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DETERMINATION OF THE THERMOELASTIC STATES OF PIECEWISE INHOMOGENEOUS THERMOSENSITIVE BODIES WITH CYLINDRICAL INTERFACES

B. V. Protsyuk

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We suggest a method for the determination of the thermoelastic state caused by plane axisymmetric temperature fields and surface loads in layered isotropic bodies with cylindrical interfaces. The temperature and coordinate dependences of the moduli of elasticity, coefficients of linear temperature expansion, and Poisson ratios are taken into account. The method is based on the solution of the systems of integral-algebraic equations for radial displacements. In the case of a cylinder, these systems are obtained from the integral representation of the solution of the problem for the ordinary differential equation with generalized derivatives. In this case, we use the Green function of the elasticity problem for a homogeneous cylinder. In the cases of a layered space with cylindrical cavity, a continuous cylinder, and the continuous space, the corresponding systems and the remaining relations required for the determination of the thermoelastic state are obtained as a result of the limit transitions. The relations for the determination of thermal stresses in the corresponding single-layer bodies are presented. The numerical investigations are performed for a three-layer cylinder with functionally gradient layer.

The solution of one-dimensional problems of elasticity and thermoelasticity for one- and multilayer cylindrical bodies with variable physicomechanical characteristics is often based on the application of analytic and numerical-analytic methods [1–10, 12, 13, 15, 17–23], including the methods used for the reduction of the corresponding problems to the solution of the integral equations for stresses. In the present work, we propose a method for the determination of the thermoelastic state caused by plane axisymmetric temperature fields and surface loads in isotropic thermosensitive inhomogeneous and piecewise inhomogeneous bodies with cylindrical interfaces. The method is based on the solution of systems of integral-algebraic equations for the radial displacements. Moreover, the required functions appear in the integral operators only in the integrands of single integrals. The method is based on the use of generalized functions and the Green function of the elasticity problem for a homogeneous cylinder.

Statement of the Problem of Thermoelasticity

Consider an elastic body formed by concentric circular hollow isotropic cylinders with different physicomechanical characteristics. It is assumed that these cylinders are in perfect contact and that the bounding cylindrical surfaces of the body are subjected to the action of uniformly applied loads σ_0 and σ_n, respectively. The end faces are subjected to the action of loads whose resultant force is equal to P and the body is placed in a temperature field described by the function

Pidstryhach Institute for Applied Problems in Mathematics and Mechanics, Ukrainian National Academy of Sciences, Lviv, Ukraine.

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$$t(r) = t_1(r) + \sum_{i=1}^{n-1} [t_{i+1}(r) - t_i(r)] S(r - r_i), \tag{1}$$

where $t_p(r)$, $p = 1, \ldots, n$, are the known distributions of temperatures for $r_{p-1} < r < r_p$; r, r_0, and r_p are, respectively, the radial coordinate, the inner radius of the first layer, and the outer radius of the pth layer related to the characteristic linear size ℓ; n is the number of layers, and $S(\zeta)$ is the Heaviside function.

We now determine the thermoelastic state of the body by assuming that the physicomechanical characteristics of the components are functions of temperature and coordinate. For this purpose, we use the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \tag{2}$$

the relations

$$\sigma_r = c(r)\frac{du}{dr} + \nu^*(r)c(r)\left(\frac{u}{r} + \varepsilon_z\right) - c^*(r)\Phi(r),$$

$$\sigma_\varphi = \nu^*(r)c(r)\left(\frac{du}{dr} + \varepsilon_z\right) + c(r)\frac{u}{r} - c^*(r)\Phi(r),$$

$$\sigma_z = c(r)\varepsilon_z + \nu^*(r)c(r)\left(\frac{du}{dr} + \frac{u}{r}\right) - c^*(r)\Phi(r), \tag{3}$$

where the radial displacement $u(r)$ related to ℓ satisfies the equation with generalized derivatives

$$\frac{d}{dr}\left[c(r)\frac{du}{dr}\right] + \frac{d}{dr}\left[\lambda(r)\frac{u}{r}\right] + 2\mu(r)\frac{d}{dr}\left(\frac{u}{r}\right) = \frac{d}{dr}[c^*(r)\Phi(r)] - \varepsilon_z\frac{d\lambda(r)}{dr}, \tag{4}$$

and the boundary conditions

$$\sigma_r\big|_{r=r_0} = -\sigma_0, \qquad \sigma_r\big|_{r=r_n} = -\sigma_n. \tag{5}$$

Here, the functions

$$c(r) = \lambda(r) + 2\mu(r),$$

$$\lambda(r) = \frac{E(t,r)\nu(t,r)}{[1+\nu(t,r)][1-2\nu(t,r)]}, \qquad \mu(r) = \frac{E(t,r)}{2[1+\nu(t,r)]},$$

$$\nu^*(r) = \frac{\nu(t,r)}{1-\nu(t,r)}, \qquad \text{and} \qquad c^*(r) = \frac{E(t,r)}{1-2\nu(t,r)}$$

have the form (1); the functions $E(t,r)$, $\nu(t,r)$, and $\Phi(r)$ coincide (within the limits of the pth layer), respectively, with the modulus of elasticity $E_p(t_p, r)$, Poisson's ratio $\nu_p(t_p, r)$, and

$$\Phi_p(r) = \int\limits_0^{t_p(r)} \alpha_{tp}(\zeta, r)\, d\zeta\,;$$

$\alpha_{tp}(t_p, r)$ are the coefficients of linear thermal expansion of the pth layer, and $\varepsilon_z = \text{const}$ is the level of axial strains (this parameter is now unknown).

Integral Representation of the Solution

We now pass from the differential statement of the problem of determination of displacements to its formulation in the integral form with the help of the Green function obtained as a special case [11]:

$$G(r,\rho) = \frac{1}{2c_0}\left\{\frac{r}{\rho}S(\rho - r) + \frac{\rho}{r}S(r-\rho) + \frac{\rho}{r_n^2 - r_0^2}\left[r\psi_0^+(\rho) + \frac{kr_0^2\psi_n^+(\rho)}{r}\right]\right\}, \tag{6}$$

which is a solution of the problem

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} - \frac{G}{r^2} = -\frac{1}{c_0\rho}\delta(r-\rho), \tag{7}$$

$$\tau_r\big|_{r=r_0} = \tau_r\big|_{r=r_n} = 0. \tag{8}$$

Here and in what follows,

$$\psi_m^{\pm}(\rho) = 1 - 2\nu_0 \pm \frac{r_m^2}{\rho^2}, \quad m = 0, n; \qquad \tau_r = c_0\frac{\partial G(r,\rho)}{\partial r} + \lambda_0\frac{G(r,\rho)}{r}; \qquad k = 1/(1-2\nu_0);$$

$c_0 = \lambda_0 + 2\mu_0$, λ_0, μ_0, and ν_0 are values from the ranges of $\lambda_1(r)$, $\mu_1(r)$, and $\nu_1(t_1, r)$, respectively, and $\delta(\zeta)$ is the Dirac delta-function.

We now multiply Eq. (4) by $rG(r,\rho)$. Integrating the obtained equation from r_0 to r_n, in view of relation (7), we get

$$\frac{c(\rho)u(\rho)}{c_0} = \left\{rG\sigma_r - \left[r\frac{\partial G}{\partial r}c(r) + \lambda(r)G\right]u\right\}\Bigg|_{r_0}^{r_n}$$

$$+ \int\limits_{r_0}^{r_n}\left[r\frac{\partial G}{\partial r}\left(\frac{dc(r)}{dr}\right)_{\text{cl}} + G\left(\frac{d\lambda(r)}{dr}\right)_{\text{cl}}\right]u\,dr$$

$$+ \sum_{i=1}^{n-1}\left(K_{ci}r_i\frac{\partial G}{\partial r}\bigg|_{r=r_i} + K_{\lambda i}\,G\big|_{r=r_i}\right)u_i(r_i)$$

$$+\int_{r_0}^{r_n}\frac{\partial}{\partial r}(rG)c^*(r)\Phi(r)\,dr-\varepsilon_z\int_{r_0}^{r_n}\frac{\partial}{\partial r}(rG)\lambda(r)\,dr, \tag{9}$$

where

$$K_{ci}=c_{i+1}(r_i)-c_i(r_i), \quad K_{\lambda i}=K_{ci}-2K_{\mu i}, \quad \text{and} \quad K_{\mu i}=\mu_{i+1}(r_i)-\mu_i(r_i).$$

The index "cl" means that the corresponding derivative is classical.

System of Integral-Algebraic Equations for Displacements

We now replace the integrals over the thickness of the cylinder in (9) by the sum of integrals over the thicknesses of layers. After appropriate transformations, in view of the boundary conditions (5) and the Green function (6), we get the following system of integral-algebraic equations for the displacements $u_p(\rho)$ of the pth layer:

$$c_p(\rho)u_p(\rho) = u_{tp}(\rho)+\frac{1}{\rho}V_{cp}(\rho)+\rho V_{\mu p}(\rho)+\rho\frac{\beta_{up}\psi_n^+(\rho)+\beta_{up}^*\psi_0^+(\rho)}{r_n^2-r_0^2}$$

$$+\rho U(\rho)+\sum_{i=1}^{n-1}g_{up}^{(i)}(\rho)u_i(r_i)+u_y(\rho)-\varepsilon_z u_{\varepsilon p}(\rho), \quad r_{p-1}<\rho<r_p, \tag{10}$$

where

$$u_{\alpha p}(\rho) = \frac{V_{\alpha p}(\rho)}{\rho}+\rho\frac{\beta_{\alpha p}\psi_n^+(\rho)+\beta_{\alpha p}^*\psi_0^+(\rho)}{r_n^2-r_0^2},$$

$$V_{\alpha p}(\rho) = \int_{r_{p-1}}^{\rho} r\Lambda_{\alpha p}(r)\,dr, \quad \alpha=t,\varepsilon,$$

$$\Lambda_{tp}(r) = c_p^*(r)\Phi_p(r), \quad \Lambda_{\varepsilon p}(r) = \lambda_p(r),$$

$$\beta_{\alpha p} = \sum_{i=1}^{p-1}V_{\alpha i}(r_i), \quad \beta_{\alpha p}^* = \sum_{i=p}^{n}V_{\alpha i}(r_i),$$

$$V_{cp}(\rho) = \int_{r_{p-1}}^{\rho} r y_{1p}(r)u_p(r)\,dr, \quad V_{\mu p}(\rho) = \int_{r_{p-1}}^{\rho}\frac{1}{r}y_{2p}(r)u_p(r)\,dr,$$

$$y_{1p}(r) = \frac{d}{dr}\left[\frac{\mu_p(r)}{1-2\nu_p(t_p,r)}\right], \quad y_{2p}(r) = \frac{d\mu_p(r)}{dr},$$

$$\beta_{\mathrm{u}p} = \sum_{i=1}^{p-1} V_{\mathrm{c}i}(r_i) - kr_0^2 \sum_{i=1}^{p-1} V_{\mu i}(r_i),$$

$$\beta_{\mathrm{u}p}^* = \sum_{i=p}^{n} V_{\mathrm{c}i}(r_i) - kr_n^2 \sum_{i=p}^{n} V_{\mu i}(r_i),$$

$$U(\rho) = \frac{\gamma_n r_n u_n(r_n)\psi_0^+(\rho) - \gamma_0 r_0 u_1(r_0)\psi_n^+(\rho)}{r_n^2 - r_0^2},$$

$$u_y(\rho) = \rho \frac{r_0^2\sigma_0\psi_n^+(\rho) - r_n^2\sigma_n\psi_0^+(\rho)}{k_0(r_n^2 - r_0^2)},$$

$$\gamma_0 = \frac{2\mu_1(r_0)}{k_0} - c_1(r_0), \quad \gamma_n = \frac{2\mu_n(r_n)}{k_0} - c_n(r_n), \quad k_0 = \frac{1-2\nu_0}{1-\nu_0},$$

$$g_{\mathrm{u}p}^{(i)}(\rho) = \frac{r_i\rho}{r_n^2 - r_0^2}\begin{cases} b_{ni}\psi_0^+(\rho), & p \le i, \\ b_{0i}\psi_n^+(\rho), & p > i, \end{cases}$$

$$b_{ni} = K_{ci} - kK_{\mu i}\psi_n^+(r_i), \quad b_{0i} = K_{ci} - kK_{\mu i}\psi_0^+(r_i).$$

In view of the structure of Eqs. (10), we seek their solution in the form of a sum

$$u_p(\rho) = u_p^t(\rho) + u_p^y(\rho) - \varepsilon_z u_p^\varepsilon(\rho), \tag{11}$$

where (in view of the fact that the physicomechanical characteristics are variable) the first term describes the displacements caused by the temperature field, whereas the second term corresponds to the surface loads σ_0 and σ_n in the cylinder with fixed ends.

Substituting (11) in (10), we obtain the corresponding systems of equations for each function $u_p^s(\rho)$, $s = t, y, \varepsilon$:

$$u_p^s(\rho) - \frac{1}{c_p(\rho)}\left[\frac{V_{\mathrm{c}p}^s(\rho)}{\rho} + \rho V_{\mu p}^s(\rho) + \rho\frac{d_{0p}^s\psi_n^+(\rho) + d_{np}^s\psi_0^+(\rho)}{r_n^2 - r_0^2} + \sum_{i=1}^{n-1} g_{\mathrm{u}p}^{(i)}(\rho)u_i^s(r_i)\right] = \frac{u_{0p}^s(\rho)}{c_p(\rho)}, \tag{12}$$

where

$$d_{0p}^s = \beta_{\mathrm{u}p}^s - \gamma_0 r_0 u_1^s(r_0), \quad d_{np}^s = \beta_{\mathrm{u}p}^{*s} + \gamma_n r_n u_n^s(r_n),$$

$$\beta_{up}^{s} = \sum_{i=1}^{p-1} V_{ci}^{s}(r_i) - kr_0^2 \sum_{i=1}^{p-1} V_{\mu i}^{s}(r_i), \qquad \beta_{up}^{*s} = \sum_{i=p}^{n} V_{ci}^{s}(r_i) - kr_n^2 \tilde{\beta}_{up}^{*s},$$

$$\tilde{\beta}_{up}^{*s} = \sum_{i=p}^{n} V_{\mu i}^{s}(r_i), \tag{13}$$

$$V_{cp}^{s}(\rho) = \int_{r_{p-1}}^{\rho} r y_{1p}(r) u_p^s(r)\, dr, \qquad V_{\mu p}^{s}(\rho) = \int_{r_{p-1}}^{\rho} \frac{1}{r} y_{2p}(r) u_p^s(r)\, dr,$$

$$u_{0p}^{t}(\rho) = u_{tp}(\rho), \qquad u_{0p}^{y}(\rho) = u_y(\rho), \qquad u_{0p}^{\varepsilon}(\rho) = u_{\varepsilon p}(\rho).$$

Relations for the Determination of Strains and Stresses

Under the assumption that the solution of Eqs. (12) is known, we now write the relations for the other components of the stress-strain state.

Differentiating (10) with regard for (11) and (13), we get the following formula for the radial strains:

$$c_p(\rho)\varepsilon_{rp}(\rho) = \varepsilon_{rp}^{t}(\rho) + \varepsilon_{rp}^{y}(\rho) - \varepsilon_z \varepsilon_{rp}^{\varepsilon}(\rho), \tag{14}$$

where

$$\varepsilon_{rp}^{s}(\rho) = e_p^s(\rho) - \frac{1}{\rho^2} V_{cp}^{s}(\rho) + V_{\mu p}^{s}(\rho) + \frac{d_{0p}^{s}\psi_n^{-}(\rho) + d_{np}^{s}\psi_0^{-}(\rho)}{r_n^2 - r_0^2} + \sum_{i=1}^{n-1} g_{\varepsilon p}^{(i)}(\rho) u_i^s(r_i),$$

$$e_p^{\alpha}(\rho) = -\frac{1}{\rho^2} V_{\alpha p}(\rho) + \Lambda_{\alpha p}(\rho) + \frac{\beta_{\alpha p}\psi_n^{-}(\rho) + \beta_{\alpha p}^{*}\psi_0^{-}(\rho)}{r_n^2 - r_0^2}, \qquad \alpha = t, \varepsilon,$$

$$e_p^{y}(\rho) = \frac{r_0^2 \sigma_0 \psi_n^{-}(\rho) - r_n^2 \sigma_n \psi_0^{-}(\rho)}{k_0 (r_n^2 - r_0^2)},$$

$$g_{\varepsilon p}^{(i)}(\rho) = \frac{r_i}{r_n^2 - r_0^2} \begin{cases} b_{ni}\psi_0^{-}(\rho), & p \le i, \\ b_{0i}\psi_n^{-}(\rho), & p > i. \end{cases}$$

Substituting (10) and (14) in the dependences for the pth layer obtained on the basis of (3), we get

$$\sigma_{\gamma p}(\rho) = \sigma_{\gamma p}^{t}(\rho) + \sigma_{\gamma p}^{y}(\rho) - \varepsilon_z \sigma_{\gamma p}^{\varepsilon}(\rho), \quad \gamma = r, \varphi, z, \tag{15}$$

where

$$\sigma_{rp}^{s}(\rho) = \sigma_p^{s-}(\rho), \qquad \sigma_{\varphi p}^{s}(\rho) = \sigma_p^{s+}(\rho),$$

$$\sigma_p^{s\mp}(\rho) = f_p^{s\mp}(\rho) \mp \frac{k_{0p}(\rho)V_{cp}^s(\rho)}{\rho^2} + k_p(\rho)V_{\mu p}^s(\rho)$$

$$+ \frac{d_{0p}^s\tilde{\psi}_{np}^{\mp}(\rho) + d_{np}^s\tilde{\psi}_{0p}^{\mp}(\rho)}{r_n^2 - r_0^2} + \sum_{i=1}^{n-1} g_p^{(i)\mp}(\rho)u_i^s(r_i),$$

$$k_{0p}(\rho) = \frac{1-2\nu_p(t_p,\rho)}{1-\nu_p(t_p,\rho)}, \qquad k_p(\rho) = \frac{1}{1-\nu_p(t_p,\rho)},$$

$$g_p^{(i)\mp}(\rho) = \frac{r_i}{r_n^2 - r_0^2}\begin{cases} b_{ni}\tilde{\psi}_{0p}^{\mp}(\rho), & p \le i, \\ b_{0i}\tilde{\psi}_{np}^{\mp}(\rho), & p > i, \end{cases}$$

$$f_p^{\alpha\mp}(\rho) = \mp\frac{k_{0p}(\rho)}{\rho^2}V_{\alpha p}(\rho) - \frac{1}{2}k_{0p}(\rho)(1\mp 1)\Lambda_{\alpha p}(\rho)$$

$$+ \frac{\beta_{\alpha p}\tilde{\psi}_{np}^{\mp}(\rho) + \beta_{\alpha p}^*\tilde{\psi}_{0p}^{\mp}(\rho)}{r_n^2 - r_0^2}, \qquad \alpha = t, \varepsilon,$$

$$f_p^{y\mp}(\rho) = \frac{r_0^2\sigma_0\tilde{\psi}_{np}^{\mp}(\rho) - r_n^2\sigma_n\tilde{\psi}_{0p}^{\mp}(\rho)}{k_0(r_n^2 - r_0^2)},$$

$$\tilde{\psi}_{mp}^{\mp}(\rho) = (1-2\nu_0)k_p(\rho) \mp k_{0p}(\rho)\frac{r_m^2}{\rho^2}, \qquad m = 0, n,$$

$$\sigma_{zp}^t(\rho) = -\zeta_p(\rho)\Phi_p(\rho) + 2\nu_p^*(\rho)[A_p^t + V_{\mu p}^t(\rho)],$$

$$\sigma_{zp}^y(\rho) = 2\nu_p^*(\rho)[A_p^y + V_{\mu p}^y(\rho)],$$

$$\sigma_{zp}^\varepsilon(\rho) = -\zeta_p^*(\rho) + 2\nu_p^*(\rho)[A_p^\varepsilon + V_{\mu p}^\varepsilon(\rho)],$$

$$kA_p^s = \frac{1}{r_n^2 - r_0^2}\left[\beta_{s1}^* + \beta_{up}^s + \beta_{up}^{*s} + \gamma_n r_n u_n^s(r_n) - \gamma_0 r_0 u_1^s(r_0) + \sum_{i=1}^{n-1} g_{zp}^{(i)}u_i^s(r_i)\right],$$

$$\zeta_p(\rho) = \frac{E_p(t_p,\rho)}{1-\nu_p(t_p,\rho)}, \qquad \zeta_p^*(\rho) = \frac{E_p(t_p,\rho)}{1-\nu_p^2(t_p,\rho)},$$

$$g_{zp}^{(i)} = r_i\begin{cases} b_{ni}, & p \le i, \\ b_{0i}, & p > i, \end{cases} \qquad \beta_{y1}^* = \frac{r_0^2\sigma_0 - r_n^2\sigma_n}{k_0}.$$

The axial strains can be found from the condition

$$\int_{r_0}^{r_n} \rho\sigma_{zz}(\rho)\,d\rho \;=\; \sum_{p=1}^{n}\int_{r_{p-1}}^{r_p} \rho\sigma_{zp}(\rho)\,d\rho \;=\; \frac{P}{2\pi}.$$

In view of relation (15), we find

$$\varepsilon_z \;=\; \varepsilon_z^P + \varepsilon_z^t + \varepsilon_z^y, \tag{16}$$

where

$$\varepsilon_z^P \;=\; \frac{P}{2\pi d_\varepsilon},$$

$$\varepsilon_z^t \;=\; \frac{1}{d_\varepsilon}\left(\sum_{p=1}^{n}\eta_{tp} - 2\sum_{p=1}^{n}\eta_{\mu p}^t - \sum_{p=1}^{n}A_p^t\eta_{\nu p}\right), \qquad \varepsilon_z^y \;=\; -\frac{1}{d_\varepsilon}\left(2\sum_{p=1}^{n}\eta_{\mu p}^y + \sum_{p=1}^{n}A_p^y\eta_{\nu p}\right),$$

$$d_\varepsilon \;=\; \sum_{p=1}^{n}\eta_p - 2\sum_{p=1}^{n}\eta_{\mu p}^\varepsilon - \sum_{p=1}^{n}A_p^\varepsilon\eta_{\nu p},$$

$$\eta_{tp} \;=\; \int_{r_{p-1}}^{r_p}\rho\zeta_p(\rho)\Phi_p(\rho)\,d\rho, \qquad \eta_{\mu p}^s \;=\; \int_{r_{p-1}}^{r_p}\rho\nu_p^*(\rho)V_{\mu p}^s(\rho)\,d\rho, \qquad \eta_{\nu p} \;=\; \int_{r_{p-1}}^{r_p}\rho\nu_p^*(\rho)\,d\rho,$$

$$\eta_p \;=\; \int_{r_{p-1}}^{r_p}\rho\zeta_p^*(\rho)\,d\rho.$$

Special Cases

If we let r_n tend to infinity in (12)–(15) and set $\varepsilon_z = 0$ and $\sigma_n = 0$, then we get the corresponding relations for a layered space with cylindrical cavity. In this case, the systems of integral-algebraic equations for the displacements $u_p^s(\rho)$, $s = t, y$, take the form

$$u_p^s(\rho) - \frac{1}{c_p(\rho)}\left[\frac{V_{cp}^s(\rho) + d_{0p}^s}{\rho} + \rho V_{\mu p}^s(\rho)\right.$$
$$\left. - \,k\tilde{\beta}_{up}^{*s}\rho\psi_0^+(\rho) + \sum_{i=1}^{n-1} g_{up}^{(i)}(\rho)u_i^s(r_i)\right] \;=\; \frac{u_{0p}^s(\rho)}{c_p(\rho)}, \tag{17}$$

and the terms in formulas (14) and (15) are given by the expressions:

$$\varepsilon^s_{rp}(\rho) = e^s_p(\rho) + V^s_{\mu p}(\rho) - k\tilde{\beta}^{*s}_{up}\psi^-_0(\rho) - \frac{V^s_{cp}(\rho) + d^s_{0p}}{\rho^2} + \sum_{i=1}^{n-1} g^{(i)}_{\varepsilon p}(\rho) u^s_i(r_i),$$

$$\sigma^{s\mp}_p(\rho) = f^{s\mp}_p(\rho) \mp k_{0p}(\rho)\frac{V^s_{cp}(\rho) + d^s_{0p}}{\rho^2}$$

$$+ k_p(\rho)V^s_{\mu p}(\rho) - k\tilde{\beta}^{*s}_{up}\tilde{\psi}^{\mp}_{0p}(\rho) + \sum_{i=1}^{n-1} g^{(i)\mp}_p(\rho) u^s_i(r_i), \qquad (18)$$

$$\sigma^t_{zp}(\rho) = -\zeta_p(\rho)\Phi_p(\rho) + 2\nu^*_p(\rho)\left[-\tilde{\beta}^{*t}_{up} + \sum_{i=1}^{n-1} g^{(i)}_{zp} u^t_i(r_i) + V^t_{\mu p}(\rho)\right],$$

$$\sigma^y_{zp}(\rho) = 2\nu^*_p(\rho)\left[-\tilde{\beta}^{*y}_{up} + \sum_{i=1}^{n-1} g^{(i)}_{zp} u^y_i(r_i) + V^y_{\mu p}(\rho)\right].$$

Here,

$$u^t_{0p}(\rho) = \frac{1}{\rho}[V_{tp}(\rho) + \beta_{tp}], \qquad u^y_{0p}(\rho) = \frac{r_0^2\sigma_0}{k_0\rho},$$

$$\tilde{\beta}^{*s}_{un} = \int_{r_{n-1}}^{\infty} \frac{1}{r} y_{2n}(r) u^s_n(r)\,dr,$$

$$e^t_p(\rho) = -\frac{V_{tp}(\rho) + \beta_{tp}}{\rho^2} + c^*_p(\rho)\Phi_p(\rho), \qquad e^y_p(\rho) = -\frac{r_0^2\sigma_0}{k_0\rho^2},$$

$$f^{t\mp}_p(\rho) = \mp k_{0p}(\rho)\left[\frac{V_{tp}(\rho) + \beta_{tp}}{\rho^2} - \frac{1}{2}(1 \mp 1)c^*_p(\rho)\Phi_p(\rho)\right],$$

$$f^{y\mp}_p(\rho) = \mp\frac{r_0^2\sigma_0}{k_0\rho^2} k_{0p}(\rho),$$

$$g^{(i)}_{up}(\rho) = \begin{cases} -kK_{\mu i}\rho\psi^+_0(\rho)/r_i, & p \le i, \\ b_{0i}r_i/\rho, & p > i, \end{cases} \qquad g^{(i)}_{\varepsilon p}(\rho) = \begin{cases} -kK_{\mu i}\psi^-_0(\rho)/r_i, & p \le i, \\ -b_{0i}r_i/\rho^2, & p > i, \end{cases}$$

$$g^{(i)\mp}_p(\rho) = \begin{cases} -kK_{\mu i}\tilde{\psi}^{\mp}_{0p}(\rho)/r_i^2, & p \le i, \\ \mp b_{0i}k_{0p}(\rho)/\rho^2, & p > i, \end{cases} \qquad g^{(i)}_{zp} = \begin{cases} -kK_{\mu i}/r_i, & p \le i, \\ 0, & p > i. \end{cases}$$

The other notation is the same as for the layered hollow cylinder.

For $n = 1$, relations (12)–(18) yield the corresponding formulas for the determination of the components of the thermoelastic state in the following one-component hollow inhomogeneous thermosensitive objects:

in a cylinder:

$$u^s(\rho) - \frac{1}{c(\rho)}\left[\frac{1}{\rho}V_c^s(\rho) + \rho V_\mu^s(\rho) + \rho\frac{d_1^s\psi_0^+(\rho) - \gamma_0 r_0 u^s(r_0)\psi_1^+(\rho)}{r_1^2 - r_0^2}\right] = \frac{u_0^s(\rho)}{c(\rho)}, \quad s = t, y, \varepsilon,$$

$$\varepsilon_r^s(\rho) = e^s(\rho) - \frac{1}{\rho^2}V_c^s(\rho) + V_\mu^s(\rho) + \frac{d_1^s\psi_0^-(\rho) - \gamma_0 r_0 u^s(r_0)\psi_1^-(\rho)}{r_1^2 - r_0^2},$$

$$\sigma_r^s(\rho) = f_r^s(\rho) - \frac{k_{01}(\rho)}{\rho^2}V_c^s(\rho) + k_1(\rho)V_\mu^s(\rho) + \frac{d_1^s\tilde{\psi}_0^-(\rho) - \gamma_0 r_0 u^s(r_0)\tilde{\psi}_1^-(\rho)}{r_1^2 - r_0^2}, \tag{19}$$

$$\sigma_\varphi^s(\rho) = f_\varphi^s(\rho) + \frac{k_{01}(\rho)}{\rho^2}V_c^s(\rho) + k_1(\rho)V_\mu^s(\rho) + \frac{d_1^s\tilde{\psi}_0^+(\rho) - \gamma_0 r_0 u^s(r_0)\tilde{\psi}_1^+(\rho)}{r_1^2 - r_0^2},$$

$$\sigma_z^t(\rho) = -\zeta(\rho)\Phi(\rho) + 2\nu^*(\rho)[A^t + V_\mu^t(\rho)],$$

$$\sigma_z^y(\rho) = 2\nu^*(\rho)[A^y + V_\mu^y(\rho)], \qquad \sigma_z^\varepsilon(\rho) = -\zeta^*(\rho) + 2\nu^*(\rho)[A^\varepsilon + V_\mu^\varepsilon(\rho)],$$

where

$$u_0^t(\rho) = \frac{1}{\rho}V_t(\rho) + \frac{V_t(r_1)}{r_1^2 - r_0^2}\rho\psi_0^+(\rho), \qquad u_0^y(\rho) = \rho\frac{r_0^2\sigma_0\psi_1^+(\rho) - r_1^2\sigma_1\psi_0^+(\rho)}{k_0(r_1^2 - r_0^2)},$$

$$u_0^\varepsilon(\rho) = \frac{V_\varepsilon(\rho)}{\rho} + \frac{V_\varepsilon(r_1)\rho\psi_0^+(\rho)}{r_1^2 - r_0^2}, \qquad V_t(\rho) = \int_{r_0}^{\rho} rc^*(r)\Phi(r)\,dr,$$

$$V_c^s(\rho) = \int_{r_0}^{\rho} r\frac{d}{dr}\left[\frac{\mu(r)}{1 - 2\nu(t, r)}\right]u^s(r)\,dr, \qquad V_\mu^s(\rho) = \int_{r_0}^{\rho}\frac{1}{r}\frac{d\mu(r)}{dr}u^s(r)\,dr,$$

$$d_1^s = V_c^s(r_1) - kr_1^2\,V_\mu^s(r_1) + \gamma_1 r_1 u^s(r_1),$$

$$\gamma_0 = 2\mu(r_0)/k_0 - c(r_0), \qquad \gamma_1 = 2\mu(r_1)/k_0 - c(r_1),$$

$$k_{01}(\rho) = \frac{1 - 2\nu(t, \rho)}{1 - \nu(t, \rho)}, \qquad k_1(\rho) = \frac{1}{1 - \nu(t, \rho)},$$

$$\zeta(\rho) = \frac{E(t, \rho)}{1 - \nu(t, \rho)}, \qquad \zeta^*(\rho) = \frac{E(t, \rho)}{1 - \nu^2(t, \rho)},$$

$$e^t(\rho) = -\frac{1}{\rho^2}V_t(\rho) + c^*(\rho)\Phi(\rho) + \frac{V_t(r_1)\psi_0^-(\rho)}{r_1^2 - r_0^2},$$

$$e^y(\rho) = \frac{r_0^2\sigma_0\psi_1^-(\rho) - r_1^2\sigma_1\psi_0^-(\rho)}{k_0(r_1^2 - r_0^2)}, \qquad e^\varepsilon(\rho) = -\frac{V_\varepsilon(\rho)}{\rho^2} + \lambda(\rho) + \frac{V_\varepsilon(r_1)\psi_0^-(\rho)}{r_1^2 - r_0^2},$$

$$f_r^t(\rho) = -\frac{k_{01}(\rho)}{\rho^2}V_t(\rho) + \frac{V_t(r_1)\tilde{\psi}_0^-(\rho)}{r_1^2 - r_0^2},$$

$$f_r^y(\rho) = \frac{r_0^2\sigma_0\tilde{\psi}_1^-(\rho) - r_1^2\sigma_1\tilde{\psi}_0^-(\rho)}{k_0(r_1^2 - r_0^2)},$$

$$f_r^\varepsilon(\rho) = -\frac{k_{01}(\rho)}{\rho^2}V_\varepsilon(\rho) + \frac{V_\varepsilon(r_1)\tilde{\psi}_0^-(\rho)}{r_1^2 - r_0^2},$$

$$f_\varphi^t(\rho) = \frac{k_{01}(\rho)}{\rho^2}V_t(\rho) - k_{01}(\rho)c^*(\rho)\Phi(\rho) + \frac{V_t(r_1)\tilde{\psi}_0^+(\rho)}{r_1^2 - r_0^2},$$

$$f_\varphi^\varepsilon(\rho) = \frac{k_{01}(\rho)}{\rho^2}V_\varepsilon(\rho) - k_{01}(\rho)\lambda(\rho) + \frac{V_\varepsilon(r_1)\tilde{\psi}_0^+(\rho)}{r_1^2 - r_0^2},$$

$$f_\varphi^y(\rho) = \frac{r_0^2\sigma_0\tilde{\psi}_1^+(\rho) - r_1^2\sigma_1\tilde{\psi}_0^+(\rho)}{k_0(r_1^2 - r_0^2)},$$

$$A^s = \frac{V_s(r_1) + V_c^s(r_1) - kr_1^2V_\mu^s(r_1) + \gamma_1 r_1 u^s(r_1) - \gamma_0 r_0 u^s(r_0)}{k(r_1^2 - r_0^2)},$$

$$V_y(\rho) = \frac{r_0^2\sigma_0 - r_1^2\sigma_1}{k_0},$$

$$\varepsilon_z^t = \frac{1}{d_\varepsilon}(\eta_t - 2\eta_\mu^t - A^t\eta_\nu), \qquad \varepsilon_z^y = -\frac{1}{d_\varepsilon}(2\eta_\mu^y + A^y\eta_\nu),$$

$$d_\varepsilon = \eta - 2\eta_\mu^\varepsilon - A^\varepsilon\eta_\nu,$$

$$\eta_t = \int_{r_0}^{r_1}\rho\zeta(\rho)\Phi(\rho)\,d\rho, \qquad \eta_\mu^s = \int_{r_0}^{r_1}\rho\nu^*(\rho)V_\mu^s(\rho)\,d\rho,$$

$$\eta_\nu = \int_{r_0}^{r_1}\rho\nu^*(\rho)\,d\rho, \qquad \eta = \int_{r_0}^{r_1}\rho\zeta^*(\rho)\,d\rho;$$

in the space:

$$u^s(\rho) - \frac{1}{c(\rho)}\left[\frac{V_c^s(\rho) - \gamma_0 r_0 u^s(r_0)}{\rho} + \rho V_\mu^s(\rho) - k\tilde{\beta}_u^{*s}\rho\psi_0^+(\rho)\right] = \frac{u_0^s(\rho)}{c(\rho)}, \qquad s = t, y,$$

$$\varepsilon_r^s(\rho) = e^s(\rho) + V_\mu^s(\rho) - k\tilde{\beta}_u^{*s}\psi_0^-(\rho) - \frac{V_c^s(\rho) - \gamma_0 r_0 u^s(r_0)}{\rho^2},$$

$$\sigma_r^s(\rho) = f_r^s(\rho) - \frac{k_{01}(\rho)}{\rho^2}[V_c^s(\rho) - \gamma_0 r_0 u^s(r_0)] + k_1(\rho)V_\mu^s(\rho) - k\tilde{\beta}_u^{*s}\tilde{\psi}_0^-(\rho), \tag{20}$$

$$\sigma_\varphi^s(\rho) = f_\varphi^s(\rho) + \frac{k_{01}(\rho)}{\rho^2}[V_c^s(\rho) - \gamma_0 r_0 u^s(r_0)] + k_1(\rho)V_\mu^s(\rho) - k\tilde{\beta}_u^{*s}\tilde{\psi}_0^+(\rho),$$

$$\sigma_z^t(\rho) = -\zeta(\rho)\Phi(\rho) + \nu^*(\rho)[-k\tilde{\beta}_u^{*t} + 2V_\mu^t(\rho)],$$

$$\sigma_z^y(\rho) = \nu^*(\rho)[-k\tilde{\beta}_u^{*y} + 2V_\mu^y(\rho)],$$

where

$$u_0^t(\rho) = \frac{1}{\rho}V_t(\rho), \qquad u_0^y(\rho) = \frac{r_0^2\sigma_0}{k_0\rho}, \qquad \tilde{\beta}_u^{*s} = \int_{r_0}^{\infty}\frac{1}{r}\frac{d\mu(r)}{dr}u^s(r)\,dr,$$

$$e^t(\rho) = -\frac{1}{\rho^2}V_t(\rho) + c^*(\rho)\Phi(\rho), \qquad e^y(\rho) = -\frac{r_0^2\sigma_0}{k_0\rho^2},$$

$$f_r^t(\rho) = -\frac{k_{01}(\rho)}{\rho^2}V_t(\rho), \qquad f_r^y(\rho) = -\frac{r_0^2\sigma_0}{k_0\rho^2}k_{01}(\rho),$$

$$f_\varphi^t(\rho) = k_{01}(\rho)\left\{\frac{1}{\rho^2}V_t(\rho) - c^*(\rho)\Phi(\rho)\right\}, \qquad \sigma_\varphi^y(\rho) = -\sigma_r^y(\rho).$$

Substituting $r_0 = 0$ in the formulas for the analyzed hollow bodies, we obtain the corresponding formulas for continuous bodies. In this case, as $\rho \to 0$ (i.e., on their axis), we find

$$\frac{1}{\rho}V_{t1}(\rho) \to 0, \qquad \frac{1}{\rho^2}V_{t1}(\rho) \to \frac{1}{2}c_1^*(0)\Phi_1(0), \qquad \frac{1}{\rho}V_{c1}(\rho) \to 0,$$

$$\frac{1}{\rho^2}V_{c1}(\rho) \to 0, \qquad V_{\mu 1}(\rho) \to 0, \qquad \frac{1}{\rho}V_{\varepsilon 1}(\rho) \to 0, \qquad \frac{1}{\rho^2}V_{\varepsilon 1}(\rho) \to \frac{1}{2}\lambda_1(0).$$

Note that, for the corresponding quasistatic problems, time appears in the obtained relations as a parameter.

Numerical Results

For the variable and constant physicomechanical characteristics, we study the static thermoelastic state of a hollow three-layer cylinder with fixed or free ends. We set a heat flux

$$q_0 = 6 \cdot 10^5 \text{ W/m}^2$$

and a load

$$\sigma_0 = 0.04947 \text{ GPa}$$

on the inner surface, whereas the values of temperature

$$t_c = 20 \text{ °C}$$

and the load

$$\sigma_n = 0.02249 \text{ GPa}$$

are set on the outer surface. The dependences of the physicomechanical characteristics of the first and third layers are chosen as follows:

$$\lambda_t^{(1)}(T_1) = \left(1.71 + 0.21 \cdot 10^{-3} T_1 + 0.116 \cdot 10^{-6} T_1^2\right) \text{ [W/(m} \cdot \text{K)]},$$

$$\alpha_t^{(1)}(T_1) = \left(13.3 \cdot 10^{-6} - 18.9 \cdot 10^{-9} T_1 + 12.7 \cdot 10^{-12} T_1^2\right) \text{ [K}^{-1}\text{]},$$

$$E_1(T_1) = (132.2 - 50.3 \cdot 10^{-3} T_1 - 8.1 \cdot 10^{-6} T_1^2) \text{ [GPa]},$$

$$\lambda_t^{(3)}(T_3) = (14.3 + 0.014\, T_3) \text{ [W/(m} \cdot \text{K))]},$$

$$\alpha_t^{(3)}(T_3) = \left(14.854 \cdot 10^{-6} + 0.0033 \cdot 10^{-6} T_3\right) \text{ [K}^{-1}\text{]},$$

$$E_3(T_3) = (206.11 - 0.07\, T_3) \text{ [GPa]}.$$

The dependences of the physicomechanical characteristics of the second layer are determined according to the relation [16]

$$p_2(T_2, \rho) = \frac{1}{2}[p_1(T_2) - p_3(T_2)] \cos\left(\pi \frac{\rho - \rho_1}{\rho_2 - \rho_1}\right) + \frac{1}{2}[p_1(T_2) + p_3(T_2)].$$

Here, $p_1(T)$ and $p_3(T)$ are the corresponding dependences for the first and third layers, $\lambda_t^{(i)}(T_i)$ are the heat-conduction coefficients, and $T_i = (t_i(\rho) + 273)$ °C.

The constant physicomechanical characteristics correspond to Poisson's ratios $\nu_1 = 0.3$, $\nu_2 = 0.23$, and $\nu_3 = 0.35$ (in what follows, this is combination **1** of Poisson's ratios). We also take the following mean values of the dependences presented above:

$$\lambda_t^{(1)} = \lambda_{s1} = 1.945 \text{ W/(m}\cdot{}^\circ\text{C)}, \quad \lambda_t^{(2)} = 13.18398 \text{ W/(m}\cdot{}^\circ\text{C)},$$

$$\lambda_t^{(3)} = 24.42 \text{ W/(m}\cdot{}^\circ\text{C)},$$

$$\alpha_t^{(1)} = \alpha_{s1} = 0.88457\cdot 10^{-5}\ {}^\circ\text{C}^{-1}, \quad \alpha_t^{(2)} = 1.30428\cdot 10^{-5}\ {}^\circ\text{C}^{-1},$$

$$\alpha_t^{(3)} = 1.72399\cdot 10^{-5}\ {}^\circ\text{C}^{-1},$$

$$E_1 = E_{s1} = 89.95874 \text{ GPa}, \quad E_2 = 122.72937 \text{ GPa}, \quad \text{and} \quad E_3 = 155.5 \text{ GPa}.$$

The distributions of temperatures $t_i(\rho)$ are obtained from the following system of integral equations [14]:

$$t_3(\rho) = t_c + t_s r_0 \int_\rho^{r_3} \frac{\lambda_{s1}}{\xi\lambda_t^{(3)}[t_3(\xi)]} d\xi,$$

$$t_2(\rho) = t_3(r_2) + t_s r_0 \int_\rho^{r_2} \frac{\lambda_{s1}}{\xi\lambda_t^{(2)}[t_2(\xi),\xi]} d\xi, \tag{21}$$

$$t_1(\rho) = t_2(r_1) + t_s r_0 \int_\rho^{r_1} \frac{\lambda_{s1}}{\xi\lambda_t^{(1)}[t_1(\xi)]} d\xi,$$

$$t_s = \frac{q_0\ell}{\lambda_{s1}}.$$

This system is solved, by analogy with system (12) (for $n = 3$), by the method of successive approximations for $r_0 = 0.8$, $r_1 = 0.82$, $r_2 = 0.9$, $r_3 = 1$, and $\ell = 0.05$ m. We choose the solution of the problem of heat conduction for the three-layer cylinder with constant heat-conduction coefficients as the zero-order approximation in the solution of Eqs. (21). At the same time, in the solution of Eqs. (12), the right-hand sides of the corresponding equations are taken as the zero-order approximation.

For constant physicomechanical characteristics in the case of fixed ends $(\varepsilon_z = 0)$, in Table 1, we present the values of displacements $\tilde{u} = u/(\alpha_{s1}t_s)$, strains $\tilde{\varepsilon}_r = \varepsilon_r/(\alpha_{s1}t_s)$, and stresses $\tilde{\sigma}_j = \sigma_j/(E_{s1}\alpha_{s1}t_s)$, $j = r, \varphi, z$, on the bounding and middle surfaces of the layers caused by the temperature field; in Table 2, we present the corresponding values caused by the surface loads. At the same time, in Table 3, we present the third terms in relations (11), (14), and (15) obtained under the simultaneous action of thermal and surface loads (the case of free ends). Here, for the sake of comparison, we show the results of calculations carried out on the basis of the exact solution [11] in the lower rows (under the bar). In this case, the values of axial strains $\varepsilon_z/(\alpha_{s1}t_s)$ computed on the basis of (16) and on the basis of the exact solution are 0.0914444 and 0.0914443, respectively. This reveals a high accuracy of determination of the quantities characterizing the thermoelastic state by using the proposed method.

Table 1

ρ	$\tilde{u}\cdot 10$	$\tilde{\varepsilon}_r\cdot 10^2$	$\tilde{\sigma}_r\cdot 10^2$	$\tilde{\sigma}_\varphi\cdot 10$	$\tilde{\sigma}_z\cdot 10$
r_0	0.1370304	6.464131	0.0	–0.3654808	–0.4972408
	0.1370305	6.464130	0.0	–0.3654807	–0.4972408
$\frac{r_0+r_1}{2}$	0.1425442	4.5715737	–0.0360969	–0.2198995	–0.3552692
	0.1425443	4.5715730	–0.0360968	–0.2198993	–0.3552691
r_1-0	0.1461890	2.7256009	–0.0537797	–0.0779015	–0.2150395
	0.1461891	2.7256002	–0.0537796	–0.0779014	–0.2150395
r_1+0	0.1461890	3.909923	–0.0537798	–0.2413138	–0.4390568
	0.1461891	3.909922	–0.0537796	–0.2413136	–0.4390567
$\frac{r_1+r_2}{2}$	0.1590021	2.519231	–0.1270620	–0.0870617	–0.2921329
	0.1590022	2.519230	–0.1270616	–0.0870615	–0.2921392
r_2-0	0.1665094	1.253597	–0.1284402	0.0533191	–0.1518898
	0.1665095	1.253597	–0.1284398	0.0533193	–0.1518898
r_2+0	0.1665095	2.2012004	–0.1284405	–0.0577990	–0.2946906
	0.1665095	2.2012001	–0.1284398	–0.0577988	–0.2946905
$\frac{r_2+r_3}{2}$	0.1740122	0.8245205	–0.1048831	0.1184741	–0.1160617
	0.1740122	0.8245203	–0.1048823	0.1184743	–0.1160616
r_3	0.1749795	–0.4170706	0.0	0.2774502	0.0534025
	0.1749795	–0.4170707	0.0	0.2774504	0.0534026

We also compare the values of relative displacements, strains, and stresses computed on the basis of [12] and the proposed method for $\nu_1 = \nu_2 = \nu_3 = 0.3$ (in what follows, this is combination **2** of Poisson's ratios) and the presented variable physicomechanical characteristics. It turns out that, in seven approximations, the corresponding values differ by at most 10^{-5}. Note that, for the identical constant Poisson's ratios and the other variable physicomechanical characteristics, the independence of the radial and circular stresses on the axial strains was established in [12]. At the same time, the determination of the thermoelastic state of layered cylinders was reduced to the solution of the systems of integral-algebraic equations for the normalized radial stresses.

The numerical data obtained for combination **1** and for the following Poisson's ratios: $\nu_1 = 0.3$, $\nu_2 = 0.27$, and $\nu_3 = 0.33$ (combination **3**) and the same other variable physicomechanical characteristics demonstrate that we need 32 and 11 approximations, respectively, in order to attain the same accuracy as for combination **2**.

The largest differences between the maximum values of displacements, strains, and radial, circular, and axial stresses for the combinations of Poisson's ratios **1** and **2** do not exceed 2, 9, 12, 20, and 3.7 %, respectively. For the analyzed three combinations of Poisson's ratios **1, 2**, and **3**, the radial changes of the investigated parameters caused by the temperature field in a cylinder with fixed ends are characterized by curves 1, 2, and 3 in Figs. 1–5, respectively. In the same figures, we also illustrate the behavior of the indicated parameters caused

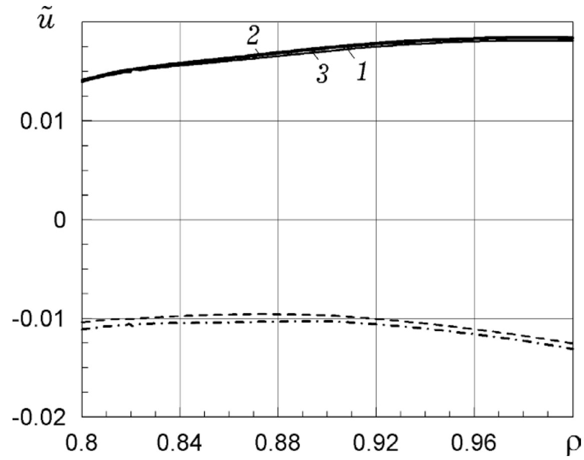

Fig. 1

Table 2

ρ	$\tilde{u} \cdot 10^2$	$\tilde{\varepsilon}_r \cdot 10^2$	$\tilde{\sigma}_r \cdot 10^2$	$\tilde{\sigma}_\varphi \cdot 10^2$	$\tilde{\sigma}_z \cdot 10^2$
r_0	0.4845818 / 0.4845822	–0.5592011 / –0.5592013	–0.4033127 / –0.4033127	0.4927860 / 0.4927865	0.0268419 / 0.0268421
$\frac{r_0+r_1}{2}$	0.4790617 / 0.4790621	–0.5449081 / –0.5449083	–0.3923181 / –0.3923180	0.4817913 / 0.4817919	0.0268419 / 0.0268421
r_1-0	0.4736819 / 0.4736823	–0.5311347 / –0.5311349	–0.3817232 / –0.3817232	0.4711965 / 0.4711197	0.0268419 / 0.0268421
r_1+0	0.4736819 / 0.4736823	–0.4139004 / –0.4139005	–0.3817232 / –0.3817232	0.7180916 / 0.7180923	0.0773647 / 0.0773649
$\frac{r_1+r_2}{2}$	0.4580482 / 0.4580486	–0.3688538 / –0.3688539	–0.3317587 / –0.3317586	0.6681270 / 0.6681277	0.0773647 / 0.0773649
r_2-0	0.4440954 / 0.4440958	–0.3296789 / –0.3296790	–0.2883068 / –0.2883067	0.6246752 / 0.6246759	0.0773647 / 0.0773649
r_2+0	0.4440954 / 0.4440958	–0.3696206 / –0.3696208	–0.2883069 / –0.2883067	0.8167744 / 0.8167753	0.1849636 / 0.1849639
$\frac{r_2+r_3}{2}$	0.4267500 / 0.4267503	–0.3253918 / –0.3253919	–0.2316753 / –0.2316751	0.7601428 / 0.7601439	0.1849636 / 0.1849639
r_3	0.4114487 / 0.4114490	–0.2876299 / –0.2876300	–0.1833242 / –0.1833239	0.7117917 / 0.7117925	0.1849636 / 0.1849639

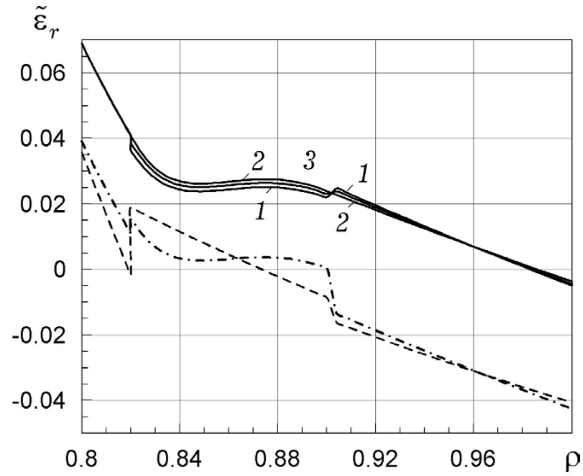

Fig. 2

Table 3

ρ	$\tilde{u} \cdot 10$	$\tilde{\varepsilon}_r \cdot 10$	$\tilde{\sigma}_r \cdot 10^3$	$\tilde{\sigma}_\varphi \cdot 10$	$\tilde{\sigma}_z$
r_0	–0.22588481 –0.22588479	–0.2708948 –0.2708944	0.0 0.0	–0.0088162 –0.0088165	0.0911799 0.0911798
$\frac{r_0 + r_1}{2}$	–0.22859446 –0.22859445	–0.2710354 –0.2710351	–0.0108171 –0.0108174	–0.0087081 –0.0087083	0.0911799 0.0911798
$r_1 - 0$	–0.23130550 –0.23130548	–0.2711709 –0.2711706	–0.0212409 –0.0212416	–0.0086038 –0.0086041	0.0911799 0.0911798
$r_1 + 0$	–0.23130547 –0.23130548	–0.1890222 –0.1890220	–0.0212361 –0.0212416	–0.1034294 –0.1034298	0.1223724 0.1223723
$\frac{r_1 + r_2}{2}$	–0.23895293 –0.23895293	–0.1932498 –0.1932496	–0.4901579 –0.4901511	–0.0987403 –0.0987406	0.1223724 0.1223723
$r_2 - 0$	–0.24675812 –0.24675811	–0.1969264 –0.1969261	–0.8979445 –0.8979524	–0.0946624 –0.0946627	0.1223724 0.1223723
$r_2 + 0$	–0.24675812 –0.24675811	–0.3479965 –0.3479961	–0.8979451 –0.8979524	0.0855424 0.0855417	0.1607477 0.1607475
$\frac{r_2 + r_3}{2}$	–0.26406081 –0.26406079	–0.3442135 –0.3442131	–0.4135531 –0.4135634	0.0806985 0.0806978	0.1607477 0.1607475
r_3	–0.28118867 –0.28118862	–0.3409836 –0.3409832	0.0 0.0	0.0765628 0.0765622	0.1607477 0.1607475

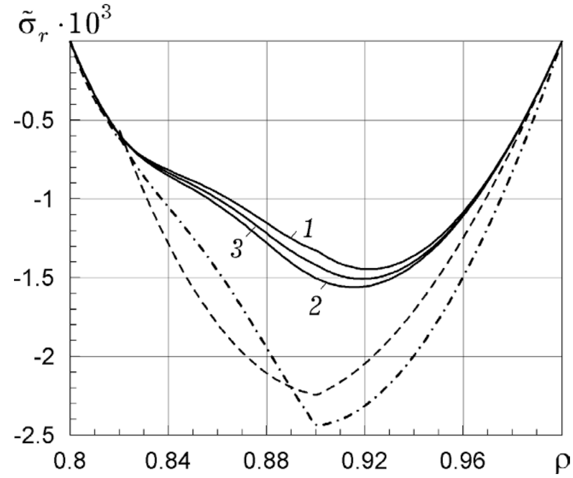

Fig. 3

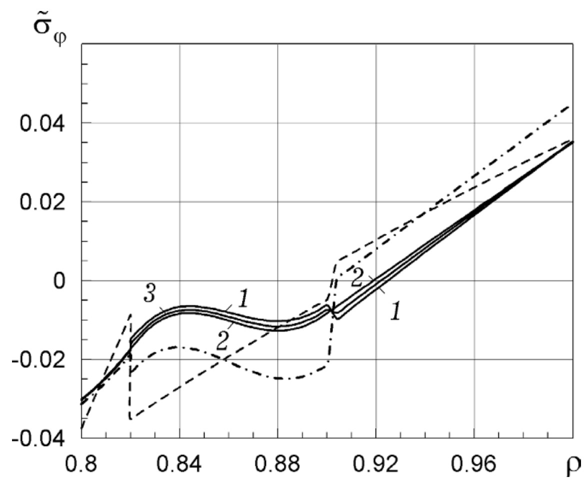

Fig. 4

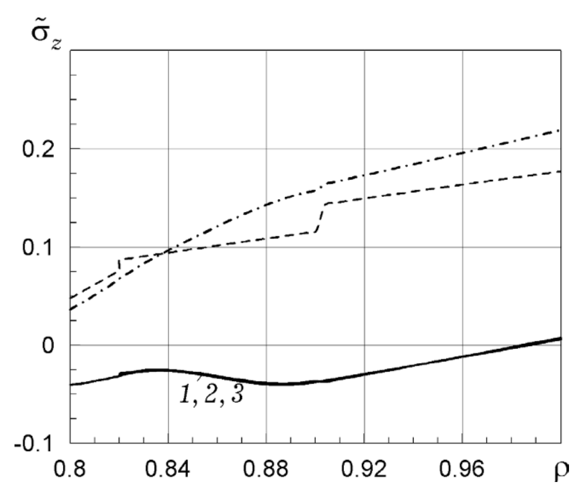

Fig. 5

by the temperature field in a cylinder with free ends for the combination of Poisson's ratios **1** with variable (dash-dotted lines) and constant (dashed lines) moduli of elasticity and the coefficients of linear thermal expansion. It is easy to see that the displacements are negative in a cylinder with free ends, unlike the case of a cylinder with fixed ends (Fig. 1). In other words, under the action of the temperature field, the inner and outer diameters of the analyzed hollow cylinder decrease and the axial stresses are tensile (Fig. 5). In the cases of free or fixed ends of the cylinder, the difference between the strains and the radial and circular stresses may be more than twofold (Figs. 2–4). The analysis of the variations of physicomechanical characteristics leads, in particular, to noticeably different distributions of strains and circular stresses in the middle layer (Figs. 2 and 4).

CONCLUSIONS

The problem of determination of the thermoelastic state caused by plane axisymmetric temperature fields and surface loads in piecewise inhomogeneous thermosensitive isotropic cylindrical bodies is reduced to the solution of the corresponding systems of integral-algebraic equations for radial displacements. The obtained relations are approved for the static problems of thermoelasticity for hollow three-layer cylinders with fixed or free ends and variable or constant physicomechanical characteristics. We demonstrate high accuracy of determination of the parameters of thermoelastic state by using the proposed method and a noticeable effect of variations of the physicomechanical characteristics and the conditions imposed at the ends on their thermomechanical behavior.

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