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ON THE SPECTRAL DENSITY OF STATIONARY PROCESSES AND RANDOM FIELDS

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In this note, we show that a stationary sequence obtained by applying a fixed deterministic function to shifts of a stationary sequence (satisfying a mild regularity condition) has a spectral density. In the multiparametric setting, we obtain a similar result for a function of a shifted i.i.d. field. Bibliography: 7 titles.

1. Introduction

Stationary processes are an important tool for modeling time series appearing in theoretical probability theory and also in real life evolutions. In many situations, correlations between variables could be viewed as a measure of dependence, and, in the Gaussian setting, they determine the distribution. Condensed information about the correlation structure of a stochastic process is contained in the so-called "*spectral measure*," and, when it exists, in its density called the "*spectral density function.*" Then, covariances between variables are obtained as the Fourier coefficients of this function. Because the spectral density function encapsulates all the information about covariances of a stochastic process, its study occupies a central place in their theory. In this note, our investigation is centered around the existence of a spectral density. Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of complex-valued mean zero random variables defined on a probability space $(\Omega, \mathcal{K}, \mathbf{P})$. We call this sequence *weakly stationary* (or *second order stationary*) if there exist complex numbers $\gamma(n)$, $n \in \mathbb{Z}$, such that

$$\operatorname{cov}(X_j, X_k) = \mathbf{E}(X_j \overline{X_k}) = \gamma(j-k)$$

for all $j, k \in \mathbb{Z}$. Note that $\gamma(-n) = \overline{\gamma(n)}$.

By the Birkhoff–Herglotz Theorem (see, e.g., Brockwell and Davis [3]), there exists a unique measure on the unit circle, or, equivalently, a nondecreasing function F, called the *spectral distribution function* on $[0, 2\pi)$, such that

$$\gamma(n) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} F(\mathrm{d}t) \quad \text{for all } n \in \mathbb{Z}. \tag{1}$$

If F is absolutely continuous with respect to the Lebesgue measure λ on $[0, 2\pi)$, then the Radon–Nikodym derivative f of F with respect to the Lebesgue measure is called the *spectral density;* in other words, $F(\mathrm{d}t) = f(t)\,\mathrm{d}t$ and

$$\gamma(n) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} f(t)\,\mathrm{d}t \quad \text{for all} \ \ n \in \mathbb{Z}.$$

The most common situation where the existence of the spectral density may be established is the case of a *regular process*, cf., e.g. [4, Chap. 7]. Recall that a process $(X_n)_{n\in\mathbb{Z}}$ is called

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regular if the tail space

$$G_{-\infty}^X := \bigcap_{n\in\mathbb{Z}} G_n^X$$

is trivial, where G_n^X is the closed linear span of $\{X_k\}_{k\le n}$.

The regularity of the process is equivalent (cf. [3, Chap. 5] or [4, Chap. 7, Theorem 13]) to the existence of the Wold representation, i.e.,

$$X_k = \sum_{j=0}^{\infty} a_j \eta_{k-j},$$

where $\{a_j\}_{j\ge 0}$ is a square summable deterministic sequence of complex numbers and $\{\eta_n\}_{n\in\mathbb{Z}}$ is an uncorrelated, zero mean, unit variance sequence of random variables such that $G_n^\eta = G_n^X$. In this case, $(X_k)_{k\in\mathbb{Z}}$ has the same scalar product (covariance) structure in $\mathbb{L}_2(\Omega, \mathcal{K}, \mathbf{P})$ as the sequence of functions $(x_k)_{k\in\mathbb{Z}}$ in $\mathbb{L}_2([0,2\pi),\lambda)$, where

$$x_k(t) := (2\pi)^{-1/2} \sum_{j=0}^{\infty} a_j \mathrm{e}^{\mathrm{i}(k-j)t} = \mathrm{e}^{\mathrm{i}kt} x_0(t);$$

therefore,

$$\gamma(k) = \int_0^{2\pi} x_k(t)\,\overline{x_0(t)}\,\mathrm{d}t = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt}\,|x_0(t)|^2\,\mathrm{d}t.$$

It follows that X has the spectral density

$$f(t) = |x_0(t)|^2 = \frac{1}{2\pi}\Big|\sum_{j=0}^{\infty} a_j \mathrm{e}^{-\mathrm{i}jt}\Big|^2, \quad t\in[0,2\pi),$$

cf. [4, Chap. 7, Corollary 5]. Moreover, by the Kolmogorov criterion [4, Chap. 7, Theorem 15], the process $(X_n)_{n\in\mathbb{Z}}$ is regular iff it has a spectral density f satisfying the condition

$$\int_0^{2\pi} \log f(t)\,\mathrm{d}t > -\infty.$$

It is not clear, however, what can we say about the density existence when a regularity condition is not necessarily satisfied, as, for example, in the case of functions of a two-sided sequence of i.i.d. random variables.

More generally, we also study the existence of a spectral density for random fields. For simplicity, we discuss only $\mathbb{Z}^2$-indexed random fields. Extension to the index set $\mathbb{Z}^d$ with $d>2$ is easy.

In the sequel, where necessary, we use the standard coordinate notation, e.g., $\mathbf{k}=(k_1,k_2)$ for $\mathbf{k}\in\mathbb{Z}^2$ and $\mathbf{k}\cdot\mathbf{t} = k_1t_1 + k_2t_2$ for $\mathbf{k}\in\mathbb{Z}^2$, $\mathbf{t}\in\mathbb{R}^2$.

We call a collection of complex-valued mean zero random variables $(X_\mathbf{k})_{\mathbf{k}\in\mathbb{Z}^2}$ *weakly stationary* (or *second order stationary*) if there exist complex numbers $\gamma(\mathbf{n})$, $\mathbf{n}\in\mathbb{Z}^2$, such that

$$\mathrm{cov}(X_\mathbf{j}, X_\mathbf{k}) = \mathbf{E}(X_\mathbf{j}\overline{X_\mathbf{k}}) = \gamma(\mathbf{j}-\mathbf{k})$$

for all $\mathbf{j},\mathbf{k}\in\mathbb{Z}^2$. In the context of weakly stationary random fields, it is known that there exists a unique measure F on $[0,2\pi)^2$ such that

$$\mathrm{cov}(X_\mathbf{k}, X_\mathbf{0}) = \int_{[0,2\pi)^2} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{t}} F(\mathrm{d}t_1,\,\mathrm{d}t_2) \quad \text{for all} \ \ \mathbf{k}\in\mathbb{Z}^2\,.$$

If F is absolutely continuous with respect to Lebesgue measure λ^2 on $[0,2\pi)^2$, then there exists the Radon–Nikodym derivative f of F with respect to λ^2, i.e., $F(\mathrm{d}t_1, \mathrm{d}t_2) = f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$. This function f is called *spectral density*, and we have the equalities

$$\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) = \int\limits_{[0,2\pi)^2} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{t}} f(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \quad \text{for all } \mathbf{k} \in \mathbb{Z}^2 .$$

For the sake of clarity, we treat separately first processes and then random fields.

1.1. Results for stationary processes. We start by pointing out a well-known characterization of the existence of a spectral density.

Theorem 1. *Let $X := (X_k)_{k\in\mathbb{Z}}$ be a mean zero, complex-valued, second order, stationary stochastic process. Then the following statements are equivalent:*

(1) *X has a spectral density.*

(2) *There are complex numbers $(a_j)_{j\in\mathbb{Z}}$ with $\sum_{j\in\mathbb{Z}} |a_j|^2 < \infty$ such that*

$$\gamma(k) := \operatorname{cov}(X_k, X_0) = \sum_{j\in\mathbb{Z}} a_j \overline{a_{j+k}}, \quad k \in \mathbb{Z}.$$

(3) *There exists a stationary process $\widetilde{X} := (\widetilde{X}_k)_{k\in\mathbb{Z}}$ equidistributed with X and such that $\widetilde{X}$ admits a representation*

$$\widetilde{X}_k = \sum_{j\in\mathbb{Z}} a_j\,\eta_{j+k} \quad \textit{for all } k \in \mathbb{Z}, \tag{2}$$

where $(a_j)_{j\in\mathbb{Z}}$ satisfies $\sum_{j\in\mathbb{Z}} |a_j|^2 < \infty$ and $(\eta_j)_{j\in\mathbb{Z}}$ is a sequence of mean zero, unit variance, uncorrelated random variables. In this case, the spectral density is

$$f(t) = \frac{1}{2\pi}\Big|\sum_{j\in\mathbb{Z}} a_j\,\mathrm{e}^{\mathrm{i}jt}\Big|^2.$$

Remark 2. If a second order, stationary stochastic process $(X_k)_{k\in\mathbb{Z}}$ is real-valued, Theorem 1 holds with a sequence $(a_n)_{n\in\mathbb{Z}}$ of real numbers and the density f is a symmetric function.

Furthermore, if the process $(X_k)_{k\in\mathbb{Z}}$ is Gaussian, then the variables $(\eta_j)_{j\in\mathbb{Z}}$ in (2) are i.i.d. standard normal. For this latter statement, see also Varadhan lectures [6, Chap. 6, Sec. 6.6].

Let $(\xi_j)_{j\in\mathbb{Z}}$ be a strictly stationary sequence of random variables defined on $(\Omega, \mathcal{K}, \mathbf{P})$ and for $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{C}$ construct

$$X_0 = g(\ldots, \xi_{-1}, \xi_0, \xi_1 \ldots), \quad X_k = X_0 \circ T^k, \tag{3}$$

where T is the shift operator on $\mathbb{R}^{\mathbb{Z}}$.

Define

$$\mathcal{F}_k = \sigma(\xi_j : j \le k),\ \mathcal{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \mathcal{F}_k,\ \text{and } \mathcal{F} = \sigma((\xi_j)_{j\in\mathbb{Z}}). \tag{4}$$

We assume the following regularity condition:

$$\mathbf{E}(X_0|\mathcal{F}_{-\infty}) = 0 \quad \text{a.s.}, \tag{5}$$

which implies that $\mathbf{E}(X_0) = 0$.

Theorem 3. *Define the strictly stationary sequence (X_k) by (3). Assume that condition (5) is satisfied and $\mathbf{E}|X_0|^2 < \infty$. Then the sequence $(X_k)_{k\in\mathbb{Z}}$ has a spectral density.*

Let us mention that condition (5) is satisfied when the left tail sigma field $\mathcal{F}_{-\infty}$ of $(\xi_k)_{k\in\mathbb{Z}}$ is trivial. This happens, for instance, when $(\xi_k)_{k\in\mathbb{Z}}$ is a sequence of i.i.d. random variables. Other examples are provided by conditions imposed on mixing coefficients.

The strong mixing coefficient is defined in the following way:

$$\alpha(\mathcal{A},\mathcal{B}) = \sup\{|\mathbf{P}(A\cap B) - \mathbf{P}(A)\,\mathbf{P}(B)| : A\in\mathcal{A},\ B\in\mathcal{B}\},$$

where $\mathcal{A}$ and $\mathcal{B}$ are two sigma fields.

The ρ-mixing coefficient, also known as the maximal coefficient of correlation, is defined as

$$\rho(\mathcal{A},\mathcal{B}) = \sup\{\mathbf{E}(XY)/\|X\|_2\|Y\|_2 : X\in\mathbb{L}^2(\mathcal{A}),\ Y\in\mathbb{L}^2(\mathcal{B}),\ \mathbf{E}X=\mathbf{E}Y=0\}.$$

For a stationary sequence of real-valued random variables $(\xi_j)_{j\in\mathbb{Z}}$, $\mathcal{F}^n$ denotes the σ-field generated by ξ_j with indices $j\ge n$, and $\mathcal{F}_k$, as before, denotes the σ-field generated by ξ_j with indices $j\le k$. Then we define the sequences of mixing coefficients

$$\alpha_n = \alpha(\mathcal{F}_0,\mathcal{F}^n) \ \text{ and } \rho_n = \rho(\mathcal{F}_0,\mathcal{F}^n)\,.$$

A sequence is called strongly mixing if $\alpha_n\to 0$. It is well known that for strongly mixing sequences, the left tail sigma field is trivial; see Claim 2.17a in Bradley [2]. Examples of this type include Harris recurrent Markov chains.

If $\rho_n<1$ for some $n\ge 1$, then the tail sigma field is also trivial according to Sec. 2.5 in Bradley [1].

Therefore, the result of Theorem 3 holds for functions of a sequence $(\xi_j)_{j\in\mathbb{Z}}$ if it is strongly mixing or satisfies the condition $\rho_n<1$ for some $n\ge 1$.

1.2. Results for stationary random fields. Similar results hold for random fields. Below, indices are in $\mathbb{Z}^2$, but we can easily formulate the results for indices in $\mathbb{Z}^d$ with d integer. Here is a generalization of Theorem 1 for random fields.

Theorem 4. *A second order, stationary, complex-valued random field* $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ *has a spectral density if and only if there exist numbers* $(a_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ *satisfying the condition* $\sum_{\mathbf{k}\in\mathbb{Z}^2}|a_{\mathbf{k}}|^2<\infty$ *such that* $\operatorname{cov}(X_{\mathbf{k}},X_{\mathbf{0}}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} a_{\mathbf{j}}\,\overline{a_{\mathbf{j}+\mathbf{k}}}$.

Our Remark 2 can be extended to random fields in an obvious way, just replacing indices in $\mathbb{Z}$ by indices in $\mathbb{Z}^2$. The extension of Theorem 3 is more delicate, because, in the multi-index setting, there is no unique interpretation of past and future. Here we restrict our considerations to functions of an i.i.d. random field. As the reader will see, this setting provides some additional useful commutativity properties for the related projection operators.

Let $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ be an i.i.d. random field defined on a probability space $(\Omega,\mathcal{K},\mathbf{P})$ and define a random variable

$$X_{\mathbf{0}} = g((\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}),$$

where $g:\mathbb{R}^{\mathbb{Z}^2}\to\mathbb{C}$ is a measurable function.

Moreover, define two translation operators on $\mathbb{R}^{\mathbb{Z}^2}$:

$$T_1((x_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^2}) = (x_{u_1+1,u_2})_{\mathbf{u}\in\mathbb{Z}^2}$$

and

$$T_2((x_{\mathbf{u}})_{\mathbf{u}\in\mathbb{Z}^2}) = (x_{u_1,u_2+1})_{\mathbf{u}\in\mathbb{Z}^2}.$$

Finally, let

$$X_{\mathbf{u}} = g(T_1^{u_1}T_2^{u_2}(\xi_{\mathbf{k}})) = g((\xi_{\mathbf{k}+\mathbf{u}})_{\mathbf{k}\in\mathbb{Z}^2}). \tag{6}$$

Theorem 5. *Let the stationary sequence* $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ *be defined by* (6) *and assume that* $\mathbf{E}(X_{\mathbf{0}})=0$ *and* $\mathbf{E}|X_{\mathbf{0}}|^2<\infty$. *Then the sequence* $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$ *has a spectral density.*

This theorem has immediate applications, for example, to Volterra-type random fields, which play an important role in the nonlinear system theory. For any $\mathbf{k} \in \mathbb{Z}^2$, define the Volterra-type expansion as follows:

$$X_{\mathbf{k}} = \sum_{\mathbf{u},\mathbf{v} \in \mathbb{Z}^2} b_{\mathbf{u},\mathbf{v}}\, \xi_{\mathbf{k}-\mathbf{u}}\, \xi_{\mathbf{k}-\mathbf{v}}\,,$$

where $b_{\mathbf{u},\mathbf{v}}$ are real numbers satisfying

$$b_{\mathbf{u},\mathbf{v}} = 0 \ \text{ if } \ \mathbf{u} = \mathbf{v} \sum_{\mathbf{u},\mathbf{v} \in \mathbb{Z}^2} b^2_{\mathbf{u},\mathbf{v}} < \infty\,,$$

and $(\xi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2}$ is an i.i.d. random field of centered and square integrable random variables. Under the above conditions, the random field $(X_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^2}$ exists, is stationary, zero mean, and square integrable. By Theorem 5, this random field has a spectral density since it is a function of an i.i.d. field.

2. Proofs

Proof of Theorem 1. (1) $\Rightarrow$ (2). Let f be the spectral density of X. Since $\sqrt{f(x)}$ is square integrable, by the Carleson Theorem (cf. [5]), we have an expansion

$$\sqrt{f(t)} = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt} \qquad \text{a.s. and in } \ L_2([0, 2\pi), \lambda)$$

with Fourier coefficients

$$a_j := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \sqrt{f(t)} \mathrm{e}^{-\mathrm{i}jt}\, \mathrm{d}t, \quad j \in \mathbb{Z},$$

satisfying $\sum_{j \in \mathbb{Z}} |a_j|^2 < \infty$. Therefore, by (1),

$$\begin{aligned}
\gamma(k) &= \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} f(t)\, \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} \Big| \sum_{j \in \mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt} \Big|^2 \mathrm{d}t \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Big(\sum_{j_1 \in \mathbb{Z}} a_{j_1}\, \mathrm{e}^{\mathrm{i}(j_1+k)t} \Big) \Big(\sum_{j_2 \in \mathbb{Z}} \overline{a_{j_2}}\, \mathrm{e}^{-\mathrm{i}j_2 t} \Big)\, \mathrm{d}t = \sum_{j \in \mathbb{Z}} a_j\, \overline{a_{j+k}}\,,
\end{aligned}$$

as required in (2).

(2) $\Rightarrow$ (1). Let

$$f(t) = \frac{1}{2\pi} \Big| \sum_{j \in \mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt} \Big|^2.$$

Then

$$\begin{aligned}
\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} f(t)\, \mathrm{d}t &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} \Big| \sum_{j \in \mathbb{Z}} a_j \mathrm{e}^{\mathrm{i}jt} \Big|^2 \mathrm{d}t \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Big(\sum_{j_1 \in \mathbb{Z}} a_{j_1} \mathrm{e}^{\mathrm{i}(j_1+k)t} \Big) \Big(\sum_{j \in \mathbb{Z}} \overline{a_{j_2}} \mathrm{e}^{-\mathrm{i}j_2 t} \Big)\, \mathrm{d}t = \sum_{j \in \mathbb{Z}} a_j\, \overline{a_{j+k}} = \gamma(k),
\end{aligned}$$

as required in the definition of spectral density.

(3) $\Rightarrow$ (2) is obvious.

For (1) ⇒ (3) see [4, Chap. 7, Theorem 10]. □

Proof of Theorem 3. For every $\ell \in \mathbb{Z}$ we define the projection operator $\mathcal{P}_\ell$ by letting

$$\mathcal{P}_\ell X = \mathbf{E}(X|\mathcal{F}_\ell) - \mathbf{E}(X|\mathcal{F}_{\ell-1})$$

for any integrable random variable $X \in \mathbb{L}_1(\Omega, \mathcal{K}, \mathbf{P})$.

Since we assumed that $\mathbf{E}(X_0|\mathcal{F}_{-\infty}) = 0$ a.s., by stationarity, $\mathbf{E}(X_k|\mathcal{F}_{-\infty}) = 0$ for all $k \in \mathbb{Z}$. Furthermore, since all the X_k are $\mathcal{F}$-measurable, we have the representation

$$X_k = \sum_{\ell\in\mathbb{Z}} \mathcal{P}_\ell X_k.$$

Let us compute the covariances. We have the equalities

$$\operatorname{cov}(X_k, X_0) = \sum_{\ell_1,\ell_2\in\mathbb{Z}} \operatorname{cov}(\mathcal{P}_{\ell_1} X_k, \mathcal{P}_{\ell_2} X_0).$$

Since the projections are orthogonal,

$$\operatorname{cov}(X_k, X_0) = \sum_{\ell\in\mathbb{Z}} \operatorname{cov}(\mathcal{P}_\ell X_k, \mathcal{P}_\ell X_0) = \sum_{\ell\in\mathbb{Z}} \operatorname{cov}(\mathcal{P}_0 X_{k-\ell}, \mathcal{P}_0 X_{-\ell}), \tag{7}$$

where, in the last equality, we used the fact that (X_k) is strictly stationary.

Let us denote $Y_\ell = \mathcal{P}_0 X_\ell$. Note that the stationarity and orthogonality of the projections imply that

$$\sum_{\ell\in\mathbb{Z}} \mathbf{E}\,|Y_\ell|^2 = \sum_{\ell\in\mathbb{Z}} \mathbf{E}\,|\mathcal{P}_0 X_\ell|^2 = \sum_{\ell\in\mathbb{Z}} \mathbf{E}\,|\mathcal{P}_{-\ell} X_0|^2 = \mathbf{E}\,|X_0|^2 < \infty. \tag{8}$$

Consider the function

$$f(t) = \frac{1}{2\pi}\,\mathbf{E}\,\Big|\sum_{\ell\in\mathbb{Z}} Y_{-\ell}\,\mathrm{e}^{\mathrm{i}\ell t}\Big|^2, \quad t \in [0, 2\pi).$$

By the Fubini theorem and (8),

$$\int_0^{2\pi} f(t)\,\mathrm{d}t = \frac{1}{2\pi}\,\mathbf{E}\int_0^{2\pi} \Big|\sum_{\ell\in\mathbb{Z}} Y_{-\ell}\,\mathrm{e}^{\mathrm{i}\ell t}\Big|^2\,\mathrm{d}t = \mathbf{E}\sum_{\ell\in\mathbb{Z}} |Y_{-\ell}|^2 < \infty.$$

Let us now compute Fourier coefficients of f. For every $k \in \mathbb{Z}$,

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kt} f(t)\,\mathrm{d}t &= \frac{1}{2\pi}\,\mathbf{E}\int_0^{2\pi} \Big(\sum_{\ell_1\in\mathbb{Z}} Y_{-\ell_1}\,\mathrm{e}^{\mathrm{i}(k+\ell_1)t}\Big)\Big(\sum_{\ell_2\in\mathbb{Z}} \overline{Y_{-\ell_2}}\,\mathrm{e}^{-\mathrm{i}\ell_2 t}\Big)\,\mathrm{d}t \\ &= \sum_{\ell_1,\ell_2\in\mathbb{Z}} \mathbf{E}\left(Y_{-\ell_1}\overline{Y_{-\ell_2}}\right)\mathbf{1}_{\{k+\ell_1=\ell_2\}} = \sum_{\ell\in\mathbb{Z}} \mathbf{E}\left(Y_{k-\ell}\,\overline{Y_{-\ell}}\right).\end{aligned}$$

By comparing this expression with (7), we see that f is the spectral density for $(X_k)_{k\in\mathbb{Z}}$. □

Proof. Proof of Theorem 4 is completely identical to that of Theorem 1, and therefore, is omitted. We only notice that the spectral density for the process satisfying

$$\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} a_{\mathbf{j}}\,\overline{a_{\mathbf{j}+\mathbf{k}}}$$

has the form

$$f(\mathbf{t}) = \frac{1}{(2\pi)^2}\Big|\sum_{\mathbf{j}\in\mathbb{Z}^2} a_{\mathbf{j}}\,\mathrm{e}^{\mathrm{i}\,\mathbf{j}\cdot\mathbf{t}}\Big|^2, \quad \mathbf{t} \in [0, 2\pi)^2.$$

□

Proof of Theorem 5. Define the sigma fields

$$\mathcal{F}_{k_1,k_2} = \sigma(\xi_{\mathbf{j}} : j_1 \le k_1,\ j_2 \le k_2).$$

Next, for $k \in \mathbb{Z}$, denote $\mathcal{F}_{k,\infty} = \vee_{k_2 \in \mathbb{Z}} \mathcal{F}_{k,k_2}$ and $\mathcal{F}_{\infty,k} = \vee_{k_1 \in \mathbb{Z}} \mathcal{F}_{k_1,k}$.

We introduce the projection operators by letting

$$\mathcal{P}_{u,\infty} X = \mathbf{E}(X|\mathcal{F}_{u,\infty}) - \mathbf{E}(X|\mathcal{F}_{u-1,\infty})$$

and

$$\mathcal{P}_{\infty,u} X = \mathbf{E}(X|\mathcal{F}_{\infty,u}) - \mathbf{E}(X|\mathcal{F}_{\infty,u-1})$$

for any integrable random variable $X \in \mathbb{L}_1(\Omega, \mathcal{K}, \mathbf{P})$. Furthermore, we define the iterated operator by

$$\mathcal{P}_{u_1,u_2} X = (\mathcal{P}_{u_1,\infty} \circ \mathcal{P}_{\infty,u_2}) X.$$

Since the variables $(\xi_{\mathbf{k}})$ are independent,

$$\mathbf{E}\left(\mathbf{E}(X|\mathcal{F}_{p_1,p_2})|\mathcal{F}_{u_1,u_2}\right) = \mathbf{E}(X|\mathcal{F}_{p_1\wedge u_1, p_2 \wedge u_2}) \quad \text{a.s.}$$

for all $-\infty \le p_1, p_2, u_1, u_2 \le \infty$. Using this property and the definition of the iterated operator, we see that for all $u_1, u_2 \in \mathbb{Z}$, almost surely,

$$\mathcal{P}_{u_1,u_2} X = \mathbf{E}(X|\mathcal{F}_{u_1,u_2}) - \mathbf{E}(X|\mathcal{F}_{u_1,u_2-1}) - \mathbf{E}(X|\mathcal{F}_{u_1-1,u_2}) + \mathbf{E}(X|\mathcal{F}_{u_1-1,u_2-1}).$$

We also obtain the same expression for $(\mathcal{P}_{\infty,u_2} \circ \mathcal{P}_{u_1,\infty}) X$; thus, we see that the operators $\mathcal{P}_{u_1,\infty}$ and $\mathcal{P}_{\infty,u_2}$ commute.

Next, we borrow an idea from Volný and Wang [7, Lemma 2.4(ii)] by claiming that $(u_1, u_2) \ne (p_1, p_2)$ yields the equality

$$\operatorname{cov}(\mathcal{P}_{u_1,u_2} X, \mathcal{P}_{p_1,p_2} Y) = \mathbf{E}[(\mathcal{P}_{u_1,u_2} X)(\overline{\mathcal{P}_{p_1,p_2} Y})] = 0$$

for all mean zero X and Y in $\mathbb{L}_2(\Omega, \mathcal{K}, \mathbf{P})$. Indeed, assume, without loss of generality, that $p_1 < u_1$. For any X, the variable $\mathcal{P}_{u_1,\infty} X$ is orthogonal to the space $H = \mathbb{L}_2(\Omega, \mathcal{F}_{u_1-1,\infty}, \mathbf{P})$. Hence, $\mathcal{P}_{u_1,u_2} X$ is also orthogonal to H, while $\mathcal{P}_{p_1,p_2} Y$ belongs to H due to the assumption $p_1 < u_1$.

Note that for all $u \in \mathbb{Z}$, the corresponding tail sigma fields defined as $\mathcal{F}_{u,-\infty} = \bigcap_{u_2 \in \mathbb{Z}} \mathcal{F}_{u,u_2}$, $\mathcal{F}_{-\infty,u} = \bigcap_{u_1 \in \mathbb{Z}} \mathcal{F}_{u_1,u}$, and $\mathcal{F}_{-\infty,-\infty} = \bigcap_{u \in \mathbb{Z}} \mathcal{F}_{u,-\infty}$ are trivial. Therefore,

$$\mathbf{E}(X|\mathcal{F}_{u,-\infty}) = 0 \text{ a.s.}, \quad \mathbf{E}(X|\mathcal{F}_{-\infty,u}) = 0 \text{ a.s.}, \quad \text{and} \quad \mathbf{E}(X|\mathcal{F}_{-\infty,-\infty}) = 0 \text{ a.s.}$$

It follows that for any mean zero X in $\mathbb{L}_2(\Omega, \mathcal{K}, \mathbf{P})$ we have the following orthogonal representation:

$$X = \sum_{u_1 \in \mathbb{Z}} \mathcal{P}_{u_1,\infty} X = \sum_{u_1 \in \mathbb{Z}} \mathcal{P}_{u_1,\infty} \Big(\sum_{u_2 \in \mathbb{Z}} \mathcal{P}_{\infty,u_2} X \Big) = \sum_{u_1,u_2 \in \mathbb{Z}} \mathcal{P}_{u_1,u_2} X \quad \text{a.s.} \tag{9}$$

Let us compute the covariances of $X_{\mathbf{k}}$ and $X_{\mathbf{0}}$. By using the above projection decomposition written for both $X_{\mathbf{k}}$ and $X_{\mathbf{0}}$, together with the orthogonality of the projections and stationarity, we see that

$$\begin{aligned}\operatorname{cov}(X_{\mathbf{k}}, X_{\mathbf{0}}) &= \operatorname{cov}\Big(\sum_{\mathbf{j}\in\mathbb{Z}^2} \mathcal{P}_{j_1,j_2} X_{\mathbf{k}}, \sum_{\mathbf{u}\in\mathbb{Z}^2} \mathcal{P}_{u_1,u_2} X_{\mathbf{0}}\Big) \\ &= \sum_{\mathbf{j}\in\mathbb{Z}^2} \operatorname{cov}(\mathcal{P}_{j_1,j_2} X_{\mathbf{k}}, \mathcal{P}_{j_1,j_2} X_{\mathbf{0}}) \\ &= \sum_{\mathbf{j}\in\mathbb{Z}^2} \operatorname{cov}(\mathcal{P}_{0,0} X_{\mathbf{k}-\mathbf{j}}, \mathcal{P}_{0,0} X_{-\mathbf{j}}) \\ &= \sum_{\mathbf{j}\in\mathbb{Z}^2} \operatorname{cov}(Y_{\mathbf{k}-\mathbf{j}}, Y_{-\mathbf{j}}) \qquad (10)\end{aligned}$$

for all $\mathbf{k} \in \mathbb{Z}^2$, where we used the notation $Y_{\mathbf{u}} = \mathcal{P}_{0,0} X_{\mathbf{u}}$. Observe also that, by taking into account (9) and stationarity, we have the relations

$$\sum_{\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}|Y_{\mathbf{u}}|^2 = \sum_{\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}|\mathcal{P}_{u_1,u_2} X_{\mathbf{0}}|^2 = \mathbf{E}|X_{\mathbf{0}}|^2 < \infty. \qquad (11)$$

Consider the function

$$f(\mathbf{t}) = \frac{1}{(2\pi)^2}\, \mathbf{E}\Big|\sum_{\mathbf{j}\in\mathbb{Z}^2} Y_{-\mathbf{j}}\, \mathrm{e}^{\mathrm{i}\,\mathbf{j}\cdot\mathbf{t}}\Big|^2, \quad \mathbf{t} \in [0, 2\pi)^2.$$

By the Fubini theorem and (11),

$$\int_{[0,2\pi)^2} f(\mathbf{t})\,\mathrm{d}t_1\,\mathrm{d}t_2 = \frac{1}{(2\pi)^2}\,\mathbf{E} \int_{[0,2\pi)^2} \Big|\sum_{\mathbf{j}\in\mathbb{Z}^2} Y_{-\mathbf{j}}\, \mathrm{e}^{\mathrm{i}\,\mathbf{j}\cdot\mathbf{t}}\Big|^2 \mathrm{d}t_1\,\mathrm{d}t_2 = \mathbf{E}\sum_{\mathbf{j}\in\mathbb{Z}^2} |Y_{-\mathbf{j}}|^2 < \infty.$$

Let us now compute the Fourier coefficients of f. For every $\mathbf{k} \in \mathbb{Z}^2$,

$$\int_{[0,2\pi)^2} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{t}} f(\mathbf{t})\,\mathrm{d}t_1\,\mathrm{d}t_2 = \frac{1}{(2\pi)^2}\,\mathbf{E} \int_{[0,2\pi)^2} A_1(\mathbf{t}) A_2(\mathbf{t})\ \mathrm{d}t_1\,\mathrm{d}t_2,$$

where

$$A_1(\mathbf{t}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} Y_{-\mathbf{j}}\ \mathrm{e}^{\mathrm{i}\,(\mathbf{k}+\mathbf{j})\cdot\mathbf{t}} \quad \text{and} \quad A_2(\mathbf{t}) = \sum_{\mathbf{u}\in\mathbb{Z}^2} \overline{Y_{-\mathbf{u}}}\ \mathrm{e}^{-\mathrm{i}\,\mathbf{u}\cdot\mathbf{t}}.$$

By using the orthogonality of the exponential functions, we obtain the equalities

$$\int_{[0,2\pi)^2} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{t}} f(\mathbf{t})\,\mathrm{d}t_1\,\mathrm{d}t_2 = \sum_{\mathbf{j},\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}\left(Y_{-\mathbf{j}}\,\overline{Y_{-\mathbf{u}}}\right) \mathbf{1}_{\{\mathbf{k}+\mathbf{j}=\mathbf{u}\}} = \sum_{\mathbf{u}\in\mathbb{Z}^2} \mathbf{E}\left(Y_{\mathbf{k}-\mathbf{u}}\,\overline{Y_{-\mathbf{u}}}\right).$$

By comparing this expression with (10), we see that f is the spectral density for $(X_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^2}$. □

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