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OPTIMAL CONTROL FOR QUASILINEAR DEGENERATE DISTRIBUTED SYSTEMS OF HIGHER ORDER

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We consider the problem for a distributed control with compromise quality functional for systems whose states are described by evolution equations that are unsolved with respect to the higher order time derivative. We establish the solvability of the problem for linear and quasilinear equations. The results are illustrated by an example. Bibliography: 12 *titles.*

1 Introduction

We assume that $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{U}$ are Banach spaces, $B \in \mathscr{L}(U;Y)$, $L \in \mathscr{L}(\mathscr{X};\mathscr{Y})$ are linear continuous operators such that $\ker L \neq \{0\}$, $M \in \mathscr{C}l(\mathscr{X};\mathscr{Y})$, i.e., the operator M is linear, closed and densely defined in $\mathscr{X}$, and $N : [t_0,T] \times \mathscr{X}^m \to \mathscr{Y}$ is a nonlinear operator. We study the solvability of the control problem

$$u \in \mathfrak{U}_\partial, \tag{1.1}$$

$$J(x,u) = \frac{1}{2}\|x-\widetilde{x}\|^2_{W_2^1(t_0,T;\mathscr{X})} + \frac{C}{2}\|u-\widetilde{u}\|^2_{L_2(t_0,T;\mathscr{U})} \to \inf \tag{1.2}$$

for a distributed system whose states are described by the quasilinear degenerate equation

$$Lx^{(m)}(t) = Mx(t) + N(t,x(t),x^{(1)}(t)\dots,x^{(m-1)}(t)) + Bu(t), \quad t \in (t_0,T), \tag{1.3}$$

with the initial conditions

$$P(x^{(k)}(t_0)-x_k) = 0, \quad k = 0,1,\dots,m-1. \tag{1.4}$$

Here, $u : (t_0,T) \to \mathscr{U}$ is the control function, $\mathfrak{U}_\partial$ is the set of admissible controls, $\widetilde{x} \in W_2^1(t_0,T;\mathscr{X})$ and $\widetilde{u} \in L_2(t_0,T;\mathscr{U})$ are given functions, $x_k \in \mathscr{X}$, $k = 0,1,\dots,m-1$, are given vectors, $C > 0$ is a constant, and P is the projection along the degeneracy subspaces for Equation (1.3) which will be introduced in terms of the operators L and M.

Many problems in mathematical physics can be reduced to the problem (1.3), (1.4); moreover, instead of the Cauchy initial condition, it is more convenient to consider the generalized

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Showalter–Sidorov conditions (1.4) (cf. [1, 2]) if the initial data are given only for the projection of the sought function onto the subspace im P (cf. [3]).

The solvability of initial-boundary value problems for different classes of quasilinear degenerate evolution equations was studied in numerous works. We note that our investigation is close to the results of [4]. In this paper, to prove the existence of a strong solution to the problem (1.3), (1.4), we use the schemes of [5]. As in [6]–[10], we use [11, Theorem 1.2.4] for proving the existence of a solution to the problem (1.1)–(1.4). In Section 5, we describe an example of an optimal control problems for a model system of partial differential equations unsolved with respect to the time derivatives.

2 Solvability of Degenerate Equation

To study the degenerate equation, we use the results of the theory of degenerate operator semigroups (cf. [3] for details).

We assume that $\mathscr{X}$, $\mathscr{Y}$ are Banach spaces and $\mathscr{L}(\mathscr{X};\mathscr{Y})$ is the Banach space of linear continuous operators from $\mathscr{X}$ to $\mathscr{Y}$. We denote by $\mathscr{C}l(\mathscr{X};\mathscr{Y})$ the set of linear closed operators with dense domains in the space $\mathscr{X}$ acting in $\mathscr{Y}$. For $\mathscr{Y}=\mathscr{X}$ we write $\mathscr{L}(\mathscr{X})$ and $\mathscr{C}l(\mathscr{X})$ respectively. Denote by D_M the domain of the operator M with the graph norm $\|\cdot\|_{D_M} = \|\cdot\|_{\mathscr{X}} + \|M\cdot\|_{\mathscr{Y}}$.

We assume that $L\in\mathscr{L}(\mathscr{X};\mathscr{Y})$ and $M\in\mathscr{C}l(\mathscr{X};\mathscr{Y})$. We introduce the L-resolvent set $\rho^L(M)=\{\mu\in\mathbb{C}:(\mu L-M)^{-1}\in\mathscr{L}(\mathscr{Y};\mathscr{X})\}$ and the L-spectrum $\sigma^L(M)=\mathbb{C}\backslash\rho^L(M)$ of an operator M. We also denote $R^L_\mu(M)=(\mu L-M)^{-1}L$, $L^L_\mu=L(\mu L-M)^{-1}$.

An operator M is said to be (L,σ)-*bounded* if the L-spectrum $\sigma^L(M)$ is bounded, i.e.,

$$\exists a>0\quad\forall\mu\in\mathbb{C}\quad(|\mu|>a)\Rightarrow(\mu\in\rho^L(M)).$$

Lemma 2.1 (cf. [3]). *We assume that M is (L,σ)-bounded and $\gamma=\{\mu\in\mathbb{C}:|\mu|=r>a\}$. Then the following operators are projections*:

$$P=\frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M)d\mu\in\mathscr{L}(\mathscr{X}),\quad Q=\frac{1}{2\pi i}\int\limits_\gamma L^L_\mu(M)d\mu\in\mathscr{L}(\mathscr{Y}).$$

We set $\mathscr{X}^0=\ker P$, $\mathscr{Y}^0=\ker Q$, $\mathscr{X}^1=\operatorname{im}P$, and $\mathscr{Y}^1=\operatorname{im}Q$. Denote by L_k (M_k) the restriction of the operator L (M) onto $\mathscr{X}^k$ ($D_{M_k}=D_M\cap\mathscr{X}^k$), $k=0,1$.

Theorem 2.1 (cf. [3]). *Let an operator M be (L,σ)-bounded. Then*
(i) $M_1\in\mathscr{L}(\mathscr{X}^1;\mathscr{Y}^1)$, $M_0\in\mathscr{C}l(\mathscr{X}^0;\mathscr{Y}^0)$, $L_k\in\mathscr{L}(\mathscr{X}^k;\mathscr{Y}^k)$, $k=0,1$,
(ii) *the operators* $M_0^{-1}\in\mathscr{L}(\mathscr{Y}^0;\mathscr{X}^0)$ *and* $L_1^{-1}\in\mathscr{L}(\mathscr{Y}^1;\mathscr{X}^1)$ *exist.*

We set $\mathbb{N}_0=\{0\}\cup\mathbb{N}$ and $H=M_0^{-1}L_0$. For $p\in\mathbb{N}_0$ an operator M is (L,p)-bounded if it is (L,σ)-bounded, $H^p\neq\mathbb{O}$, $H^{p+1}=\mathbb{O}$.

We consider a nonlinear operator $N:[t_0,T]\times\mathscr{X}^m\to\mathscr{Y}$. By a *strong solution* to the problem

$$Lx^{(m)}(t)=Mx(t)+N(t,x(t),x^{(1)}(t),\dots,x^{(m-1)}(t))+f(t) \tag{2.1}$$

$$P(x^{(k)}(t_0)-x_k)=0,\quad k=0,1,\dots,m-1, \tag{2.2}$$

we mean a function $x \in W_q^m(t_0, T; \mathscr{X})$, $q \in (1, \infty)$, such that (2.2) holds and (2.1) is satisfied for almost all $t \in (t_0, T)$. Arguing in the same way as in [5], we obtain the conditions of the existence and uniqueness of a strong solution to the problem (2.1), (2.2).

In the linear case, the following assertion holds.

Theorem 2.2. *We assume that M is an (L,p)-bounded operator, $N \equiv 0$, $Qf \in L_q(0,T;\mathfrak{Y})$, $H^k M_0^{-1}(I-Q)f \in W_q^{m(k+1)}(0,T;\mathscr{X})$, $k = 0, 1, \dots, p$. Then for any $x_0, x_1, \dots, x_{m-1} \in \mathscr{X}$ the problem* (2.1), (2.2) *has a unique strong solution on (t_0, T).*

A mapping $S(t_0, T) \times \mathscr{X}^m \to \mathscr{Y}$ with variables t, v_0, $v_1, \dots, v_{m-1}$ is said to be *uniformly Lipschitz* with respect to $\overline{v} = (v_0, v_1, \dots, v_{m-1})$ if there exists $l > 0$ such that for all $\overline{v} = (v_0, v_1, \dots, v_{m-1})$, $\overline{w} = (w_0, w_1, \dots, w_{m-1}) \in \mathscr{X}^m$ and almost all $t \in (t_0, T)$

$$\|N(t, v_0, v_1, \dots, v_{m-1}) - N(t, w_0, w_1, \dots, w_{m-1})\|_{\mathscr{Y}}^2 \leqslant l^2 \sum_{k=0}^{m-1} \|v_k - w_k\|_{\mathscr{X}}^2.$$

In the nonlinear case, the following assertion holds.

Theorem 2.3. *We assume that $p \in \mathbb{N}_0$, M is an (L,p)-bounded operator, and $N : [t_0, T] \times \mathscr{X}^m \to \mathscr{Y}$ is such that $QN \in C^{m(p+1)-1}([t_0, T] \times \mathscr{X}^m; \mathscr{Y})$ is uniformly Lipschitz with respect to $\overline{v} = (v_0, \dots, v_{m-1})$, $H^k M_0^{-1}(I-Q)N \in C^{m(k+1)}([t_0, T] \times \mathscr{X}^m; \mathscr{X})$, $k = 0, 1, \dots, p$, and for all $(t, \overline{v}) \in [t_0, T] \times \mathscr{X}^m$*

$$N(t, v_0, v_1, \dots, v_{m-1}) = N(t, Pv_0, Pv_1, \dots, Pv_{m-1}), \tag{2.3}$$

$Qf \in W_q^{m(p+1)-1}(t_0, T; \mathscr{Y})$, $H^k M_0^{-1}(I-Q)f \in W_q^{m(k+1)}(t_0, T; \mathscr{X})$ for $k = 0, 1, \dots, p$. Then for any $x_0, x_1, \dots, x_{m-1} \in \mathscr{X}$ the problem (2.1), (2.2) *has a unique strong solution on (t_0, T).*

If the condition (2.3) fails, but im $N \subset \mathscr{Y}^1$, the following assertion holds.

Theorem 2.4. *We assume that $p \in \mathbb{N}_0$, M is an (L,p)-bounded operator, $N : (t_0, T) \times \mathscr{X}^m \to \mathscr{Y}$ for all $v_0, v_1, \dots, v_{m-1} \in \mathscr{X}$ is measurable on (t_0, T), and uniformly Lipschitz with respect to $\overline{v}$ for some $\overline{z} \in \mathscr{X}^m$ $N(\cdot, \overline{z}) \in L_q(t_0, T; \mathscr{Y})$,* im $N \subset \mathscr{Y}^1$, *$Qf \in L_q(t_0, T; \mathscr{Y})$, $H^k M_0^{-1}(I-Q)f \in W_q^{m(k+1)}(t_0, T; \mathscr{X})$, $k = 0, 1, \dots, p$. Then for any $x_0, x_1, \dots, x_{m-1} \in \mathscr{X}$ the problem* (2.1), (2.2) *has a unique strong solution on (t_0, T).*

3 Linear Equation

We assume that $\mathscr{U}$, $\mathscr{X}$, $\mathscr{Y}$ are Hilbert spaces, $L \in \mathscr{L}(\mathscr{X}; \mathscr{Y})$, $B \in \mathscr{L}(\mathscr{U}; \mathscr{Y})$, and $M \in \mathscr{Cl}(\mathscr{X}; \mathscr{Y})$. We consider the problem for a distributed control

$$Lx^{(m)}(t) = Mx(t) + Bu(t) + y(t), \quad t \in (0, T), \tag{3.1}$$

$$P(x^{(k)}(t_0) - x_k) = 0, \quad k = 0, 1, \dots, m-1, \tag{3.2}$$

$$u \in \mathfrak{U}_\partial, \tag{3.3}$$

$$J(x, u) = \frac{1}{2}\|x - \widetilde{x}\|_{W_2^m(t_0, T; \mathscr{X})}^2 + \frac{C}{2}\|u - \widetilde{u}\|_{L_2(t_0, T; \mathscr{U})}^2 \to \inf. \tag{3.4}$$

Here, the nonempty convex closed subset $\mathfrak{U}_\partial$ of the space of controls $L_2(t_0,T;\mathscr{U})$ is the set of admissible controls, $y \in L_2(t_0,T;\mathscr{Y})$, $\widetilde{x} \in W_2^m(t_0,T;\mathscr{X})$, $\widetilde{u} \in L_2(t_0,T;\mathscr{U})$ are given functions, and x_k, $k = 0,1,\dots,m-1$, are given vectors.

Lemma 3.1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces. Then $\mathscr{Z} = \{z \in W_2^m(t_0,T;\mathscr{X}) : Lx^{(m)} - Mx \in L_2(t_0,T;\mathscr{Y})\}$ is a Banach space with respect to the norm $\|x\|^2_{\mathscr{Z}} = \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + \|Lx^{(m)} - Mx\|^2_{L_2(t_0,T;\mathscr{Y})}$.*

Proof. It is obvious that the norm axioms hold. We show that the space $\mathscr{Z}$ is complete relative to this norm. We choose a Cauchy sequence $\{x_n\}$ in $\mathscr{Z}$. Since the space $W_2^m(t_0,T;\mathscr{X})$ is complete, there exists $x \in W_2^m(t_0,T;\mathscr{X})$ such that $\|x_n - x\|_{W_2^m(t_0,T;\mathscr{X})} \to 0$. Furthermore, there exists $z = \lim\limits_{n\to\infty}(Lx_n^{(m)} - Mx_n)$ in $L_2(t_0,T;\mathscr{Y})$. Since $Lx_n^{(m)} \to Lx^{(m)}$ in $L_2(t_0,T;\mathscr{Y})$ as $n \to \infty$, we have $Mx_n \to Lx^{(m)} - z$ in $L_2(t_0,T;\mathscr{Y})$ as $n \to \infty$.

Thus, the set of $t \in (t_0,T)$ such that $x_n(t)$ does not converge to $x(t)$ in $\mathscr{X}$ or $Mx_n(t)$ does not converge to $Lx^{(m)}(t) - z(t)$ in $\mathscr{Y}$ has measure zero. Since M is a closed operator, we can conclude that for almost all $t \in (0,T)$ we have $x(t) \in \operatorname{dom} M$ and $Mx = Lx^{(m)} - z$ in $L_2(t_0,T;\mathscr{Y})$. Hence $Lx^{(m)} - Mx = z \in L_2(t_0,T;\mathscr{Y})$. □

Remark 3.1. If $\mathscr{X}$ and $\mathscr{Y}$ are Hilbert spaces, then $\mathscr{Z}$ is also al Hilbert space equipped with the inner product $\langle x,z\rangle_{\mathscr{Z}} = \langle x,z\rangle_{W_2^m(t_0,T;\mathscr{X})} + \langle Lx^{(m)} - Mx, Lz^{(m)} - Mz\rangle_{L_2(t_0,T;\mathscr{Y})}$.

We introduce the operators $\gamma_k : W_2^m(t_0,T;\mathscr{X}) \to \mathscr{X}$, $\gamma_k x = x^{(k)}(0)$, $k = 0,1,\dots,m-1$. By the Sobolev embedding theorem, the operators $\gamma_k : W_2^m(t_0,T;\mathscr{X}) \to \mathscr{X}$, $k = 0,1,\dots,m-1$, are continuous. Therefore, $\gamma_k : \mathscr{Z} \to \mathscr{X}$, $k = 0,1,\dots,m-1$, are also continuous.

The set $\mathfrak{W}$ of pairs $(x,u) \in \mathscr{Z}_2 \times L_2(t_0,T;\mathscr{U})$ satisfying (3.1)–(3.3) is called the *set of admissible pairs* for the problem (3.1)–(3.4). The problem (3.1)–(3.4) consists in finding pairs $(\widehat{x},\widehat{u}) \in \mathfrak{W}$ minimizing the cost functional $J(x,u)$:

$$J(\widehat{x},\widehat{u}) = \inf_{(x,u)\in\mathfrak{W}} J(x,u).$$

We recall that a functional $J(x,u)$ is *coercive* if for any $R > 0$ the set $\{(x,u) \in \mathfrak{W} : J(x,u) \leqslant R\}$ is bounded in $\mathscr{Z} \times L_2(t_0,T;\mathscr{U})$.

Theorem 3.1. *We assume that M is strongly (L,p)-bounded and $\mathfrak{U}_\partial \cap W_2^{m(p+1)}(t_0,T;\mathscr{U}) \neq \varnothing$. Then there exists a unique solution $(\widehat{x},\widehat{u}) \in \mathscr{Z} \times \mathfrak{U}_\partial$ to the problem (3.1)–(3.4).*

Proof. We use Theorem 1.2.3 in [11]. We set $\mathfrak{U} = L_2(t_0,T;\mathscr{U})$, $\mathfrak{V} = L_2(t_0,T;\mathscr{Y}) \times \mathscr{X}^m$, $\mathfrak{Y} = W_2^m(t_0,T;\mathscr{X})$, $\mathfrak{Y}_1 = \mathscr{Z}$, and $\mathfrak{F}_0 = (-y,-x_0,-x_1,\dots,-x_{m-1}) \in \mathfrak{V}$. As was already noted, the continuous embedding of $\mathfrak{Y}_1$ into $\mathfrak{Y}$ follows from the construction of $\mathscr{Z}$. It is obvious that the operator $\mathfrak{L} : \mathfrak{Y}_1 \times \mathfrak{U} \to \mathfrak{V}$ is linear and $\mathfrak{L}(x,u) = (Lx^{(m)} - Mx - Bu, \gamma_0 x, \gamma_1 x, \dots, \gamma_{m-1}x)$. Let us prove the continuity of this operator. We have

$$\begin{aligned}
&\|(Lx^{(m)} - Mx - Bu, \gamma_0 x, \gamma_1 x, \dots, \gamma_{m-1}x)\|^2_{L_2(t_0,T;\mathscr{Y})\times\mathscr{X}^m}\\
&\quad= \|Lx^{(m)} - Mx - Bu\|^2_{L_2(t_0,T;\mathscr{Y})} + \sum_{k=0}^{m-1}\|\gamma_k x\|^2_{\mathscr{X}}\\
&\quad\leqslant 2\|Lx^{(m)} - Mx\|^2_{L_2(t_0,T;\mathscr{Y})} + 2\|Bu\|^2_{L_2(t_0,T;\mathscr{Y})} + C_1\|x\|^2_{\mathscr{Z}}\\
&\quad\leqslant (2 + C_1)\|x\|^2_{\mathscr{Z}} + 2\|B\|^2_{\mathscr{L}(\mathscr{U};\mathscr{Y})}\|u\|^2_{L_2(t_0,T;\mathscr{U})} = C\|(x,u)\|^2_{\mathscr{Z}\times L_2(t_0,T;\mathscr{U})}.
\end{aligned}$$

It is obvious that J is strictly convex and continuous. Let us prove that J is coercive. We have

$$\|x\|^2_{\mathscr{Z}} + \|u\|^2_{L_2(t_0,T;\mathscr{U})} = \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + \|Bu + y\|^2_{L_2(t_0,T;\mathscr{Y})} + \|u\|^2_{L_2(t_0,T;\mathscr{U})}$$
$$\leqslant \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + C_2\|u\|^2_{L_2(t_0,T;\mathscr{U})} + 2\|y\|^2_{L_2(t_0,T;\mathscr{Y})} \leqslant C_3 J(x,u) + C_4 \leqslant C_3 R + C_4,$$

where we used the fact that x is a solution to Equation (3.1).

Since $\mathfrak{U}_\partial \cap W_2^{m(p+1)}(t_0,T;\mathscr{U})$ is nonempty, there exists a control $u \in \mathfrak{U}_\partial \cap W_2^{m(p+1)}(t_0,T;\mathscr{U})$ such that $f \equiv Bu$ satisfies the assumptions of Theorem 2.2 on the solvability of the problem (3.1), (3.2). Thus, all the assumptions of Theorem 1.2.3 in [11] hold. □

Remark 3.2. A sufficient condition for the relation $\mathfrak{U}_\partial \cap W_2^{m(p+1)}(t_0,T;\mathscr{U}) \neq \varnothing$ is the existence of an interior point of the set $\mathfrak{U}_\partial$ in the topology of the space $L_2(t_0,T;\mathscr{U})$.

Remark 3.3. The condition that the spaces $\mathscr{U}$, $\mathscr{X}$, and $\mathscr{Y}$ are Hilbert spaces was used only for proving the strict convexity of the quality functional, which sufficient for the uniqueness of a solution. The existence of a solution in $\mathscr{Z} \times L_2(t_0,T;\mathscr{U})$ can be proved in the same way as in the case of Banach spaces $\mathscr{U}$, $\mathscr{X}$, and $\mathscr{Y}$.

4 Quasilinear Degenerate Equation of Higher Order

We assume that $\mathscr{U}$, $\mathscr{X}$, $\mathscr{Y}$ are Banach spaces, $L \in \mathscr{L}(\mathscr{X};\mathscr{Y})$, $B \in \mathscr{L}(\mathscr{U};\mathscr{Y})$, $M \in \mathscr{C}l(\mathscr{X};\mathscr{Y})$, $N : [t_0,T] \times \mathscr{X}^m \to \mathscr{Y}$, $m \in \mathbb{N}$. We consider the optimal control problem

$$Lx^{(m)}(t) = Mx(t) + N(t, x(t), x^{(1)}(t), \ldots, x^{(m-1)}(t)) + Bu(t), \tag{4.1}$$

$$P(x^{(k)}(t_0) - x_k) = 0, \quad k = 0, 1, \ldots, m-1, \tag{4.2}$$

$$u \in \mathfrak{U}_\partial, \tag{4.3}$$

$$J(x,u) = \frac{1}{2}\|x - \widetilde{x}\|^2_{W_2^m(t_0,T;\mathscr{X})} + \frac{C}{2}\|u - \widetilde{u}\|^2_{L_2(t_0,T;\mathscr{U})} \to \inf. \tag{4.4}$$

Theorem 4.1. *We assume that $p \in \mathbb{N}_0$, M is an (L,p)-bounded operator, $N : [t_0,T] \times \mathscr{X}^m \to \mathscr{Y}$ is such that $QN \in C^{m(p+1)-1}([t_0,T] \times \mathscr{X}^m;\mathscr{Y})$ is uniformly Lipschitz with respect to $\overline{v} = (v_0, \ldots, v_{m-1})$, $H^k M_0^{-1}(I - Q)N \in C^{m(k+1)}([t_0,T] \times \mathscr{X}^m;\mathscr{X})$, $k = 0, 1, \ldots, p$, for all $(t, \overline{v}) \in [t_0,T] \times \mathscr{X}^m$ we have the equality $N(t, v_0, v_1, \ldots, v_{m-1}) = N(t, Pv_0, Pv_1, \ldots, Pv_{m-1})$, $\mathfrak{U}_\partial$ is a nonempty closed convex subset of the space $L_2(t_0,T;\mathscr{U})$, $\mathfrak{U}_\partial \cap W_2^{m(p+1)}(t_0,T;\mathscr{U})$, x_0, x_1, $\ldots, x_{m-1} \in \mathscr{X}$. Then there exists a solution $(\widehat{x}, \widehat{u}) \in \mathscr{Z} \times \mathfrak{U}_\partial$ to the problem* (4.1)–(4.4).

Proof. By Theorem 2.3, the Cauchy problem (4.1), (4.2) has a unique solution for every $u \in \mathfrak{U}_\partial \cap W_2^{m(p+1)}(t_0,T;\mathscr{U})$. Therefore, the set $\mathfrak{W}$ of admissible pairs is nonempty. We use Theorem 1.2.4 in [11]. We set $\mathfrak{Y} = W_2^m(t_0,T;\mathscr{X})$, $\mathfrak{Y}_1 = \mathscr{Z}$, $\mathfrak{U} = L_2(t_0,T;\mathscr{U})$, $\mathfrak{V} = L_2(t_0,T;\mathscr{Y}) \times \mathscr{X}^m$, $\mathfrak{F}(x(\cdot)) = (-N(t, x(t), x^{(1)}(t), \ldots, x^{(m-1)}(t)), x_0, x_1, \ldots, x_{m-1})$, $\mathfrak{L}(x,u) = (Lx^{(m)} - Mx - Bu, \gamma_0 x, \gamma_1 x, \ldots, \gamma_{m-1} x)$. The continuity and linearity of the operator $\mathfrak{L} : \mathfrak{Y}_1 \times \mathfrak{U} \to \mathfrak{V}$ are proved in Theorem 3.1.

Using the Sobolev embedding theorem the joint continuity of N, we find

$$\|x\|^2_{\mathscr{Z}} + \|u\|^2_{L_2(t_0,T;\mathscr{U})} = \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + \|Lx^{(m)} - Mx\|^2_{L_2(t_0,T;\mathscr{Y})} + \|u\|^2_{L_2(t_0,T;\mathscr{U})}$$
$$\leqslant \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + 2\|N(\cdot, x(\cdot), x^{(1)}(\cdot), \ldots, x^{(m-1)}(\cdot))\|^2_{L_2(t_0,T;\mathscr{Y})} + 2\|Bu\|^2_{L_2(t_0,T;\mathscr{Y})}$$

$$+ \|u\|^2_{L_2(t_0,T;\mathscr{U})} \leqslant \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + C_2\|u\|^2_{L_2(t_0,T;\mathscr{U})}$$
$$+ 2(T-t_0) \max_{t\in[t_0,T]} \|N(t, x(t), x^{(1)}(t), \dots, x^{(m-1)}(t))\|^2_{\mathscr{Y}} \leqslant C_3 J(x,u) + C_4.$$

From $\|v_n - v_0\|_{\mathscr{Z}} \to 0$ it follows that

$$\int_{t_0}^{T} \|N(t, v_n(t), v_n^{(1)}(t), \dots, v_n^{(m-1)}(t)) - N(t, v_0(t), v_0^{(1)}(t), \dots, v_0^{(m-1)}(t))\|^2_{\mathscr{Y}} dt$$
$$\leqslant l^2 \|v_n - v\|^2_{W_2^{m-1}(t_0,T;\mathfrak{X})} \to 0, \quad n \to \infty.$$

Therefore, the operator $\mathfrak{F} : \mathscr{Z} \to L_2(t_0,T;\mathscr{Y}) \times \mathscr{X}^m$ is continuous. We choose $\mathfrak{Y}_{-1} = W_2^{m-1}(t_0,T;\mathscr{X})$ and verify the assumptions of Theorem 1.2.4 in [11]. The conditions (1.2.20), (1.2.21) in this theorem are satisfied by the compactness of the embedding of $W_2^m(t_0,T;\mathscr{X})$ into $W_2^{m-1}(t_0,T;\mathscr{X})$ and the continuity of the embedding of $\mathscr{Z}$ into $W_2^m(t_0,T;\mathscr{X})$.

To verify the condition (1.2.22) in [11], we consider the dense subspace $S = L_2(t_0,T;\mathscr{Y})$ of $W_2^{m-1}(t_0,T;\mathscr{X})$. Since N is uniformly Lipschitz with respect to $\overline{v}$, for $w \in L_2(t_0,T;\mathscr{Y})$ we have

$$\langle N(\cdot, v_n(\cdot), \dots, v_n^{(m-1)}(\cdot)) - N(\cdot, v(\cdot), \dots, v^{(m-1)}(\cdot)), w(\cdot)\rangle_{L_2(t_0,T;\mathscr{Y})}$$
$$\leqslant l\|w\|_{L_2(t_0,T;\mathscr{Y})} \|v_n - v\|_{W_2^m(t_0,T;\mathscr{X})},$$

which implies the continuity of the extended functional $\langle \mathfrak{F}(\cdot), v\rangle$ from $\mathscr{Z}$ to $W_2^{m-1}(t_0,T;\mathscr{X})$. □

Theorem 4.2. *We assume that M is (L,p)-bounded, $N : (t_0,T) \times \mathscr{X}^m \to \mathscr{Y}$ for all $v_0, v_1, \dots, v_{m-1} \in \mathscr{X}$ is measurable on (t_0,T) and uniformly Lipschitz with respect to $\overline{v}$ for some $\overline{z} \in \mathscr{X}^m$ $N(\cdot,\overline{z}) \in L_2(t_0,T;\mathscr{Y})$, $\operatorname{im} N \subset \mathscr{Y}^1$, $\mathfrak{U}_\partial$ is a nonempty convex closed subset of the space $L_2(t_0,T;\mathscr{U})$, $\mathfrak{U}_\partial \cap W_2^{m(p+1)}(t_0,T;\mathscr{U}) \neq \varnothing$, $x_0, x_1, \dots, x_{m-1} \in \mathscr{X}^1$. Then there exists a solution $(\widehat{x},\widehat{u}) \in \mathscr{Z} \times \mathfrak{U}_\partial$ to the problem (4.1)–(4.4).*

Proof. By Theorem 2.4, the set of admissible pairs is nonempty, The remaining part of the proof differs by only the arguments concerning the coercivity of the functional J.

For almost all $t \in (t_0,T)$ and all $\overline{x} = (x_0, x_1, \dots, x_{m-1}) \in \mathscr{X}^m$ we have

$$\|N(t,\overline{x})\|^2_{\mathscr{Y}} \leqslant 2\|N(t,\overline{x}) - N(t,\overline{z})\|^2_{\mathscr{Y}} + 2\|N(t,\overline{z})\|^2_{\mathscr{Y}}$$
$$\leqslant 2l^2\|\overline{x} - \overline{z}\|^2_{\mathscr{X}^m} + 2\|N(t,\overline{z})\|^2_{\mathscr{Y}} \leqslant C_1(1 + \|\overline{x}\|^2_{\mathscr{X}^m}) + 2\|N(t,\overline{z})\|^2_{\mathscr{Y}}.$$

Hence $N(\cdot, x(\cdot), x^{(1)}(\cdot), \dots, x^{(m-1)}(\cdot)) \in L_2(t_0,T;\mathscr{Y})$, $x \in W_2^{m-1}(t_0,T;\mathscr{X})$. Using this inequality, we obtain

$$\|x\|^2_{\mathscr{Z}} + \|u\|^2_{L_2(t_0,T;\mathscr{U})} = \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + \|Lx^{(m)} - Mx\|^2_{L_2(t_0,T;\mathscr{Y})} + \|u\|^2_{L_2(t_0,T;\mathscr{U})}$$
$$\leqslant \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + 2\|N(\cdot, x(\cdot), x^{(1)}(\cdot), \dots, x^{(m-1)}(\cdot))\|^2_{L_2(t_0,T;\mathscr{Y})}$$
$$+ 2\|Bu\|^2_{L_2(t_0,T;\mathscr{Y})} + \|u\|^2_{L_2(t_0,T;\mathscr{U})}$$
$$\leqslant \|x\|^2_{W_2^m(t_0,T;\mathscr{X})} + C_2\|u\|^2_{L_2(t_0,T;\mathscr{U})} + 2C_1((T-t_0) + \|x\|^2_{W_2^m(t_0,T;\mathscr{X})}) + 4\|N(t,\overline{z})\|^2_{L_2(t_0,T;\mathscr{Y})}$$
$$\leqslant C_3 J(x,u) + C_4.$$

The theorem is proved. □

5 Example

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^∞, and let $\nu \in \mathbb{R}$. We consider the initial-boundary value problem

$$\frac{\partial^k}{\partial t^k} x_1(s, t_0) = x_{10}^k(s), \quad s \in \Omega, \quad k = 0, 1, \ldots, m-1, \tag{5.1}$$

$$x_i(s, t) = 0, \quad (s, t) \in \partial\Omega \times (t_0, T), \quad i = 1, 2, 3, \tag{5.2}$$

$$\begin{aligned} &\frac{\partial^m}{\partial t^m} x_1 = x_1 + g_1\left(s, t, x_1, x_2, x_3, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_3\right), \quad (s, t) \in \Omega \times (t_0, T), \\ &\triangle \frac{\partial^m}{\partial t^m} x_3 = x_2 + g_2\left(s, t, x_1, x_2, x_3, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_3\right), \quad (s, t) \in \Omega \times (t_0, T), \\ &0 = \triangle x_3 + g_3\left(s, t, x_1, x_2, x_3, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_3\right), \quad (s, t) \in \Omega \times (t_0, T), \end{aligned} \tag{5.3}$$

where the functions g_i, $i = 1, 2, 3$, depend on the sought functions $x_1 = x_1(s, t)$, $x_2 = x_2(s, t)$, $x_3 = x_3(s, t)$ and on their derivatives with respect to t of order up to $m - 1$.

We denote by A the Laplace operator with the domain $W_{2,0}^2(\Omega) = \{z \in W_2^2(\Omega) : z(s) = 0, s \in \partial\Omega\} \subset L_2(\Omega)$ and by $\{\varphi_k\}$ the orthonormal in $L_2(\Omega)$ system of their eigenfunctions corresponding to the system $\{\lambda_k\}$ of the eigenvalues of the operator A enumerated in non-ascending order with taken into account their multiplicity.

We reduce the problem (5.1)–(5.3) to the problem (4.1), (4.2). For this purpose we set $\mathscr{X} = W_{2,0}^2(\Omega) \times L_2(\Omega) \times W_{2,0}^2(\Omega)$, $\mathscr{Y} = (L_2(\Omega))^3$,

$$L = \begin{pmatrix} \triangle & 0 & 0 \\ 0 & 0 & \triangle \\ 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \triangle \end{pmatrix}.$$

It is easy to verify that

$$(\mu L - M)^{-1} = \sum_{k=1}^{\infty} \begin{pmatrix} 1/\mu\lambda_k - 1 & 0 & 0 \\ 0 & -1 & -\mu \\ 0 & 0 & -1/\lambda_k \end{pmatrix} \langle \cdot, \varphi_k \rangle_{L_2(\Omega)} \varphi_k \in \mathscr{L}(\mathscr{Y}; \mathscr{X})$$

for $|\mu| > |\lambda_1|^{-1}$, and the projections take the form

$$P = Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathscr{X}^1 = W_{2,0}^2(\Omega) \times \{0\} \times \{0\}$, $\mathscr{X}^0 = \{0\} \times L_2(\Omega) \times W_{2,0}^2(\Omega)$, $\mathscr{Y}^1 = L_2(\Omega) \times \{0\} \times \{0\}$, $\mathscr{Y}^0 = \{0\} \times L_2(\Omega) \times L_2(\Omega)$,

$$H = \sum_{k=1}^{\infty} \begin{pmatrix} 0 & \lambda_k \\ 0 & 0 \end{pmatrix} \langle \cdot, \varphi_k \rangle_{L_2(\Omega)} \varphi_k.$$

Consequently, $H^2 = \mathbb{O}$ and the operator M is $(L,1)$-bounded. It is clear that (5.1) implies (4.2). In this case, Theorem 4.1 can be used for control systems of the form

$$\begin{aligned}
&\triangle \frac{\partial^m}{\partial t^m} x_1 = x_1 + g_1\left(s,t,x_1,\frac{\partial}{\partial t}x_1,\dots,\frac{\partial^{m-1}}{\partial t^{m-1}}x_1\right) + u_1(s,t), \quad (s,t) \in \Omega \times (t_0,T),\\
&\triangle \frac{\partial^m}{\partial t^m} x_3 = x_2 + g_2\left(s,t,x_1,\frac{\partial}{\partial t}x_1,\dots,\frac{\partial^{m-1}}{\partial t^{m-1}}x_1\right) + u_2(s,t), \quad (s,t) \in \Omega \times (t_0,T),\\
&0 = \triangle x_3 + g_3\left(s,t,x_1,\frac{\partial}{\partial t}x_1,\dots,\frac{\partial^{m-1}}{\partial t^{m-1}}x_1\right) + u_3(s,t), \quad (s,t) \in \Omega \times (t_0,T),
\end{aligned} \tag{5.4}$$

if its nonlinear part depends only on the function x_1 and its derivatives with respect to t of order up to $m-1$. We consider the optimal control problem

$$\sum_{i=1}^{3} \|u_i\|^2_{L_2(t_0,T;L_2(\Omega))} \leqslant R^2, \tag{5.5}$$

$$\frac{1}{2}\sum_{i=1,3} \|x_i - \widetilde{x}_i\|^2_{W_2^1(t_0,T;W_2^2(\Omega))} + \frac{1}{2}\|x_2 - \widetilde{x}_2\|^2_{W_2^1(t_0,T;L_2(\Omega))} + \frac{C}{2}\sum_{i=1}^{3}\|u_i - \widetilde{u}_i\|^2_{L_2(t_0,T;L_2(\Omega))} \to \inf. \tag{5.6}$$

In this problem, $\mathscr{X} = W_2^m(t_0,T;W_{2,0}^2(\Omega) \times L_2(\Omega) \times W_{2,0}^2(\Omega))$. We denote by $B_R(v_0;\mathscr{V})$ the ball of radius R and center $v_0 \in \mathscr{V}$ in the Banach space $\mathscr{V}$.

Theorem 5.1. *We assume that $n = 1$, $g_i \in C^\infty(\overline{\Omega} \times [t_0,T] \times \mathbb{R}^m;\mathbb{R})$ is uniformly with respect to $(t,x) \in \overline{\Omega} \times [t_0,T]$ and Lipschitz with respect to $\overline{v} = (v_0,\dots,v_{m-1}) \in \mathbb{R}^m$, $i = 1,2,3$, $x_{10}^k \in W_{2,0}^2(\Omega)$, $k = 0,1,\dots,m-1$. Then there exists a solution $(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3,\widehat{u}_1,\widehat{u}_2,\widehat{u}_3) \in \mathscr{X} \times B_R(0,L_2(t_0,T;(L_2(\Omega))^3))$ to the problem* (5.1), (5.2), (5.4)–(5.6).

Proof. It suffices to prove the assumption of Theorem 4.1 concerning the smoothness of the operator N defined by the functions g_i, $i = 1,2,3$. Since the last m arguments of these functions belong, at least, to $W_2^1(\Omega)$, from [12] it follows that $N \in C^\infty([t_0,T] \times (W_2^1(\Omega))^m;(W_2^1(\Omega))^3)$. □

Remark 5.1. If g_i, $i = 1,2,3$, are independent of the derivative of order $m-1$, then $N \in C^\infty([t_0,T] \times (W_2^2(\Omega))^m;(W_2^2(\Omega))^3)$ for $n < 4$. If g_i, $i = 1,2,3$, are independent of the derivatives of the $(m-2)$th and $(m-1)$th order, then $N \in C^\infty([t_0,T] \times (W_2^3(\Omega))^m;(W_2^3(\Omega))^3)$ for $n < 6$ and so on.

Similarly, using Theorem 4.2, one can study the optimal control problem (5.5), (5.6) for the distributed system described by the equations

$$\begin{aligned}
&\triangle \frac{\partial^m}{\partial t^m} x_1 = x_1 + g_1\left(s,t,x_1,x_2,x_3,\frac{\partial}{\partial t}x_1,\dots,\frac{\partial^{m-1}}{\partial t^{m-1}}x_3\right) + u_1(s,t),\\
&\triangle \frac{\partial^m}{\partial t^m} x_3 = x_2 + u_2(s,t),\\
&0 = \triangle x_3 + u_3(s,t).
\end{aligned}$$

The nonlinear function is contained only in the first equation (corresponds to the condition $\operatorname{im} N \subset \mathscr{Y}^1$), but depends on x_1, x_2, x_3 and their derivatives of order up to $m-1$ with respect to t.

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References

1. R. E. Showalter, "Nonlinear degenerate evolution equations and partial differential equations of mixed type," *SIAM J. Math. Anal.* **6**, No. 1, 25–42 (1975).

2. N. A. Sidorov, "A class of degenerate differential equations with convergence" [in Russian], *Mat. Zametki* **35**, No. 4, 569–578 (1984); English transl.: *Math. Notes* **35**, 300–305 (1984).

3. G. A. Sviridyuk and V. E. Fedorov, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*, VSP, Utrecht etc. (2003).

4. A. A. Zamyshlyaeva and E. V. Bychkov, "The phase space of the modified Boussinesq equation" [in Russian], *Vestn. Yuzhno-Ural. Gos. Univ., Ser. Mat. Model. Program* **18**, No. 2, 13–19 (2012).

5. M. V. Plekhanova, "Quasilinear equations that are not solved for the higher-order time derivative" [in Russian], *Sib. Mat. Zh.* **56**, No. 4, 909–921 (2015); English transl.: *Sib. Math. J.* **56**, No. 4, 725–735 (2015).

6. M. V. Plekhanova and V. E. Fedorov, "An optimal control problem for a class of degenerate equations" [in Russian], *Izv. Akad. Nauk, Teor. Sist. Upr.* No. 5, 40–44 (2004); English transl.: *J. Comput. Syst. Sci. Intern.* **43**, No. 5, 698–702 (2004).

7. V. E. Fedorov and M. V. Plekhanova, "Optimal control of Sobolev type linear equations" [in Russian], *Differ. Uravn.* **40**, No. 11, 1548–1556 (2004); English transl.: *Differ. Equ.* **40**, No. 11, 1627–1637 (2004).

8. V. E. Fedorov and M. V. Plekhanova, "The problem of start control for a class of semilinear distributed systems of Sobolev type" [in Russian], *Tr. Inst. Mat. Mekh. (Ekaterinburg)* **17**, No. 1, 259–267 (2011); English transl.: *Proc. Steklov Inst. Math.* **275**, Suppl. 1, S40-S48 (2011).

9. M. V. Plekhanova and V. E. Fedorov, "On the existence and uniqueness of solutions of optimal control problems of linear distributed systems which are not solved with respect to the time derivative" [in Russian], *Izv. Ross. Akad. Nauk, Ser. Mat.* **75**, No. 2, 177–194 (2011); English transl.: *Izv. Math.* **75**, No. 2, 395–412 (2011).

10. M. V. Plekhanova and A. F. Islamova, "Problems with a robust mixed control for the linearized Boussinesq equation" [in Russian], *Differ. Uravn.* **48**, No. 4, 565–576 (2012); English transl.: *Differ. Equ.* **48**, No. 4, 574–585 (2012).

11. A. V. Fursikov, *Optimal Control of Distributed Systems. Theory and Applications*, Am. Math. Soc., Providence, RI (2000).

12. B. D. Hassard, N. D. Kazarinoff, and Y.-H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge (1981).

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