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PROPAGATION OF NONLINEAR SURFACE GRAVITY WAVES ON THE BASIS OF A MODEL DEGENERATED IN THE PARAMETER OF DISPERSION

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We obtain equations generalizing the previously known results on the propagation of nonlinear waves in water of variable depth. To this end, we use the method of power series, which enables us to decrease the dimensionality of the problem and asymptotically construct some weakly dispersive but strongly nonlinear models close to the hyperbolic models of propagation of waves on water. The model has a broader range of application as compared with the available experimental and numerical results.

1. Introduction

In the present work, unlike the traditional approaches based on the smallness both of the parameter of nonlinearity α and of the parameter of dispersion β, the nonlinear parameter α is assumed to be arbitrary. This leads to the construction of quasidegenerate models with small dispersion β and high nonlinearity α, i.e., to models close to the hyperbolic models ($\beta = 0$).

In fact, we remove the restriction imposed on the parameter of nonlinearity α. Thus, the system of evolutionary equations obtained in this case describes the propagation of strongly nonlinear waves. As a result, the influence of these nonlinear effects and the variations of depth leads to perturbations of propagating solitary waves and to the formation of "tails," unlike the cases of pure soliton waves under the conditions of balance of the nonlinear and dispersive effects for $\alpha \sim \beta$ and breaking waves for $\beta = 0$.

2. Statement of the Problem

To solve the posed problem, we apply the same approach as in the case of construction of the asymptotic approximations in the theory of vibration of the plates. We consider the problem of propagation of surface gravity waves. It is known that, in most cases, this problem is well described by the model of potential flow of an ideal incompressible liquid in its potential motion. As a result, the problem of determination of the vector field is reduced to a scalar problem for the velocity potential ϕ and the deviations of the free surface η. The problem is solved in the completely nonlinear statement for a liquid of variable depth with the unperturbed free surface $z = 0$ in a Cartesian coordinate system $x,\ y,\ z$.

In what follows, we consider the plane problem, i.e., the solutions do not depend on the coordinate y. The problem is characterized by three determining dimensionless parameters

$$\alpha = a/H_0, \quad \beta = (H_0/l)^2, \quad \gamma = \mathrm{tg}\theta = H_0/l, \quad Ur = \alpha/\beta,$$

where θ is the angle of bottom deviation , Ur is the Ursell number (derived parameter), H_0 is the depth (vertical scale), l is a characteristic horizontal scale, and a is the maximum deviation of the free surface (amplitude).

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The problem is considered in a region

$$\Omega = \left\{(x, y, z) \in R^3 | \tilde{x} \le x < \infty,\ -\infty < y < \infty,\ -H(x) \le z \le \alpha\eta(x,t)\right\}, \tag{1}$$

where $x = \tilde{x}$ is a line in front of the zone of wave breaking.

In the dimensionless variables

$$x^* = \frac{x}{l}, \quad z^* = \frac{z}{H_0}, \quad t^* = \frac{c_0}{l}t = \frac{\sqrt{gH_0}}{l}t, \quad \varphi^* = \frac{c_0}{gla}\varphi, \quad \eta^* = \frac{\eta}{a},$$

in view of (1), the problem is formulated for the required functions ϕ and η in the following way (in what follows, the asterisks are omitted):

$$\beta\varphi_{xx} + \varphi_{zz} = 0 \quad \text{in the region} \quad \Omega, \tag{2}$$

$$z = -H(x) : \varphi_z + \beta H_x \varphi_x = 0, \tag{3}$$

$$z = \alpha\eta\colon \eta_t + \alpha\eta_x\varphi_x - \beta^{-1}\varphi_z = 0, \quad \eta + \varphi_t + (\alpha/2)\varphi_x^2 + (\alpha/2\beta)\varphi_z^2 = 0, \tag{4}$$

$$t = 0\colon \varphi(x,z,t) = f_1(x,z), \quad \varphi_t(x,z,t) = f_2(x,z). \tag{5}$$

It is worth noting that we have introduced three scaling parameters H_0, l, and a instead of a single parameter (which is sufficient). This is necessary for our asymptotic analysis and agrees with the extended Huntley analysis [6, 16].

We do not know any solution of problem (2)–(5) for propagating waves in the completely nonlinear statement. This problem describes a nonlinearly dispersive system in which the propagation of solitary waves is typical. In this case, the approximate analysis is performed by the asymptotic method [8, 17], which enables us to reduce the problem to the investigation of a system of two evolutionary equations. It is assumed that the parameter of dispersion β and the gradient of the bottom surface γ are small. At the same time, the nonlinear parameter α is regarded as arbitrary, contrary to the traditional soliton-based approaches based on the assumption of balance between the nonlinear and dispersive effects: $\alpha \sim \beta$.

To deduce the evolutionary equations for a shallow liquid, we apply the method of power series. In other words, we expand the required functions in a small thickness coordinate (depth) by using the algorithm developed in the theory of elastic bodies of small thickness starting from the works by Cauchy and Poisson [2–4, 11].

3. Construction of the Evolutionary Equations for Propagating Nonlinear Weakly Dispersive Waves Over the Inhomogeneous Bottom

The two-dimensional problem (in the coordinates x and z) is reduced to a one-dimensional problem with respect to x by the method of power series, which reduces the analyzed problem to an infinite system containing the terms β^q and α^n (where q and n are finite integers) and their products. The assumptions concerning the smallness of the parameters β and γ enable us to perform the reduction of infinite systems by preserving the terms of the orders $\beta, \beta^2, \ldots$ corresponding to the long-wave approximations. Further, in the infinite system, we preserve solely the terms of the first order in β and all terms with the nonlinear parameter α^n. This corresponds to a strongly nonlinear weakly dispersive model degenerate in the parameter of dispersion β [12].

The opposite limiting case $\beta >> \alpha$ leads to parabolic models in which the dispersive effects are taken into account completely, whereas the nonlinear effects are insignificant and neglected.

We now represent the function φ in the form of expansion

$$\varphi(x,z,t)=\sum_{n=0}^{\infty}(z+H)^n\beta^n f_n(x,t). \tag{6}$$

It is easy to see that expansion (6) is symmetric in the parameters β and $z+H$.

We now substitute expansion (6) in system (2)–(5). As a result of the substitution in Eq. (2), we get the following recurrence relation:

$$\frac{1}{\beta}f_{xx}^{(k)}=2(k+1)H_x f_x^{(k+1)}+(k+1)H_{xx}f^{(k+1)}.$$

The condition imposed on the bottom (3) allows us to express $f^{(1)}$ in terms of $f^{(0)}=f$:

$$f^{(1)}=-H_x f^{(0)}-\beta H_x^3 f^{(0)}+O(\beta^2).$$

The final expression for φ takes the form

$$\varphi=f-\beta\left[(z+H)H_x f_x+\frac{(z+H)^2}{2}f_{xx}\right]$$
$$+\beta^2\left[-(z+H)H_x^3 f_x+\frac{3}{2}(z+H)^2(H_xH_{xx}f_x+H_x^2f_{xx})\right]$$
$$=\frac{(z+H)^3}{2}\left[\frac{1}{3}H_{xxx}f_x+\ldots+\frac{(z+H)^4}{24}f_{xxxx}\right]+O(\beta^3). \tag{7}$$

Substituting expression (7) in the first boundary condition (4), preserving the terms of the orders β, and $\alpha^n\beta$ and taking into account the equalities $h=H+\alpha\eta$ and $\omega=f_x$, we arrive at the equation

$$\eta_t+h_x\omega+h\omega_x-\beta\left[h_x\left(\frac{3}{2}H_{xx}\omega_x+\frac{3}{2}H_x\omega_{xx}+\frac{1}{2}H_{xxx}\omega+\frac{\alpha}{2}\eta_x\omega_{xx}\right)\right.$$
$$+h\left(\alpha\eta_xH_{xx}\omega+3H_x^2\omega_x+2\alpha\eta_xH_x\omega_x+3H_xH_{xx}\omega\right)$$
$$\left.+\frac{1}{6}h^3\omega_{xxx}+\alpha\eta_xH_x^2\omega\right]=O(\beta^2). \tag{8}$$

As a result of the differentiation with respect to x and substitution of expression (7), the second condition in (4) is reduced to the following equation:

$$\omega_t+\eta_x+\alpha\frac{1}{2}\left(\omega_x^2\right)_x-\beta\left\{\frac{1}{2}h^2\left[\omega_{xt}+\alpha(\omega\omega_{xx}-\omega_x^2)\right]+\omega_t(hH_x)_x+\omega_{xt}hH_x\right.$$
$$+\alpha\omega\omega_x(h_xH_x+H_x^2+3hH_{xx})+\alpha hH_x\omega_x^2$$

$$\left. + \alpha\omega\omega_{xx} h H_x + \omega^2 \left[H_{xx}(h_x + H_x) + h H_{xxx}\right] \right\}_x = O(\beta^2). \tag{9}$$

The evolutionary equations (8) and (9) form a closed system of coupled equations independent of z.

Further, we introduce the mean velocity (averaged over the depth)

$$u = \frac{1}{h} \int_{-H}^{\alpha\eta_1} \varphi_z dz = \omega - \beta \left[\frac{1}{2} h H_{xx}\omega + h H_x \omega_x + H_x^2 \omega + \frac{1}{6} H^2 \omega_{xx}\right] + O(\beta^2). \tag{10}$$

After necessary transformations performed with the use of expression (10), the evolutionary equations (8) and (9) take the form

$$h_t + (hu)_x = 0, \tag{11}$$

$$\begin{aligned} u_t + \eta_x + \alpha u u_x = {} & \beta \left(\frac{H^3}{3} u_{xxt} + H H_x u_{xt} + \frac{H}{2} H_{xx} u_t\right) \\ & + \alpha\beta \left[(\eta H)_x u_{xt} + H H_x u u_{xx} + \frac{2}{3}\eta H u_{xxt} + \frac{H^2}{3} u u_{xxx} - \frac{H^2}{3} u_x u_{xx} \right. \\ & \left. + \frac{H}{2} H_{xx} u_t + \frac{3}{2} H H_{xx} u u_x + \frac{H}{2} H_{xxx} u^2 + \eta_x H_x u_t \right] + L_1 + O(\beta^2), \end{aligned} \tag{12}$$

where L_1 is the operator taking into account the nonlinearities of higher orders, i.e.,

$$O(\alpha^2\beta, \alpha^3\beta, \alpha^4\beta), \quad \text{and} \quad h = H(x) + \alpha h.$$

4. Analysis of the Results

As indicated in Section 2, the evolutionary equations for shallow liquid are deduced with the use of the method of power series, i.e., by expanding the required functions with respect to a small thickness coordinate (depth), with the help of an algorithm developed in the theory of elastic bodies of small thickness starting from the works by Cauchy and Poisson [4, 11]. Thus, the system of evolutionary equations (11), (12) is obtained from the completely nonlinear statement of the problem by the asymptotic method for the case of propagating plane waves [12, 13].

This system describes both the process of propagation of solitary waves in the case where dispersive effects

$$\beta = \left(\frac{H_0}{l}\right)^2 \ll 1$$

are small as compared with the nonlinear effects of order

$$\alpha = \frac{a}{H_0},$$

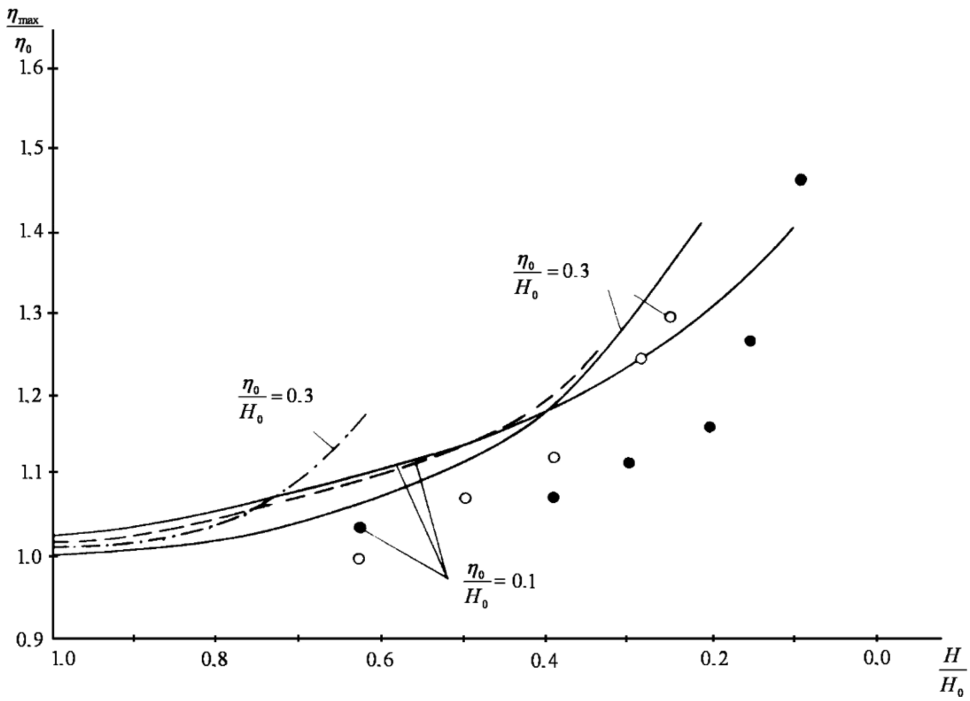

Fig. 1. Dependences of the wave amplitudes $\eta_{\max}/\eta_0$ on the depth of liquid H/H_0 over an even inclined coast $\gamma = 1/20$ for the ratios $\eta_0/H_0 = 0.1$ and 0.3. The experimental data: (○) (0.1) and (•) (0.3) [7]. The numerical results: the dashed curve corresponds to (0.1) and the dash-dotted curve corresponds to (0.3) [9]. The numerical solution of Eqs. (11) and (12) is presented by the continuous curve.

where H_0 is depth, l is the horizontal scale, and

$$a = |\eta|_{\max}$$

is the amplitude, and the process of propagation of nonlinear waves in the absence of flow.

System (11), (12) can be represented in the form of the sum of three operators: the Korteweg–de Vries operator including the terms of the order $\alpha \sim \beta \ll 1$, the operator taking into account the inhomogeneity of the bottom surface of the order $\beta \ll 1$ [10], and the operator taking into account the nonlinearities of the order $\alpha\beta$ [15]:

$$L_g = \underset{\alpha\sim\beta}{L_{kdv}} + \underset{\alpha\sim\beta}{L_{inh}} + \underset{\alpha\beta}{L_{inh}}\,. \tag{13}$$

Operator (13) includes the well-known equations as special cases. The system of evolutionary equations (11), (12) and operator (13) cannot be reduced to a single resolving equation even for the simplest types of inhomogeneities. The presented equations were applied to the investigation of the runup of solitons into the region of shallower water. As a result of the numerical analysis accompanied by the comparison with the experimental data, it is shown that the generalized evolutionary equations (11) and (12) describe the process of propagation of surface waves with higher amplitudes as compared with the Korteweg–de-Vries equation [15] (see Fig. 1). We also revealed a distortion of the shape of pulses and the appearance of tails as a result of taking into account the terms of the order $\alpha\beta$.

It is worth noting that the runup of waves onto the inclined coast was studied by using different models in numerous works (see [1]).

5. Special Cases

In the case where the parameter of nonlinearity α is small and has the same order as the parameter of dispersion β, $\alpha \sim \beta \ll 1$, the system of equations (11), (12) is reduced to the well-known equations [10]

$$h_t + (hu)_x = 0,$$

$$u_t + \eta_x + \alpha u u_x = \beta \left(\frac{H^3}{3} u_{xxt} + \frac{H H_x u_{xt} + \frac{H}{2} H_{xx} u_t}{L_{inh}} \right) + O(\beta^2).$$

For $\beta = 0$, system (11), (12) is reduced to a system of quasilinear equations for waves on the shallow water

$$u_t + \eta_x + \alpha u u_x = 0, \quad \eta_t + (hu)_x = 0.$$

For $\alpha = 0$, this system yields the linearized equations

$$u_t = -\eta_x, \quad \eta_t = -(Hu)_x.$$

Denoting $u = \partial\phi/\partial x$, we can represent it in the form of a wave equation:

$$\frac{\partial}{\partial x}\left(H \frac{\partial \varphi}{\partial x}\right) - \frac{\partial^2 \varphi}{\partial t^2} = 0.$$

6. Conclusions

The evolutionary equations (11), (12) are of interest for the analysis of the propagation of waves in the coastal zone, where the characteristic distances are quite small. Therefore, the dispersive effects are not accumulated, whereas the nonlinear effects are significant [14]. The analysis of the initial-boundary-value problem posed for Eqs. (11), (12) and used to describe the transformations of solitary waves running up onto the coast indicates that the influence of significant nonlinear effects and the variations depth on the propagation of solitons leads to a distortion of the wave profile and the appearance of oscillating tails [13].

Note that, following [5], the approach based on the method of power series and leading to the system of evolutionary equations (11), (12) can be generalized to the case of stationary flow over the curved bottom.

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