N$ such that

$$1 + \alpha r + \ldots + (\alpha r)^{k_0-2} \geq \frac{1}{1-\alpha r} - \frac{\varepsilon}{(1-\gamma)(\alpha-\delta)r}.$$

Thus, by ($4.14_\ell$), (4.16), and (4.17), it is clear that ($4.2_\ell$) holds for $k = k_0$. The proof the corollary is complete.

## 5. Differential Equations with Property B

**Theorem 5.1.** *Let conditions (1.2) and (1.3) be satisfied, let, for any $\ell \in \{1, \ldots, n\}$ with even $\ell+n$, conditions ($2.9_{\ell,c}$) and ($3.1_\ell$) hold, and let ($2.9_{1,c}$) be satisfied for even $n$. Moreover, let, for any large $t_* \in R_+$ and $\ell \in \{1, \ldots, n-2\}$ with even $\ell+n$, condition ($4.1_\ell$) be true for some $k \in N$ if $\alpha = 1$ or, for some $k \in N$, $\gamma \in (1, +\infty)$, and $\delta \in (1, \alpha]$, relation ($4.2_\ell$) hold when $\alpha > 1$. Then equation (1.1) possesses Property* **B**, *where $\alpha$ is defined by the first condition in (2.1) and $\rho^{\alpha)}_{k,\ell,t_*}$ is given by (2.11)–(2.13).*

***Proof.*** Assume that equation (1.1) has a proper nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$ [the case $u(t) < 0$ is similar]. Then, by (1.2), (1.3), and Lemma 2.1, there exists $\ell \in \{1, \ldots, n\}$ such that $\ell + n$ is even and condition ($2.3_\ell$) holds. Since, for any $\ell \in \{1, \ldots, n-2\}$ with even $\ell + n$, the conditions of Theorem 4.1 are satisfied, we have $\ell \notin \{1, \ldots, n-2\}$. Let $\ell = n$. Thus, by ($2.3_n$), it is clear that there exists $c \in (0, 1]$ such that, for large $t$,

$$u(\tau(t)) \geq c\tau^{n-1}(t).$$

Hence, by ($2.9_{n,c}$), it follows from (1.1) that

$$u^{(n-1)}(t) \geq \int_{t_1}^{t} (c\tau^{n-1}(s))^{\mu(s)} |p(s)| ds \to +\infty \quad \text{for} \quad t \to +\infty,$$

where $t_1$ is a sufficiently large number. This means that condition (1.4) is satisfied. We now assume that $\ell = 0$, $n$ is even, and there exists $c \in (0, 1]$ such that $u(t) \geq c$ for $t \geq t_2$, where $t_2$ is a sufficiently large number. According to $(2.3_0)$, from (1.1), we get

$$\sum_{i=0}^{n-1}(n-i-1)!\,t_1|u^{(i)}(t_1)| \ge \int_{t_1}^{t} s^{n-1}c^{\mu(s)}|p(s)|ds \quad \text{for} \quad t \ge t_2.$$

The last inequality contradicts conditions $(2.9_{1,c})$. The obtained contradiction proves that condition (1.5) holds and, therefore, equation (1.1) possesses Property **B**.

**Theorem 5.1′.** *Let conditions (1.2) and (1.3) be satisfied, let conditions $(2.9_{\ell,1})$ and $(3.1_\ell)$ be satisfied for any $\ell \in \{1,\dots,n\}$ with even $\ell+n$, and let $(2.9_{1,1})$ hold for even $n$. Moreover, let $\beta < +\infty$ and let, for any large $t_* \in R_+$ and $\ell \in \{1,\dots,n-2\}$ with even $\ell+n$, condition $(4.1_\ell)$ be satisfied for some $k \in N$ if $\alpha = 1$ or relation $(4.2_\ell)$ hold for some $k \in N$, $\gamma \in (1,+\infty)$, and $\delta \in (1,\alpha]$ if $\alpha > 1$. Then equation (1.1) possesses Property* **B**, *where $\alpha$ and $\beta$ are defined by the first condition in (2.1) and $\rho^{(\alpha)}_{k,\ell,t_*}$ is given by $(2.11_\ell)$–$(2.13_\ell)$.*

***Proof.*** Since $\beta < +\infty$, by $(2.9_{\ell,1})$ for any $\ell \in \{1,\dots,n\}$ with even $\ell+n$ , condition $(2.9_{\ell,c})$ is satisfied. This means that conditions of Theorem 5.1 are satisfied, which proves the validity of the theorem.

**Theorem 5.2.** *Let $\alpha > 1$, let conditions (1.2), (1.3), $(2.9_{1,c})$, $(3.1_1)$ be satisfied, and let*

$$\liminf_{t\to+\infty}\frac{(\tau(t))^{\mu(t)}}{t} > 0. \tag{5.1}$$

*If, in addition, for some $\delta \in (1,\alpha)$,*

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta}|p(\xi)|d\xi\, ds = +\infty, \tag{5.2}$$

*when $n$ is odd, or*

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-3-\delta}(\tau(\xi))^{\delta+\mu(\xi)}|p(\xi)|d\xi\, ds = +\infty, \tag{5.3}$$

*when $n$ is even, then equation (1.1) possesses Property* **B**, *where $\alpha$ is given by the first condition in (2.1).*

***Proof.*** According to $(2.9_{1,c})$, $(3.1_1)$, and (5.1), it is obvious that, for any $\ell \in \{1,\dots,n\}$, conditions $(2.9_{\ell,c})$ and $(3.1_\ell)$ hold. On the other hand, by (5.1), (5.2), and (5.3), for any $\ell \in \{1,\dots,n-2\}$ with even $\ell+n$, condition $(4.2_\ell)$ is satisfied. This means that if $\alpha > 1$, then all conditions of Theorem 5.1 hold, which proves the validity of the theorem.

**Theorem 5.2′.** *Let $\alpha > 1$, let $\beta < +\infty$, and let conditions (1.2), (1.3), $(2.9_{1,1})$, $(3.1_1)$, and (5.1) be satisfied. Moreover, assume that, for some $\delta \in (1,\alpha]$, condition (5.2) holds if $n$ is odd and condition (5.3) holds if $n$ is even. Then equation (1.1) possesses Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

***Proof.*** Since $\beta < +\infty$, by virtue of $(2.9_{1,1})$, it is clear that, for any $c \in (0,1]$, condition $(2.9_{1,c})$ is satisfied. Thus, all conditions of Theorem 5.2 are satisfied, which proves the validity of the theorem.

***Corollary 5.1.*** *Let $\alpha > 1$, let conditions (1.2), (1.3), ($2.9_{1,c}$), ($3.1_1$), and (5.1) be satisfied, and let*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-3}\tau(s)|p(s)|ds > 0. \tag{5.4}$$

*Moreover, if, for some $\delta \in (1,\alpha]$ and $\gamma > 0$,*

$$\int_0^{+\infty}\int_s^{+\infty} \xi^{n-2-\delta}(\tau(\xi))^{\delta+\gamma(\mu(\xi)-\delta)}|p(\xi)|d\xi\, ds = +\infty, \tag{5.5}$$

*then equation (1.1) possesses Property* **B**, *where $\alpha$ is defined by the first condition in (2.1).*

***Proof.*** Since $\alpha > 1$, by (5.4), ($2.11_1$), and ($2.13_1$), for any $\gamma > 0$, there exists $t_\gamma \in R_+$ such that

$$\rho^{(\alpha)}_{1,1,t_*}(t) \ge \ell\,!\,t^\gamma$$

for $t \ge t_\gamma$. Therefore, by (5.4), (5.5), and (5.1), for any $\ell \in \{1,\dots,n-2\}$ condition ($4.2_\ell$) holds. Thus, for $\alpha > 1$, all conditions of Theorem 5.1$'$ are satisfied. Hence, according to the same theorem, equation (1.1) has Property **B**.

By Corollary 5.1, Theorem 5.2$'$ can be proved similarly.

***Corollary 5.1$'$.*** *Let $\alpha > 1$, let $\beta < +\infty$, and let conditions (1.2), (1.3), ($2.9_{1,1}$), ($3.1_1$), (5.1), and (5.4) be satisfied. Moreover, if, for some $\delta \in (1,\alpha]$ and $\gamma > 0$, condition (5.5) is true, then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

***Corollary 5.2.*** *Let $\alpha > 1$, let conditions (1.2), (1.3), ($2.9_{1,c}$), ($3.1_1$), (5.1), and (5.4) be satisfied, and let there exist $m \in N$ such that condition (4.7) holds. Then equation (1.1) has Property* **B**, *where $\alpha$ is defined by the first condition in (2.1).*

***Proof.*** By (5.1), ($2.9_{1,c}$), ($3.1_1$), and (5.4), it is clear that, for any $\ell \in \{1,\dots,n\}$, conditions ($2.9_{\ell,c}$), ($3.1_\ell$), and ($4.6_\ell$) are satisfied.

Assume that equation (1.1) has a nonoscillatory proper solution $u : (t_0,+\infty) \to (0,+\infty)$. Then, by (1.2), (1.3), and Lemma 2.1, there exists $\ell \in \{1,\dots,n\}$ such that $\ell + n$ is even and condition ($2.3_\ell$) is satisfied. By Corollary 4.4, $\ell \notin \{1,\dots,n-2\}$. If $\ell = n$ (if $n$ is even and $\ell = 0$), then, by ($2.9_{n,c}$) [($2.9_{1,c}$)] as in Theorem 5.1, we can show that condition (1.4) [condition (1.5)] holds, i.e., equation (1.1) has Property **B**.

***Corollary 5.2$'$.*** *Let $\alpha > 1$ and $\beta < +\infty$ and let conditions (1.2), (1.3), ($2.9_{1,1}$), ($3.1_1$), (5.1), and (5.4) be satisfied. If, in addition, there exists $m \in N$ such that condition* (4.10) *holds, then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

***Corollary 5.3.*** *Let $\alpha > 1$ and let conditions (1.2), (1.3), ($2.9_{1,c}$), ($3.1_1$), and (5.1) be satisfied. Assume, moreover, that there exist $\gamma \in (0,1)$ and $r \in (0,1)$ such that conditions ($4.14_1$) and (4.15) hold and, at least one of the conditions (4.16) or $r\,\alpha < 1$ and, for some $\varepsilon > 0$ and $\delta \in (1,\alpha)$, ($4.17_1$) are satisfied. Then equation (1.1) has Property* **B**, *where $\alpha$ is given by the first condition in (2.1).*

***Proof.*** Suppose that equation (1.1) has a proper nonoscillatory solution $u : (t_0, +\infty) \to (0, +\infty)$. Then, by (1.2), (1.3), and Lemma 2.1, there exists $\ell \in \{1, \dots, n\}$ such that $\ell + n$ is even and condition $(2.3_\ell)$ holds. Since by $(2.9_{1,c})$, $(3.1_1)$, $(4.14_1)$, and (5.1), for any $\ell \in \{1, \dots, n-2\}$, conditions $(2.9_{\ell,c})$, $(3.1_\ell)$ and $(4.14_\ell)$ are satisfied, according to Corollary 4.6, we get $\ell \notin \{1, \dots, n-2\}$. On the other hand, by analogy with Theorem 5.1, we can show that if $\ell = 0$ $(\ell = n)$, then condition (1.4) [(1.5)] is satisfied, i.e., equation (1.1) has Property **B**.

***Corollary 5.3′.*** *Let $\alpha > 1$ and $\beta < +\infty$, let conditions (1.2), (1.3), $(2.9_{1,1})$, $(3.1_1)$, and (5.1) be satisfied, and let conditions $(4.14_1)$ and (4.15) hold for some $\gamma \in (0,1)$ and $r \in (0,1)$. Then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

**Theorem 5.3.** *Let $\alpha > 1$, let conditions (1.2), (1.3), $(2.9_{n,c})$, and $(3.1_{n-1})$ be satisfied, and let*

$$\limsup_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t} < +\infty. \tag{5.6}$$

*If, in addition, for some $\delta \in (1, \alpha]$,*

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{1-\delta} (\tau(\xi))^{\delta+(n-3)\mu(\xi)} |p(\xi)| d\xi\, ds = +\infty, \tag{5.7}$$

*then equation (1.1) has Property* **B**, *where $\alpha$ is given by the first condition in (2.1).*

***Proof.*** According to $(2.9_{n,c})$, $(3.1_{n-1})$, and (5.6), it is obvious that, for any $\ell \in \{1, \dots, n-1\}$, conditions $(2.9_{\ell,c})$ and $(3.1_\ell)$ are satisfied. On the other hand, by (5.6) and (5.7), for any $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$, condition $(4.2_\ell)$ holds. Hence, if $\alpha > 1$, then all conditions of Theorem 5.1 are satisfied, which proves the validity of the theorem.

**Theorem 5.3′.** *Let $\alpha > 1$, let $\beta < +\infty$, let conditions (1.2), (1.3), $(2.9_{n,1})$, $(3.1_{n-1})$, and (5.6) be satisfied, and let condition (5.7) hold for some $\delta \in (1, \alpha)$. Then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

***Proof.*** Since $\beta < +\infty$, by $(2.9_{n,1})$ it is obvious that, for any $c \in (0, 1]$, conditions $(2.9_{n,c})$ hold. Thus, all conditions of Theorem 5.3 are satisfied, which proves the validity of the theorem.

***Corollary 5.4.*** *Let $\alpha > 1$, let conditions (1.2), (1.3), $(2.9_{n,c})$, $(3.1_{n-1})$ and (5.6) be satisfied, and let*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} (\tau(s))^{1+(n-3)\mu(s)} |p(s)| ds > 0. \tag{5.8}$$

*If, moreover,*

$$\int_0^{+\infty} \int_s^{+\infty} \xi^{-1-\delta} (\tau(\xi))^{\delta+(n-3)\mu(\xi)+\gamma(\mu(\xi)-\delta)} |p(\xi)| d\xi\, ds = +\infty \tag{5.9}$$

*for some $\delta \in (1, \alpha]$ and $\gamma > 0$, then equation (1.1) has Property* **B**, *where $\alpha$ is given by the first condition in (2.1).*

***Proof.*** Since $\alpha > 1$, by (5.8), ($2.11_{n-2}$), and ($2.13_{n-2}$), for any $\gamma > 0$ there exists $t_* \in R_+$ such that

$$\rho^{(\alpha)}_{1,n-2,t_*}(t) \geq \ell\,!\,t^{\gamma}$$

for $t \geq t_\gamma$. Therefore, by (5.6), (5.8), and (5.9), for any $\ell \in \{1, \dots, n-2\}$ conditions ($4.2_\ell$) are satisfied. Hence, according to the same theorem, equation (1.1) has Property **B**.

***Corollary 5.4′.*** *Let $\alpha > 1$, let $\beta < +\infty$, and let conditions (1.2), (1.3), ($2.9_{n,1}$), ($3.1_{n-1}$), (5.8), and (5.9) be satisfied. Then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

In view of (5.6), by repeating the arguments used in Corollary 5.3, we easily prove that the following corollary is true:

***Corollary 5.5.*** *Let $\alpha > 1$ and let conditions (1.2), (1.3), ($2.9_{n,c}$), ($3.1_{n-1}$), and (5.6) be satisfied. Moreover, assume that there exist $\gamma \in (0,1)$ and $r \in (0,1)$ such that conditions ($4.14_{n-2}$), (4.15), and at least one of the conditions (4.16) or $r\,\alpha < 1$ and, for some $\varepsilon > 0$ and $\delta \in (1, \alpha]$, ($4.17_{n-2}$) are satisfied. Then equation (1.1) has Property* **B**, *where $\alpha$ is given by the first condition in (2.1).*

***Corollary 5.5′.*** *Let $\alpha > 1$, let $\beta < +\infty$, and let conditions (1.2), (1.3), ($2.9_{n,1}$), ($3.1_{n-1}$), and (5.6) be satisfied. Moreover, assume that there exist $\gamma \in (0,1)$ and $r \in (0,1)$ such that conditions ($4.14_{n-2}$), (4.15), and at least one of conditions (4.16) or $r\,\alpha < 1$ and, for some $\varepsilon > 0$ and $\delta \in (1, \alpha]$, ($4.17_{n-2}$) are satisfied. Then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

**Theorem 5.4.** *Let $\alpha = 1$ and let conditions (1.2), (1.3), ($2.9_{1,c}$), ($3.1_1$) and (5.1) be satisfied. If, in addition,*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-2} |p(\xi)| d\xi\, ds > 0, \tag{5.10}$$

*when $n$ is odd, and*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-3} (\tau(\xi))^{\mu(\xi)} |p(\xi)| d\xi\, ds > 0, \tag{5.11}$$

*when $n$ is even, then equation (1.1) has Property* **B**, *where $\alpha$ is given by the first condition in (2.1).*

***Proof.*** According to ($2.9_{1,c}$), ($3.1_1$), and (5.1), for any $\ell \in \{1, \dots, n\}$, conditions ($2.9_{\ell,c}$) and ($3.1_\ell$) are satisfied. On the other hand, by (5.1), (5.10), and (5.11), condition ($4.1_\ell$) holds for any $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$. Hence, if $\alpha = 1$, then all conditions of Theorem 5.1 are satisfied, which proves the validity of the theorem.

**Theorem 5.4′.** *Let $\alpha = 1$, let $\beta < +\infty$, and let conditions (1.2), (1.3), ($2.9_{1,1}$), ($3.1_1$), (5.1), (5.10), and (5.11) be satisfied. Then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

***Proof.*** Since $\beta < +\infty$, by ($2.9_{1,1}$), condition ($2.9_{1,c}$) holds for any $c \in (0,1]$. Thus, all conditions of Theorem 5.4 are satisfied, which proves the validity of the theorem.

**Theorem 5.5.** *Let $\alpha = 1$ and let conditions (1.2), (1.3), ($2.9_{1,c}$), ($3.1_1$), and (5.1) be satisfied. If, in addition,*

$$\liminf_{t\to+\infty} t \int_0^t \int_s^{+\infty} \xi^{n-3}\tau(\xi)|p(\xi)|d\xi\, ds > \max\left(\frac{\ell\,!(n-\ell)!}{\omega^{\ell-1}}, \ell \in \{1, 2, \dots, n-2\}\right) \tag{5.12}$$

*and*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi^{n-2}(\tau(\xi))^{\mu(\xi)}|p(\xi)|d\xi\, ds > 0, \tag{5.13}$$

*then equation (1.1) has Property* **B**, *where*

$$\omega = \liminf_{t\to+\infty} \frac{(\tau(t))^{\mu(t)}}{t}. \tag{5.14}$$

***Proof.*** By (5.12), (5.14), and ($2.11_\ell$), it is obvious that, for large $t$, we get

$$\rho^{(1)}_{1,\ell,t_*}(t) \geq \ell\,!\,t, \quad \ell \in \{1, \dots, n-2\}. \tag{5.15}$$

On the other hand, according to ($2.9_{1,c}$), ($3.1_1$), (5.1), (5.14), (5.15), and (5.13), for any $\ell \in \{1, \dots, n-1\}$, conditions ($2.9_{\ell,c}$), ($3.1_\ell$) and ($4.1_\ell$) are satisfied. Thus, if $\alpha = 1$, then all conditions of Theorem 5.1 are true, which proves the validity of the theorem.

The proof of Theorem 5.4 has been used as a guide in proving Theorem 5.4′. In exactly the same way, the proof of Theorem 5.5 is used as a guide in proving the next theorem .

**Theorem 5.5′.** *Let $\alpha = 1$, let $\beta < +\infty$, and let conditions (1.2),(1.3), ($2.9_{1,1}$), ($3.1_1$), (5.1), (5.12), and (5.13) be satisfied. Then equation (1.1) has Property* **B**, *where $\omega$ is given by condition (5.14).*

**Theorem 5.6.** *Let $\alpha = 1$ and let conditions (1.2), (1.3), ($2.9_{n,c}$), ($3.1_{n-1}$) and (5.6) be satisfied. If, in addition,*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi(\tau(\xi))^{(n-3)r(\xi)}|p(\xi)|d\xi\, ds > 0, \tag{5.16}$$

*then equation (1.1) has Property* **B**, *where $\alpha$ is given by the first condition in (2.1).*

***Proof.*** According to ($2.9_{n,c}$), ($3.1_{n-1}$), and (5.6), conditions ($2.9_{\ell,c}$) and ($3.1_\ell$) hold for any $\ell \in \{1, \dots, n-1\}$. On the other hand, by (5.6) and (5.16), condition ($4.1_\ell$) holds for any $\ell \in \{1, \dots, n-2\}$ with even $\ell + n$. Hence, if $\alpha = 1$, then all conditions of Theorem 5.1 are satisfied, which proves the validity of the theorem.

**Theorem 5.6′.** *Let $\alpha = 1$, let $\beta < +\infty$, and let conditions (1.2), (1.3), ($2.9_{n,1}$), ($3.1_{n-1}$), (5.6), and (5.16) be satisfied. Then equation (1.1) has Property* **B**, *where $\alpha$ and $\beta$ are given by (2.1).*

***Proof.*** Since $\beta < +\infty$, it suffices to show that, by ($2.9_{n,1}$), for any $c \in (0, 1]$ condition ($2.9_{n,c}$) is satisfied.

**Theorem 5.7.** *Let $\alpha = 1$ and let conditions (1.2), (1.3), (2.9$_{n,c}$), (3.1$_{n-1}$), and (5.6) be satisfied. If, in addition,*

$$\liminf_{t\to+\infty} t \int_0^t \int_s^{+\infty} (\tau(\xi)^{1+(n-3)\mu(\xi)}|p(\xi)|d\xi\, ds$$
$$> \max\left(\frac{\ell\,!(n-\ell)!}{\omega^{n-\ell-2}}, \ell \in \{1,2,\dots,n-2\}\right), \tag{5.17}$$

*then the condition*

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \int_s^{+\infty} \xi(\tau(\xi))^{(n-2)\mu(\xi)}|p(\xi)|d\xi\, ds > 0, \tag{5.18}$$

*is sufficient for equation (1.1) to have Property* **B**, *where*

$$\omega = \liminf_{t\to+\infty} \frac{t}{(\tau(t))^{\mu(t)}}\,. \tag{5.19}$$

***Proof.*** By (5.17), (5.19), and (2.11), it is clear that, for large $t$, condition (5.15) holds.

On the other hand, according to (2.9$_{n,c}$), (3.1$_{n-1}$), (5.6), (5.15), (5.18), and (5.19), for any $\ell \in \{1,\dots,n-1\}$, conditions (2.9$_{\ell,c}$), (3.1$_\ell$), and (4.1$_\ell$) are satisfied. Thus, if $\alpha = 1$, then all conditions of Theorem 5.1 are true, which proves the validity of the theorem.

**Theorem 5.7′.** *Let $\alpha = 1$, let $\beta < +\infty$, and let conditions (1.2), (1.3), (2.9$_{n,1}$), (3.1$_{n-1}$), and (5.6) be satisfied. If, moreover, conditions (5.17) and (5.18) hold, then equation (1.1) has Property* **B**, *where $\alpha$, $\beta$, and $\omega$ are given by (2.1) and (5.19).*

***Proof.*** Since $\beta < +\infty$, it suffices to show that, by (2.9$_{n,1}$), conditions (2.9$_{n,c}$) are satisfied for any $c \in (0,1]$.

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