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ON THE INVARIANT SOLUTIONS OF SOME FIVE-DIMENSIONAL D'ALEMBERT EQUATIONS

V. I. Fedorchuk UDC 519.46: 517.944

By using the invariants of nonconjugate subgroups of the Poincaré group $P(1,4)$ [conjugation is considered with respect to the group $P(1,4)$], we propose ansatzes that reduce some linear and nonlinear five-dimensional d'Alembert equations to ordinary differential equations. On the basis of the solutions of the reduced equations, we construct the invariant solutions of these five-dimensional d'Alembert equations.

Introduction

In the solution of various problems of differential geometry, theory of nonlinear waves, and theoretical and mathematical physics in the spaces of different dimensions, it is customary to use linear and nonlinear Klein–Gordon equations, sine-Gordon equations, sinh-Gordon equations, and Liouville equations, etc.

The linear and nonlinear Klein–Gordon equations in spaces of different dimensions are used for the construction and investigation of the models of field theory.

In the monograph [7], one can find various applications of the Klein–Gordon equations to the five-dimensional field theory. In the five-dimensional Minkowski space $M(1,4)$, linear Klein–Gordon equations appear in the theory of fields with fundamental length [5].

The translation-invariant and spherically symmetric analytic solutions of the nonlinear multidimensional Klein–Gordon equations with polynomial nonlinearities were constructed and analyzed in [22].

Some other models of the field theory connected with nonlinear multidimensional Klein–Gordon equations were described in [20, 21].

The sine-Gordon equations in spaces of different dimensions are extensively used in physics and mathematics. Thus, the two-dimensional sine-Gordon equation is applied, in particular, to the description of propagation of dislocations in crystals, of the motion of Bloch walls in magnetic crystals, and of the surfaces with constant negative curvature. It is also used in the unitary theory of elementary particles, in the Thirring model of the classical and quantum field theories, etc. (see [1, 4, 11, 25, 29, 34] and the references therein).

In the spaces of higher dimensions, the sine-Gordon equation is also used in physics. It has been thoroughly investigated in [10, 12, 26, 35].

The soliton solutions of the two-dimensional sine-Gordon equations are well known [6].

The multiparameter families of exact solutions of the sine-Gordon equations in spaces of different dimensions were constructed in [14, 17–20].

The sinh-Gordon equation in spaces of different dimensions is extensively used in physics and mathematics. In particular, this equation appears in analyzing some problems of the field theory [27].

Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, Ukrainian National Academy of Sciences, Lviv, Ukraine.

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The analysis and physical interpretation of the solutions of two-dimensional sinh-Gordon equation can be found in [32].

The multiparameter families of exact solutions for the sinh-Gordon equation in the spaces of different dimensions are constructed in [16, 18].

Singular solutions of essentially nonlinear Liouville and sinh-Gordon equations were constructed and studied in [33]. A physical interpretation of singular solutions was also proposed in the cited work.

The Liouville equation appears in the problems of differential geometry, theory of nonlinear waves, and quantum field theory [3].

The general solution of the Liouville equation in the two-dimensional case was constructed by Liouville in 1853.

The symmetry reduction for the three-dimensional Liouville equation was carried out and some exact solutions of this equation were obtained in [17, 18].

The symmetry reduction for the Liouville equation in the Minkowski space $\mathbb{R}_{1,n}$ was carried out in [2]. Some exact solutions of this equation were constructed in the same paper.

Singular solutions of the Liouville equation were constructed and investigated in [13, 23, 33].

In the present work, we consider the following five-dimensional partial differential equations:

$$\Box_5 u = \lambda u, \qquad \lambda \in \mathbb{R}, \tag{1}$$

$$\Box_5 u = \sin u, \tag{2}$$

$$\Box_5 u = e^u, \tag{3}$$

$$\Box_5 u = \sinh u, \tag{4}$$

where

$$\Box_5 \equiv \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}$$

is the d'Alembert operator in the five-dimensional Minkowski space $M(1,4)$.

To study these equations, we use the regular method of construction of (partial) exact solutions of differential equations discovered many years ago by S. Lie (see, e.g., [30, 31]).

Equations (1)–(4) are invariant under the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is a group of rotations and translations of the space $M(1,4)$. The nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ [the operation of conjugation was considered with respect to the group $P(1,4)$] were described in [8, 9, 15].

In the present work, we apply the subgroup structure of the group $P(1,4)$ and the invariants of its nonconjugate subgroups to perform the symmetry reduction of Eqs. (1)–(4) and construct some classes of their invariant solutions. These solutions are presented in next sections without details of calculations.

1. Some Invariant Solutions of the Linear Five-Dimensional d'Alembert Equation

Consider an equation

$$\Box_5 u = \lambda u, \qquad \lambda \in \mathbb{R}.$$

In what follows, we present some invariant solutions of this equation.

(a) Case $\lambda \neq 0$.

1. $$u(x) = c_1 \exp\left(\sqrt{\frac{\lambda}{k}}\,\omega\right) + c_2 \exp\left(-\sqrt{\frac{\lambda}{k}}\,\omega\right),$$

where ω are the one-dimensional invariants of nonconjugate subgroups of the group $P(1,4)$ and k is a constant.

The following ω and k are possible:

$$\omega = x_0, \quad k = 1, \qquad \omega = x_2, \quad k = -1,$$

$$\omega = x_3, \quad k = -1, \qquad \omega = x_4, \quad k = -1,$$

$$\omega = x_2 - a\ln(x_0 + x_4), \quad k = -1,$$

$$\omega = x_3 - a\ln(x_0 + x_4), \quad k = -1,$$

$$\omega = 2x_2 - (x_0 + x_4)^2, \quad k = -4,$$

$$\omega = (x_0 + x_4)^2 + 2\alpha_0 x_3, \quad k = -4\alpha_0^2,$$

$$\omega = \mu((x_0 + x_4)^2 - 2x_1) + 2x_3, \quad k = -4(\mu^2 + 1),$$

$$\omega = 2(\delta x_2 - \gamma x_3) - \delta(x_0 + x_4)^2, \quad k = -4(\delta^2 + \gamma^2).$$

2. $$u(x) = c_1 J_0(\sqrt{-\lambda\varepsilon}\,\omega) + c_2 Y_0(\sqrt{-\lambda\varepsilon}\,\omega),$$

where J and Y are Bessel functions of the first and second kind, respectively. The following invariants ω and parameters ε are possible:

$$\omega = (x_0^2 - x_4^2)^{1/2}, \quad \varepsilon = 1,$$

$$\omega = (x_1^2 + x_2^2)^{1/2}, \quad \varepsilon = -1,$$

$$\omega = (x_3^2 + x_4^2)^{1/2}, \quad \varepsilon = -1.$$

3. $$u(x) = \frac{c_1}{\omega}\sinh(\sqrt{\lambda\varepsilon}\,\omega) + \frac{c_2}{\omega}\cosh(\sqrt{\lambda\varepsilon}\,\omega),$$

where ω and ε have the following form:

$$\omega = (x_0^2 - x_3^2 - x_4^2)^{1/2}, \qquad \varepsilon = 1,$$

$$\omega = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \qquad \varepsilon = -1.$$

4. $$u(x) = \frac{c_1}{\omega} J_1(\sqrt{-\lambda\varepsilon}\,\omega) + \frac{c_2}{\omega} Y_1(\sqrt{-\lambda\varepsilon}\,\omega),$$

where ω and ε have the following form:

$$\omega = (x_0^2 - x_1^2 - x_2^2 - x_4^2)^{1/2}, \qquad \varepsilon = 1,$$

$$\omega = (x_0^2 - x_1^2 - x_2^2 - x_3^2)^{1/2}, \qquad \varepsilon = 1,$$

$$\omega = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}, \qquad \varepsilon = -1.$$

5. $$u(x) = \frac{c_1}{\omega^3} \exp(\sqrt{\lambda}\,\omega)(\sqrt{\lambda} - \lambda\omega) + \frac{c_2}{\omega^3} \exp(-\sqrt{\lambda}\,\omega)(\lambda\omega + \sqrt{\lambda}),$$

where

$$\omega = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}.$$

6. $$u(x) = f_1(\omega_2)\sin(\sqrt{\lambda}\,\omega_1) + f_2(\omega_2)\cos(\sqrt{\lambda}\,\omega_1),$$

where

$$\omega_1 = x_2, \qquad \omega_2 = x_0 + x_4,$$

$$\omega_1 = x_3, \qquad \omega_2 = x_0 + x_4.$$

7. $$u(x) = f_1(\omega_1)\sin\left(\frac{\sqrt{\lambda}\,\omega_1\omega_2}{\sqrt{\omega_1^2 + 1}}\right) + f_2(\omega_1)\cos\left(\frac{\sqrt{\lambda}\,\omega_1\omega_2}{\sqrt{\omega_1^2 + 1}}\right),$$

where

$$\omega_1 = x_0 + x_4, \qquad \omega_2 = \frac{x_3}{x_0 + x_4} + x_2,$$

$$\omega_1 = x_0 + x_4, \qquad \omega_2 = x_3 + \frac{x_1}{x_0 + x_4}.$$

8. $$u(x) = f_1(\omega_2)J_0(\sqrt{\lambda}\,\omega_1) + f_2(\omega_2)Y_0(\sqrt{\lambda}\,\omega_1),$$

where f_1 and f_2 are arbitrary smooth functions, J and Y are the Bessel functions of the first and second kind, respectively, and ω_1 and ω_2 are the two-dimensional invariants of nonconjugate subgroups of the group $P(1,4)$. These invariants are given by the formulas

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_0 + x_4.$$

9. $$u(x) = \frac{f_1(\omega_1)}{\omega_2}\sinh(\sqrt{-\lambda}\,\omega_2) + \frac{f_2(\omega_1)}{\omega_2}\cosh(\sqrt{-\lambda}\,\omega_2),$$

where

$$\omega_1 = x_0 + x_4, \quad \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

(b) Case $\lambda = 0$.

1. $$u(x) = c_1\omega + c_2,$$

where c_1 and c_2 are arbitrary constants and ω are the one-dimensional invariants of nonconjugate subgroups of the group $P(1,4)$ given by the formulas

$$x_0, \quad x_2, \quad x_3, \quad x_4, \quad x_2 - a\ln(x_0 + x_4), \quad x_3 - a\ln(x_0 + x_4),$$

$$2x_2 - (x_0 + x_4)^2, \quad (x_0 + x_4)^2 + 2\alpha_0 x_3, \quad \mu((x_0 + x_4)^2 - 2x_1) + 2x_3,$$

$$2(\delta x_2 - \gamma x_3) - \delta(x_0 + x_4)^2.$$

2. $$u(x) = c_1\ln(\omega) + c_2,$$

where ω is one of the one-dimensional invariants of nonconjugate subgroups of the group $P(1,4)$ given by the formulas

$$(x_0^2 - x_4^2)^{1/2}, \quad (x_1^2 + x_2^2)^{1/2}, \quad (x_3^2 + x_4^2)^{1/2}.$$

3. $$u(x) = \frac{c_1}{\omega} + c_2,$$

where ω is one of the invariants

$$(x_0^2 - x_3^2 - x_4^2)^{1/2}, \quad (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

4. $$u(x) = \frac{c_1}{\omega^2} + c_2,$$

where ω is one of the following invariants:

$$(x_0^2 - x_1^2 - x_2^2 - x_4^2)^{1/2}, \quad (x_0^2 - x_1^2 - x_2^2 - x_3^2)^{1/2}, \quad (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}.$$

5. $$u(x) = \frac{c_1}{\omega^3} + c_2,$$

where

$$\omega = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}.$$

6. $$u(x) = \omega_1 f_1(\omega_2) + f_2(\omega_2),$$

where f_1 and f_2 are arbitrary smooth functions and ω_1 and ω_2 are the two- dimensional invariants of non-conjugate subgroups of the group $P(1,4)$ given by the formulas

$$\omega_1 = x_2, \quad \omega_2 = x_0 + x_4,$$

$$\omega_1 = x_3, \quad \omega_2 = x_0 + x_4,$$

$$\omega_1 = \frac{x_3}{x_0 + x_4} + x_2, \quad \omega_2 = x_0 + x_4,$$

$$\omega_1 = x_3 + \frac{x_1}{x_0 + x_4}, \quad \omega_2 = x_0 + x_4.$$

7. $$u(x) = \ln(\omega_1) f_1(\omega_2) + f_2(\omega_2),$$

where

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = x_0 + x_4.$$

8. $$u(x) = \frac{1}{\omega_2} f_1(\omega_1) + f_2(\omega_1),$$

where

$$\omega_1 = x_0 + x_4, \quad \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

2. Some Invariant Solutions of the Five-Dimensional Sine-Gordon Equation

Consider an equation

$$\Box_5 u = \sin u.$$

Some invariant solutions of this equation have the form:

1. $$u(x) = 4\arctan(\alpha e^{\varepsilon_0 \omega}) - \frac{1}{2}(1 - \varepsilon)\pi.$$

2. $$u(x) = 2\arccos[\mathrm{dn}(\omega+\alpha,m)]+\frac{1}{2}(1+\varepsilon)\pi, \quad 0<m<1.$$

3. $$u(x) = 2\arccos\left[\mathrm{cn}\left(\frac{\omega+\alpha}{m},m\right)\right]+\frac{1}{2}(1+\varepsilon)\pi, \quad 0<m<1, \quad \alpha=\mathrm{const},$$

where $\mathrm{dn}(\omega+\alpha,m)$ and $\mathrm{cn}\left(\frac{\omega+\alpha}{m},m\right)$ are the Jacobi elliptic functions, $\varepsilon_0=\pm1$, $\varepsilon=\pm1$, $\alpha\in\mathbb{R}$, and ω and ε have the form

$$\omega=x_0, \quad \varepsilon=1, \qquad \omega=x_2, \quad \varepsilon=-1,$$

$$\omega=x_3, \quad \varepsilon=-1, \qquad \omega=x_4, \quad \varepsilon=-1,$$

$$\omega=x_2-a_2\ln(x_0+x_4), \quad \varepsilon=-1,$$

$$\omega = x_3-a\ln(x_0+x_4), \quad \varepsilon=-1.$$

4. $$u(x) = 4\arctan\left(\tanh\frac{\omega}{2}\right),$$

where ω takes one of the following forms:

$$x_2, \quad x_3, \quad x_4, \quad x_2-a_2\ln(x_0+x_4), \quad x_3-a\ln(x_0+x_4).$$

Solutions of the indicated form were obtained for the sine-Gordon equations in spaces of different dimensions in [14, 17, 20]. In particular, solutions 1–3 with $\omega=x_0$ and $\varepsilon=1$ can be found in [18, 20].

3. Some Invariant Solutions of Five-Dimensional Liouville Equation

Consider an equation

$$\Box_5 u = e^u.$$

We now present some invariant solutions for this equation.

1. $$u(x) = \ln\left(\frac{c_1}{2}\left(\tan^2\left(\sqrt{\frac{c_1}{4k}}(\omega+c_2)\right)+1\right)\right),$$

where

$$\omega=x_0, \quad k=1, \qquad \omega=x_2, \quad k=-1,$$

$$\omega=x_3, \quad k=-1, \qquad \omega=x_4, \quad k=-1,$$

$$\omega = x_2 - a_2 \ln(x_0 + x_4), \qquad k = -1,$$

$$\omega = x_3 - a_3 \ln(x_0 + x_4), \qquad k = -1, ,$$

$$\omega = 2x_2 - (x_0 + x_4)^2, \qquad k = -4,$$

$$\omega = (x_0 + x_4)^2 + 2a_0 x_3, \qquad k = -4\alpha_0^2, \qquad \alpha_0 < 0,$$

$$\omega = \mu((x_0 + x_4)^2 - 2x_1) + 2x_3, \qquad k = -4(\mu^2 + 1), \qquad \mu > 0,$$

$$\omega = 2(\delta x_2 - \gamma x_3) - \delta(x_0 + x_4)^2, \qquad k = -4(\delta^2 + \gamma^2), \qquad \gamma > 0.$$

2.
$$u(x) = \ln\left(\frac{\varepsilon(c_1 - 4)}{2\omega^2}\left(\tan^2\left(\frac{1}{2\varepsilon}\sqrt{c_1 - 4}\,(\ln\omega - c_2)\right) + 1\right)\right),$$

where

$$\omega = (x_0^2 - x_4^2)^{1/2}, \qquad \varepsilon = 1,$$

$$\omega = (x_1^2 + x_2^2)^{1/2}, \qquad \varepsilon = -1,$$

$$\omega = (x_3^2 + x_4^2)^{1/2}, \qquad \varepsilon = -1.$$

3.
$$u(x) = \ln\left(-\frac{1}{2f_1^2(\omega_2)}\left(\tanh^2\left(\frac{f_2(\omega_2) + \omega_1}{2f_1(\omega_2)}\right) - 1\right)\right),$$

where

$$\omega_1 = x_2, \qquad \omega_2 = x_0 + x_4, \qquad \omega_1 = x_3, \qquad \omega_2 = x_0 + x_4.$$

4.
$$u(x) = \ln\left(-\frac{f_1(\omega_1)}{2\omega_1^2}\left(\tanh^2\left(\frac{\sqrt{f_1(\omega_1)}\,(f_2(\omega_1) + \omega_2)}{2\sqrt{\omega_1^2 + 1}}\right) - 1\right)\right),$$

where

$$\omega_1 = x_0 + x_4, \qquad \omega_2 = \frac{x_3}{x_0 + x_4} + x_2,$$

$$\omega_1 = x_0 + x_4, \qquad \omega_2 = x_3 + \frac{x_1}{x_0 + x_4}.$$

5. $$u(x) = \ln\left(\frac{4 - f_1(\omega_2)}{2\omega_1^2}\left(\tan^2\left(\frac{1}{2}\sqrt{f_1(\omega_2) - 4}\,(f_2(\omega_2) - \ln(\omega_1))\right) + 1\right)\right),$$

where

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \qquad \omega_2 = x_0 + x_4.$$

4. Some Invariant Solutions of the Five-Dimensional Sinh-Gordon Equation

The next equation under consideration has the form

$$\Box_5 u = \sinh u.$$

We now present some invariant solutions of this equation:

1. $$u(x) = 2\operatorname{arctanh}(\sin\omega).$$

2. $$u(x) = 2\operatorname{arctanh}(\operatorname{sn}(z,k)), \qquad z = \frac{\sqrt{c+2}}{2}\omega, \qquad k^2 = \frac{c-2}{c+2}, \qquad c > 2,$$

where $\operatorname{sn}(z,k)$ is the Jacobi elliptic function.

3. $$u(x) = 4\operatorname{arctanh}(e^{\omega}), \qquad c = 2,$$

where $\omega = x_0$.

4. $$u(x) = \operatorname{arccosh}\left(\frac{c}{2}\operatorname{cn}^2(z,k) + \operatorname{sn}^2(z,k)\right),$$

$$z = \frac{\sqrt{c+2}}{2}\omega, \qquad k^2 = \frac{c-2}{c+2}, \qquad c > 2,$$

where ω is one of the following functions:

$$x_2, \quad x_3, \quad x_4, \quad x_2 - a\ln(x_0 + x_4), \quad x_3 - a\ln(x_0 + x_4).$$

The multiparameter families of the exact solutions of the sinh-Gordon equations in spaces of different dimensions were constructed in [16, 18]. In particular, solutions 1–3 with $\omega = x_0$ can be found in [18].

Hence, it is shown that some results established in [16, 18] by using the generalized Lie approach can be obtained within the framework of the classical Lie method for the sinh-Gordon equation in the $(1+4)$-dimensional Minkowski space $M(1,4)$.

Final Remarks

In the present work, we present a collection of invariant solutions of Eqs. (1)–(4). These solutions are obtained by using the standard Lie algorithm and the invariants of nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$. Some of these solutions can be useful for the construction of five-dimensional relativistic models. Information about the exact solutions of Maxwell equations and equations of axion electrodynamics can be found in [24] and [28].

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