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IDEMPOTENT ELEMENTS OF THE SEMIGROUP $B_X(D)$ DEFINED BY SEMILATTICES OF THE CLASS $\Sigma_2(X, 8)$

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Abstract. A complete semigroup of binary relations is defined by semilattices of the class $\Sigma_2(X, 8)$. A description of idempotent elements of this semigroup is given. For the case where X is a finite set and $Z_7 \cap Z_6 \neq \varnothing$, formulas are derived by calculating the number of idempotent elements of the semigroup.

1. Let X be an arbitrary nonempty set, D be an X-semilattice of unions, i.e., a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operation of unification of elements from D, and f be an arbitrary mapping from X into D. To each mapping f, a binary relation α_f on the set X corresponds; it satisfies the condition

$$\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x)).$$

The set of all such α_f $(f : X \to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations and we call it a complete semigroup of binary relations defined by an X-semilattice of unions D.

We denote by $\varnothing$ an empty binary relation or an empty subset of the set X. The condition $(x, y) \in \alpha$ will be written in the form $x\alpha y$. Further, let

$$x, y \in X, \quad Y \subseteq X, \quad \alpha \in B_X(D), \quad T \in D, \quad \varnothing \neq D' \subseteq D, \quad t \in \breve{D} = \bigcup_{Y \in D} Y.$$

Then we introduce the following sets:

$$y\alpha = \{x \in X \mid y\alpha x\}, \quad Y\alpha = \bigcup_{y \in Y} y\alpha, \quad V(D, \alpha) = \{Y\alpha \mid Y \in D\},$$
$$X^* = \{T \mid \varnothing \neq T \subseteq X\}, \quad D'_t = \{Z' \in D' \mid t \in Z'\},$$
$$Y_T^\alpha = \{x \in X \mid x\alpha = T\}.$$

We use the symbol $\wedge(D, D_t)$ to denote the exact lower bound of the set D' in the semilattice D.

Definition 1.1. Let $\varepsilon \in B_X(D)$. If $\varepsilon \circ \varepsilon = \varepsilon$ or $\alpha \circ \varepsilon = \alpha$ for any $\alpha \in B_X(D)$, then ε is called an idempotent element or a right unit of the semigroup $B_X(D)$, respectively.

Definition 1.2. We say that a complete X-semilattice of unions D is an XI-semilattice of unions if it satisfies the following two conditions:

(a) $\wedge(D, D_t) \in D$ for any $t \in \breve{D}$;

(b) $Z = \bigcup_{t \in Z} \wedge(D, D_t)$ for any nonempty element Z of D.

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 97, Proceedings of the International Conference "Lie Groups, Differential Equations, and Geometry," June 10–22, 2013, Batumi, Georgia, Part 2, 2015.

Definition 1.3. Let D be an arbitrary complete X-semilattice of unions, $\alpha \in B_X(D)$. If

$$V[\alpha] = \begin{cases} V(X^*, \alpha), & \text{if } \varnothing \notin D, \\ V(X^*, \alpha), & \text{if } \varnothing \in V(X^*, \alpha), \\ V(X^*, \alpha) \cup \{\varnothing\}, & \text{if } \varnothing \notin V(X^*, \alpha) \text{ and } \varnothing \in D, \end{cases}$$

then it is obvious that any binary relation α of the semigroup $B_X(D)$ can always be written in the form

$$\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T).$$

In the sequel, such a representation of a binary relation α is said to be quasinormal.

Note that for the quasinormal representation of a binary relation α, not all sets Y_T^α $(T \in V[\alpha])$ may be different from an empty set. For this representation, the following conditions are always fulfilled:

(a) $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$ for any $T, T' \in D$ and $T \neq T'$;

(b) $X = \bigcup_{T \in V[\alpha]} Y_T^\alpha$.

Definition 1.4. Denote by the symbol $\Sigma'_{\mathrm{XI}}(X, D)$ the set of all XI-subsemilattices of the X-semilattice of unions D. Every element of this set contains an empty set if $\varnothing \in D$ or is the set of all XI-subsemilatices of D.

Further, let

$$D, D' \in \Sigma'_{\mathrm{XI}}(X, D), \quad \vartheta_{\mathrm{XI}} \subseteq \Sigma'_{\mathrm{XI}}(X, D) \times \Sigma'_{\mathrm{XI}}(X, D).$$

Assume that $D \vartheta_{\mathrm{XI}} D'$ if and only if there exists a complete isomorphism φ between the semilattices D and D'. One can easily verify that the binary relation ϑ_{XI} is an equivalence relation on the set $\Sigma'_{\mathrm{XI}}(X, D)$.

Further, if Q is an XI-subsemilattice of unions, then the symbol $Q\vartheta_{\mathrm{XI}}$ stands for the ϑ_{XI}-equivalence class of the set $\Sigma'_{\mathrm{XI}}(D)$, where for each of its elements there exists a complete isomorphism on the semilattice Q.

2. We denote by the symbol $\Sigma_2(X, 8)$ the class of all X-semilattices of unions, every element of which is isomorphic to an X-semilattice of the form

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$

where

$$\begin{gathered} Z_3 \subset Z_1 \subset \breve{D}, \quad Z_4 \subset Z_1 \subset \breve{D}, \quad Z_4 \subset Z_2 \subset \breve{D}, \quad Z_5 \subset Z_2 \subset \breve{D}, \\ Z_6 \subset Z_3 \subset Z_1 \subset \breve{D}, \quad Z_6 \subset Z_4 \subset Z_1 \subset D, \quad Z_6 \subset Z_4 \subset Z_2 \subset \breve{D}, \\ Z_7 \subset Z_4 \subset Z_1 \subset \breve{D}, \quad Z_7 \subset Z_4 \subset Z_2 \subset \breve{D}, \quad Z_7 \subset Z_5 \subset Z_2 \subset \breve{D}, \\ Z_1 \setminus Z_2 \neq \varnothing, \quad Z_2 \setminus Z_1 \neq \varnothing, \quad Z_3 \setminus Z_4 \neq \varnothing, \quad Z_4 \setminus Z_3 \neq \varnothing, \\ Z_3 \setminus Z_5 \neq \varnothing, \quad Z_5 \setminus Z_3 \neq \varnothing, \quad Z_4 \setminus Z_5 \neq \varnothing, \quad Z_5 \setminus Z_4 \neq \varnothing, \\ Z_6 \setminus Z_7 \neq \varnothing, \quad Z_7 \setminus Z_6 \neq \varnothing. \end{gathered} \tag{1}$$

The semilattice satisfying conditions (1) is shown in Fig. 1. Let

$$C(D) = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$$

be the family sets, where P_0, P_1, P_2, P_3, P_4, P_5, P_6, and P_7 are pairwise disjoint subsets of the set X and

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 \end{pmatrix}$$

is the mapping of the semilattice D on the family of sets $C(D)$. Then the formal equalities of the semilattice D are written in the form

$$\begin{aligned}
\breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_3 &= P_0 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_4 &= P_0 \cup P_3 \cup P_5 \cup P_6 \cup P_7,\\
Z_5 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_6 \cup P_7,\\
Z_6 &= P_0 \cup P_5 \cup P_7,\\
Z_7 &= P_0 \cup P_3 \cup P_6.
\end{aligned}$$

Here the elements P_1, P_2, P_3, and P_5 are basis sources and the elements P_0, P_4, P_6, and P_7 are completeness sources of the semilattice D. Therefore, $|X| \geq 4$ and $\delta = 4$.

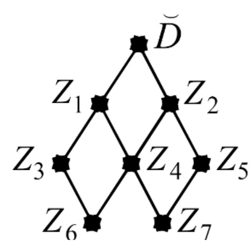

Fig. 1.

Lemma 2.1. *Let $D \in \Sigma_2(X, 8)$, $|\Sigma_2(X, 8)| = s$ and $|X| \geq \delta \geq 4$. If X is a finite set, then*

$$s = \frac{1}{2} \cdot (9^n - 4 \cdot 8^n + 6 \cdot 7^n - 4 \cdot 6^n + 5^n).$$

Example 2.1. Let $n = 4$, 5, 6, 7, 8, 9, or 10. Then

$$s = 24,\ 840,\ 17760,\ 147000,\ 2099412,\ 27156780,\ 327284760$$

and

$$|B_X(D)| = 4096,\ 32768,\ 262144,\ 2097152,\ 16777216,\ 134217728,\ 1073741824.$$

We are going to find all subsemilattices of the semilattice D.

Lemma 2.2. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8), \quad Z_7 \cap Z_6 \neq \varnothing.$$

Then the following sets exhaust all XI-*subsemilattices of the considered semilattice* D:

(1) $\{\breve{D}\}$, $\{Z_1\}$, $\{Z_2\}$, $\{Z_3\}$, $\{Z_4\}$, $\{Z_5\}$, $\{Z_6\}$, $\{Z_7\}$ (*see diagram* 1 *in Fig.* 2);

(2) $\{Z_7, Z_5\}$, $\{Z_7, Z_4\}$, $\{Z_7, Z_2\}$, $\{Z_7, Z_1\}$, $\{Z_7, \breve{D}\}$, $\{Z_6, Z_4\}$, $\{Z_6, Z_3\}$, $\{Z_6, Z_2\}$, $\{Z_6, Z_1\}$, $\{Z_6, \breve{D}\}$, $\{Z_5, Z_2\}$, $\{Z_5, \breve{D}\}$, $\{Z_4, Z_2\}$, $\{Z_4, Z_1\}$, $\{Z_4, \breve{D}\}$, $\{Z_3, Z_1\}$, $\{Z_3, \breve{D}\}$, $\{Z_2, \breve{D}\}$, $\{Z_1, \breve{D}\}$ (*see diagram* 2 *in Fig.* 2);

(3) $\{Z_7, Z_5, Z_2\}$, $\{Z_7, Z_5, \breve{D}\}$, $\{Z_7, Z_4, Z_2\}$, $\{Z_7, Z_4, Z_1\}$, $\{Z_7, Z_4, \breve{D}\}$, $\{Z_7, Z_2, \breve{D}\}$, $\{Z_7, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_2\}$, $\{Z_6, Z_4, \breve{D}\}$, $\{Z_6, Z_4, Z_1\}$, $\{Z_6, Z_2, \breve{D}\}$, $\{Z_6, Z_3, Z_1\}$, $\{Z_6, Z_3, \breve{D}\}$, $\{Z_6, Z_1, \breve{D}\}$, $\{Z_5, Z_2, \breve{D}\}$, $\{Z_4, Z_2, \breve{D}\}$, $\{Z_4, Z_1, \breve{D}\}$, $\{Z_3, Z_1, \breve{D}\}$ (*see diagram* 3 *in Fig.* 2);

(4) $\{Z_7, Z_5, Z_2, \breve{D}\}$, $\{Z_7, Z_4, Z_2, \breve{D}\}$, $\{Z_7, Z_4, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_2, \breve{D}\}$, $\{Z_6, Z_4, Z_1, \breve{D}\}$, $\{Z_6, Z_3, Z_1, \breve{D}\}$ (*see diagram* 4 *in Fig.* 2);

(5) $\{Z_7, Z_5, Z_4, Z_2\}$, $\{Z_7, Z_5, Z_1, \breve{D}\}$, $\{Z_7, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_3, Z_1\}$, $\{Z_6, Z_3, Z_2, \breve{D}\}$, $\{Z_6, Z_2, Z_1, \breve{D}\}$, $\{Z_4, Z_2, Z_1, \breve{D}\}$ (*see diagram* 5 *in Fig.* 2);

(6) $\{Z_7, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_2, Z_1, \breve{D}\}$ (*see diagram* 6 *in Fig.* 2);

(7) $\{Z_7, Z_5, Z_4, Z_2, \breve{D}\}$, $\{Z_6, Z_4, Z_3, Z_1, \breve{D}\}$ (*see diagram* 7 *in Fig.* 2);

(8) $\{Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$, $\{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ (*see diagram* 8 *in Fig.* 2).

Lemma 2.3. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8), \quad Z_7 \cap Z_6 = \varnothing, \quad Z_7 \cap Z_3 \neq \varnothing, \quad Z_6 \cap Z_5 \neq \varnothing.$$

Then the semilattices from Lemma 2.2 *and the following sets exhaust all* XI-*subsemilattices of the considered semilattice* D:

(1) $\{Z_7, Z_6, Z_4\}$ (*see diagram* 9 *in Fig.* 2);

(2) $\{Z_7, Z_6, Z_4, Z_2\}$, $\{Z_7, Z_6, Z_4, Z_1\}$, $\{Z_7, Z_6, Z_4, \breve{D}\}$ (*see diagram* 10 *in Fig.* 2);

(3) $\{Z_7, Z_6, Z_4, Z_2, \breve{D}\}$, $\{Z_7, Z_6, Z_4, Z_1, \breve{D}\}$ (*see diagram* 11 *in Fig.*2);

(4) $\{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\}$ (*see diagram* 12 *in Fig.* 2).

Lemma 2.4. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8), \quad Z_7 \cap Z_6 = \varnothing, \quad Z_7 \cap Z_3 = \varnothing, \quad Z_6 \cap Z_5 \neq \varnothing.$$

Then the semilattices from Lemma 2.2 *and the following sets exhaust all* XI-*subsemilattices of the considered semilattice* D:

(1) $\{Z_7, Z_3, Z_1\}$ (*see diagram* 9 *in Fig.* 2);

(2) $\{Z_7, Z_3, Z_1, \breve{D}\}$ (*see diagram* 10 *in Fig.* 2);

(3) $\{Z_7, Z_6, Z_4, Z_3, Z_1\}$ (*see diagram* 13 *in Fig.* 2);

(4) $\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$ (*see diagram* 14 *in Fig.* 2);

(5) $\{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ (*see diagram* 15 *in Fig.* 2).

Lemma 2.5. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8), \quad Z_7 \cap Z_6 = \varnothing, \quad Z_6 \cap Z_5 = \varnothing, \quad Z_7 \cap Z_3 \neq \varnothing.$$

Then the semilattices from Lemma 2.2 *and the following sets exhaust all* XI-*subsemilattices of the considered semilattice* D:

(1) $\{Z_6, Z_5, Z_2\}$ (*see diagram* 9 *in Fig.* 2);

(2) $\{Z_6, Z_5, Z_2, \breve{D}\}$ (*see diagram* 10 *in Fig.* 2);

(3) $\{Z_7, Z_6, Z_5, Z_4, Z_2\}$ (*see diagram* 13 *in Fig.* 2);

(4) $\{Z_7, Z_6, Z_5, Z_4, Z_2, \breve{D}\}$ (*see diagram* 14 *in Fig.* 2);

(5) $\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$ (*see diagram* 15 *in Fig.* 2).

Lemma 2.6. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8), \quad Z_7 \cap Z_6 = \varnothing, \quad Z_7 \cap Z_3 = \varnothing, \quad Z_6 \cap Z_5 = \varnothing, \quad Z_5 \cap Z_3 \neq \varnothing.$$

Then the semilattices from Lemmas 2.4 *and* 2.5 *are* XI-*subsemilattices of the considered semilattice* D.

Lemma 2.7. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_2(X, 8), \quad Z_5 \cap Z_3 = \varnothing.$$

Then all semilattices from Lemma 2.6 *and the following sets exhaust all* XI*-subsemilattices of the considered semilattice* D:

(1) $\{Z_5, Z_3, \breve{D}\}$ *(see diagram* 9 *in Fig.* 2*)*;

(2) $\{Z_6, Z_5, Z_3, Z_2, \breve{D}\}$, $\{Z_7, Z_5, Z_3, Z_1, \breve{D}\}$ *(see diagram* 13 *in Fig.* 2*)*;

(3) $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ *(see diagram* 16 *in Fig.* 2*)*.

3.

Lemma 3.1. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}.$$

If the equality $Z_5 \cap Z_3 = \varnothing$ *is fulfilled, then the following equalities are valid:*

$$P_3 = Z_7, \quad P_5 = Z_6, \quad P_2 = Z_3 \setminus Z_2, \quad P_1 = Z_5 \setminus Z_1.$$

Theorem 3.1. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}.$$

If the equality $Z_5 \cap Z_3 = \varnothing$ *is fulfilled, then the binary relation*

$$\varepsilon = (Z_7 \times Z_7) \cup (Z_6 \times Z_6) \cup \big((Z_5 \setminus Z_1) \times Z_5\big) \cup \big((Z_3 \setminus Z_2) \times Z_3\big) \cup \big((X \setminus \breve{D}) \times \breve{D}\big)$$

is the largest right unit of the semigroup $B_X(D)$.

Lemma 3.2. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$$

be an XI*-subsemilattice of the* X*-semilattice* D. *The binary relation* α *having a quasinormal representation of the form*

$$\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \\ \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where Y_7^α, Y_6^α, Y_5^α, $Y_3^\alpha \notin \{\varnothing\}$, *is a right unit of the semigroup* $B_X(D)$ *if and only if the binary relation* α *satisfies the following conditions:*

$$Y_7^\alpha \supseteq Z_7, \quad Y_6^\alpha \supseteq Z_6, \quad Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5, \quad Y_6^\alpha \cup Y_3^\alpha \supseteq Z_3, \\ Y_5^\alpha \cap Z_5 \neq \varnothing, \quad Y_3^\alpha \cap Z_3 \neq \varnothing.$$

Theorem 3.2. *Let*

$$D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$$

and $E_X^{(r)}(D)$ *be the set of all right units of the semigroup* $B_X(D)$. *If* X *is a finite set, then the following formula holds*:

$$|E_X^{(r)}(D)| = (2^{|Z_5 \setminus Z_1|} - 1) \cdot (2^{|Z_3 \setminus Z_2|} - 1) \cdot 8^{|X \setminus \breve{D}|}.$$

We denote by Q_i, $i = 1, 2, \ldots, 16$, the following sets:

(a) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 in Fig. 2);

(b) $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$ (see diagram 2 in Fig. 2);

(c) $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 in Fig. 2);

(d) $Q_4 = \{T, T', T'', \breve{D}\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T'' \subset \breve{D}$ (see diagram 4 in Fig. 2);

(e) $Q_5 = \{T, T', T'', T' \cup T''\}$, where $T, T', T'' \in D$, $T \subset T''$, $T \subset T'$ and $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ (see diagram 5 in Fig. 2);

(f) $Q_6 = \{T, Z_4, Z, Z', \breve{D}\}$, where $T \in \{Z_7, Z_6\}$, $Z, Z' \in \{Z_2, Z_1\}$, $Z \neq Z'$ (see diagram 6 in Fig. 2);

(g) $Q_7 = \{T, T', T'', T' \cup T'', \breve{D}\}$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$ and $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ (see diagram 7 in Fig. 2);

(h) $Q_8 = \{T, T', Z_4, Z_4 \cup T', Z, \breve{D}\}$, where $T \in \{Z_7, Z_6\}$, $T' \in \{Z_5, Z_3\}$, $Z_4 \cup T'$, $Z \in \{Z_2, Z_1\}$, $Z_4 \cup T' \neq Z$, $T \subset T'$ and $T' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T' \neq \varnothing$, $(Z_4 \cup T') \setminus Z \neq \varnothing$, $Z \setminus (Z_4 \cup T') \neq \varnothing$ (see diagram 8 in Fig. 2);

(i) $Q_9 = \{T, T', T \cup T'\}$, where $T, T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$ and $T \cap T' = \varnothing$ (see diagram 9 in Fig. 2);

(j) $Q_{10} = \{T, T', T \cup T', T''\}$, where $T, T', T'' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cap T' = \varnothing$ and $T \cup T' \subset T''$ (see diagram 10 in Fig. 2);

(k) $Q_{11} = \{Z_7, Z_6, Z_4, Z, \breve{D}\}$, where $Z \in \{Z_2, Z_1\}$ and $Z_7 \cap Z_6 = \varnothing$ (see diagram 11 in Fig. 2);

(l) $Q_{12} = \{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\}$, where $Z_7 \cap Z_6 = \varnothing$ (see diagram 12 in Fig. 2);

(m) $Q_{13} = \{T, T', T \cup T', T'', Z\}$, where $T, T', T'', Z \in D$, $(T \cup T') \setminus T'' \neq \varnothing$, $T'' \setminus (T \cup T') \neq \varnothing$, $T \cap T'' = \varnothing$ and $(T \cup T') \subset Z$, $T' \subset T'' \subset Z$ (see diagram 13 in Fig. 2);

(n) $Q_{14} = \{T, T', Z_4, Z, Z', \breve{D}\}$, where $T, T', Z, Z' \in D$, $Z_4 \setminus Z \neq \varnothing$, $Z \setminus Z_4 \neq \varnothing$, $T \cap Z = \varnothing$ and $(T \cup T') \subset Z'$, $T' \subset Z \subset Z' \subset \breve{D}$ (see diagram 14 in Fig. 2);

(o) $Q_{15} = \{T, T', Z_4, T'', Z, T'' \cup Z_4, \breve{D}\}$, where $T, T' \in \{Z_7, Z_6\}$, $T \neq T'$, $T \subset T''$, $T'' \in \{Z_5, Z_3\}$, $Z_4 \subset Z$, $Z \cup T'' \cup Z_4 = \breve{D}$, $(T'' \cup Z_4) \setminus Z \neq \varnothing$, $Z \setminus (T'' \cup Z_4) \neq \varnothing$, $T'' \setminus Z_4 \neq \varnothing$, $Z_4 \setminus T'' \neq \varnothing$ and $T' \cap T'' = \varnothing$ (see diagram 15 in Fig. 2);

(p) $Q_{16} = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, where $Z_5 \cap Z_3 = \varnothing$ (see diagram 16 in Fig. 2).

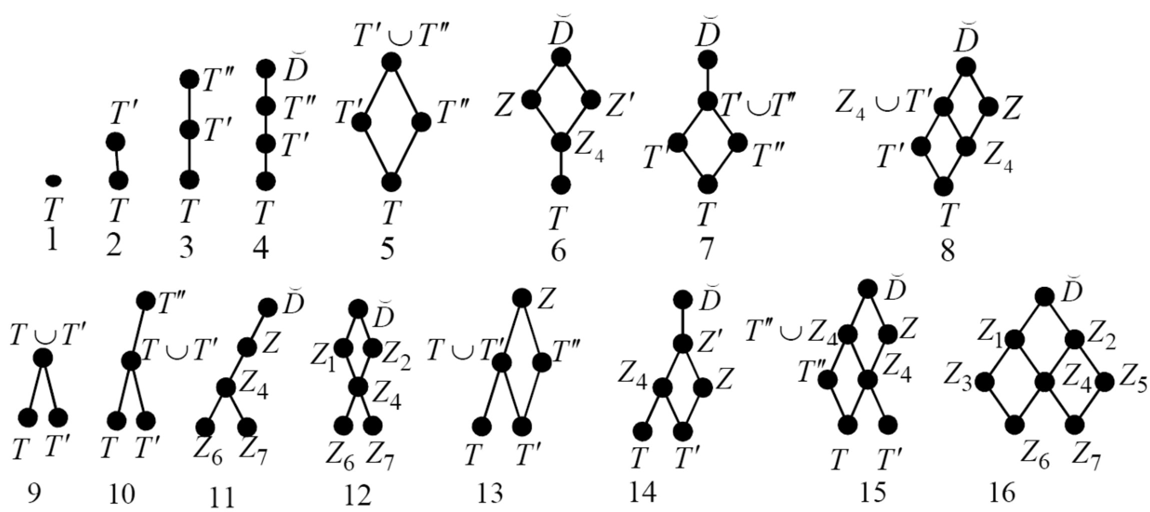

Fig. 2.

Let D' be an XI-subsemilattice of the semilattice D. Denote by $I(D')$ the set of all right units of the semigroup $B_X(D')$ and take

$$|I^*(Q_i)| = \sum_{D' \in Q_i \vartheta_{\mathrm{XI}}} |I(D')|,$$

where $i = 1, 2, \dots, 16$.

The set all right units of the complete semigroup $B_X(D')$ of binary relations defined by the complete X-semilattice of unions D' will sometimes be denoted by the symbol $E_X^{(r)}(D')$.

Lemma 3.3. *If $D \in \Sigma_2(X, 8)$, then the following equalities are valid*:

(a) $|I(Q_1)| = 1$;

(b) $|I(Q_2)| = (2^{|T' \setminus T|} - 1) \cdot 2^{|X \setminus T'|}$;

(c) $|I(Q_3)| = (2^{|T' \setminus T|} - 1) \cdot (3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}) \cdot 3^{|X \setminus T''|}$;

(d) $|I(Q_4)| = (2^{|T' \setminus T|} - 1) \cdot (3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}) \cdot (4^{|\breve{D} \setminus T''|} - 3^{|\breve{D} \setminus T''|}) \cdot 4^{|X \setminus \breve{D}|}$;

(e) $|I(Q_5)| = (2^{|T' \setminus T''|} - 1) \cdot (2^{|T'' \setminus T'|} - 1) \cdot 4^{|X \setminus (T' \cup T'')|}$;

(f) $|I(Q_6)| = (2^{|Z_4 \setminus T|} - 1) \cdot 2^{|(Z \cap Z') \setminus Z_4|} \cdot (3^{|Z \setminus Z'|} - 2^{|Z \setminus Z'|}) \cdot (3^{|Z' \setminus Z|} - 2^{|Z' \setminus Z|}) \cdot 5^{|X \setminus \breve{D}|}$;

(g) $|I(Q_7)| = (2^{|T' \setminus T''|} - 1) \cdot (2^{|T'' \setminus T'|} - 1) \cdot (5^{|\breve{D} \setminus (T' \cup T'')|} - 4^{|\breve{D} \setminus (T' \cup T'')|}) \cdot 5^{|X \setminus \breve{D}|}$;

(h) $|I(Q_8)| = (2^{|T' \setminus Z|} - 1) \cdot (2^{|Z_4 \setminus T'|} - 1) \cdot (3^{|Z \setminus (Z_4 \cup T')|} - 2^{|Z \setminus (Z_4 \cup T')|}) \cdot 6^{|X \setminus \breve{D}|}$;

(i) $|I(Q_9)| = 3^{|X \setminus (T \cup T')|}$;

(j) $|I(Q_{10})| = (4^{|T'' \setminus (T \cup T')|} - 3^{|T'' \setminus (T \cup T')|}) \cdot 4^{|X \setminus T''|}$;

(k) $|I(Q_{11})| = (4^{|Z \setminus Z_4|} - 3^{|Z \setminus Z_4|}) \cdot (5^{|\breve{D} \setminus Z|} - 4^{|\breve{D} \setminus Z|}) \cdot 5^{|X \setminus \breve{D}|}$;

(l) $|I(Q_{12})| = 3^{|(Z_2 \cap Z_1) \setminus Z_4|} (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus \breve{D}|}$;

(m) $|I(Q_{13})| = (2^{|T'' \setminus (T' \cup T)|} - 1) \cdot 5^{|X \setminus Z|}$;

(n) $|I(Q_{14})| = (2^{|Z \setminus Z_4|} - 1) \cdot (6^{|\breve{D} \setminus Z'|} - 5^{|\breve{D} \setminus Z'|}) \cdot 6^{|X \setminus \breve{D}|}$;

(o) $|I(Q_{15})| = (2^{|T'' \setminus Z|} - 1) \cdot (4^{|Z \setminus (T'' \cup Z_4)|} - 3^{|Z \setminus (T'' \cup Z_4)|}) \cdot 7^{|X \setminus \breve{D}|}$;

(p) $|I(Q_{16})| = (2^{|Z_5 \setminus Z_1|} - 1) \cdot (2^{|Z_3 \setminus Z_2|} - 1) \cdot 8^{|X \setminus \breve{D}|}$.

4.

Theorem 4.1. *Let*

$$D \in \Sigma_2(X, 8), \quad Z_7 \cap Z_6 \neq \varnothing, \quad \alpha \in B_X(D).$$

The binary relation α is an idempotent relation of the semigroup $B_X(D)$ if and only if the binary relation α satisfies one of the following conditions:

(a) $\alpha = X \times T$, *where* $T \in D$;

(b)

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T'),$$

where

$$T, T' \in D, \quad T \subset T', \quad Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_T^\alpha \supseteq T, \quad Y_{T'}^\alpha \cap T' \neq \varnothing;$$

(c)

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T''),$$

where

$$T, T', T'' \in D, \quad T \subset T' \subset T'', \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_T^\alpha \supseteq T, \quad Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T', \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing;$$

(d)

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_0^\alpha \times \breve{D}),$$

where

$$T, T', T'' \in D, \quad T \subset T' \subset T'' \subset \breve{D}, \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_T^\alpha \supseteq T, \quad Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T', \quad Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T'',$$
$$Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing, \quad Y_0^\alpha \cap \breve{D} \neq \varnothing;$$

(e)

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')),$$

where

$$T, T', T'' \in D, \quad T \subset T'', \quad T \subset T', \quad T' \setminus T'' \neq \varnothing, \quad T'' \setminus T' \neq \varnothing, \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T', \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T'', \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing;$$

(f)

$$\alpha = (Y_T^\alpha \times T) \cup (Y_4^\alpha \times Z_4) \cup (Y_{Z'}^\alpha \times Z') \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \breve{D}),$$

where

$$T \in \{Z_7, Z_6\}, \quad Z \setminus Z' \neq \varnothing, \quad Z' \setminus Z \neq \varnothing, \quad Z, Z' \subset \breve{D}, \quad Y_T^\alpha, Y_4^\alpha, Y_{Z'}^\alpha, Y_Z^\alpha, Y_0^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_T^\alpha \supseteq T, \quad Y_T^\alpha \cup Y_4^\alpha \supseteq Z_4, \quad Y_T^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z,$$
$$Y_T^\alpha \cup Y_4^\alpha \cup Y_{Z'}^\alpha \supseteq Z', \quad Y_4^\alpha \cap Z_4 \neq \varnothing, \quad Y_Z^\alpha \cap Z \neq \varnothing, \quad Y_{Z'}^\alpha \cap Z' \neq \varnothing;$$

(g)

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')) \cup (Y_0^\alpha \times \breve{D}),$$

where

$$T, T', T'' \in D, \quad T \subset T', \quad T \subset T'', \quad T' \setminus T'' \neq \varnothing, \quad T'' \setminus T' \neq \varnothing, \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T', \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T'', \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing, \quad Y_0^\alpha \cap \breve{D} \neq \varnothing;$$

(h)

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_4^\alpha \times Z_4) \cup (Y_{T' \cup Z_4}^\alpha \times (T' \cup Z_4)) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \breve{D}),$$

where

$$T \in \{Z_7, Z_6\}, \quad T' \in \{Z_5, Z_3\}, \quad Z_4 \cup T', \quad Z \in \{Z_2, Z_1\}, \quad Z_4 \cup T' \neq Z, \quad T \subset T',$$
$$T' \setminus Z_4 \neq \varnothing, \quad Z_4 \setminus T' \neq \varnothing, \quad (Z_4 \cup T') \setminus Z \neq \varnothing, \quad Z \setminus (Z_4 \cup T') \neq \varnothing,$$
$$Y_T^\alpha, Y_{T'}^\alpha, Y_4^\alpha, Y_Z^\alpha, Y_0^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T', \quad Y_T^\alpha \cup Y_4^\alpha \supseteq Z_4, \quad Y_T^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z, \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_4^\alpha \cap Z_4 \neq \varnothing, \quad Y_Z^\alpha \cap Z \neq \varnothing.$$

Lemma 4.1. *Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_1)|$ can be calculated by the formula*

$$|I^*(Q_1)| = 8.$$

Lemma 4.2. *Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_2)|$ can be calculated by the formula*

$$\begin{aligned}
|I^*(Q_2)| = {} & (2^{|\breve{D}\setminus Z_1|}-1)\cdot 2^{|X\setminus\breve{D}|} + (2^{|\breve{D}\setminus Z_2|}-1)\cdot 2^{|X\setminus\breve{D}|} + (2^{|\breve{D}\setminus Z_3|}-1)\cdot 2^{|X\setminus\breve{D}|} \\
& + (2^{|\breve{D}\setminus Z_4|}-1)\cdot 2^{|X\setminus\breve{D}|} + (2^{|\breve{D}\setminus Z_5|}-1)\cdot 2^{|X\setminus\breve{D}|} + (2^{|\breve{D}\setminus Z_6|}-1)\cdot 2^{|X\setminus\breve{D}|} \\
& + (2^{|\breve{D}\setminus Z_7|}-1)\cdot 2^{|X\setminus\breve{D}|} + (2^{|Z_1\setminus Z_3|}-1)\cdot 2^{|X\setminus Z_1|} + (2^{|Z_1\setminus Z_4|}-1)\cdot 2^{|X\setminus Z_1|} \\
& + (2^{|Z_1\setminus Z_6|}-1)\cdot 2^{|X\setminus Z_1|} + (2^{|Z_1\setminus Z_7|}-1)\cdot 2^{|X\setminus Z_1|} + (2^{|Z_2\setminus Z_4|}-1)\cdot 2^{|X\setminus Z_2|} \\
& + (2^{|Z_2\setminus Z_5|}-1)\cdot 2^{|X\setminus Z_2|} + (2^{|Z_2\setminus Z_6|}-1)\cdot 2^{|X\setminus Z_2|} + (2^{|Z_2\setminus Z_7|}-1)\cdot 2^{|X\setminus Z_2|} \\
& + (2^{|Z_3\setminus Z_6|}-1)\cdot 2^{|X\setminus Z_3|} + (2^{|Z_4\setminus Z_6|}-1)\cdot 2^{|X\setminus Z_4|} + (2^{|Z_4\setminus Z_7|}-1)\cdot 2^{|X\setminus Z_4|} \\
& + (2^{|Z_5\setminus Z_7|}-1)\cdot 2^{|X\setminus Z_5|}.
\end{aligned}$$

Lemma 4.3. *Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_3)|$ can be calculated by the formula*

$$\begin{aligned}
|I^*(Q_3)| = {} & (2^{|Z_1\setminus Z_3|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_1\setminus Z_4|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|} \\
& + (2^{|Z_2\setminus Z_4|}-1)\cdot(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_2\setminus Z_5|}-1)\cdot(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|})\cdot 3^{|X\setminus\breve{D}|} \\
& + (2^{|Z_1\setminus Z_6|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_2\setminus Z_6|}-1)\cdot(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|})\cdot 3^{|X\setminus\breve{D}|} \\
& + (2^{|Z_3\setminus Z_6|}-1)\cdot(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_4\setminus Z_6|}-1)\cdot(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|})\cdot 3^{|X\setminus\breve{D}|} \\
& + (2^{|Z_1\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_2\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|})\cdot 3^{|X\setminus\breve{D}|} \\
& + (2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_5\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|})\cdot 3^{|X\setminus\breve{D}|} \\
& + (2^{|Z_3\setminus Z_6|}-1)\cdot(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|})\cdot 3^{|X\setminus Z_1|} + (2^{|Z_4\setminus Z_6|}-1)\cdot(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|})\cdot 3^{|X\setminus Z_1|} \\
& + (2^{|Z_4\setminus Z_6|}-1)\cdot(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|})\cdot 3^{|X\setminus Z_2|} + (2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|})\cdot 3^{|X\setminus Z_1|} \\
& + (2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|})\cdot 3^{|X\setminus Z_2|} + (2^{|Z_5\setminus Z_7|}-1)\cdot(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|})\cdot 3^{|X\setminus Z_2|}.
\end{aligned}$$

Lemma 4.4. *Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_4)|$ can be calculated by the formula*

$$\begin{aligned}
|I^*(Q_4)| = {} & (2^{|Z_5\setminus Z_7|}-1)\cdot(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|})\cdot(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|})\cdot 4^{|X\setminus\breve{D}|} \\
& + (2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|})\cdot(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|})\cdot 4^{|X\setminus\breve{D}|} \\
& + (2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|})\cdot(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|})\cdot 4^{|X\setminus\breve{D}|} \\
& + (2^{|Z_4\setminus Z_6|}-1)\cdot(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|})\cdot(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|})\cdot 4^{|X\setminus\breve{D}|} \\
& + (2^{|Z_4\setminus Z_6|}-1)\cdot(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|})\cdot(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|})\cdot 4^{|X\setminus\breve{D}|} \\
& + (2^{|Z_3\setminus Z_6|}-1)\cdot(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|})\cdot(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|})\cdot 4^{|X\setminus\breve{D}|}.
\end{aligned}$$

Lemma 4.5. *Let $D \in \Sigma_2(X,8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_5)|$ can be calculated by the formula*

$$|I^*(Q_5)| = 3\cdot(2^{|Z_2\setminus Z_1|}-1)\cdot(2^{|Z_1\setminus Z_2|}-1)\cdot 4^{|X\setminus\breve{D}|} + (2^{|Z_5\setminus Z_4|}-1)\cdot(2^{|Z_4\setminus Z_5|}-1)\cdot 4^{|X\setminus Z_2|}$$

$$+ (2^{|Z_5\setminus Z_1|} - 1) \cdot (2^{|Z_1\setminus Z_5|} - 1) \cdot 4^{|X\setminus \breve{D}|} + (2^{|Z_4\setminus Z_3|} - 1) \cdot (2^{|Z_3\setminus Z_4|} - 1) \cdot 4^{|X\setminus Z_1|}$$
$$+ (2^{|Z_3\setminus Z_2|} - 1) \cdot (2^{|Z_2\setminus Z_3|} - 1) \cdot 4^{|X\setminus \breve{D}|}.$$

Lemma 4.6. *Let $D \in \Sigma_2(X, 8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_6)|$ can be calculated by the formula*

$$|I^*(Q_6)| = (2^{|Z_4\setminus Z_7|} - 1) \cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|} \cdot (3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}) \cdot (3^{|Z_2\setminus Z_1|} - 2^{|Z_2\setminus Z_1|}) \cdot 5^{|X\setminus \breve{D}|}$$
$$+ (2^{|Z_4\setminus Z_6|} - 1) \cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|} \cdot (3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}) \cdot (3^{|Z_2\setminus Z_1|} - 2^{|Z_2\setminus Z_1|}) \cdot 5^{|X\setminus \breve{D}|}.$$

Lemma 4.7. *Let $D \in \Sigma_2(X, 8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_7)|$ can be calculated by the formula*

$$|I^*(Q_7)| = (2^{|Z_4\setminus Z_5|} - 1) \cdot (2^{|Z_5\setminus Z_4|} - 1) \cdot (5^{|\breve{D}\setminus Z_2|} - 4^{|\breve{D}\setminus Z_2|}) \cdot 5^{|X\setminus \breve{D}|}$$
$$+ (2^{|Z_3\setminus Z_4|} - 1) \cdot (2^{|Z_4\setminus Z_3|} - 1) \cdot (5^{|\breve{D}\setminus Z_1|} - 4^{|\breve{D}\setminus Z_1|}) \cdot 5^{|X\setminus \breve{D}|}.$$

Lemma 4.8. *Let $D \in \Sigma_2(X, 8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_8)|$ can be calculated by the formula*

$$|I^*(Q_8)| = (2^{|Z_3\setminus Z_2|} - 1) \cdot (2^{|Z_4\setminus Z_3|} - 1) \cdot (3^{|Z_2\setminus Z_1|} - 2^{|Z_2\setminus Z_1|}) \cdot 6^{|X\setminus \breve{D}|}$$
$$+ (2^{|Z_5\setminus Z_1|} - 1) \cdot (2^{|Z_4\setminus Z_5|} - 1) \cdot (3^{|Z_2\setminus Z_1|} - 2^{|Z_2\setminus Z_1|}) \cdot 6^{|X\setminus \breve{D}|}.$$

Let us assume that

$$k_1 = \sum_{i=1}^{8} |I^*(Q_i)|.$$

Theorem 4.2. *Let $D \in \Sigma_2(X, 8)$ and $Z_7 \cap Z_6 \neq \varnothing$. If X is a finite set and I_D is the set of all idempotent elements of the semigroup $B_X(D)$, then $|I_D| = k_1$.*

No.	Set X	Semilattice D	Number of elements of the semigroup $B_X(D)$	Number of idempotents of the semigroup $B_X(D)$
1	$X = \{1,2,3,4,5\}$	$D = \{\{1,4\},\{1,5\},\{1,2,4\},\{1,4,5\},\{1,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{1,2,3,4,5\}\}$	32768	164

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