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GALERKIN APPROXIMATIONS IN PROBLEMS WITH p-LAPLACIAN

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We study the elliptic problem with p-Laplacian and construct a system of Galerkin approximations. We estimate the difference between an exact and approximate solutions in the case of constant or variable exponent p. Bibliography: 3 *titles.*

1 Constant Exponent. Estimates

We consider the Dirichlet problem

$$-\text{div}\,(|\nabla u|^{p-2}\nabla u) = f, \quad u\big|_{\partial\Omega} = 0, \tag{1.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, where $p > 1$, $X = W_0^{1,p}$ is the Sobolev space, $u \in X$, and f is a linear continuous functional on X, i.e., $f \in X^*$. The left-hand side of the equation in (1.1) is called the *p-Laplacian* and is denoted by $\Delta_p u$. The norm in the space X is defined by

$$\|u\|_X = \|\nabla u\|_p = \Big(\int\limits_\Omega |\nabla u|^p\,dx \Big)^{\frac{1}{p}}.$$

Definition 1.1. By a *solution to the problem* (1.1) we mean a function $u \in X$ such that

$$\int\limits_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx = (f, \varphi) \quad \forall \varphi \in X. \tag{1.2}$$

As is known, the problem (1.2) is the Euler equation for the variational problem

$$\min_{u\in X} \int\limits_\Omega \Big(\frac{|\nabla u|^p}{p} - g\cdot u\Big)\,dx.$$

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A solution to this problem exists and is unique.

To prove the solvability of the problem (1.2), it is convenient to write it in the operator form. For this purpose we consider the operator $A : X \to X^*$ from a space X to the dual X^* such that

$$(Au, \varphi) = \int\limits_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \quad \forall \varphi \in X.$$

The functional Au is continuous since $|\nabla u|^{p-2}\nabla u \in L^{p'}(\Omega)$ and $\nabla \varphi \in L^p(\Omega)$. Thus, the problem (1.2) can be written as

$$Au = f.$$

Since the operator A is monotone and coercive, we can apply the method of monotone operators. We construct approximate solutions by the Galerkin method. Let $X_1 \subset X_2 \subset \ldots \subset X_n$ be an expanding sequence of finite-dimensional subspaces of X such that their union is dense in X. The Galerkin approximations are found as solutions $u_n \in X_n$ to the system

$$u_n \in X_n, \quad \int\limits_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \, dx = (f, \varphi) \quad \forall \varphi \in X_n. \tag{1.3}$$

Similarly, we introduce the operator $A_n : X_n \to X_n^*$ by the rule

$$(A_n u, \varphi) = \int\limits_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \quad \forall \varphi \in X_n.$$

Then the problem (1.3) for Galerkin approximations can be written as

$$A_n u_n = f,$$

where the operator A_n is monotone.

To solve the problems (1.2) and (1.3), we derive some estimates. Setting $\varphi = u$ in (1.2) and $\varphi = u_n$ in (1.3), we get

$$\int\limits_\Omega |\nabla u|^p \, dx = (f, u) \leqslant \|f\|_{X^*} \|u\|_X \,,$$

$$\|u\|_X \leqslant \|f\|_{X^*}^{\frac{1}{p-1}} \tag{1.4}$$

$$\|u_n\|_X \leqslant \|f\|_{X^*}^{\frac{1}{p-1}} \,. \tag{1.5}$$

Theorem 1.1. *The following estimate holds*:

$$\|u - u_n\|_X \begin{cases} = \operatorname{dist}(u, X_n), & p = 2, \\ \leqslant C(\operatorname{dist}(u, X_n))^{\frac{p}{2}} \, \|f\|_{X^*}^{\frac{2-p}{2(p-1)}}, & 1 < p \leqslant 2, \\ \leqslant C(\operatorname{dist}(u, X_n))^{\frac{2}{p}} \, \|f\|_{X^*}^{\frac{p-2}{p(p-1)}}, & p \geqslant 2, \end{cases}$$

where C is a constant depending only on p and d.

Proof. 1. *Case* $p = 2$. In this case, it is obvious that the p-Laplacian is the Laplace operator. Equations (1.2) and (1.3) take the form

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = (f, \varphi) \quad \forall \varphi \in X,$$

$$\int_\Omega \nabla u_n \cdot \nabla \varphi \, dx = (f, \varphi) \quad \forall \varphi \in X_n.$$

Subtracting the second identity from the first one, we get

$$\int_\Omega (\nabla u - \nabla u_n) \cdot \nabla \varphi \, dx = 0 \quad \forall \varphi \in X_n. \tag{1.6}$$

Since φ is arbitrary, we can set $\varphi = u_n$ and write

$$\int_\Omega (\nabla u - \nabla u_n) \cdot \nabla u_n \, dx = 0.$$

Subtracting $\int_\Omega (\nabla u - \nabla u_n) \cdot \nabla u \, dx$ from both sides of the last equality, we find

$$\int_\Omega (\nabla u - \nabla u_n)^2 \, dx = \int_\Omega (\nabla u - \nabla u_n) \cdot \nabla u \, dx. \tag{1.7}$$

By the definition of distance, there exists $w_n \in X_n$ such that $\|u - w_n\|_X = \operatorname{dist}(u, X_n)$. Setting $\varphi = w_n$ in (1.6) and subtracting from (1.7), we get

$$\begin{aligned} \int_\Omega |\nabla u - \nabla u_n|^2 \, dx &= \int_\Omega (\nabla u - \nabla u_n)(\nabla u - \nabla w_n) \, dx \\ &\leqslant \left(\int_\Omega |\nabla u - \nabla u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u - \nabla w_n|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

where we used the Cauchy–Bunyakowsky inequality at the last step. Therefore,

$$\int_\Omega |\nabla u - \nabla u_n|^2 \, dx \leqslant \int_\Omega |\nabla u - \nabla w_n|^2 \, dx,$$

$$\|u - u_n\|_X \leqslant \operatorname{dist}(u, X_n).$$

By the definition of $\operatorname{dist}(u, X_n)$, the last estimate implies the equality $\|u - u_n\|_X = \operatorname{dist}(u, X_n)$. In the case $p \neq 2$, the identities

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = (f, \varphi) \quad \forall \varphi \in X,$$

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \, dx = (f, \varphi) \quad \forall \varphi \in X_n$$

imply

$$\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n)\cdot\nabla\varphi\,dx = 0 \quad \varphi\in X_n, \tag{1.8}$$

$$\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n)\cdot(\nabla u - \nabla u_n)\,dx$$
$$= \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n)\cdot\nabla u\,dx. \tag{1.9}$$

There exists $w_n \in X_n$ such that $\|u - w_n\|_X = \operatorname{dist}(u, X_n)$. Setting $\varphi = w_n$ in (1.8) and subtracting from (1.9), we find

$$\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n)(\nabla u - \nabla u_n)\,dx$$
$$= \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_n|^{p-2}\nabla u_n)(\nabla u - \nabla w_n)\,dx. \tag{1.10}$$

Denoting

$$l(a) = |a|^{p-2}\,a, \quad a\in\mathbb{R}^d, \qquad D = D(a,b) = (l(b) - l(a))\cdot(b-a), \tag{1.11}$$

we can write the identity (1.10) in the form

$$\int_\Omega D(\nabla u, \nabla u_n)\,dx = \int_\Omega (l(\nabla u) - l(\nabla u_n))\cdot(\nabla u - \nabla w_n)\,dx. \tag{1.12}$$

2. *Case* $1 < p < 2$. We use the following inequalities proved in [1, 2]:

$$|l(b) - l(a)|^{p'} \leqslant D(a,b) = D \quad \forall a,b\in\mathbb{R}^d, \tag{1.13}$$

$$|b-a|^2 \leqslant CD(a,b)(|a|^{2-p} + |b|^{2-p}) \quad \forall a,b\in\mathbb{R}^d \tag{1.14}$$

By the Hölder inequality, from (1.12) it follows that

$$\int_\Omega D\,dx \leqslant \left(\int_\Omega (l(\nabla u) - l(\nabla u_n))^{p'}dx\right)^{\frac{1}{p'}}(|\nabla u - \nabla w_n|^p)^{\frac{1}{p}}.$$

By (1.13),

$$\int_\Omega D\,dx \leqslant \left(\int_\Omega D\,dx\right)^{\frac{1}{p'}}\operatorname{dist}(u, X_n), \qquad \int_\Omega D\,dx \leqslant (\operatorname{dist}(u, X_n))^p \tag{1.15}$$

By (1.14),

$$|b-a|^p \leqslant CD^{\frac{p}{2}}(|a|^{2-p} + |b|^{2-p})^{\frac{p}{2}} \leqslant CD^{\frac{p}{2}}(|a|^{\frac{(2-p)p}{2}} + |b|^{\frac{(2-p)p}{2}})$$

which, together with the Hölder inequality with exponents $2/p$ and $2/(2-p)$, implies

$$\int_\Omega |\nabla u - \nabla u_n|^p\,dx \leqslant C\left(\int_\Omega D dx\right)^{\frac{p}{2}}\left(\int_\Omega (|\nabla u|^p + |\nabla u_n|^p)dx\right)^{\frac{2-p}{2}}$$
$$\leqslant C\operatorname{dist}(u, X_n)^{p\frac{p}{2}}\,\|f\|_{X^*}^{\frac{p(2-p)}{2(p-1)}}, \tag{1.16}$$

where we used the estimates (1.15), (1.5), and (1.6). It remains to take the pth degree root:

$$\|u-u_n\|_X \leqslant C\operatorname{dist}(u,X_n)^{\frac{p}{2}} \|f\|_{X^*}^{\frac{2-p}{2(p-1)}} .$$

3. *Case* $p>2$. In this case, we have (cf. the proof in [1])

$$C|b-a|^p \leqslant D, \tag{1.17}$$

$$|l(b)-l(a)| \leqslant CD^{\frac{1}{2}}(|a|+|b|)^{\frac{p-2}{2}}. \tag{1.18}$$

From (1.12) it follows that

$$\int\limits_\Omega D\,dx = \int\limits_\Omega (l(\nabla u)-l(\nabla u_n))(\nabla u-\nabla w_n)\,dx \leqslant C\int\limits_\Omega D^{\frac{1}{2}}(|\nabla u|+|\nabla u_n|)^{\frac{p-2}{2}}|\nabla u-\nabla w_n|\,dx.$$

We apply the Hölder inequality with exponents 2, $2p/(p-2)$, and p:

$$\int\limits_\Omega Ddx \leqslant C\bigg(\int\limits_\Omega Ddx\bigg)^{\frac{1}{2}}\bigg(\int\limits_\Omega |\nabla u|^p+|\nabla u_n|^p dx\bigg)^{\frac{p-2}{2p}} \operatorname{dist}(u,X_n),$$

$$\int\limits_\Omega Ddx \leqslant C\bigg(\int\limits_\Omega |\nabla u|^p+|\nabla u_n|^p dx\bigg)^{\frac{p-2}{p}} (\operatorname{dist}(u,X_n))^2 \quad \Rightarrow$$

$$\|u-u_n\|_X^p \leqslant \int\limits_\Omega Ddx \leqslant C\bigg(\int\limits_\Omega |\nabla u|^p+|\nabla u_n|^p dx\bigg)^{\frac{p-2}{p}} (\operatorname{dist}(u,X_n))^2 \quad \Rightarrow \tag{1.9}$$

$$\|u-u_n\|_X \leqslant C\,\|f\|_{X^*}^{\frac{p-2}{p(p-1)}} (\operatorname{dist}(u,X_n))^{\frac{2}{p}}.$$

If the union of X_n is dense in X, then $\operatorname{dist}(u,X_n)\to 0$ as $n\to\infty$. □

2 Variable Exponent. The Sobolev–Orlicz Space

We consider the functional

$$F[u] = \int\limits_\Omega f(x,\nabla u)\,dx, \tag{2.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$, $f(x,\xi)$ is measurable in $x\in\Omega$, convex in $\xi\in\mathbb{R}^d$ and satisfies the standard growth condition

$$c_1|\xi|^\alpha - 1 \leqslant f(x,\xi) \leqslant c_2|\xi|^\alpha + 1 \quad (\alpha>0, c_1>0). \tag{2.2}$$

The functional F on the Sobolev space $W_0^{1,\alpha}$ is convex, lower semicontinuous, and coercive:

$$F[u] \geqslant c_1\int\limits_\Omega |\nabla u|^\alpha\,dx - |\Omega|.$$

Consequently, the variational Dirichlet problem (Problem E_1)

$$E_1 = \min_{u\in W_0^{1,\alpha}} F[u] \tag{2.3}$$

has a solution, which is guaranteed by the left estimate in (2.2). From the right estimate in (2.2) it follows that the functional F is locally bounded and, consequently, is continuous on $W_0^{1,\alpha}$. Therefore, for Problem E_1 there exists a smooth minimizing sequence, i.e.,

$$E_1 = \inf_{u\in C_0^\infty} F[u]. \tag{2.4}$$

We replace the standard growth condition with the following more general one:

$$-c_0(x) + c_1|\xi|^\alpha \leqslant f(x,\xi) \leqslant c_0(x) + c_2|\xi|^\beta, \tag{2.5}$$

where $c_0 \in L^1(\Omega)$, $1 < \alpha \leqslant \beta$. As above, the functional F is convex, lower semicontinuous, and coercive on $W_0^{1,\alpha}$. Hence Problem E_1 has a solution in $W_0^{1,\alpha}$. However, the functional F is not necessarily continuous on $W_0^{1,\alpha}$, and (2.4) is not guaranteed. It can happen that

$$E_1 = \min_{u\in W_0^{1,\alpha}} F[u] < \inf_{u\in C_0^\infty} F[u] = E_2. \tag{2.6}$$

Similar inequalities are referred to as the *Lavrent'ev gap*. Thus, no sufficiently smooth minimizing sequence exists for Problem E_1. Therefore, in addition to the original problem, it is necessary to study the minimization problem, referred to as the *relaxation problem*, over only smooth functions.

In what sequel, we need the Sobolev spaces with variable exponent: the Sobolev–Orlicz spaces. We denote by $L^{p(\,\cdot\,)}(\Omega)$ the class of measurable vector-valued functions $v : \Omega \to \mathbb{R}^d$ such that

$$\int\limits_\Omega |v(x)|^{p(x)}\,dx < \infty$$

and introduce the *Luxemburg norm*

$$\|v\|_{p(\cdot)} = \inf\left\{\lambda > 0,\ \int\limits_\Omega \left|\frac{v}{\lambda}\right|^p \leqslant 1\right\}. \tag{2.7}$$

Let $1 < \alpha \leqslant p(x) \leqslant \beta < \infty$. Then

$$\|v\|_{p(\cdot)}^\alpha - 1 \leqslant \int\limits_\Omega |v|^p dx \leqslant \|v\|_{p(\cdot)}^\beta + 1. \tag{2.8}$$

In the variable $L^{p(\cdot)}$ Sobolev–Orlicz spaces, the Hölder inequality takes the form

$$\int\limits_\Omega fg dx \leqslant 2\,\|f\|_{L^{p(\cdot)}}\,\|g\|_{L^{p'(\cdot)}} \tag{2.9}$$

We introduce the Sobolev–Orlicz space

$$W_0^{1,p(\cdot)} = \left\{u \in W_0^{1,1},\ \int\limits_\Omega |\nabla u|^p dx < \infty\right\} \tag{2.10}$$

equipped with the norm $\|u\|_{W_0^{1,p(\cdot)}} = \|\nabla u\|_{p(\cdot)}$.

In what follows, $H = H_0^{1,p(\cdot)}$ is the closure of the set $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}$.

Definition 2.1. An exponent $p(x)$ is *regular* if $C_0^\infty(\Omega)$ is dense in the space $W_0^{1,p(\cdot)}$.

Theorem 2.1 (cf. [3])**.** *If the logarithmic condition*

$$|p(x) - p(y)| \leqslant w(|x-y|) \equiv \frac{k}{\ln \frac{1}{|x-y|}}, \quad x, y \in \Omega, \quad |x-y| \leqslant \frac{1}{4}$$

holds, then the exponent p is regular.

3 Variable Exponent. Estimates

We consider the Dirichlet problem

$$-\mathrm{div}\,(|\nabla u|^{p(x)-2}\,\nabla u) = f, \quad u\big|_{\partial\Omega} = 0 \tag{3.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, where the exponent p is a measurable function, $1 < \alpha \leqslant p(x) \leqslant \beta < \infty$. The right-hand side is a linear functional on H, i.e., $f \in H^*$.

Definition 3.1. By a *solution to the problem* (3.1) we mean a function $u \in H$ such that

$$\int\limits_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx = (f, \varphi) \quad \forall \varphi \in H. \tag{3.2}$$

As in the case of a constant exponent p, one can show that (3.2) is the Euler equation for the variational problem

$$\min_{u \in H} \int\limits_\Omega \Big(\frac{|\nabla u|^p}{p} - g \cdot u \Big) dx$$

which has a unique solution.

To write the problem (3.2) in the operator form, we define the operator $A : H \to H^*$ from a space H to the dual H^* by the formula

$$(Au, \varphi) = \int\limits_\Omega |\nabla u|^{p(x)-2}\,\nabla u \cdot \nabla \varphi dx \quad \forall \varphi \in H.$$

The functional Au is continuous. Indeed, $|\nabla u|^{p(x)-2}\,\nabla u \in L^{p'(\cdot)}(\Omega)$, $\nabla \varphi \in L^{p(\cdot)}(\Omega)$. Applying the Hölder inequality, we obtain the estimate

$$|(Au, \varphi)| \leqslant 2\,\|\nabla u\|_{p(\cdot)} \cdot \|\nabla \varphi\|_{p(\cdot)}\,.$$

Thus, the problem (3.2) can be written as

$$Au = f.$$

Since the operator A is monotone and coercive, we can apply the method of monotone operators. Let $H_1 \subset H_2 \subset \ldots \subset H_n$ be an expanding sequence of finite-dimensional subspaces of H such that their union is dense in H. Galerkin approximations are found as solutions $u_n \in H_n$ to the system

$$u_n \in H_n, \quad \int\limits_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi dx = (f, \varphi) \quad \forall \varphi \in H_n. \tag{3.3}$$

As above, we introduce the operator $A_n : H_n \to H_n^*$ by the rule

$$(A_n u, \varphi) = \int\limits_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx \quad \forall \varphi \in H_n.$$

Then the problem (3.3) can be written as

$$A_n u_n = f,$$

where the operator A_n is monotone. Setting $\varphi = u$ in (3.2) and $\varphi = u_n$ in (3.3), we find

$$\int\limits_{\Omega} |\nabla u|^p dx = (f, u) \leqslant \|f\|_{H^*} \|u\|_H , \tag{3.4a}$$

$$\int\limits_{\Omega} |\nabla u_n|^p dx = (f, u_n) \leqslant \|f\|_{H^*} \|u_n\|_H . \tag{3.4b}$$

Introduce the notation

$$\begin{aligned} &l(\psi) = |\psi|^{p(\cdot)-2} \psi, \quad \bar{l}(\psi) = |\psi|^{p'(\cdot)-2} \psi, \\ &D(a,b) = (|b|^{p(x)-2} b - |a|^{p(x)-2} a, b - a) = (l(b) - l(a), b - a), \\ &\overline{D}(a,b) = (|b|^{p'(\cdot)-2} b - |a|^{p'(\cdot)-2} a, b - a) = (\bar{l}(b) - \bar{l}(a), b - a). \end{aligned} \tag{3.5}$$

In this problem, the value of $p(x)$ depends on points of Ω. We divide Ω into two subdomains $\Omega^+ = \{x \mid p(x) \geqslant 2\}$ and $\Omega^- = \{x \mid p(x) < 2\}$. It suffices to consider the case $p > 2$ on Ω^+ and use similar arguments with the exponent p' in the case of Ω^-. We introduce the flow $z = |\nabla u|^{p(\cdot)-2} \nabla u$. Then

$$\nabla u = |z|^{\frac{2-p(\cdot)}{p(\cdot)-1}} z = |z|^{p'(\cdot)-2} z = \bar{l}(z). \tag{3.6}$$

Since $p'(x) > 2$ on Ω^-, the replacement of gradients with flows leads to the dual relation

$$\int\limits_{\Omega^-} (l(\nabla u) - l(\nabla u_n)) \cdot (\nabla u - \nabla u_n) dx = \int\limits_{\Omega^-} (z - z_n) \cdot (\bar{l}(z) - \bar{l}(z_n)) dx. \tag{3.7}$$

Theorem 3.1. *Let $\alpha > 1$ satisfy (2.8), and let $p > 1$. Then*

$$\begin{aligned} &\int\limits_{\Omega^+} |\nabla u - \nabla u_n|^p dx + \int\limits_{\Omega^-} |z - z_n|^{p'} dx \\ &\leqslant 2(M + \overline{M})([1 + \|f\|_{H^*} \|u\|_H]^{\frac{1}{\alpha}} + [1 + \|f\|_{H^*} \|u_n\|_H]^{\frac{1}{\alpha}}) \operatorname{dist}(u, H_n), \end{aligned}$$

where $z = |\nabla u|^{p(\cdot)-2} \nabla u$, $z_n = |\nabla u_n|^{p(\cdot)-2} \nabla u_n$, $M = \left\|2^{p-2}\right\|_{p'(\cdot)}$, and $\overline{M} = \|2^{p'-2}\|$.

Proof. For u and u_n we have

$$\int\limits_{\Omega^+} l(\nabla u)\nabla\varphi\,dx + \int\limits_{\Omega^-} l(\nabla u)\nabla\varphi\,dx = (f,\varphi), \quad \varphi \in H,$$

$$\int\limits_{\Omega^+} l(\nabla u_n)\nabla\varphi\,dx + \int\limits_{\Omega^-} l(\nabla u_n)\nabla\varphi\,dx = (f,\varphi), \quad \varphi \in H_n.$$

Subtracting, we find

$$\int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n)) \cdot \nabla\varphi\,dx = 0, \quad \varphi \in H_n, \tag{3.8}$$

$$\int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n)) \cdot \nabla u_n\,dx = 0.$$

Subtracting $\int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n)) \cdot \nabla u\,dx$ from the last equality, we get

$$\int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n)) \cdot (\nabla u - \nabla u_n)\,dx = \int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n)) \cdot \nabla u\,dx. \tag{3.9}$$

There is $w_n \in H_n$ such that $\|u - w_n\|_H = \operatorname{dist}(u, H_n)$. We substitute $\varphi = w_n$ into (3.8) and subtract the obtained expression from (3.9):

$$\int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n))(\nabla u - \nabla u_n)\,dx = \int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n))(\nabla u - \nabla w_n)\,dx. \tag{3.10}$$

Using (3.5), we can write (3.10) in the form

$$\int\limits_{\Omega} D(\nabla u, \nabla u_n)\,dx = \int\limits_{\Omega} (l(\nabla u) - l(\nabla u_n))(\nabla u - \nabla w_n)\,dx. \tag{3.11}$$

We use the inequalities (cf. the proof in [1])

$$|\nabla u - \nabla u_n|^p \leqslant 2^{p-2} \cdot D(\nabla u, \nabla u_n), \quad p \geqslant 2.$$

By (3.11) and (3.7), we have

$$\int\limits_{\Omega^+} |\nabla u - \nabla u_n|^p\,dx + \int\limits_{\Omega^-} |z - z_n|^{p'}\,dx \leqslant \int\limits_{\Omega^+} 2^{p-2}\,D(\nabla u, \nabla u_n)\,dx + \int\limits_{\Omega^-} 2^{p'-2}\,\overline{D}(z, z_n)\,dx. \tag{3.12}$$

Let us estimate the integral over Ω^+:

$$\begin{aligned}\int\limits_{\Omega^+} 2^{p-2}\,D(\nabla u, \nabla u_n)\,dx &= \int\limits_{\Omega^+} 2^{p-2}\,(l(\nabla u) - l(\nabla u_n))(\nabla u - \nabla u_n)\,dx \\ &= \int\limits_{\Omega^+} 2^{p-2}\,(l(\nabla u) - l(\nabla u_n))(\nabla u - \nabla w_n)\,dx \\ &\leqslant 2\left\|2^{p-2}\,(l(\nabla u) - l(\nabla u_n))\right\|_{p'(\cdot)} \cdot \|\nabla u - \nabla w_n\|_{p(\cdot)}.\end{aligned}$$

Setting $M = \left\|2^{p-2}\right\|_{p'(\cdot)}$, we get

$$2M \left\|(l(\nabla u) - l(\nabla u_n))\right\|_{p'(\cdot)} \operatorname{dist}(u, H_n) \leqslant 2M \left\|(l(\nabla u) + l(\nabla u_n))\right\|_{p'(\cdot)} \operatorname{dist}(u, H_n)$$
$$\leqslant 2M(\left\|l(\nabla u)\right\|_{p'(\cdot)} + \left\|l(\nabla u_n)\right\|_{p'(\cdot)})\operatorname{dist}(u, H_n).$$

By (2.8), we have

$$\|l(\nabla u)\|^{\alpha}_{p'(\cdot)} - 1 \leqslant \int\limits_{\Omega^+} |\nabla u|^{(p-1)p'}\, dx = \int\limits_{\Omega^+} |\nabla u|^p\, dx \leqslant \|f\|_{H^*} \|u\|_H\,,$$

$$\left\|l(\nabla u)\right\|_{p'(\cdot)} \leqslant [1 + \|f\|_{H^*} \|u\|_H]^{\frac{1}{\alpha}}. \tag{3.13a}$$

Similarly,

$$\left\|l(\nabla u_n)\right\|_{p'(\cdot)} \leqslant [1 + \|f\|_{H^*} \|u_n\|_H]^{\frac{1}{\alpha}}. \tag{3.13b}$$

We have

$$\int\limits_{\Omega^+} 2^{p-2}\, D(\nabla u, \nabla u_n)\, dx \leqslant 2\, M\big([1 + \|f\|_{H^*} \|u\|_H]^{\frac{1}{\alpha}} + [1 + \|f\|_{H^*} \|u_n\|_H]^{\frac{1}{\alpha}}\big)\operatorname{dist}(u, H_n). \tag{3.14}$$

For the integral over Ω^- the required estimate is proved in a similar way. Namely, setting $\theta_n = l(\nabla w_n)$ and $\overline{M} = \|2^{p'-2}\|$, we find

$$\int\limits_{\Omega^+} 2^{p'-2}\, \overline{D}(z, z_n)\, dx = \int\limits_{\Omega^-} 2^{p'-2}\, (z - z_n)(\bar{l}(z) - \bar{l}(\theta_n))\, dx \leqslant 2\overline{M}(\|z\|_{p'(\cdot)} + \|z_n\|_{p'(\cdot)})\operatorname{dist}(u, H_n)$$
$$\leqslant 2\, \overline{M}\big([1 + \|f\|_{H^*} \|u\|_H]^{\frac{1}{\alpha}} + [1 + \|f\|_{H^*} \|u_n\|_H]^{\frac{1}{\alpha}}\big)\operatorname{dist}(u, H_n). \tag{3.15}$$

Adding (3.14) and (3.15), we obtain the required estimate. □

4 Estimates in Anisotropic Case

The above results can be extended to a larger class of elliptic problems. Let us consider the Dirichlet problem

$$-\operatorname{div}(|\nabla u|^{p-2}\, A\nabla u) = f, \quad u\big|_{\partial\Omega} = 0 \tag{4.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, where $p > 1$ is a constant and $A = A(x)$ is a measurable bounded positive definite matrix. The right-hand side f is a linear functional on $X = W^{1,p}_0$, i.e., $f \in X^*$.

Definition 4.1. By a *solution to the problem* (4.1) we mean a function $u \in X$ such that

$$\int\limits_{\Omega} |\nabla u|^{p-2}\, A\nabla u \cdot \nabla\varphi\, dx = (f, \varphi) \quad \forall \varphi \in X. \tag{4.2}$$

We denote

$$\begin{aligned} &l_A(\psi) = |\psi|^{p-2}\, A\psi, \quad l^*_A(\zeta) = |\zeta|^{p'-2} A^{-1}\zeta,\\ &D_A(\psi, \zeta) = (l_A(\psi) - l_A(\zeta)) \cdot (\psi - \zeta),\\ &D^*_A(\psi, \zeta) = (l^*_A(\psi) - l^*_A(\zeta)) \cdot (\psi - \zeta). \end{aligned} \tag{4.3}$$

Then a solution to the problem (4.1) is a function $u \in X$ such that

$$\int_\Omega l_A(\nabla u) \cdot \nabla \varphi \, dx = (f, \varphi) \quad \forall \varphi \in X.$$

Let l_A satisfy the boundedness and coercivity conditions

$$|l_A(\zeta)| \leqslant C_0 |\zeta|^{p-1} \tag{4.4}$$

$$l_A(\zeta) \cdot \zeta \geqslant C_1 |\zeta|^p, \quad p > 1, \tag{4.5}$$

To prove the uniqueness of a solution to the problem (4.2), we need the monotonicity of l_A:

$$D_A(\zeta, \eta) = (l_A(\zeta) - l_A(\eta)) \cdot (\zeta - \eta) \geqslant 0. \tag{4.6}$$

To provide these properties, we impose additional conditions on the matrix A.

Lemma 4.1. *Let A be a positive definite symmetric matrix, and let*

$$\mu(A) = \sup_{|x|=1} \frac{|Ax|}{(Ax, x)}.$$

If $\mu(A) < p/(|p-2|)$, then

$$|l_A(b) - l_A(a)|^{p'} \leqslant C D_A(b, a), \quad 1 < p < 2; \tag{4.7a}$$

$$C|b - a|^p \leqslant D_A(b, a), \quad p > 2, \tag{4.7b}$$

$$|l_A(b) - l_A(a)|^2 \leqslant C D_A(b, a) \cdot (|a| + |b|)^{p-2}, \quad p > 2, \tag{4.7c}$$

where $C = C(A, p, d) > 0$ (cf. [2]). Furthermore, the relation (4.6) holds.

The coefficient C in Lemma 4.1 depends on the maximal and minimal eigenvalues of the matrix A and on $\mu(A)$, which allows us to deal with flows of the form l_A with a bounded measurable symmetric positive definite matrix A such that $\mu(A) < p/|p-2|$.

It is important to note that even a simple monotonicity condition (4.6) for flows l_A can fail if the above conditions on the matrix A are not satisfied. Indeed (cf. [2]), let $A = \operatorname{diag}(\lambda_1, \lambda_2)$, $\eta(t, 0)$, $\zeta(1, 1)$. Then $D_A(\zeta, \eta) = \lambda_1 (2^{\frac{p-2}{2}} - t^{p-1})(1 - t) + 2^{\frac{p-2}{2}} \lambda_2$. It is easy to see that the first term is negative for $t \in (2^{\frac{p-2}{2p-2}}, 1)$ if $p < 2$ and for $t \in (1, 2^{\frac{p-2}{2p-2}})$ if $p > 2$. Dividing the expression by λ_2, we find $D_A(\zeta, \eta) < 0$.

By the coercivity condition (4.4), we have

$$\int_\Omega |\nabla u|^p \, dx \leqslant \frac{1}{C_1} \int_\Omega l_A(\nabla u) \cdot \nabla u \, dx \leqslant \frac{1}{C_1} \|f\|_{X^*} \|u\|_X ,$$

$$\|\nabla u\|_X^p \leqslant \frac{1}{C_1} \|f\|_{X^*}^{p'} . \tag{4.8a}$$

Let $X_1 \subset X_2 \subset \ldots \subset X_n$ be an expanding sequence of finite-dimensional subspaces of X such that their union is dense in X. The Galerkin approximations are found as solutions $u_n \in X_n$ to the system

$$u_n \in X_n, \quad \int_\Omega l_A(\nabla u_n) \cdot \nabla \varphi \, dx = (f, \varphi) \quad \forall \varphi \in X_n. \tag{4.9}$$

It is obvious that the solvability of the problem (4.9) can be established in different ways. Furthermore,

$$\|\nabla u_n\|_X^p \leqslant \frac{1}{C_1} \|f\|_{X^*}^{p'} . \tag{4.8b}$$

For the sake of brevity we write z instead of l_A.

In the case $1 < p < 2$, we estimate the norm with respect to the flow z and conjugate exponent p'. Therefore, we have

$$\mu^*(A^{-1}) = \sup_{|z|=1} \frac{|A^{-1}z|}{(A^{-1}z, z)} < \frac{p'}{|p'-2|} \tag{4.10}$$

and the monotonicity property in p':

$$|\psi - \zeta|^{p'} \leqslant C^*(l_A^*(\psi) - l_A^*(\zeta), \psi - \zeta), \tag{4.11}$$

where $C^* = C(A^{-1}, p', n)$.

We note that for conjugate exponents p and p'

$$\frac{p}{p-2} \equiv \frac{p'}{2-p'}, \quad p \neq 2.$$

We show that $\mu^*(A^{-1}) = \sup\limits_{|z|=1} \frac{|A^{-1}z|}{(A^{-1}z,z)}$ can be expressed in terms of the matrix A.

Lemma 4.2. *Under the assumptions of Lemma* 4.1,

$$\mu^*(A^{-1}) = \sup_{|x|=1} \frac{1}{(Ax, x)}.$$

Proof. In the above notation, $z = |x|^{p(\cdot)-2} Ax$ and $x = |z|^{p'(\cdot)-2} A^{-1}z$. Then $A^{-1}z = A^{-1}(|x|^{p(\cdot)-2} Ax) = |x|^{p(\cdot)-2} Ix = |x|^{p(\cdot)-2} x$ and $|A^{-1}z| = |x|^{p(\cdot)-1}$. We write the denominator in the expression for $\overline{\mu}(A^{-1})$ as follows:

$$(A^{-1}z, z) = |x|^{p(\cdot)-2} x \cdot |x|^{p(\cdot)-2} Ax = |x|^{2(p(\cdot)-2)}(Ax, x).$$

Thus,

$$\mu^*(A^{-1}) = \sup_{|z|=1} \frac{|A^{-1}z|}{(A^{-1}z, z)} = \sup_{|z|=1} \frac{|x|^{p(\cdot)-1}}{|x|^{2(p(\cdot)-2)}(Ax, x)} = \sup_{|z|=1} \frac{1}{|x|^{p(\cdot)-3}(Ax, x)}.$$

For $|x| = 1$ we get $\mu^*(A^{-1}) = \sup\limits_{|x|=1} \frac{1}{(Ax,x)}$, which is required. □

There is $w_n \in H_n$ such that $\|u - w_n\|_H = \operatorname{dist}(u, H_n)$. Arguing as above, we find

$$\int\limits_\Omega (l_A(\nabla u) - l_A(\nabla u_n)) \cdot (\nabla u - \nabla u_n) dx = \int\limits_\Omega (l_A(\nabla u) - l_A(\nabla u_n)) \cdot (\nabla u - \nabla w_n) dx. \tag{4.12}$$

Theorem 4.1. *Under the assumptions of Lemma* 4.1 *on the matrix* A,

$$\int\limits_\Omega |z - z_n|^{p'} dx \leqslant C^* \ \|f\|_{X^*}^{\frac{p'-2}{p'(p'-1)}} (\operatorname{dist}(u, X_n))^{\frac{2}{p'}}, \quad 1 < p < 2,$$

$$\|\nabla u - \nabla u_n\|_X \leqslant C \ \|f\|_{X^*}^{\frac{p-2}{p(p-1)}} (\operatorname{dist}(u, X_n))^{\frac{2}{p}}, \quad p \geqslant 2,$$

where $C(A, p, d)$, $C^*(A^{-1}, p', d)$, $z = |\nabla u|^{p-2} A\nabla u$, *and* $z_n = |\nabla u_n|^{p-2} A\nabla u_n$.

Proof. 1. *Case* $p \geqslant 2$. Without loss of generality we can apply estimates in Lemma 4.1. We can verify the following estimate of type (1.19):

$$\|\nabla u - \nabla u_n\|_X^p \leqslant \int\limits_\Omega C\, D_A\, dx \leqslant C\biggl(\int\limits_\Omega |\nabla u|^p + |\nabla u_n|^p\, dx\biggr)^{\frac{p-2}{p}} (\operatorname{dist}(u, X_n))^2,$$

where $C = C(A, p, d)$. By (4.8a) and (4.8b),

$$\|\nabla u - \nabla u_n\|_X \leqslant C^{\frac{1}{p}}\, \|f\|_{X^*}^{\frac{p-2}{p(p-1)}} (\operatorname{dist}(u, X_n))^{\frac{2}{p}}.$$

2. *Case* $1 < p < 2$. From Lemma 4.1 for the exponent p' and flows z, z_n we have

$$\int\limits_\Omega |z - z_n|^{p'}\, dx \leqslant \int\limits_\Omega C^*\, D_A^*\, dx \leqslant \overline{C}\biggl(\int\limits_\Omega |z|^{p'} + |z_n|^{p'}\, dx\biggr)^{\frac{p'-2}{p'}} (\operatorname{dist}(u, X_n))^2,$$

where $C^* = C(A^{-1}, p', d)$. By (4.7a) and (4.7b),

$$\int\limits_\Omega |z - z_n|^{p'}\, dx \leqslant (C^*)^{\frac{1}{p'}}\, \|f\|_{X^*}^{\frac{p'-2}{p'(p'-1)}} (\operatorname{dist}(u, X_n))^{\frac{2}{p'}}.$$

The theorem is proved. □

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References

1. P. Lindqvist, *Notes on the p-Laplace Equation*, Report No. 102, University Jyväskylä, Jyväskylä (2006).
2. M. D. Surnachev and V. V. Zhikov, "On existence and uniqueness classes for the Cauchy problem for parabolic equations of the p-Laplace type," *Commun. Pure Appl. Anal.* **12**, No. 4, 1783–1812 (2013).
3. V. V. Zhikov, "On variational problems and nonlinear elliptic equations with nonstandard growth condition" [in Russian]], *Probl. Mat. Anal.* **54**, 23–112 (2011); English transl.: *J. Math. Sci., New York* **173**, No. 5, 463-570 (2011).

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