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IDEMPOTENT ELEMENTS OF THE SEMIGROUP $B_X(D)$ DEFINED BY SEMILATTICES OF THE CLASS $\Sigma_3(X,8)$ WHEN $Z_7 = \varnothing$

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Abstract. The paper presents a full description of idempotent elements of the semigroup of binary relations $B_X(D)$, which are defined by semilattices of the class $\Sigma_3(X,8)$. For the case where X is a finite set and $Z_7 = \varnothing$, we derive formulas for calculating the number of idempotent elements of the respective semigroup.

Let X and $\Sigma_3(X,8)$ be respectively a nonempty set and the class of all X-semilattices of unions where every element is isomorphic to some semilattice of unions $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ that satisfies the conditions

$$\begin{gathered} Z_7 \subset Z_6 \subset Z_4 \subset Z_2 \subset \breve{D}, \quad Z_7 \subset Z_6 \subset Z_4 \subset Z_1 \subset \breve{D}, \\ Z_7 \subset Z_5 \subset Z_4 \subset Z_2 \subset \breve{D}, \quad Z_7 \subset Z_5 \subset Z_4 \subset Z_1 \subset \breve{D}, \quad Z_7 \subset Z_5 \subset Z_3 \subset Z_1 \subset \breve{D}; \\ Z_1 \setminus Z_2 \neq \varnothing, \quad Z_2 \setminus Z_1 \neq \varnothing, \quad Z_3 \setminus Z_2 \neq \varnothing, \quad Z_2 \setminus Z_3 \neq \varnothing, \quad Z_3 \setminus Z_4 \neq \varnothing, \\ Z_4 \setminus Z_3 \neq \varnothing, \quad Z_3 \setminus Z_6 \neq \varnothing, \quad Z_6 \setminus Z_3 \neq \varnothing, \quad Z_5 \setminus Z_6 \neq \varnothing, \quad Z_6 \setminus Z_5 \neq \varnothing. \end{gathered} \tag{1}$$

A semilattice satisfying conditions (1) is shown in Fig. 1.

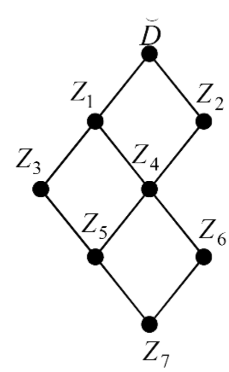

Fig. 1.

Further, assume that $C(D) = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ is the set of pairwise nonitersecting subsets of the set X. Then formal equalities for an element of the considered semilattice are written

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in the form

$$
\begin{aligned}
\breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_3 &= P_0 \cup P_2 \cup P_4 \cup P_5 \cup P_6 \cup P_7,\\
Z_4 &= P_0 \cup P_3 \cup P_5 \cup P_6 \cup P_7,\\
Z_5 &= P_0 \cup P_6 \cup P_7,\\
Z_6 &= P_0 \cup P_3 \cup P_5 \cup P_7,\\
Z_7 &= P_0,
\end{aligned}
\tag{2}
$$

where

$$|P_0| \geq 0, \quad |P_4| \geq 0, \quad |P_5| \geq 0, \quad |P_7| \geq 0, \quad |P_1| \geq 1, \quad |P_2| \geq 1, \quad |P_3| \geq 1, \quad |P_6| \geq 1.$$

We call the elements P_0, P_4, P_5 and P_7 the sources of completeness, and the elements P_1, P_2, P_3, P_6 the basis sources of the X-semilattice of unions D.

Lemma 1. *Let $D \in \Sigma_3(X, 8)$ and $Z_7 = \varnothing$. Then the following sets are all* XI*-subsemilattices of the semilattice D:*

(1) $\{\varnothing\}$;

(2) $\{\varnothing, Z_6\}$, $\{\varnothing, Z_5\}$, $\{\varnothing, Z_4\}$, $\{\varnothing, Z_3\}$, $\{\varnothing, Z_2\}$, $\{\varnothing, Z_1\}$, $\{\varnothing, \breve{D}\}$;

(3) $\{\varnothing, Z_6, Z_4\}$, $\{\varnothing, Z_6, Z_2\}$, $\{\varnothing, Z_6, Z_1\}$, $\{\varnothing, Z_6, \breve{D}\}$, $\{\varnothing, Z_5, Z_4\}$, $\{\varnothing, Z_5, Z_3\}$, $\{\varnothing, Z_5, Z_2\}$, $\{\varnothing, Z_5, Z_1\}$, $\{\varnothing, Z_5, \breve{D}\}$, $\{\varnothing, Z_4, Z_2\}$, $\{\varnothing, Z_4, Z_1\}$, $\{\varnothing, Z_4, \breve{D}\}$, $\{\varnothing, Z_3, Z_1\}$, $\{\varnothing, Z_3, \breve{D}\}$, $\{\varnothing, Z_2, \breve{D}\}$, $\{\varnothing, Z_1, \breve{D}\}$;

(4) $\{\varnothing, Z_6, Z_4, Z_2\}$, $\{\varnothing, Z_6, Z_4, Z_1\}$, $\{\varnothing, Z_6, Z_4, \breve{D}\}$, $\{\varnothing, Z_6, Z_2, \breve{D}\}$, $\{\varnothing, Z_6, Z_1, \breve{D}\}$, $\{\varnothing, Z_5, Z_4, Z_2\}$, $\{\varnothing, Z_5, Z_4, Z_1\}$, $\{\varnothing, Z_5, Z_4, \breve{D}\}$, $\{\varnothing, Z_5, Z_3, Z_1\}$, $\{\varnothing, Z_5, Z_3, \breve{D}\}$, $\{\varnothing, Z_5, Z_2, \breve{D}\}$, $\{\varnothing, Z_5, Z_1, \breve{D}\}$, $\{\varnothing, Z_4, Z_2, \breve{D}\}$, $\{\varnothing, Z_4, Z_1, \breve{D}\}$, $\{\varnothing, Z_3, Z_1, \breve{D}\}$;

(5) $\{\varnothing, Z_6, Z_4, Z_2, \breve{D}\}$, $\{\varnothing, Z_6, Z_4, Z_1, \breve{D}\}$, $\{\varnothing, Z_5, Z_4, Z_2, \breve{D}\}$, $\{\varnothing, Z_5, Z_4, Z_1, \breve{D}\}$, $\{\varnothing, Z_5, Z_3, Z_1, \breve{D}\}$;

(6) $\{\varnothing, Z_6, Z_5, Z_4\}$, $\{\varnothing, Z_6, Z_3, Z_1\}$, $\{\varnothing, Z_4, Z_3, Z_1\}$, $\{\varnothing, Z_3, Z_2, \breve{D}\}$, $\{\varnothing, Z_2, Z_1, \breve{D}\}$;

(7) $\{\varnothing, Z_6, Z_2, Z_1, \breve{D}\}$, $\{\varnothing, Z_5, Z_4, Z_3, Z_1, \}$, $\{\varnothing, Z_5, Z_3, Z_2, \breve{D}\}$, $\{\varnothing, Z_5, Z_2, Z_1, \breve{D}\}$, $\{\varnothing, Z_4, Z_2, Z_1, \breve{D}\}$;

(8) $\{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\}$, $\{\varnothing, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$;

(9) $\{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$;

(10) $\{\varnothing, Z_6, Z_5, Z_4, Z_2\}$, $\{\varnothing, Z_6, Z_5, Z_4, Z_1\}$, $\{\varnothing, Z_6, Z_5, Z_4, \breve{D}\}$, $\{\varnothing, Z_4, Z_3, Z_1, \breve{D}\}$, $\{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}$;

(11) $\{\varnothing, Z_6, Z_5, Z_4, Z_2, \breve{D}\}$, $\{\varnothing, Z_6, Z_5, Z_4, Z_1, \breve{D}\}$;

(12) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1\}$, $\{\varnothing, Z_6, Z_3, Z_2, Z_1, \breve{D}\}$, $\{\varnothing, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$;

(13) $\{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$;

(14) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$;

(15) $\{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$;

(16) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$.

We denote the semilattices as follows:

(1) $Q_1 = \{\varnothing\}$;

(2) $Q_2 = \{\varnothing, T\}$, where $\varnothing \neq T \in D$;

(3) $Q_3 = \{\varnothing, T, T'\}$, where $\varnothing \neq T$, $T' \in D$, $T \subset T'$;

(4) $Q_4 = \{\varnothing, T, T', T''\}$, where $\varnothing \neq= T$, $T', T'' \in D$, $T \subset T' \subset T''$;

(5) $Q_5 = \{\varnothing, T, T', T'', \breve{D}\}$, where $\varnothing \neq T$, $T', T'', \breve{D} \in D$, $T \subset T' \subset T'' \subset \breve{D}$;

(6) $Q_6 = \{\varnothing, T, T', T \cup T'\}$, where $\varnothing \neq T$, $T' \in D$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$;

(7) $Q_7 = \{\varnothing, T, T', T'', T' \cup T''\}$, where, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$;

(8) $Q_8 = \{\varnothing, T, Z_4, Z_2, Z_1, \breve{D}\}$, where $T \in \{Z_6, Z_5\}$;

(9) $Q_9 = \{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$;

(10) $Q_{10} = \{\varnothing, T, T', T \cup T', T''\}$, where $T' \setminus T \neq \varnothing$, $T \setminus T' \neq \varnothing$, $T \cup T' \subset T''$;

(11) $Q_{11} = \{\varnothing, Z_6, Z_5, Z_4, T, \breve{D}\}$, where $T \in \{Z_2, Z_1\}$;

(12) $Q_{12} = \{\varnothing, T, T', T \cup T', T'', T \cup T' \cup T''\}$, where $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T' \subset T''$, $(T \cup T') \setminus T'' \neq \varnothing$, $T'' \setminus (T \cup T') \neq \varnothing$;

(13) $Q_{13} = \{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$;

(14) $Q_{14} = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$;

(15) $Q_{15} = \{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$;

(16) $Q_{16} = \{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$.

Theorem 1. *Let $D \in \Sigma_3(X, 8)$, $Z_7 = \varnothing$ and $\alpha \in B_X(D)$. A binary relation α is an idempotent element of the semigroup $B_X(D)$ if and only if a binary relation α satisfies only one of the following conditions*:

(1) $\alpha = \varnothing$ *or* $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T)$, *where*

$$\varnothing \neq T \in D, \quad Y_T^\alpha \notin \{\varnothing\},$$

and satisfies the conditions

$$Y_7^\alpha \supseteq \varnothing, \quad Y_T^\alpha \cap T \neq \varnothing;$$

(2) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$, *where*

$$\varnothing \neq T \subset T' \in D, \quad Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\},$$

and satisfies the conditions

$$Y_7^\alpha \supseteq \varnothing, \quad Y_7^\alpha \cup Y_T^\alpha \supseteq T, \quad Y_T^\alpha \cap T \neq \varnothing, \quad Y_{T'}^\alpha \cap T' \neq \varnothing;$$

(3) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$, *where*

$$\varnothing \neq T \subset T' \subset T'' \in D, \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\},$$

and satisfies the conditions

$$Y_7^\alpha \supseteq \varnothing, \quad Y_7^\alpha \cup Y_T^\alpha \supseteq T, \quad Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T',$$
$$Y_T^\alpha \cap T \neq \varnothing, \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing;$$

(4) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_0^\alpha \times \breve{D})$, *where*

$$\varnothing \neq T \subset T' \subset T'' \subset \breve{D}, \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha \supseteq \varnothing, \quad Y_7^\alpha \cup Y_T^\alpha \supseteq T, \quad Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T',$$
$$Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T'', \quad Y_T^\alpha \cap T \neq \varnothing, \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing, \quad Y_0^\alpha \cap \breve{D} \neq \varnothing;$$

(5) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T\cup T'}^\alpha \times (T \cup T'))$, *where*

$$\varnothing \neq T, T' \in D, \quad T' \setminus T \neq \varnothing, \quad T \setminus T' \neq \varnothing, \quad Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha \cup Y_T^\alpha \supseteq T, \quad Y_7^\alpha \cup Y_{T'}^\alpha \supseteq T', \quad Y_T^\alpha \cap T \neq \varnothing, \quad Y_{T'}^\alpha \cap T' \neq \varnothing;$$

(6) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T'\cup T''}^\alpha \times (T' \cup T''))$, *where*

$$\varnothing \neq T \subset T', \quad \varnothing \neq T \subset T'', \quad T' \setminus T'' \neq \varnothing, \quad T'' \setminus T' \neq \varnothing, \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha \supseteq \varnothing, \quad Y_7^\alpha \cup Y_T^\alpha \supseteq T, \quad Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T',$$
$$Y_7^\alpha \cup Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T'', \quad Y_T^\alpha \cap T \neq \varnothing, \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing.$$

(7) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_T^\alpha \times T) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, *where*

$$T \in \{Z_6, Z_5\}, \quad Y_T^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha \supseteq \varnothing, \quad Y_7^\alpha \cup Y_T^\alpha \supseteq T, \quad Y_7^\alpha \cup Y_T^\alpha \cup Y_4^\alpha \supseteq Z_4,$$
$$Y_7^\alpha \cup Y_T^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2, \quad Y_7^\alpha \cup Y_T^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq Z_1, \quad Y_T^\alpha \cap T \neq \varnothing,$$
$$Y_4^\alpha \cap Z_4 \neq \varnothing, \quad Y_2^\alpha \cap Z_2 \neq \varnothing, \quad Y_1^\alpha \cap Z_1 \neq \varnothing;$$

(8) $\alpha = (Y_7^\alpha \times Z_7) \cup Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, *where*

$$Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha \supseteq \varnothing, \quad Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5, \quad Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3, \quad Y_7^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \supseteq Z_4,$$
$$Y_5^\alpha \cap Z_5 \neq \varnothing, \quad Y_3^\alpha \cap Z_3 \neq \varnothing, \quad Y_4^\alpha \cap Z_4 \neq \varnothing, \quad Y_0^\alpha \cap \breve{D} \neq \varnothing;$$

(9) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T\cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$, *where*

$$\varnothing \neq T, T'', \quad T \setminus T' \neq \varnothing, \quad T' \setminus T \neq \varnothing, \quad T \cup T' \subset T'', \quad Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha \cup Y_T^\alpha \supseteq T, \quad Y_7^\alpha \cup Y_{T'}^\alpha \supseteq T', \quad Y_T^\alpha \cap T \neq \varnothing, \quad Y_{T'}^\alpha \cap T' \neq \varnothing, \quad Y_{T''}^\alpha \cap T'' \neq \varnothing;$$

(10) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_T^\alpha \times T) \cup (Y_0^\alpha \times \breve{D})$, *where*

$$T \in \{Z_2, Z_1\}, \quad Y_6^\alpha, Y_5^\alpha, Y_T^\alpha, Y_0^\alpha \notin \{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6, \quad Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5, \quad Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_T^\alpha \supseteq T,$$
$$Y_6^\alpha \cap Z_6 \neq \varnothing, \quad Y_5^\alpha \cap Z_5 \neq \varnothing, \quad Y_T^\alpha \cap T \neq \varnothing, \quad Y_0^\alpha \cap \breve{D} \neq \varnothing;$$

(11) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_{T\cup T'}^\alpha\times(T\cup T'))\cup(Y_{T''}^\alpha\times T'')\cup(Y_{T\cup T'\cup T''}^\alpha\times(T\cup T'\cup T''))$, *where*

$$\varnothing\neq T,T',\quad T\setminus T'\neq\varnothing,\quad T'\setminus T\neq\varnothing,\quad T\subset T'',$$
$$(T\cup T')\setminus T''\neq\varnothing,\quad T''\setminus(T\cup T')\neq\varnothing,\quad Y_T^\alpha,Y_{T'}^\alpha,Y_{T''}^\alpha,Y_{T\cup T'\cup T''}^\alpha\notin\{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha\cup Y_T^\alpha\supseteq T,\quad Y_7^\alpha\cup Y_{T'}^\alpha\supseteq T',\quad Y_7^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T'',$$
$$Y_T^\alpha\cap T\neq\varnothing,\quad Y_{T'}^\alpha\cap T'\neq\varnothing,\quad Y_{T''}^\alpha\cap T''\neq\varnothing;$$

(12) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, *where*

$$Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha\supseteq\varnothing,\quad Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5,\quad Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3,$$
$$Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4,\quad Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1,$$
$$Y_5^\alpha\cap Z_5\neq\varnothing,\quad Y_3^\alpha\cap Z_3\neq\varnothing,\quad Y_4^\alpha\cap Z_4\neq\varnothing,\quad Y_1^\alpha\cap Z_1\neq\varnothing;$$

(13) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_6^\alpha\times Z_6)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, *where*

$$Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5,\quad Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6,\quad Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3,$$
$$Y_5^\alpha\cap Z_5\neq\varnothing,\quad Y_6^\alpha\cap Z_6\neq\varnothing,\quad Y_3^\alpha\cap Z_3\neq\varnothing,\quad Y_0^\alpha\cap\breve{D}\neq\varnothing;$$

(14) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_6^\alpha\times Z_6)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, *where*

$$Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6,\quad Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5,\quad Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2,$$
$$Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1,\quad Y_6^\alpha\cap Z_6\neq\varnothing,\quad Y_5^\alpha\cap Z_5\neq\varnothing,$$
$$Y_2^\alpha\cap Z_2\neq\varnothing,\quad Y_1^\alpha\cap Z_1\neq\varnothing;$$

(15) $\alpha=(Y_7^\alpha\times\varnothing)\cup(Y_6^\alpha\times Z_6)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, *where*

$$Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$$

and satisfies the conditions

$$Y_7^\alpha\supseteq Z_7,\quad Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5,\quad Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6,\quad Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3,$$
$$Y_7^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2,\quad Y_5^\alpha\cap Z_5\neq\varnothing,\quad Y_6^\alpha\cap Z_6\neq\varnothing,\quad Y_3^\alpha\cap Z_3\neq\varnothing,\quad Y_2^\alpha\cap Z_2\neq\varnothing.$$

Lemma 2. *Let $D\in\Sigma_3(X,8)$ and $Z_7=\varnothing$. If X is a finite set, then the number $|I^*(Q_1)|$ is calculated by the formula $|I^*(Q_1)|=1$.*

Lemma 3. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_2)|$ is calculated by the formula*

$$|I^*(Q_2)| = (2^{|\breve{D}|}-1)\cdot 2^{|X\setminus\breve{D}|} + (2^{|Z_6|}-1)\cdot 2^{|X\setminus Z_6|} + (2^{|Z_5|}-1)\cdot 2^{|X\setminus Z_5|} + (2^{|Z_4|}-1)\cdot 2^{|X\setminus Z_4|} + \\ + (2^{|Z_3|}-1)\cdot 2^{|X\setminus Z_3|} + (2^{|Z_2|}-1)\cdot 2^{|X\setminus Z_2|} + (2^{|Z_1|}-1)\cdot 2^{|X\setminus Z_1|}.$$

Lemma 4. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_3)|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_3)| = {} & (2^{|Z_1|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_2|}-1)\cdot(3^{|\breve{D}\setminus Z_2|}-2^{|\breve{D}\setminus Z_2|})\cdot 3^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_3|}-1)\cdot(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_4|}-1)\cdot(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|})\cdot 3^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|\breve{D}\setminus Z_5|}-2^{|\breve{D}\setminus Z_5|})\cdot 3^{|X\setminus\breve{D}|} + (2^{|Z_6|}-1)\cdot(3^{|\breve{D}\setminus Z_6|}-2^{|\breve{D}\setminus Z_6|})\cdot 3^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_6|}-1)\cdot(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|})\cdot 3^{|X\setminus Z_4|} + (2^{|Z_6|}-1)\cdot(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|})\cdot 3^{|X\setminus Z_2|}\\
&+ (2^{|Z_6|}-1)\cdot(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|})\cdot 3^{|X\setminus Z_1|} + (2^{|Z_5|}-1)\cdot(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|})\cdot 3^{|X\setminus Z_4|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|})\cdot 3^{|X\setminus Z_3|} + (2^{|Z_5|}-1)\cdot(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|})\cdot 3^{|X\setminus Z_2|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|})\cdot 3^{|X\setminus Z_1|} + (2^{|Z_4|}-1)\cdot(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|})\cdot 3^{|X\setminus Z_2|}\\
&+ (2^{|Z_4|}-1)\cdot(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|})\cdot 3^{|X\setminus Z_1|} + (2^{|Z_3|}-1)\cdot(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|})\cdot 3^{|X\setminus Z_1|}.\end{aligned}$$

Lemma 5. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_4)|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_4)| = {} & (2^{|Z_6|}-1)\cdot(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|})\cdot(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_6|}-1)\cdot(3^{|Z_2\setminus Z_6|}-2^{|Z_2\setminus Z_6|})\cdot(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_6|}-1)\cdot(3^{|Z_1\setminus Z_6|}-2^{|Z_1\setminus Z_6|})\cdot(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|})\cdot(4^{|\breve{D}\setminus Z_4|}-3^{|\breve{D}\setminus Z_4|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|})\cdot(4^{|\breve{D}\setminus Z_3|}-3^{|\breve{D}\setminus Z_3|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_2\setminus Z_5|}-2^{|Z_2\setminus Z_5|})\cdot(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_1\setminus Z_5|}-2^{|Z_1\setminus Z_5|})\cdot(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_4|}-1)\cdot(3^{|Z_2\setminus Z_4|}-2^{|Z_2\setminus Z_4|})\cdot(4^{|\breve{D}\setminus Z_2|}-3^{|\breve{D}\setminus Z_2|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_4|}-1)\cdot(3^{|Z_1\setminus Z_4|}-2^{|Z_1\setminus Z_4|})\cdot(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_3|}-1)\cdot(3^{|Z_1\setminus Z_3|}-2^{|Z_1\setminus Z_3|})\cdot(4^{|\breve{D}\setminus Z_1|}-3^{|\breve{D}\setminus Z_1|})\cdot 4^{|X\setminus\breve{D}|}\\
&+ (2^{|Z_6|}-1)\cdot(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|})\cdot(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|})\cdot 4^{|X\setminus Z_2|}\\
&+ (2^{|Z_6|}-1)\cdot(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|})\cdot(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|})\cdot 4^{|X\setminus Z_1|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|})\cdot(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|})\cdot 4^{|X\setminus Z_2|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|})\cdot(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|})\cdot 4^{|X\setminus Z_1|}\\
&+ (2^{|Z_5|}-1)\cdot(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|})\cdot(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|})\cdot 4^{|X\setminus Z_1|}.\end{aligned}$$

Lemma 6. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_5)|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_5)| &= (2^{|Z_6|}-1)\cdot(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|})\cdot(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|})\cdot(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_6|}-1)\cdot(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|})\cdot(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|})\cdot(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_5|}-1)\cdot(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|})\cdot(4^{|Z_2\setminus Z_4|}-3^{|Z_2\setminus Z_4|})\cdot(5^{|\breve{D}\setminus Z_2|}-4^{|\breve{D}\setminus Z_2|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_5|}-1)\cdot(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|})\cdot(4^{|Z_1\setminus Z_4|}-3^{|Z_1\setminus Z_4|})\cdot(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_5|}-1)\cdot(3^{|Z_3\setminus Z_5|}-2^{|Z_3\setminus Z_5|})\cdot(4^{|Z_1\setminus Z_3|}-3^{|Z_1\setminus Z_3|})\cdot(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}.\end{aligned}$$

Lemma 7. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_6)|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_6)| &= (2^{|Z_1\setminus Z_2|}-1)\cdot(2^{|Z_2\setminus Z_1|}-1)\cdot 4^{|X\setminus\breve{D}|}+(2^{|Z_5\setminus Z_6|}-1)\cdot(2^{|Z_6\setminus Z_5|}-1)\cdot 4^{|X\setminus Z_4|}\\
&+(2^{|Z_3\setminus Z_6|}-1)\cdot(2^{|Z_6\setminus Z_3|}-1)\cdot 4^{|X\setminus Z_1|}+(2^{|Z_3\setminus Z_4|}-1)\cdot(2^{|Z_4\setminus Z_3|}-1)\cdot 4^{|X\setminus Z_1|}\\
&+(2^{|Z_3\setminus Z_2|}-1)\cdot(2^{|Z_2\setminus Z_3|}-1)\cdot 4^{|X\setminus\breve{D}|}.\end{aligned}$$

Lemma 8. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_7)|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_7)| &= (2^{|Z_4|}-1)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_4|}\cdot(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|})\cdot(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_6|}-1)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_6|}\cdot(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|})\cdot(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_5|}-1)\cdot 2^{|(Z_1\cap Z_2)\setminus Z_5|}\cdot(3^{|Z_1\setminus Z_2|}-2^{|Z_1\setminus Z_2|})\cdot(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_5|}-1)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}\cdot(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|})\cdot(3^{|Z_2\setminus Z_3|}-2^{|Z_2\setminus Z_3|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_5|}-1)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\cdot(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|})\cdot(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|})\cdot 5^{|X\setminus Z_1|}.\end{aligned}$$

Lemma 9. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_8)|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_8)| &= (2^{|Z_6|}-1)\cdot(3^{|Z_4\setminus Z_6|}-2^{|Z_4\setminus Z_6|})\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\\
&\qquad\times(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|})\cdot(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|})\cdot 6^{|X\setminus\breve{D}|}\\
&+(2^{|Z_5|}-1)\cdot(3^{|Z_4\setminus Z_5|}-2^{|Z_4\setminus Z_5|})\cdot 3^{|(Z_1\cap Z_2)\setminus Z_4|}\\
&\qquad\times(4^{|Z_1\setminus Z_2|}-3^{|Z_1\setminus Z_2|})\cdot(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|})\cdot 6^{|X\setminus\breve{D}|}.\end{aligned}$$

Lemma 10. *Let $D \in \Sigma_3(X,8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_9)|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_9)| &= (2^{|Z_5|}-1)\cdot 2^{|(Z_3\cap Z_4)\setminus Z_5|}\\
&\qquad\times(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|})\cdot(3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|})\cdot(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|})\cdot 6^{|X\setminus\breve{D}|}.\end{aligned}$$

Lemma 11. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{10})|$ is calculated by the formula*

$$\begin{aligned}|I^*(Q_{10})| &= (2^{|Z_6\setminus Z_5|}-1)\cdot(2^{|Z_5\setminus Z_6|}-1)\cdot(5^{|\breve{D}\setminus Z_4|}-4^{|\breve{D}\setminus Z_4|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_6\setminus Z_3|}-1)\cdot(2^{|Z_3\setminus Z_6|}-1)\cdot(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}\\
&+(2^{|Z_4\setminus Z_3|}-1)\cdot(2^{|Z_3\setminus Z_4|}-1)\cdot(5^{|\breve{D}\setminus Z_1|}-4^{|\breve{D}\setminus Z_1|})\cdot 5^{|X\setminus\breve{D}|}\end{aligned}$$

$$+ (2^{|Z_6\setminus Z_5|}-1)\cdot(2^{|Z_5\setminus Z_6|}-1)\cdot(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|})\cdot 5^{|X\setminus Z_2|}$$
$$+ (2^{|Z_6\setminus Z_5|}-1)\cdot(2^{|Z_5\setminus Z_6|}-1)\cdot(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|})\cdot 5^{|X\setminus Z_1|}.$$

Lemma 12. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{11})|$ is calculated by the formula*

$$|I^*(Q_{11})| = (2^{|Z_6\setminus Z_5|}-1)\cdot(2^{|Z_5\setminus Z_6|}-1)\cdot(5^{|Z_2\setminus Z_4|}-4^{|Z_2\setminus Z_4|})\cdot(6^{|\breve{D}\setminus Z_2|}-5^{|\breve{D}\setminus Z_2|})\cdot 6^{|X\setminus \breve{D}|}$$
$$+ (2^{|Z_6\setminus Z_5|}-1)\cdot(2^{|Z_5\setminus Z_6|}-1)\cdot(5^{|Z_1\setminus Z_4|}-4^{|Z_1\setminus Z_4|})\cdot(6^{|\breve{D}\setminus Z_1|}-5^{|\breve{D}\setminus Z_1|})\cdot 6^{|X\setminus \breve{D}|}.$$

Lemma 13. *Let $D \in \Sigma_3(X,8)$ and $Z_7 \neq \varnothing$. If X is a finite set, then the number $|I^*(Q_{12})|$ is be calculated by the formula*

$$|I^*(Q_{12})| = (2^{|Z_6\setminus Z_3|}-1)\cdot(2^{|Z_5\setminus Z_6|}-1)\cdot(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|})\cdot 6^{|X\setminus Z_1|}$$
$$+ (2^{|Z_3\setminus Z_2|}-1)\cdot(2^{|Z_6\setminus Z_3|}-1)\cdot(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|})\cdot 6^{|X\setminus \breve{D}|}$$
$$+ (2^{|Z_3\setminus Z_2|}-1)\cdot(2^{|Z_4\setminus Z_3|}-1)\cdot(3^{|Z_2\setminus Z_1|}-2^{|Z_2\setminus Z_1|})\cdot 6^{|X\setminus \breve{D}|}.$$

Lemma 14. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{13})|$ is calculated by the formula*

$$|I^*(Q_{13})| = (2^{|Z_5|}-1)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_5|}$$
$$\times (3^{|Z_4\setminus Z_3|}-2^{|Z_4\setminus Z_3|})\cdot(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|})\cdot(4^{|Z_2\setminus Z_1|}-3^{|Z_2\setminus Z_1|})\cdot 7^{|X\setminus \breve{D}|}.$$

Lemma 15. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{14})|$ is calculated by the formula*

$$|I^*(Q_{14})| = (2^{|Z_5\setminus Z_6|}-1)\cdot(2^{|Z_6\setminus Z_3|}-1)\cdot(3^{|Z_3\setminus Z_4|}-2^{|Z_3\setminus Z_4|})\cdot(7^{|\breve{D}\ setminus Z_1|}-6^{|\breve{D}\setminus Z_1|})\cdot 7^{|X\setminus \breve{D}|}.$$

Lemma 16. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{15})|$ is calculated by the formula*

$$|I^*(Q_{15})| = (2^{|Z_5\setminus Z_6|}-1)\cdot(2^{|Z_6\setminus Z_5|}-1)\cdot 4^{|(Z_1\cap Z_2)\setminus Z_4|}$$
$$\times (5^{|Z_2\setminus Z_1|}-4^{|Z_2\setminus Z_1|})\cdot(5^{|Z_1\setminus Z_2|}-4^{|Z_1\setminus Z_2|})\cdot 7^{|X\setminus \breve{D}|}.$$

Lemma 17. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I^*(Q_{16})|$ is calculated by the formula*

$$|I^*(Q_{16})| = (2^{|Z_6\setminus Z_3|}-1)\cdot 2^{|(Z_3\cap Z_2)\setminus Z_4|}$$
$$\times (2^{|Z_5\setminus Z_6|}-1)\cdot(3^{|Z_3\setminus Z_2|}-2^{|Z_3\setminus Z_2|})\cdot(5^{|Z_2\setminus Z_1|}-4^{|Z_2\setminus Z_1|})\cdot 8^{|X\setminus \breve{D}|}.$$

Theorem 2. *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. If X is a finite set, then the number $|I(D)|$ is calculated by the formula*

$$|I(D)| = \sum_{i=1}^{16} |I^*(Q_i)|.$$

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Table 1.

No.	Set X	Semilattice D	Number of elements of the semigroup $B_X(D)$	Number of idempotent elements of the semigroup $B_X(D)$
1	$X = \{1, 2, 3, 4\}$	$D = \Big\{\{1,2,3,4\}, \{2,3,4\}, \{1,3,4\}, \{2,4\}, \{3,4\}, \{4\}, \{3\}, \{\varnothing\}\Big\}$	4096	448

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