DOI 10.1007/s10958-016-3140-3
Journal of Mathematical Sciences, Vol. 219, No. 5, December, 2016

ON THE CONVEX HULL AND WINDING NUMBER OF SELF-SIMILAR PROCESSES

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It is well known that for a standard Brownian motion (BM) $\{B(t),\ t \geq 0\}$ with values in $\mathbf{R}^d$, its convex hull $V(t) = \text{conv}\{B(s),\ s \leq t\}$ with probability 1 for each $t > 0$ contains 0 as an interior point. We also know that the winding number of a typical path of a two-dimensional BM is equal to $+\infty$. The aim of this paper is to show that these properties are not specifically "Brownian," but hold for a much larger class of d-dimensional self-similar processes. This class contains, in particular, d-dimensional fractional Brownian motions and (concerning convex hulls) strictly stable Lévy processes. Bibliography: 10 titles.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a basic probability space. Consider a d-dimensional random process $X = \{X(t),\ t \geq 0\}$ defined on Ω that is self-similar of index $H > 0$. This means that for each constant $c > 0$, the process $\{X(ct),\ t \geq 0\}$ has the same distribution as $\{c^H X(t),\ t \geq 0\}$.

Let $L = \{L(u),\ u \in \mathbf{R}^1\}$ be the strictly stationary process obtained from X by the Lamperti transformation:

$$L(u) = e^{-Hu} X(e^u), \quad u \in \mathbf{R}^1. \tag{1}$$

Equivalently,

$$X(t) = t^H L(\log t), \quad t \in \mathbf{R}_*^+.$$

Let $\Theta = \{0,1\}^d$ be the set of all dyadic sequences of length d. Denote by $D_\theta,\ \theta \in \Theta$, the quadrant

$$D_\theta = \prod_{i=1}^{d} \mathbf{R}_{\theta_i},$$

where $\mathbf{R}_{\theta_i} = [0, \infty)$ if $\theta_i = 1$ and $\mathbf{R}_{\theta_i} = (-\infty, 0]$ if $\theta_i = 0$. The positive quadrant $D_{(1,1,\ldots,1)}$ for simplicity is denoted by D.

We say that the process X is *nondegenerate* if

$$\mathbf{P}\{X(1) \in D_\theta\} > 0$$

for all $\theta \in \Theta$.

Two important examples of self-similar processes are **fractional Brownian motion** and **stable Lévy process**.

Definition 1. *We call a self-similar (of index $H > 0$) process B^H fractional Brownian motion (FBM) if for each $e \in \mathbf{R}^d$, the scalar process $t \to \langle B^H(t), e\rangle$ is a standard one-dimensional FBM of index H up to a constant $c(e)$.*

It is easy to see that in this case, $c^2(e) = \langle Qe, e\rangle$, where Q is the covariance matrix of $B^H(1)$, and hence,

$$\mathbf{E}\langle B^H(t), e\rangle\langle B^H(s), e\rangle = \langle Qe, e\rangle \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geq 0;\ e \in \mathbf{R}^d.$$

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Published in *Zapiski Nauchnykh Seminarov POMI*, Vol. 441, 2015, pp. 154–162. Original article submitted October 30, 2015.
1072-3374/16/2195-0707

The process B^H is nondegenerate iff the rank of the matrix Q is equal to d. If $H = \frac{1}{2}$ and $Q = I_d$, then B^H is a standard Brownian motion.

(See Xiao [8], Račkauskas and Suquet [6], and Lavancier et al. [4] and references therein for more general definitions of an operator self-similar FBM.)

Definition 2. *We call $S = \{S(t),\ t \in \mathbf{R}_+\}$ an α-strictly stable Lévy process (StS) if*

(1) *$S(1)$ has a α-strictly stable distribution in $\mathbf{R}^d$;*
(2) *it has independent and stationary increments;*
(3) *it is continuous in probability.*

Then for each $t \in \mathbf{R}_+$, the random variable $S(t)$ has the same distribution as $t^{\frac{1}{\alpha}} S(1)$.

The cadlag version of S on $[0,1]$ can be obtained with the help of LePage series representation (see [7] for more details). If $\alpha \in (0,1)$ or if $\alpha \in (1,2)$ and $\mathbf{E}\, X(1) = 0$, then

$$\{S(t),\ t \in [0,1]\} \overset{\mathcal{L}}{=} \{c \sum_1^\infty \Gamma_k^{-1/\alpha} \varepsilon_k \mathbf{1}_{[0,t]}(\eta_k),\ \ t \in [0,1]\}, \tag{2}$$

where $\overset{\mathcal{L}}{=}$ means equality in law, c is a constant, $\Gamma_k = \sum_1^k \gamma_j$, $\{\gamma_j\}$ is a sequence of i.i.d. random variables with common standard exponential distribution, $\{\varepsilon_k\}$ is a sequence of i.i.d. random variables with common distribution σ concentrated on the unit sphere $\mathbf{S}^{d-1}$, $\{\eta_k\}$ is a sequence of $[0,1]$-uniformly distributed i.i.d. random variables, and the three sequences $\{\gamma_j\}$, $\{\varepsilon_k\}$, $\{\eta_k\}$ are assumed to be independent.

The measure σ is called a spectral measure of S. It is easy to see that if (2) takes place, then the process X is nondegenerate iff $\operatorname{vect}\{\operatorname{supp}\sigma\} = \mathbf{R}^d$.

In Sec. 2, the object of our interest is the convex hull process $V = \{V(t)\}$ associated with X. We show that under very sharp conditions, with probability 1 for all $t > 0$, the convex set $V(t)$ contains 0 as its interior point. From this result some interesting corollaries are deduced.

Section 3 is devoted to studying the winding numbers of two-dimensional self-similar processes. As a corollary of our main result, we show that for a typical path of a standard two-dimensional FBM, the number of its clockwise and anti-clockwise winds around 0 in a neighborhood of zero or at infinity is equal to ∞.

2. Convex hulls

For a Borel set $A \subset \mathbf{R}^d$ we denote by $\operatorname{conv}(A)$ the closed convex hull of A and define the convex hull process related to X:

$$V(t) = \operatorname{conv}\{\, X(s),\ \ s \le t\}.$$

Theorem 1. *Let X be a nondegenerate self-similar process such that the strictly stationary process L generating X is ergodic. Then with probability 1 for all $t > 0$, the point 0 is an interior point of $V(t)$.*

Application to FBM. Let B^H be a FBM with index H. The next properties follow from the definition without difficulties.

(1) **Continuity.** The process X has a continuous version.
Below we always assume B^H to be continuous.
(2) **Reversibility.** If the process Y is defined by

$$Y(t) = B^H(1) - B^H(1-t), \quad t \in [0,1],$$

then $\{Y(t),\ t \in [0,1]\} \overset{\mathcal{L}}{=} \{B^H(t),\ t \in [0,1]\}$.

(3) **Ergodicity.** Let $L = \{L(u),\ u \in \mathbf{R}^1\}$ be the strictly stationary Gaussian process obtained from B^H by the Lamperti transformation (1).

Then L is ergodic (see Cornfeld et al. [1], Chap. 14, Sec. 2, Theorems 1 and 2).

It is assumed below that the process B^H is nondegenerate.

Corollary 1. *Let V be the convex hull process related to B^H. Then with probability* 1 *for all $t > 0$, the point* 0 *is an interior point of $V(t)$.*

This follows immediately from Theorem 1.

Corollary 2. *Let V be the convex hull process related to B^H. Then for each $t > 0$ with probability* 1*, the point $B^H(t)$ is an interior point of $V(t)$.*

Proof of Corollary 2. Denote by A° the interior of A. By the self-similarity of the process B^H, it is sufficient to state this property for $t = 1$. Then, due to the reversibility of B^H, by Theorem 1, a.s.

$$0 \in [\operatorname{conv}\{ B^H(1) - B^H(1-t), \quad t \in [0,1]\}]^\circ. \tag{3}$$

Since

$$\operatorname{conv}\{ B^H(1) - B^H(1-t),\ t \in [0,1]\} = B^H(1) - \operatorname{conv}\{ B^H(1-s),\ s \in [0,1]\},$$

relation (3) is equivalent to

$$B^H(1) \in [\operatorname{conv}\{ B^H(s),\ s \in [0,1]\}]^\circ,$$

which concludes the proof. □

Let $\mathcal{K}_d$ be the family of all compact convex subsets of $\mathbf{R}^d$. It is well known that $\mathcal{K}_d$ equipped with the Hausdorff metric is a Polish space.

We say that a function $f : [0,1] \to \mathcal{K}_d$ is *increasing* if $f(t) \subset f(s)$ for $0 \le t < s \le 1$.

We say that a function $f : [0,1] \to \mathcal{K}_d$ is *almost everywhere constant* if f is such that for almost every $t \in [0,1]$ there exists an interval $(t - \varepsilon, t + \varepsilon)$ on which f is constant.

We say that a function $f : [0,1] \to \mathcal{K}_d$ is a *Cantor-staircase* (C-S) if f is continuous, increasing, and almost everywhere constant.

The next statement follows easily from Corollary 2.

Corollary 3. *Let V be the convex hull process related to B^H. Then with probability* 1*, the paths of the process $t \to V(t)$ are C-S functions.*

Remark 1. Let $h : \mathcal{K} \to \mathbf{R}^1$ be an increasing continuous function. Then almost all paths of the process $t \to h(V(t))$ are C-S real-valued functions. This obvious fact may be applied to all reasonable geometrical characteristics of $V(t)$, such as volume, surface area, diameter, … .

Application to StS. Let now S be an StS process with exponent $\alpha < 2$. The following properties are more or less known.

(1) **Right continuity.** The process S has a *cadlag* version (see the remark above just after the definition).

(2) **Reversibility.** Let

$$Y(t) = S(1) - S(1-t), \quad t \in [0,1].$$

Then $\{Y(t),\ t \in [0,1]\} \stackrel{\mathcal{L}}{=} \{S(t),\ t \in [0,1]\}$.

(3) **Self-similarity.** The process S is self-similar of index $H = \frac{1}{\alpha}$.

(4) **Ergodicity.** Let $L = \{L(u),\ u \in \mathbf{R}^1\}$ be the strictly stationary process obtained from S by the Lamperti transformation (1). Then L is ergodic.

We assume that the law of $S(1)$ is nondegenerate.

Corollary 4. *Let V be the convex hull process related to S. Then with probability 1 for all $t > 0$, the point 0 is an interior point of $V(t)$.*

Corollary 5. *Let V be the convex hull process related to S. Then for each $t > 0$ with probability 1, the point $X(t)$ is an interior point of $V(t)$.*

Corollary 6. *Let V be the convex hull process related to S. Then with probability 1, the paths of the process $t \to V(t)$ are right continuous almost everywhere constant functions.*

We omit proofs of these statements since they are similar to those of Corollaries 1–3.

Proof of Theorem 1. We first show that

$$p \stackrel{\text{def}}{=} \mathbf{P}\{\text{there exists } \ t \in (0,1] \mid X(t) \in D^\circ\} = 1. \tag{4}$$

Remark that p is strictly positive:

$$p \ \geq\ \mathbf{P}\{X(1) \in D^\circ\} > 0 \tag{5}$$

due to the hypothesis that the law of $X(1)$ is nondegenerate.

By the self-similarity,

$$\mathbf{P}\big\{D^\circ \cap \{X(t),\ t \in [0,T]\} = \varnothing\big\} = 1 - p$$

for every $T > 0$. Since the sequence of events $(A_n)_{n\in\mathbf{N}}$,

$$A_n = \big\{D^\circ \ \cap \{X(t), t \in [0,n]\} = \varnothing\big\},$$

is decreasing, it follows that

$$1 - p = \lim \mathbf{P}(A_n) = \mathbf{P}(\cap_n A_n) = \mathbf{P}\{X(t) \notin D^\circ \quad \text{for all} \quad t \geq 0\}.$$

In terms of the stationary process L from the Lamperti representation, this means that

$$\mathbf{P}\{L(s) \notin D^\circ \quad \text{for all} \quad s \in \mathbf{R}^1\} = 1 - p.$$

Since this event is invariant, by the ergodicity of L and due to (5), we see that the value $p = 1$ is the only one possible.

Applying similar arguments to another quadrants D_θ, $\theta \in \Theta$, we conclude that with probability 1 there exist points $t_\theta \in (0,1]$ such that $X(t_\theta) \in D_\theta^\circ$, $\theta \in \Theta$. Now, to complete the proof, it is sufficient to remark that

$$V(1)^\circ = \operatorname{conv}\{X(t),\ t \in [0,1]\}^\circ \supset \operatorname{conv}\{X(t_\theta),\ \theta \in \Theta\}^\circ$$

and that the last set obviously contains 0. □

3. Winding numbers

Let now $X = \{X(t),\ t \geq 0\}$ be a two-dimensional self-similar process. It is assumed that the following properties are fulfilled:

(1) The process X is continuous.
(2) The process X is nondegenerate.
(3) The process X is symmetric: X and $-X$ have the same law.
(4) The stationary process L associated with X is ergodic.
(5) Starting from 0, the process X with probability 1 never comes back:

$$\mathbf{P}\{X(t) \neq 0 \quad \text{for all} \quad t > 0\} = 1. \tag{6}$$

Due to the last hypothesis, considering $\mathbf{R}^2$ as the complex plane, we can define the winding numbers (around 0) $\nu[s,t]$, $0 < s < t$, in the usual way (see [5, Chap. 5]):

$$\nu[s,t] = \arg(X(t)) - \arg(X(s)).$$

We set

$$\nu_+(0,t] = \limsup_{s\downarrow 0} \nu[s,t], \quad \nu_-(0,t] = \liminf_{s\downarrow 0} \nu[s,t],$$
$$\nu_+[s,\infty) = \limsup_{t\to\infty} \nu[s,t], \quad \nu_-[s,\infty) = \liminf_{t\to\infty} \nu[s,t].$$

The values $\nu_+(0,t]$ and $-\nu_-(0,t]$ represent, respectively, the number of clockwise and anti-clockwise winds around 0 in a neighborhood of the starting point, while $\nu_+[s,\infty)$ and $-\nu_-[s,\infty)$ are the similar winding numbers at infinity.

Theorem 2. *Let X be a two-dimensional self-similar process with properties* (1)–(5) *mentioned above. Then, with probability one for all $t > 0$,*

$$\nu_+(0,t] = \nu_+[t,\infty) = -\nu_-(0,t] = -\nu_-[t,\infty) = +\infty. \tag{7}$$

Corollary 7. *Let B^H be a two-dimensional standard FBM and assume that $H \in [1/2,\, 1)$. Then with probability one for all $t > 0$, equalities* (7) *take place.*

Proof. The case $H = 1/2$ is well known, see [5, Chap. 5], which give us exhaustive information on Brownian winding numbers.

If $H \in (1/2,\, 1)$, we apply Theorem 2 since all the hypotheses (1)–(5) are fulfilled; indeed, properties (1)–(3) are obvious; the ergodicity of L, $L(t) = (L_1(t), L_2(t))$, follows from the fact that $\mathbf{E}\, L_1(t)L_1(0) \to 0$ as $t \to \infty$ (see [1, Chap. 14, Sec. 2, Theorem 2]). Property (5) can be deduced from Theorem 11 of [8] (see also [9, Theorem 4.2] and [10, Theorem 2.6]). □

Remark 2. If $H \in (0, \frac{1}{2})$, the process $t \to \arg B^H(t) - \arg B^H(0)$ is not continuous with positive probability since the set $\{t \in (0,1] \mid B^H(t) = 0\}$ is not empty (see [8, Theorem 11)]). This means that in this case, the winding numbers can only be defined for the excursions of B^H, and we need more sophisticated methods for their study.

Proof of Theorem 2. By (5),

$$\mathbf{P}\big\{L(t) \neq 0 \quad \text{for all} \quad t \in \mathbf{R}^1\big\} = 1.$$

Hence, as above, we can define for L the winding numbers $\nu^L_{\substack{+\\-}}(-\infty, t]$ and $\nu^L_{\substack{+\\-}}[t,\infty)$, and, in addition,

$$\nu^L_{\substack{+\\-}}(-\infty,t] = \nu_{\substack{+\\-}}(0,e^t] \quad \text{and} \quad \nu^L_{\substack{+\\-}}[t,\infty) = \nu_{\substack{+\\-}}[e^t,\infty).$$

Therefore, from now on we can work with the process L and omit the index L in the notation of winding numbers.

Let us show that

$$\mathbf{P}\big\{|\nu_{\substack{+\\-}}[t,\infty)| = \infty \quad \text{for all} \quad t \in \mathbf{R}^1\big\} = 1. \tag{8}$$

By symmetry (property (3)), it is sufficient to state that

$$\mathbf{P}\big\{\nu_+[t,\infty) = \infty \quad \text{for all} \quad t \in \mathbf{R}^1\big\} = 1. \tag{9}$$

Using the arguments from the proof of Theorem 1, we remark that the process L visits infinitely often each of four basic quadrants. It follows by continuity that at least one of the two events A and B,

$$A = \big\{\text{there exists} \ \ t > 0 \quad \text{such that} \arg X(t) - \arg X(0) > \pi/2\big\}$$

and

$$B = \{\text{there exists } \ t > 0 \quad \text{such that} \arg X(t) - \arg X(0) < \pi/2\},$$

has probability 1. By the symmetry (property (3)), $\mathbf{P}(A) = \mathbf{P}(B)$. Thus,

$$\mathbf{P}\{\text{there exists } \ t > 0 \quad \text{such that} \arg X(t) - \arg X(0) > \pi/2\} = 1.$$

From this it follows by the stationarity that for all $s \in \mathbf{R}^1$,

$$\mathbf{P}\{\text{there exists } \ t > s \quad \text{such that} \arg X(t) - \arg X(s) > \pi/2\} = 1.$$

The set

$$E = \{(s, \omega) \in \mathbf{R}^1 \times \Omega \mid \text{there exists } \ t > s \quad \text{such that} \arg X(t) - \arg X(s) > \pi/2\}$$

is measurable since the process $s \to \sup_{t>s}(\arg X(t) - \arg X(s))$ is continuous.

Based on the aforementioned and due to the Fubini theorem, the set E is such that

$$\lambda \times \mathbf{P}(E^{\complement}) = 0,$$

where λ is the Lebesgue measure. Therefore, there exists $\Omega' \subset \Omega$, $\mathbf{P}(\Omega') = 1$, such that for each $\omega \in \Omega'$ and for almost all $s \in \mathbf{R}^1$ there exists $t > s$ for which $\arg X(t) - \arg X(s) > \pi/2$. Take $\omega \in \Omega'$. Let us denote by E_ω the corresponding ω-section of E. Without loss of generality, we may assume that for each $\omega \in \Omega'$, the point 0 belongs to E_ω. Since $\lambda(E_\omega^{\complement}) = 0$, E_ω is dense in $\mathbf{R}^1$. Let $u > 0$ be such that $\arg X(u) - \arg X(0) > \pi/2$. By continuity, $\arg X(t) - \arg X(0) > \pi/2$ for all t in a sufficiently small neighborhood of u and, therefore, there exists $t_1 \in E_\omega$ for which $\arg X(t_1) - \arg X(0) > \pi/2$. Repeating this reasoning, we can construct an increasing sequence (t_n) such that $t_1 = 0$ and $t_n \in E_\omega$. Since $\arg X(t_n) - \arg X(t_{n-1}) > \pi/2$ for each n, we get the relation

$$\sup_{t>0}(\arg X(t) - \arg X(0)) = \infty.$$

Thus, it is proved that for each t,

$$\mathbf{P}\{\nu_+[t, \infty) = \infty\} = 1. \tag{10}$$

Now to show that

$$\mathbf{P}\{\nu_+[t, \infty) = \infty \quad \text{for all} \quad t \in \mathbf{R}^1\} = 1,$$

it is sufficient to remark that for each ω from Ω', the ω-section $E_\omega = \mathbf{R}^1$. Indeed, assuming that there exists $u \in E_\omega^{\complement}$, we should have

$$\arg X(s) - \arg X(u) \leq \pi/2$$

for each $s > t$, but that is in contradiction with the existence of $t \in E_\omega$, $t > u$, for which (10) holds. Thus, (9) is proved. Applying the previous reasonings to the process $\{L(-t),\ t \in \mathbf{R}^1\}$, we prove the remaining equalities in (7). □

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