DOI 10.1007/s10958-016-3085-6
Journal of Mathematical Sciences, Vol. 219, No. 1, November, 2016

NONWANDERING SETS OF C^1-SMOOTH SKEW PRODUCTS OF INTERVAL MAPS WITH COMPLICATED DYNAMICS OF QUOTIENT MAP

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We obtain exact formulas describing the nonwandering set of a C^1-smooth skew product of interval maps with Ω-stable quotient map of type $\succ 2^\infty$. Bibliography: 16 *titles.*

1 Introduction

An important role in the study of properties of a dynamical system with compact phase space is played by the nonwandering set (cf. the definition in [1]). The nonwandering set of a skew product of interval maps with a closed set of periodic points of the quotient map was studied in [2]–[6]. The structure of the nonwandering sets of continuous skew products of interval maps with a closed set of periodic points was independently studied in [7] and, in a particular case, in [8]. In this paper, we obtain formulas that explain the mechanism of formation of the nonwandering set of a C^1-smooth skew product of interval maps with Ω-stable quotient map having a complicated dynamics. We introduce some notions which will be used below. Basic facts from topology can be found, for example, in [9].

We consider a skew product $F : I \to I$ of interval maps, i.e. a map of the form

$$F(x, y) = (f(x), g_x(y)), \quad g_x(y) = g(x, y), \quad (x; y) \in I, \tag{1.1}$$

where $I = I_1 \times I_2$ is a rectangle in the plane, I_1 and I_2 are segments. By (1.1), for any $n > 1$

$$F^n(x, y) = (f^n(x), g_{x,n}(y)), \quad g_{x,n} = g_{f^{n-1}(x)} \circ \ldots \circ g_x. \tag{1.2}$$

A map $g_{x,n}$, where x is a periodic point of f ($x \in \mathrm{Per}\,(f)$) and n is its (least) period, will be denoted by $\widetilde{g}_x$. Introduce the notation:

$T^0(I)$ ($T^1(I)$) is the space of continuous (C^1-smooth) skew products of maps of I with the standard C^0-norm (C^1-norm),

$C^1(I_k)$, $k = 1, 2$, is the space of C^1-smooth maps of I_k to itself,

$C^1_{\partial_k}(I_k)$, $k = 1, 2$, is the subspace of $C^1(I_k)$ of maps $\psi \in C^1(I_k)$ satisfying the condition of the ψ-invariance of the boundary ∂I_k of I_k, i.e., $\psi(\partial I_k) \subset \partial I_k$,

Translated from *Problemy Matematicheskogo Analiza* **85**, June 2016, pp. 83-94.

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$C^1_\omega(I_k)$, $k = 1, 2$, is the space of Ω-stable in $C^1_{\partial_k}(I_k)$ maps of I_k into itself.

Proposition 1.1 (cf. [10])**.** *If $f \in C^1_\omega(I_1)$, then one of the following assertions holds*:

(a) *f is a map of type $\prec 2^\infty$ (i.e., the set of (least) periods of periodic points of the map f coincides with the set $\{2^i\}_{i=0}^{i=\mu} = \{1, 2, \ldots, 2^\mu\}$ for some $0 \leqslant \mu < +\infty$), where the nonwandering set $\Omega(f)$ is finite and consists of hyperbolic periodic points,*

(b) *f is a map of type $\succ 2^\infty$ (i.e., there exists an f-periodic point $x \in \mathrm{Per}\,(f)$ of period $n(x) \notin \{2^i\}_{i \geqslant 0}$), where the nonwandering set $\Omega(f)$ is the union of finitely many hyperbolic periodic points and finitely many locally maximal quasiminimal (i.e., maximal quasiminimal sets in some its neighborhood) hyperbolic perfect nowhere dense sets.*

The set $C^1_\omega(I_1)$ is open and everywhere dense in $C^1_{\partial_1}(I_1)$.

Note that a map $f \in C^1_\omega(I_1)$ of type $\succ 2^\infty$ has complicated dynamics on any locally maximal quasiminimal set. In particular, such a set contains an everywhere dense subset of periodic points with an unbounded set, continuum of other quasiminimal sets, continuum of minimal sets, etc.

Let $T^1_*(I)$ be the subspace of $T^1(I)$ of skew products of interval maps with quotient maps in $C^1_\omega(I_1)$, equipped with the C^1-norm. In this paper, we describe the nonwandering set of a skew product of maps in $T^1_*(I)$ with quotient map of type $\succ 2^\infty$. We use the technique developed in [11]–[13] and based on the following multivalued functions:

the Ω-*function of a skew product* $F \in T^0(I)$, i.e., the function whose graph in the phase space I coincides with the nonwandering set $\Omega(F)$ of F,

the *auxiliary function* $\eta_n : \Omega(f) \to 2^{I_2}$ for the Ω-function of a map $F \in T^1_*(I)$, i.e., the function defined for $x \in \Omega(f)$ by the equality $\eta_n(x) = \Omega(g_{x,n})$, where $\Omega(\cdot)$ is the nonwandering set and 2^{I_2} is the space of closed subsets of I_2 equipped with the exponential topology,

the *suitable function* $\overline{\eta}_n : \Omega(f) \to 2^{I_2}$, $n \geqslant 1$, to the Ω-function of a map $F \in T^1_*(I)$, i.e., the function whose graph in I is the closure $\overline{\eta}_n$ of the graph of the auxiliary function η_n.

Following [13], we represent the iteration F^n of a skew product $F \in T^0(I)$ in the form

$$F^n = F_{n,1} \circ F_n, \tag{1.3}$$

where

$$F_n(x, y) = (id(x), g_{x,n}(y)), \tag{1.4}$$

$$F_{n,1}(x, y) = (f^n(x), id(y)). \tag{1.5}$$

Here, $id(x)$ and $id(y)$ are the identity maps of I_1 and I_2 respectively.

After we have introduced the auxiliary functions η_n (the suitable functions $\overline{\eta}_n$) for all $n \geqslant 1$, we should move each point $(x; y) \in \eta_n$ or $\overline{\eta}_n$ to the point $(f^n(x); y)$ by using the direct product $F_{n,1}$ (cf. (1.3)–(1.4)). In the natural way, we obtain the multifunctions $\eta_{n,1} : \Omega(f) \to 2^{I_2}$ ($\overline{\eta}_{n,1} : \Omega(f) \to 2^{I_2}$), $n \geqslant 1$, so that $\eta_{n,1}(x) = (F_{n,1}(\eta_n))(x)$ ($\overline{\eta}_{n,1}(x) = (F_{n,1}(\overline{\eta}_n))(x)$) for any $x \in \Omega(f)$, where η_n ($\overline{\eta}_n$) is the graph of the corresponding multifunction in I and $(F_{n,1}(\eta_n))(x)$ ($(F_{n,1}(\overline{\eta}_n))(x)$) is the cut (the projection of the section to the Oy-axis) of the set $F_{n,1}(\eta_n)$ ($F_{n,1}(\overline{\eta}_n)$) along a fiber over the point $x \in \Omega(f)$.

Let $F \in T^1_*(I)$ be an arbitrary skew product of interval maps with quotient map of type $\succ 2^\infty$. By Proposition 1.1 (b), the perfect part $\Omega_p(f)$ of the set $\Omega(f)$ is nonempty. In this paper, we give a description of the nonwandering set of a map F acting in the fibers over points in $\Omega_p(f)$. The possibility to solve this problem is provided by the decomposition theorem for the space of skew products in $T^1_*(I)$ with quotient maps of type $\succ 2^\infty$ in the union of four nonempty pairwise disjoint subspaces $T^1_{*,\,j}(I)$, $j = 1, 2, 3, 4$ (cf. [12, 13]). To describe these subspaces, one uses the return times for the trajectories of points of the nonempty set $\Omega_p(f)$ in an arbitrary neighborhood of each of these points. These return times for the trajectories of points of the set $\Omega_p(f)$ are determined by periods of periodic points of $f_{|\Omega_p(f)}$. We denote by $\tau(f_{|\Omega_p(f)})$ the set of (least) periods of periodic points of $f_{|\Omega_p(f)}$. There are natural numbers m_*, n_*, and i_* such that for any $i \geqslant i_*$

$$m_* n_* i \in \tau(f_{|\Omega_p(f)}) \tag{1.6}$$

(we refer to [13] for details). We set

$$l^*_i = m_* n_* i. \tag{1.7}$$

Following [13], we introduce the following subspaces:

$T^1_{*,\,1}(I)$ is the subspace of skew products in $T^1_*(I)$ with quotient maps of type $\succ 2^\infty$ that have continuous auxiliary functions $\eta_{l^*_i}$ for all $i \geqslant i^*$ and some $i^* \geqslant i_*$,

$T^1_{*,\,2}(I)$ is the subspace of skew products that do not belong to $T^1_{*,\,1}(I)$ and have continuous suitable functions $\overline{\eta}_{l^*_i}$ for all $i \geqslant i^*$ and some $i^* \geqslant i_*$,

$T^1_{*,\,3}(I)$ and $T^1_{*,\,4}(I)$ are the subspaces of maps in $T^1_*(I)$ such that a sequence of suitable functions $\{\overline{\eta}_{l_i}\}_{i\geqslant 0}$ contains countably many discontinuous functions, but the Ω-function of any map in $T^1_{*,\,3}(I)$ is continuous, whereas the Ω-function of any map in $T^1_{*,\,4}(I)$ is discontinuous.

Theorems on the structure of the nonwandering set of maps in $T^1_{*,\,1}(I)$ (the result was announced in [14]) and in $T^1_{*,\,2}(I)$ are proved in Section 2. In Section 3, we describe the nonwandering sets of skew products in the spaces $T^1_{*,\,3}(I)$ and $T^1_{*,\,4}(I)$.

2 Nonwandering Sets of Skew Products in $T^1_{*,\,1}(I)$ and $T^1_{*,\,2}(I)$

Let F belong to $T^1_{*,1}(I)$ or $T^1_{*,2}(I)$. In both cases, we use the same subsequence $\{l_i\}_{i\geqslant i^*}$, $l_i = m_* n_* i!$, of the sequence of natural numbers $\{l^*_i\}_{i\geqslant i^*}$ defined by the equality (1.7). The natural number $i!$ for $i > 1$ can be represented in the form

$$i! = 2^{j(i)}(2j'(i) + 1), \quad j(i) \geqslant 0, \quad j'(i) \geqslant 1.$$

To avoid difficulties caused by the possible failure of the identity (cf. [15])

$$\Omega(\widetilde{g}_x^{m_* n_* 2^j (2k+1)}) = \Omega(\widetilde{g}_x^{m_* n_* 2^{j-1}(2k+1)}),$$

we define the multivalued functions

$$\overline{\eta}'_{l_i} = \bigcup_{\gamma=0}^{j(i)} \overline{\eta}_{2^{-\gamma} l_i}, \quad \overline{\eta}'_{l_i,\,1} = \bigcup_{\gamma=0}^{j(i)} \overline{\eta}_{2^{-\gamma} l_i,\,1} \tag{2.1}$$

on the nonwandering set $\Omega(f)$ of f. The functions defined by (2.1) should be understood in the following sense:

$$\overline{\eta}'_{l_i}(x) = \bigcup_{\gamma=0}^{j(i)} \overline{\eta}_{2^{-\gamma} l_i}(x), \quad \overline{\eta}'_{l_i,\,1}(x) = \bigcup_{\gamma=0}^{j(i)} \overline{\eta}_{2^{-\gamma} l_i,\,1}(x) \quad \forall\ x \in \Omega(f).$$

Let $\mathrm{Per}_p(f)$ be the set of periodic points in $\Omega_p(f)$ (in view of [10], we have $\overline{\mathrm{Per}_p(f)} = \Omega_p(f)$), and let $\mathrm{Per}^*_p(f)$ be an arbitrary invariant everywhere dense in $\Omega_p(f)$ subset of $\mathrm{Per}_p(f)$ (possibly, coinciding with $\mathrm{Per}_p(f)$). We use the notation $(\overline{\eta}_{l_i})^{P^*}$ for the restriction of $\overline{\eta}_{l_i}$ on $\mathrm{Per}^*_p(f)$ and its graph I as well. We set

$$(\overline{\eta}_{l_i,\,1})^{P^*} = F_{l_i,\,1|\mathrm{Per}^*_p(f)\times I_2}((\overline{\eta}_{l_i})^{P^*}). \tag{2.2}$$

In (2.2), we used the graphs of functions $(\overline{\eta}_{l_i})^{P^*}$ and $(\overline{\eta}_{l_i,\,1})^{P^*}$. We denote by $\mathrm{Per}_p(f,\,n)$ ($\mathrm{Per}^*_p(f,\,n)$) a finite set of points in $\mathrm{Per}_p(f)$ ($\mathrm{Per}^*_p(f)$) whose (least) periods divide $n \in \tau(f_{|\Omega_p(f)})$. For any $i \geqslant i^*$ we use the restrictions of functions defined by (2.1):

$$\overline{\eta}'_{l_i\,|\mathrm{Per}^*_p(f,\,l_i)} = \bigcup_{\gamma=0}^{j(i)} \overline{\eta}_{2^{-\gamma}l_i\,|\mathrm{Per}^*_p(f,\,2^{-\gamma}l_i)}, \tag{2.3}$$

$$(\overline{\eta}'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,l_i)} = \bigcup_{\gamma=0}^{j(i)} (\overline{\eta}_{2^{-\gamma}l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,2^{-\gamma}l_i)}. \tag{2.4}$$

The equalities (2.3) and (2.4) are understood in accordance to (2.1) (cf. also [13, 14]).

We note that the sequence of natural numbers $\{l_i, \dots 2^{-j(i)}l_i\}_{i\geqslant i^*}$ is a subsequence of $\{l^*_i\}_{i\geqslant i^*}$. Therefore, for any map F in $T^1_{*,1}(I)$ or $T^1_{*,\,2}(I)$ the multivalued functions $\overline{\eta}_{2^{-\gamma}l_i}$ are continuous on the set $\Omega(f)$ for every $0 \leqslant \gamma \leqslant j(i)$, $i \geqslant i^*$.

In what follows, it suffices to use the natural extensions η^{ex}_n and $\eta^{ex}_{n,1}$ of η_n and $\eta_{n,1}$ on I_1 and $f^n(I_1)$ respectively (in the case under consideration, $\Omega(f) \neq I_1$). Then for all $n \geqslant 1$

$$\begin{aligned}
&\eta^{ex}_n(x) = \Omega(g_{x,n}) \quad \forall\ x \in I_1,\\
&\eta^{ex}_{n,1}(x) = (F_{n,\,1}(\eta^{ex}_n))(x) \quad \forall\ x \in f^n(I_1),
\end{aligned}$$

where $\eta^{ex}_n(x)$ means the value of the function η^{ex}_n at the point x in the first identity and the graph of the corresponding multivalued function in the second identity, whereas $(F_{n,\,1}(\eta^{ex}_n))(x)$ is the section of the set $F_{n,\,1}(\eta^{ex}_n)$ along the fiber over the point x.

An important role will be played by the following functions defined on the set $\bigcap\limits_{\gamma=0}^{\overline{j}(i)} f^{2^{-\gamma}l^*_i}(I_1)$:

$$\eta^{ex'}{}_{l^*_i,1} = \bigcup_{\gamma=0}^{\overline{j}(i)} \eta^{ex}_{2^{-\gamma}l^*_i,1}, \tag{2.5}$$

where $i = 2^{\overline{j}(i)}(2\overline{j}'(i)+1)$, $\overline{j}(i) \geqslant 0$, $\overline{j}'(i) \geqslant 1$. The equality (2.5) is understood as follows:

$$\eta^{ex'}{}_{l^*_i,1}(x) = \bigcup_{\gamma=0}^{\overline{j}(i)} \eta^{ex}_{2^{-\gamma}l^*_i,1}(x) \quad \forall\ x \in \bigcap_{\gamma=0}^{\overline{j}(i)} f^{2^{-\gamma}l^*_i}(I_1).$$

Theorem 2.1. *Assume that $F \in T^1_{*,\,1}(I)$ and $\mathrm{Per}^*_p(f)$ is an invariant everywhere dense in $\Omega_p(f)$ subset of the set $\mathrm{Per}_p(f)$. Then the topological limit $\underset{i\to+\infty}{\mathrm{Lim}}\,(\eta'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,l_i)}$ exists and is independent of $\mathrm{Per}^*_p(f)$; moreover,*

$$\zeta^{F^{m_*n_*}_{|\Omega^*_p(F)}} = \underset{i\to+\infty}{\mathrm{Ls}}\,\eta'_{l_i,\,1} = \underset{i\to+\infty}{\mathrm{Ls}}\,(\eta'_{l_i,\,1})^{P^*}$$

$$= \operatorname*{Lim}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}_p^*(f,l_i)} = \overline{\bigcup_{x\in\mathrm{Per}_p^*(f)} \{x\}\times\Omega(\widetilde{g}_x)}, \tag{2.6}$$

where $\Omega_p^*(F) = \Omega_p(f)\times I_2$, $\zeta^{F^{m_*n_*}_{|\Omega_p^*(F)}}$ *is the graph of the* Ω*-function of the map* $F^{m_*n_*}_{|\Omega_p^*(F)}$ *in* I, $\eta'_{l_i,1}$, $(\eta'_{l_i,1})^{P^*}$, $(\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}_p^*(f,l_i)}$ *are the graphs of the corresponding functions in* I, $\operatorname*{Ls}_{i\to+\infty}(\cdot)_i$ *is the upper topological limit of a sequence of sets. Furthermore, the value* $\zeta^{F^{m_*n_*}}(x)$ *of the* Ω*-function of the map* $F^{m_*n_*}$ *at any point* $x\in\Omega_p(f)$ *is defined by*

$$\zeta^{F^{m_*n_*}}(x) = \operatorname*{Ls}_{i\to+\infty} \eta^{ex'}_{m^*n^*i,1|U_{1,\varepsilon_i}(x)}, \tag{2.7}$$

where $U_{1,\varepsilon_i}(x)$ *is an arbitrary* ε_i*-neighborhood of the point* $x\in\Omega_p(f)$ *in* I_1 *and* $\lim_{i\to+\infty}\varepsilon_i = 0$.

To prove Theorem 2.1, we need some auxiliary assertions.

Since the set $\mathrm{Per}_p^*(f)$ is everywhere dense in $\Omega_p(f)$ and the functions $\eta_{l_i,1}$ are continuous for $i\geqslant i^*$, from (2.2) we obtain the following assertion.

Lemma 2.1. *If* $F\in T^1_{*,1}(I)$ *and an invariant set* $\mathrm{Per}_p^*(f)$ *is everywhere dense in* $\Omega_p(f)$, *then the closure* $\overline{(\eta_{l_i,1})^{P^*}}$ *of the graphs of functions* $(\eta_{l_i,1})^{P^*}$ *in* I *coincides with the graph of the function* $\eta_{l_i,1}$, $i\geqslant i^*$.

From Lemma 2.1, the properties of the closure of a finite union of sets (cf. (2.1) and (2.2)), and properties of the upper limit of a sequence of sets, we obtain the following assertion.

Corollary 2.1. *For an arbitrary map* $F\in T^1_{*,1}(I)$ *satisfying the assumptions of Theorem* 2.1 *the following equality holds*:

$$\operatorname*{Ls}_{i\to+\infty}\eta'_{l_i,1} = \operatorname*{Ls}_{i\to+\infty}(\eta'_{l_i,1})^{P^*}.$$

For any $i\geqslant i^*$ we consider the set $(\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}_p^*(f,l_i)}$. Let x be an arbitrary point in $\mathrm{Per}_p^*(f,2^{-\gamma}l_i)$ for some $0\leqslant\gamma\leqslant j(i)$. Then a unique preimage of x under the map $(f_{|\mathrm{Per}_p^*(f)})^{l_i}$ coincides with x. Using this property and the identities (2.2), (2.4), we find

$$(\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}_p^*(f,l_i)} = \eta'_{l_i|\mathrm{Per}_p^*(f,l_i)}. \tag{2.8}$$

Lemma 2.2. *Let the assumptions of Theorem* 2.1 *be satisfied. Then the topological limit* $\operatorname*{Lim}_{i\to+\infty}({\eta'}_{l_i,1})^{P^*}{}_{|\mathrm{Per}_p^*(f,l_i)}$ *exists and is independent of the choice of the set* $\mathrm{Per}_p^*(f)$; *moreover,*

$$\operatorname*{Lim}_{i\to+\infty}({\eta'}_{l_i,1})^{P^*}{}_{|\mathrm{Per}_p^*(f,l_i)} = \overline{\bigcup_{x\in\mathrm{Per}_p^*(f)}\{x\}\times\Omega(\widetilde{g}_x)}. \tag{2.9}$$

Proof. By (2.8), it suffices to consider $\{{\eta'}_{l_i|\mathrm{Per}_p^*(f,l_i)}\}_{i\geqslant i^*}$. Since $l_i = m^*n^*i!$, from (2.3) we find

$${\eta'}_{l_i|\mathrm{Per}_p^*(f,l_i)} \subset {\eta'}_{l_{i+1}|\mathrm{Per}_p^*(f,l_{i+1})}, \tag{2.10}$$

which implies the existence of the topological limit $\operatorname*{Lim}_{i\to+\infty}{\eta'}_{l_i|\mathrm{Per}_p^*(f,l_i)}$ and the equality

$$\operatorname*{Lim}_{i\to+\infty}{\eta'}_{l_i|\mathrm{Per}_p^*(f,l_i)} = \overline{\bigcup_{x\in\mathrm{Per}_p^*(f)}\{x\}\times\Omega(\widetilde{g}_x)}.$$

By (2.8), the topological limit $\operatorname*{Lim}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l_i)}$ exists and the equality (2.9) holds. Since $\mathrm{Per}^*_p(f)$ is everywhere dense in $\mathrm{Per}_p(f)$ and $\mathrm{Per}_p(f)$ is everywhere dense in $\Omega_p(f)$ (cf. [10]), from (2.9) it follows that the topological limit $\operatorname*{Lim}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l_i)}$ is independent of the choice of the invariant set $\mathrm{Per}^*_p(f)$ and coincides with $\overline{\bigcup_{x\in\mathrm{Per}_p(f)} \{x\}\times\Omega(\widetilde{g}_x)}$. □

Lemma 2.3. *Let the assumptions of Theorem* 2.1 *hold. Then*

$$\operatorname*{Ls}_{i\to+\infty} (\eta'_{l_i,1})^{P^*} = \operatorname*{Lim}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l_i)}. \tag{2.11}$$

Proof. Since for any $i \geqslant i^*$

$$(\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l_i)} \subset (\eta'_{l_i,1})^{P^*}, \tag{2.12}$$

we have

$$\operatorname*{Lim}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l_i)} \subseteq \operatorname*{Ls}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}. \tag{2.13}$$

We prove the inverse inclusion

$$\operatorname*{Ls}_{i\to+\infty} (\eta'_{l_i,1})^{P^*} \subseteq \operatorname*{Lim}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l_i)}. \tag{2.14}$$

Indeed, let $(x;\, y) \in \operatorname*{Ls}_{i\to+\infty} (\eta'_{l_i,1})^{P^*}$ be an arbitrary point, i.e., there exists a sequence of points $(x_{i_\nu},\, y_{i_\nu}) \in (\eta'_{l_{i_\nu},1})^{P^*}$ $(\nu \geqslant 1)$ converging to $(x;\, y)$.

Using the compactness of I, we apply the Bolzano–Weierstrass lemma to the sequence of sets $\{(\eta'_{l_{i_\nu},1})^{P^*}\}_{\nu\geqslant 1}$ (if it is not converging). From the above sequence we extract a converging subsequence $\{(\eta'_{l_{i_{\nu(s)}},1})^{P^*}\}_{s\geqslant 1}$ (the limit of this subsequence can be the empty set). Lemma 2.2 and (2.12) imply

$$\operatorname*{Lim}_{s\to+\infty} (\eta'_{l_{i_{\nu(s)}},1})^{P^*} \neq \varnothing, \quad (x;\, y) \in \operatorname*{Lim}_{s\to+\infty} (\eta'_{l_{i_{\nu(s)}},1})^{P^*}.$$

We fix $\varepsilon > 0$. By the Cauchy criterion, for any $\varepsilon > 0$ there exists $s_0 \geqslant 1$ such that for any $s',\, s'' \geqslant s_0$

$$\mathrm{dist}_I((\eta'_{l_{i_{\nu(s')}},1})^{P^*},\, (\eta'_{l_{i_{\nu(s'')}},1})^{P^*}) < \varepsilon, \tag{2.15}$$

where dist_I is the Hausdorff metric in the space of closed subsets of I.

Let $s \geqslant s_0$. Then $(x_{i_{\nu(s)}},\, y_{i_{\nu(s)}}) \in (\eta'_{l_{i_{\nu(s)}},1})^{P^*}$. By the choice of $\{l_i\}_{i\geqslant i^*}$,

$$\mathrm{Per}^*_p(f) = \bigcup_{i=i^*}^{+\infty} \mathrm{Per}^*_p(f,\, l_i).$$

Therefore, for any $s \geqslant s_0$ there exists $s' \geqslant s$ such that

$$x_{i_{\nu(s)}} \in \mathrm{Per}^*_p(f,\, l_{i_{\nu(s')}}). \tag{2.16}$$

Using the uniform continuity (with respect to the Hausdorff metric dist_{I_2} in the space of closed subsets of I_2) of the functions $\eta'_{l_{i_{\nu(s')}},1}$ on the compact set $\Omega_p(f)$, for any $\varepsilon > 0$ we can find $0 < \delta(s') \leqslant \varepsilon$ such that for any $x',\, x'' \in \Omega_p(f)$, $|x' - x''| < \delta(s')$,

$$\mathrm{dist}_{I_2}(\eta'_{l_{i_{\nu(s')}},1}(x'),\, \eta'_{l_{i_{\nu(s')}},1}(x'')) < \varepsilon. \tag{2.17}$$

By (2.15), there exists a point $(x',\, y') \in ({\eta'}_{l_{i_{\nu(s')}},1})^{P^*}$ such that $|x_{i_{\nu(s)}} - x'| < \delta(s') \leqslant \varepsilon$ and $|y_{i_{\nu(s)}} - y'| < \varepsilon$. Using (2.17), we find a point $(x_{i_{\nu(s)}},\, y'') \in \eta'_{l_{i_{\nu(s')}},1}$ such that $|y'' - y'| < \varepsilon$. We set $x_{i_{\nu(s)}} = x_{i_{\nu(s')}}$ and $y'' = y_{i_{\nu(s')}}$. By (2.16), we have $(x_{i_{\nu(s')}},\, y_{i_{\nu(s')}}) \in (\eta'_{l_{i_{\nu(s')}},1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_{i_{\nu(s')}})}$ and $|y_{i_{\nu(s)}} - y_{i_{\nu(s')}}| < 2\varepsilon$. Thus, $(x, y) = \lim\limits_{s' \to +\infty} (x_{i_{\nu(s')}},\, y_{i_{\nu(s')}})$ and $(x, y) \in \operatorname*{Lim}\limits_{i \to +\infty} (\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_i)}$ in view of Lemma 2.2. The inclusion (2.14) is proved. From (2.14) and (2.13) we obtain (2.11). $\square$

Lemma 2.4. *Let the assumptions of Theorem* 2.1 *be satisfied. Then*

$$\zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}} = \operatorname*{Lim}_{i \to +\infty} (\eta'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_i)}. \tag{2.18}$$

Proof. By Lemma 2.2,

$$\operatorname*{Lim}_{i \to +\infty} (\eta'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_i)} \subset \zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}}. \tag{2.19}$$

We show the opposite inclusion

$$\zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}} \subset \operatorname*{Lim}_{i \to +\infty} (\eta'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_i)}. \tag{2.20}$$

For this purpose we show that for points $(x, y) \in \Omega^*_p(F)$ such that $(x, y) \notin \operatorname*{Lim}\limits_{i \to +\infty} (\eta'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_i)}$ we have $(x, y) \notin \zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}}$. Indeed, let a neighborhood $U((x, y))$ of a point $(x, y) \in \Omega^*_p(F)$ and a neighborhood $U(L^*)$ of the closed set $L^* = \operatorname*{Lim}\limits_{i \to +\infty} (\eta'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_i)}$ be such that

$$U((x, y)) \bigcap U(L^*) = \varnothing.$$

Then $U((x, y))$ can intersect only a finite number of sets of the sequence $\{(\eta'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\, l_i)}\}_{i \geqslant i^*}$. Using Corollary 2.1 and Lemma 2.3, we choose a neighborhood $U'((x, y)) \subseteq U((x, y))$ of (x, y) that does not intersect any set of the above sequence and any set of the sequence $\{{\eta'}_{l_i,\,1}\}_{i \geqslant i^*}$.

It can happen that ${F_{|\Omega^*_p(F)}}^{-m_* n_* \widetilde{i}}(U'((x, y))) = \varnothing$ for some $\widetilde{i} \geqslant i^*$ for the complete preimage of order $m_* n_* \widetilde{i}$ of a neighborhood $U'((x, y))$ under the map $F_{|\Omega^*_p(F)}$. Then for all $i \geqslant \widetilde{i}$

$$U'((x, y)) \cap {F_{|\Omega^*_p(F)}}^{-m_* n_* i}(U'((x, y))) = \varnothing, \quad (x, y) \notin \zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}}.$$

Let ${F_{|\Omega^*_p(F)}}^{-m_* n_* i}(U'((x, y))) \neq \varnothing$ for all $i \geqslant i^*$. Then $(F_{l_i,\,1|\Omega^*_p(F)})^{-1}(U'((x, y)))$ is nonempty and open in $\Omega^*_p(F)$. By the choice of the neighborhood $U'((x, y))$, this set does not intersect η'_{l_i} for any $i \geqslant i^*$ and, consequently, consists of wandering points of each map $F_{l_i|\Omega^*_p(F)}$. Therefore, by the continuity of η'_{l_i}, for $i \geqslant i^*$ there is a universal neighborhood $U''((x, y)) = U''_1(x) \times U''_2(y)$, $U''((x, y)) \subseteq U'((x, y))$, of the point (x, y) such that

$$(F_{l_i,\,1|\Omega^*_p(F)})^{-1}(U''((x, y))) \bigcap F_{l_i|\Omega^*_p(F)}((F_{l_i,\,1|\Omega^*_p(F)})^{-1}(U''((x, y)))) = \varnothing. \tag{2.21}$$

We apply the map $F_{l_i,\,1|\Omega^*_p(F)}$ to both sides of (2.21) and use formulas (1.4), (1.5). Then for all $x'' \in \Omega_p(f) \cap f^{-l_i}(U''_1(x))$, $x' = f^{l_i}(x'')$ $(x' \in \Omega_p(f) \times U''_1(x))$, $i \geqslant i^*$,

$$U''_2(y) \bigcap g_{x,''\,2l_i}(U''_2(y)) = U''_2(y) \bigcap g_{x',\, l_i}(U''_2(y)) = \varnothing.$$

Hence $(x, y) \notin \zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}}$. Thus, we have (2.20) which, together with (2.19), implies Lemma 2.4. $\square$

Lemma 2.5. *Let the assumptions of Theorem* 2.1 *be satisfied. Then the equality* (2.7) *holds.*

Proof. Let us verify that for any $x \in \Omega_p(f)$

$$\operatorname*{Ls}_{i\to+\infty} \eta^{ex'}_{m^*n^*i,1|U_{1,\,\varepsilon_i}(x)} \subset \zeta^{F^{m_*n_*}}(x). \tag{2.22}$$

We have $\operatorname*{Ls}_{i\to+\infty} \eta'_{l_i,1} = \zeta^{F^{m_*n_*}_{|\Omega^*_p(F)}} \subseteq \zeta^{F^{m_*n_*}}$. Therefore, we assume that the set $\operatorname*{Ls}_{i\to+\infty} \eta^{ex'}_{m^*n^*i,1|U_{1,\,\varepsilon_i}(x)} \setminus \operatorname*{Ls}_{i\to+\infty} \eta'_{l_i,1}$ is nonempty. We prove (2.22) for points

$$(x, y) \in \operatorname*{Ls}_{i\to+\infty} \eta^{ex'}_{m^*n^*i,1|U_{1,\,\varepsilon_i}(x)} \setminus \operatorname*{Ls}_{i\to+\infty} \eta'_{l_i,1}, \tag{2.23}$$

where $x \in \Omega_p(f)$ and $\{\varepsilon_i\}_{i\geqslant i^*}$ is an infinitely small sequence of positive numbers. We show that for any neighborhood $U_\varepsilon((x, y))$ of a point (x, y), $x \in \Omega_p(f)$, in I there exists a natural number $r = r(\varepsilon)$ and a point $(x_r,\, y_r) \in U_\varepsilon((x, y))$ such that for some $j = j(r)$

$$(x_r,\, y_r),\ F^{m^*n^*j}(x_r,\, y_r) \in U_\varepsilon((x, y)). \tag{2.24}$$

Indeed, by [10], $\Omega_p(f)$ is a perfect nowhere dense invariant hyperbolic set such that there are $\alpha = \alpha(f) > 0$ and $c = c(f) > 1$ such that $|(f^n(x))'| > \alpha c^n$ for any $x \in \Omega_p(f)$ and $n \geqslant 1$. Therefore, there exists $\bar{i} \geqslant i^*$ such that

$$\inf_{x\in\Omega_p(f)} \{|(f^{\bar{i}}(x))'|\} > 1. \tag{2.25}$$

Using the inequality (2.25) and C^1-smoothness of f, we find a neighborhood $U_1(\Omega_p(f))$ of the set $\Omega_p(f)$ such that for all $k \geqslant 1$

$$(f_{|U_1(\Omega_p(f))})^{-k\bar{i}}(\overline{U_1(\Omega_p(f))}) \subset U_1(\Omega_p(f)). \tag{2.26}$$

By (2.26), for $k = m^*n^*i$ we have

$$\begin{aligned} &\bigcap_{i=i^*}^{+\infty} (f_{|U_1(\Omega_p(f))})^{(-m^*n^*\bar{i})i}(U_1(\Omega_p(f))) \\ &= \operatorname*{Lim}_{i\to+\infty} (f_{|U_1(\Omega_p(f))})^{(-m^*n^*\bar{i})i}(U_1(\Omega_p(f))) = \Omega_p(f). \end{aligned} \tag{2.27}$$

Using (2.23) and (2.27), we find a sequence $\{(x_{i_r},\, y_{i_r})\}_{r\geqslant1}$ converging to (x, y) and such that $x_{i_r} \notin \Omega_p(f)$,

$$x_{i_r} \in (f_{|U_1(\Omega_p(f))})^{(-m^*n^*\bar{i})i_r}(U_1(\Omega_p(f))), \quad y_{i_r} \in \eta^{ex'}_{(m^*n^*\bar{i})i_r,1|U_{1,\,\varepsilon_{i_r}}}(x_{i_r}). \tag{2.28}$$

By the uniform continuity of $F^{m^*n^*\bar{i}}$ on $\overline{U_1(\Omega_p(f))} \times I_2$ with respect to $\varepsilon > 0$, there exists a positive number δ such that for any $(x',\, y')$, $(x,''\, y'') \in \overline{U_1(\Omega_p(f))} \times I_2$, $|x' - x''|$, $|y' - y''| < \delta$,

$$|f^{m^*n^*\bar{i}}(x') - f^{m^*n^*\bar{i}}(x'')|,\ |g_{x',\, m^*n^*\bar{i}}(y') - g_{x,''\, m^*n^*\bar{i}}(y'')| < \frac{\varepsilon}{3}. \tag{2.29}$$

For the sake of definiteness, we assume that the first and second inequalities in (2.29) are valid only for $\delta < \varepsilon/3$ (for any $\varepsilon > 0$). For $\delta > 0$ we find $\bar{r} \geqslant 1$ such that $(x_{i_r},\, y_{i_r}) \in U_{\delta/3}((x, y))$

for all $r \geqslant \overline{r}$, where $U_{\delta/3}((x,y))$ is a $\delta/3$-neighborhood of the point (x,y) in I. By the second relation in (2.28), we have $y_{i_r} \in \Omega(g_{x_{i_r},\,k(r)})$, where $k(r) = 2^{-\gamma(r)}(m^*n^*\overline{i})i_r$, $i_r = 2^{\overline{j}(r)}(2\overline{j}'(r)+1)$ $(\overline{j}(r) \geqslant 0,\ \overline{j}'(r) \geqslant 1)$, $0 \leqslant \gamma(r) \leqslant \overline{j}'(r)$ (cf. (2.5)). Using (1.6), we choose a sufficiently large number $r \geqslant \overline{r}$ such that a $\delta/3$-neighborhood $U_{1,\,\delta/3}(x)$ of the point x in I_1 contains a periodic point $\widetilde{x}$ with the (least) period $m(\widetilde{x})$ that is a divisor of the number $k(r)$, multiple to $m^*n^*\overline{i}$. Moreover,

$$|x_{i_r} - \widetilde{x}| < \frac{2}{3}\delta < \delta. \tag{2.30}$$

By the choice of $\widetilde{x}$, for any $y \in I_2$ we have $F_{k_r}(x,y) = F_{m(\widetilde{x})}^{k(r)/m(\widetilde{x})}(x,y)$ (cf. (1.4)). Since $y_{i_r} \in \Omega(g_{x_{i_r},\,k(r)})$, in any neighborhood $U_{2,\,\theta}(y_{i_r})$ $(0 < \theta < \delta/3)$ of the point y_{i_r} in I_2 there is a point y'_{i_r} such that for some $q = q(\theta)$, $q \geqslant 1$,

$$g^q_{x_{i_r},\,k(r)}(y'_{i_r}) = y_{i_r} \tag{2.31}$$

(cf. [16]). By (2.29)–(2.31) and the inequality $\delta < \varepsilon/3$, there exist segments of the negative semitrajectories of the points $(x_{i_r},\, y_{i_r})$ and $(\widetilde{x},\, y_{i_r})$ relative to F_{k_r} that consist of preimages of these points of order up to q and approximate each other up to $\varepsilon/3$.

Let a point $\widetilde{y}_{i_r} \in I_2$ be such that $g^q_{\widetilde{x},\,k(r)}(\widetilde{y}_{i_r}) = y_{i_r}$ and $|y'_{i_r} - \widetilde{y}_{i_r}| < \varepsilon/3$. Since $|y'_{i_r} - y_{i_r}| < \theta$ and $|y_{i_r} - y| < \varepsilon/3$ simultaneously, we have $|\widetilde{y}_{i_r} - y| < \varepsilon$. Thus, $(\widetilde{x},\, \widetilde{y}_{i_r}) \in U_\varepsilon((x,y))$ and $F^{k_r q}(\widetilde{x},\, \widetilde{y}_{i_r}) = (\widetilde{x},\, y_{i_r}) \in U_\varepsilon((x,y))$, i.e., (2.24) holds for $(x_r,\, y_r) = (\widetilde{x},\, \widetilde{y}_{i_r})$ and $j = 2^{-\gamma(r)}\overline{i}i_r q$. Consequently, $(x,y) \in \zeta^{F^{m^*n^*}}$ and (2.22) is proved.

As in the proof of Lemma 2.4, we verify that for any $x \in \Omega_p(f)$ the opposite inclusion to (2.22) holds,$\zeta^{F^{m_*n_*}}(x) \subset \operatorname*{Ls}_{i\to+\infty} \eta^{ex'}_{m^*n^*i,1|U_{1,\,\varepsilon_i}(x)}$, which implies (2.7). Lemma 2.5 is proved. □

Proof of Theorem 2.1. The equalities (2.6) follow from Lemmas 2.1–2.4 and Corollary 2.1. The equality (2.7) is established in Lemma 2.5. □

We note that Theorem 2.1 fails for maps in $T^1_{*,\,2}(I)$.

We introduce the notion of weakly nonwandering points with respect to the family of fiber maps, which generalizes the definition in [6].

Definition 2.1. A point $(x,y) \in I$ is *weakly nonwandering* relative to the family of maps acting in the fibers over points of a set $A \subseteq I_1$ of the skew product of $F \in T^0(I)$ if $x \in \Omega(f) \bigcap \overline{A}$, and for any neighborhood $U_\varepsilon((x,y)) = U_{1,\,\varepsilon}(x) \times U_{2,\,\varepsilon}(y)$ of (x,y) in I there exists a point $(x_\varepsilon,\, y_\varepsilon) \in U_\varepsilon((x,y))$, $x_\varepsilon \in A$, and a natural number $i = i(\varepsilon)$ such that $g_{x_\varepsilon,\,i}(y_\varepsilon) \in U_{2,\,\varepsilon}(y)$ for $f^i(x_\varepsilon) \in U_{1,\,\varepsilon}(x)$.

We note that for any $1 \leqslant j \leqslant 4$ there exists a skew product $F_j \in T^1_{*,\,j}(I)$ possessing weakly nonwandering points with respect to the family of maps acting in the fibers over points of I_1, but nonwandering with respect to the family of maps acting in the fibers over points of $\Omega(f)$.

Theorem 2.2. *Assume that $F \in T^1_{*,\,2}(I)$ and the set $\mathrm{Per}^*_p(f)$ is the same as in Theorem* 2.1. *Then the topological limit $\operatorname*{Lim}_{i\to+\infty} (\overline{\eta}'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,l_i)}$ exists and is independent of $\mathrm{Per}^*_p(f)$; moreover,*

$$\begin{aligned}\zeta^{F^{m_*n_*}_{|\Omega^*_p(F)}} &= \operatorname*{Ls}_{i\to+\infty} \overline{\eta}'_{l_i,\,1} = \operatorname*{Ls}_{i\to+\infty} (\overline{\eta}'_{l_i,\,1})^{P^*} \\ &= \operatorname*{Lim}_{i\to+\infty} (\overline{\eta}'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,l_i)} = \overline{\bigcup_{x\in\mathrm{Per}^*_p(f)} \{x\} \times B_{\Omega_p}(\widetilde{g}_x)},\end{aligned} \tag{2.32}$$

where $\zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}}$, $\overline{\eta}'_{l_i,1}$, $(\overline{\eta}'_{l_i,1})^{P^*}$, $(\overline{\eta}'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,l_i)}$ *are the graphs of the corresponding functions in* I, $B_{\Omega_p}(\widetilde{g}_x)$ *is the set of points* $y \in I_2$ *such that any point* (x, y) *is weakly nonwandering with respect to the family of maps acting in the fibers over points in* $\Omega_p(f)$. *Furthermore, the value* $\zeta^{F^{m_* n_*}}(x)$ *of the* Ω*-function of the map* $F^{m_* n_*}$ *at any point* $x \in \Omega_p(f)$ *is defined by the equality* (2.7) *for arbitrary neighborhoods* $U_{1,\,\varepsilon_i}(x)$ *of* $x \in \Omega_p(f)$ *in* I_1, *where* $\lim\limits_{i\to+\infty} \varepsilon_i = 0$.

To prove Theorem 2.2, we need a number of auxiliary results.

Lemma 2.6. *Assume that* $F \in T^1_{*,\,2}(I)$ *and the set* $\mathrm{Per}^*_p(f)$ *is the same as in Theorem* 2.1. *Then*

$$\operatorname*{Ls}_{i\to+\infty} \overline{\eta}'_{l_i,\,1} = \operatorname*{Ls}_{i\to+\infty} (\overline{\eta}'_{l_i,\,1})^{P^*}. \tag{2.33}$$

Proof. We verify the identity

$$\overline{(\overline{\eta}_{l_i,\,1})^{P^*}} = \overline{\eta}_{l_i,\,1}. \tag{2.34}$$

Since $F \in T^1_{*,\,2}(I)$, the functions $\overline{\eta}_{l_i,\,1}$ are continuous for any $i \geqslant i^*$. We choose a number $i \geqslant i^*$, a point $(x;\, y)$ on the graph $\overline{\eta}_{l_i,1}$, and a rectangular ε-neighborhood $U_\varepsilon((x;\, y)) = U_{1,\,\varepsilon}(x) \times U_{2,\,\varepsilon}(y)$ of $(x; y)$ in I. To prove (2.34), it suffices to verify that

$$(\overline{\eta}_{l_i,\,1})^{P^*} \bigcap U_\varepsilon((x;\, y)) \neq \varnothing. \tag{2.35}$$

Indeed, by the uniform continuity of $F_{l_i,\,1}$ (cf. (1.5)), for $\varepsilon > 0$ there exists $0 < \delta_i \leqslant \varepsilon$ such that for any $(x'; y')$, $(x''; y'') \in I$, $|x' - x''|$, $|y' - y''| < \delta_i$,

$$|f^{l_i}(x') - f^{l_i}(x'')| < \varepsilon \tag{2.36}$$

(the inequality $|id(y') - id(y'')| = |y' - y''| < \varepsilon$ is valid by the choice of δ_i).

By the uniform continuity of the fitting function $\overline{\eta}_{l_i}$ on the compact set $\Omega_p(f)$, for $\delta_i > 0$ there exists $0 < \vartheta_i \leqslant \delta_i$ such that for any $\overline{x}$, $\overline{x}' \in \Omega_p(f)$, $|\overline{x} - \overline{x}'| < \vartheta_i$,

$$\mathrm{dist}_{I_2}(\overline{\eta}_{l_i}(\overline{x}),\, \overline{\eta}_{l_i}(\overline{x}')) < \delta_i. \tag{2.37}$$

Since the point $(x;\, y)$ lies on the graph of the function $\overline{\eta}_{l_i,\,1}$, by the definition of $\overline{\eta}_{l_i,\,1}$, there is the preimage $(\overline{x};\, y)$ of the point $(x; y)$ (under the map $F_{l_i,\,1}$) such that $\overline{x} \in \{(f_{|\Omega_p(f)})^{-l_i}(x)\}$, $y \in \overline{\eta}_{l_i}(\overline{x})$. Since the set $\mathrm{Per}^*_p(f)$ is everywhere dense in $\Omega_p(f)$, for the point $\overline{x}'$ such that $|\overline{x} - \overline{x}'| < \vartheta_i$ we take an arbitrary point in $\mathrm{Per}^*_p(f)$ lying in the ϑ_i-neighborhood of the point $\overline{x}$. Using (2.37), we find a point $y' \in \overline{\eta}_{l_i}(\overline{x}')$ such that $|y - y'| < \delta_i$. We set $x' = \overline{x}'$, $x'' = \overline{x}$, $y'' = y$. Since $\vartheta_i \leqslant \delta_i$, from (2.36) it follows that $F_{l_i,\,1}(x',\, y') \in U_\varepsilon((x;\, y))$. By the choice of the point $\overline{x}'$, we have $F_{l_i,\,1}(x',\, y') \in (\overline{\eta}_{l_i,\,1})^{P^*}$ (here, $(\overline{\eta}_{l_i,1})^{P^*}$ is the graph of the corresponding multivalued function). Thus, the inequality (2.35) and, consequently, the equality (2.34), is proved.

By (2.1)–(2.2) and (2.34), we have $\overline{(\overline{\eta}'_{l_i,\,1})^{P^*}} = \overline{\eta}'_{l_i,\,1}$. Taking into account properties of the upper topological limit of a sequence of sets, we obtain (2.33). □

The following assertion is proved in the same way as Lemmas 2.2–2.4.

Lemma 2.7. *Let the assumptions of Theorem* 2.2 *be satisfied. Then the topological limit* $\operatorname*{Lim}\limits_{i\to+\infty} (\overline{\eta}'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,l_i)}$ *exists and is independent of the choice of the set* $\mathrm{Per}^*_p(f)$ *and*

$$\zeta^{F^{m_* n_*}_{|\Omega^*_p(F)}} = \operatorname*{Ls}_{i\to+\infty} (\overline{\eta}'_{l_i,\,1})^{P^*} = \operatorname*{Lim}_{i\to+\infty} (\overline{\eta}'_{l_i,\,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,\,l_i)}.$$

Corollary 2.2. *Under the assumptions of Theorem* 2.2, *the following equality holds*:

$$\operatorname*{Lim}_{i\to+\infty} (\overline{\eta}'_{l_i,1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l_i)} = \overline{\bigcup_{x\in\mathrm{Per}^*_p(f)} \{x\}\times B_{\Omega_p}(\widetilde{g}_x)}. \tag{2.38}$$

Proof. It suffices to verify that for any $x \in \mathrm{Per}^*_p(f,\, l_i)$, $i \geqslant i^*$,

$$\overline{\eta}'_{l_i}(x) = B_{\Omega_p}(\widetilde{g}_x). \tag{2.39}$$

Let y be an arbitrary point of the set $B_{\Omega_p}(\widetilde{g}_x)$. By Definition 2.1, we have $(x,y)\in\zeta^{F^{m_*n_*}_{|\Omega^*_p(F)}}$. Using Lemma 2.7 and the results of [13], we find a strictly increasing sequence of natural numbers $\{i(k)\}_{k\geqslant1}$ and for every $k\geqslant1$ converging to (x,y) the sequence of points $\{(x_n(k),y_n(k))\}_{n\geqslant1}$ such that $(x_n(k),y_n(k))\in(\eta'_{l_{i(k)},1})^{P^*}$. Moreover, the sequence of (least) periods of points $x_n(k)\in\mathrm{Per}^*_p(f)$ is not bounded [10] (with respect to n for every $k\geqslant1$). Using Lemma 2.7, we find a sequence of natural numbers $\{i'(k)\}_{k\geqslant1}$ ($i'(k)\geqslant i(k)$) and converging (for every $k\geqslant1$) to (x,y) sequence of points $\{(x_n(k),\,y'_n(k))\}_{n\geqslant1}$ such that $(x_n(k),\,y'_n(k))\in(\eta'_{l_{i'(k)},1})^{P^*}{}_{|\mathrm{Per}^*_p(f,l'_i(k))}$. By (2.8), $(x_n(k),\,y'_n(k))\in(\eta'_{l_{i'(k)}})^{P^*}{}_{|\mathrm{Per}^*_p(f,l'_i(k))}$. Therefore, $B_{\Omega_p}(\widetilde{g}_x)\subseteq\overline{\eta}'_{l_i}(x)$. At the same time, we have $\overline{\eta}'_{l_i}(x)\subseteq B_{\Omega_p}(\widetilde{g}_x)$. Hence (2.39) holds, which implies (2.38). □

Proof of Theorem 2.2. The equalities (2.32) follows from Lemma 2.6, 2.7 and Corollary 2.2. The equality (2.7) for an arbitrary map $F\in T^1_{*,2}(I)$ is established in the same way as in Lemma 2.5 for $F\in T^1_{*,1}(I)$. Theorem 2.2 is proved. □

We note that Theorem 2.2 fails for skew products in $T^1_{*,3}(I)$ and $T^1_{*,4}(I)$.

3 Nonwandering Sets of Skew Products in $T^1_{*,3}(I)$ and $T^1_{*,4}(I)$

Let F be an arbitrary skew product of interval maps in $T^1_{*,3}(I)\bigcup T^1_{*,4}(I)$, and let $\{\overline{\eta}_{l^*_{i_k},1}\}_{k\geqslant1}$ be a subsequence of all discontinuous functions of the sequence $\{\overline{\eta}_{l^*_i,1}\}_{i\geqslant i^*}$. We denote by $S_d(\overline{\eta}_{l^*_{i_k},1})$ the set of points of discontinuity (of the first Baire category) of the upper semicontinuous function $\overline{\eta}_{l^*_{i_k},1}$ ($k\geqslant1$) and by $S_c(\overline{\eta}_{l^*_i})$ ($S_c(\overline{\eta}_{l^*_i,1})$) the set of points of continuity (of the second Baire category) of the upper semicontinuous multivalued function $\overline{\eta}_{l^*_i}$ ($\overline{\eta}_{l^*_i,1}$) ($i\geqslant i^*$). If a point $x\in\Omega_p(f)$ is such that $\{(f_{|\Omega_p(f)})^{-l^*_i}(x)\}\subset S_c(\overline{\eta}_{l^*_i})$, where $\{(f_{|\Omega_p(f)})^{-l^*_i}(x)\}$ is the complete preimage of the point x under the map $(f_{|\Omega_p(f)})^{l^*_i}$ (consisting of finitely many points for any map $f\in C^1_\omega(I_1)$), then $x\in S_c(\overline{\eta}_{l^*_i,1})$.

We construct the set of continuity points of all functions $\overline{\eta}_{l^*_i,1}$ ($i\geqslant i^*$) which is independent of i. For this purpose we introduce an everywhere dense in $\Omega_p(f)$ nonempty set of the second Baire category

$$S_{c,\,P(f)} = \bigcap_{i=i^*}^{+\infty}\Big(\bigcap_{r=0}^{+\infty}(f_{|\Omega_p(f)})^{-l^*_r}(P(f^{l^*}))\bigcap S_c(\overline{\eta}_{l^*_i})\Big),$$

where $l^*_0=0$, $l^*_1=l^*=m_*n_*$, $(f_{|\Omega_p(f)})^{-l^*_r}(P(f^{l^*}))$ is the complete preimage of order l^*_r under the map $f_{|\Omega_p(f)}$ of the set $P(f^{l^*})$ of nonperiodic Poisson stable points of the map f^{l^*}. By the definition of the set $S_{c,\,P(f)}$, the f^{l^*}-trajectory of an arbitrary point $x \in S_{c,\,P(f)}$ (denoted by $O(x,\,f^{l^*})$) belongs to $S_c(\overline{\eta}_{l_i^*,1})$ for any $i \geqslant i^*$. We set

$$S_c^* = \bigcup_{x \in S_{c,\,P(f)}} O(x,\,f^{l^*}).$$

Theorem 3.1. *Let $F \in T^1_{*,\,3}(I) \bigcup T^1_{*,\,4}(I)$. Then*

$$\zeta^{F^{m^*n^*}_{|\Omega_p^*(F)}} = \operatorname*{Ls}_{i\to+\infty} \overline{\eta}_{l_i^*,1} = \operatorname*{Ls}_{k\to+\infty} \overline{\eta}_{l_{i_k}^*,1_{|S_d(\overline{\eta}_{l_{i_k}^*,1})}} \bigcup \operatorname*{Ls}_{i\to+\infty} \overline{\eta}_{l_i^*,1_{|S_c^*}}$$

$$= \operatorname*{Ls}_{k\to+\infty} \overline{\eta}_{l_{i_k}^*,1_{|S_d(\overline{\eta}_{l_{i_k}^*,1})}} \bigcup \overline{\bigcup_{x \in \operatorname{Per}_p(f)} \{x\} \times B_{S_c^*}(\widetilde{g}_x)},$$

where $\zeta^{F^{m^*n^*}_{|\Omega_p^*(F)}}$, $\overline{\eta}_{l_{i_k}^*,1_{|S_d(\overline{\eta}_{l_{i_k}^*,1})}}$, $\overline{\eta}_{l_i^*,1_{|S_c^*}}$ are the graphs of the corresponding multivalued functions in I and $B_{S_c^*}(\widetilde{g}_x)$ is the set of points $y \in I_2$ such that each point (x,y) is weakly nonwandering with respect to the family of maps acting in the fibers over points of S_c^*. For any point $x \in \Omega_p(f)$ and a neighborhood $U_{1,\,\varepsilon_i}(x)$ such that $\lim\limits_{i\to+\infty} \varepsilon_i = 0$ we have

$$\zeta^{F^{m^*n^*}}(x) = \operatorname*{Ls}_{i\to+\infty} \eta^{ex'}_{l_i^*,1_{|U_{1,\,\varepsilon_i}(x)}};$$

moreover, if $F \in T^1_{*,3}(I)$, then

$$\zeta^{F^{m^*n^*}} = \overline{\bigcup_{x \in \operatorname{Per}(f)} \{x\} \times B_{U_1(\Omega_p(f))}(\widetilde{g}_x)},$$

where $B_{U_1(\Omega_p(f))}(\widetilde{g}_x)$ is the set of points $y \in I_2$ such that each point (x,y) is weakly nonwandering with respect to the family of maps acting in the fibers over points of an arbitrary neighborhood $U_1(\Omega_p(f))$ of the set $\Omega_p(f)$ in I_1.

The proof of Theorem 3.1 is based on the same ideas as the proof of Theorems 2.1 and 2.2.

Acknowledgments

The work is supported by the Ministry of Education and Science of the Russian Federation (grant No. 14-10).

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Submitted on December 25, 2015