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BOOLEAN-VALUED ANALYSIS OF ORDER-BOUNDED OPERATORS

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Abstract. This is a survey of some recent applications of Boolean-valued models of set theory to the study of order-bounded operators in vector lattices.

Introduction

The term *Boolean-valued analysis* signifies the technique of studying properties of an arbitrary mathematical object by comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, we usually take the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean-valued universe in which the conventional set-theoretic concepts and propositions acquire bizarre interpretations. Use of two models for studying a single object is a family feature of the so-called *nonstandard methods of analysis*. For this reason, Boolean-valued analysis means an instance of nonstandard analysis in common parlance.

Proliferation of Boolean-valued analysis stems from the celebrated achievement of P. J. Cohen who proved at the beginning of the 1960s that the negation of the continuum hypothesis, CH, is consistent with the axioms of Zermelo–Fraenkel set theory, ZFC. This result by Cohen, together with the consistency of CH with ZFC established earlier by K. Gödel, proves that CH is independent of the conventional axioms of ZFC.

The first applications of Boolean-valued models to functional analysis were given by E. I. Gordon for Dedekind complete vector lattices and positive operators in [22–24] and G. Takeuti for self-adjoint operators in Hilbert spaces and harmonic analysis in [72–74]. The further developments and corresponding references are presented in [46, 47].

The aim of the paper is to survey some recent applications of Boolean-valued models of set theory to the study of order-bounded operators in vector lattices. Section 1 contains a sketch of the adaptation of the main constructions and principles of Boolean-valued models of set theory to analysis. The three subsequent sections treat the classes of operators in vector lattices: multiplication type operators, weighted shift type operators, and conditional expectation type operators.

The reader can find the necessary information on Boolean algebras in [68, 76], on the theory of vector lattices, in [10, 34, 39, 77, 80], on Boolean-valued models of set theory, in [11, 32, 75], and on Boolean-valued analysis, in [45–47].

Everywhere below, $\mathbb{B}$ denotes a complete Boolean algebra, while $\mathbb{V}^{(\mathbb{B})}$ stands for the corresponding Boolean-valued universe (the universe of $\mathbb{B}$-valued sets). A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ with $\bigvee\limits_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = 0$ for $\xi \neq \eta$.

By a vector lattice throughout the sequel we will mean a real Archimedean vector lattice, unless specified otherwise. We let $:=$ denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ symbolize the natural numbers, the rationals, the reals, and the complex numbers. We denote the Boolean algebras of bands and band projections in a vector lattice X by $\mathbb{B}(X)$ and $\mathbb{P}(X)$; and we let X^{u} stand for the universal completion of a vector lattice X.

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The ideal center $\mathscr{Z}(X)$ of a vector lattice X is an f-algebra. Let $\mathrm{Orth}(X)$ and $\mathrm{Orth}^\infty(X)$ stand for the f-algebras of orthomorphisms and extended orthomorphisms, respectively. Then $\mathscr{Z}(X) \subset \mathrm{Orth}(X) \subset \mathrm{Orth}^\infty(X)$. The space of all order-bounded linear operators from X to Y is denoted by $L^\sim(X,Y)$. The Riesz–Kantorovich theorem tells us that if Y is a Dedekind complete vector lattice, then so is $L^\sim(X,Y)$.

1. Boolean-Valued Analysis

1.1. Boolean-Valued Models. We start by recalling some auxiliary facts about the construction and treatment of Boolean-valued models. Some more detailed presentation can be found in [11, 46, 47]. In the sequel, ZFC := ZF + AC, where ZF stands for the Zermelo–Fraenkel set theory and AC for the axiom of choice. A formula of the language of the Zermelo–Fraenkel set theory is referred to as "a formula of ZFC" and a formula of ZFC provable in ZFC is phrased as "a theorem of ZFC."

1.1.1. Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$\mathbb{V}_\alpha^{(\mathbb{B})} := \big\{x\colon x \text{ is a function} \wedge (\exists\,\beta)\big(\beta < \alpha \wedge \mathrm{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})} \wedge \mathrm{Im}(x) \subset \mathbb{B}\big)\big\}.$$

After this recursive definition, the *Boolean-valued universe* $\mathbb{V}^{(\mathbb{B})}$ or, in other words, the *class of $\mathbb{B}$-sets* is introduced by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In the case of the two-element Boolean algebra $\mathbb{2} := \{0, \mathbb{1}\}$, this procedure yields a version of the classical *von Neumann universe*

$$\mathbb{V} := \bigcup_{\alpha\in\mathrm{On}},$$

where

$$\mathbb{V}_0 := \varnothing, \quad \mathbb{V}_{\alpha+1} := \mathscr{P}(\mathbb{V}_\alpha), \quad \mathbb{V}_\beta := \bigcup_{\alpha<\beta} \mathbb{V}_\alpha,$$

β is a limit ordinal (see [47, Theorem 4.2.8]).

1.1.2. Let $\varphi(u_1,\dots,u_n)$ be an arbitrary formula of ZFC. Then for arbitrary $x_1,\dots,x_n \in \mathbb{V}^{(\mathbb{B})}$ *Boolean truth value* $[\![\varphi(x_1,\dots,x_n)]\!] \in \mathbb{B}$ is introduced by induction on the complexity of φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ (for instance, $[\![\forall\, x\ \varphi(x)]\!] := \bigwedge\big\{[\![\varphi(x)]\!]\colon x \in \mathbb{V}^{(\mathbb{B})}\big\}$ and $[\![\varphi_1 \vee \varphi_2]\!] := [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$) and taking into consideration the way in which a formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$ (with x, y assumed to be elements of $\mathbb{V}^{(\mathbb{B})}$) are defined by means of the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t\in\mathrm{dom}(y)} (y(t) \wedge [\![t = x]\!]), \quad [\![x = y]\!] = \bigvee_{t\in\mathrm{dom}(x)} (x(t) \Rightarrow [\![t \in y]\!]) \wedge \bigvee_{t\in\mathrm{dom}(y)} (y(t) \Rightarrow [\![t \in x]\!]).$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i.e., $(a \Rightarrow b) := (a^* \vee b)$, where a^* is as usual the *complement* of a in $\mathbb{B}$. The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following is fulfilled.

1.1.3. The transfer principle. *Whenever $\varphi(u_1,\dots,u_n)$ is a theorem of* ZFC, *then*

$$(\forall x_1,\dots,x_n \in \mathbb{V}^{(\mathbb{B})})[\![\varphi(x_1,\dots,x_n)]\!] = \mathbb{1}$$

is also a theorem of ZFC. *This is also phrased by saying that $\mathbb{V}^{(\mathbb{B})}$ is a Boolean-valued model of* ZFC *or, in short, $\mathbb{V}^{(\mathbb{B})} \models$* ZFC.

We enter into the next agreement. If $\varphi(x)$ is a formula of ZFC, then, on assuming x to be an element of $\mathbb{V}^{(\mathbb{B})}$, the phrase "$x$ satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$" or, briefly, "$\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$" means that $[\![\varphi(x)]\!] = \mathbb{1}$. This is sometimes written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

1.1.4. There is a natural equivalence relation $x \sim y \iff [\![x = y]\!] = \mathbb{1}$ in the class $\mathbb{V}^{(\mathbb{B})}$. Choosing a representative of the least rank in each equivalence class or, more exactly, using the so-called "Frege–Russell–Scott trick," we obtain a *separated Boolean-valued universe* $\bar{\mathbb{V}}^{(\mathbb{B})}$ for which $x = y \iff [\![x = y]\!] = \mathbb{1}$. It is easy to see that the Boolean truth value of a formula remains unaltered if we replace in it each element of $\mathbb{V}^{(\mathbb{B})}$ by one of its equivalents (see [47, Sec. 4.5]). In this connection, from now on we take $\mathbb{V}^{(\mathbb{B})} := \bar{\mathbb{V}}^{(\mathbb{B})}$ without further specification.

1.1.5. Given $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, define the function $bx\colon z \mapsto b \wedge x(z)$ $(z \in \mathrm{dom}(x))$. Here we presume that $b\varnothing := \varnothing$ for all $b \in \mathbb{B}$. Observe that in $\mathbb{V}^{(\mathbb{B})}$ the element bx is defined correctly for $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$ (see [47, Sec. 4.3]).

1.1.6. The mixing principle. *Let* $(b_\xi)_{\xi\in\Xi}$ *be a* partition of unity *in* $\mathbb{B}$, *i.e.,* $\sup\limits_{\xi\in\Xi} b_\xi = \mathbb{1}$ *and* $\xi \neq \eta \implies b_\xi \wedge b_\eta = 0$. *For each family* $(x_\xi)_{\xi\in\Xi}$ *in* $\mathbb{V}^{(\mathbb{B})}$ *there exists a unique element* x *in* $\mathbb{V}^{(\mathbb{B})}$ *such that* $[\![x = x_\xi]\!] \geq b_\xi$ *for all* $\xi \in \Xi$.

This x is called the *mixing* of $(x_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ and is denoted by $\mathrm{mix}_{\xi\in\Xi}\, b_\xi x_\xi$.

1.1.7. The maximum principle. *For a formula* $\varphi(u_0, u_1, \dots, u_n)$ *of* ZFC *the following is a theorem of* ZFC: *for every collection* $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ *there exists* $x_0 \in \mathbb{V}^{(\mathbb{B})}$ *satisfying*

$$[\![(\exists\, x)\ \varphi(x, x_1, \dots, x_n)]\!] = [\![\varphi(x_0, x_1, \dots, x_n)]\!].$$

In particular, if it is true within $\mathbb{V}^{(\mathbb{B})}$ that "there is an x for which $\varphi(x)$," then there is an element x_0 in $\mathbb{V}^{(\mathbb{B})}$ (in the sense of $\mathbb{V}$) with $[\![\varphi(x_0)]\!] = \mathbb{1}$. In symbols,

$$\big(\mathbb{V}^{(\mathbb{B})} \models (\exists\, x)\ \varphi(x)\big) \implies \big((\exists\, x_0)\ \mathbb{V}^{(\mathbb{B})} \models \varphi(x_0)\big).$$

1.2. Escher Rules. Now, we present a remarkable interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$, which is based on the operations of canonical embedding, descent, and ascent.

1.2.1. We start with the canonical embedding of the von Neumann universe into the Boolean-valued universe. Given $x \in \mathbb{V}$, we denote by $x^\wedge$ the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$, i.e., the element defined by the following recursion schema:

$$\varnothing^\wedge := \varnothing, \quad \mathrm{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \mathrm{Im}(x^\wedge) := \{\mathbb{1}\}.$$

Henceforth, working in the separated universe $\bar{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol $x^\wedge$ for the distinguished element of the class corresponding to x. The map $x \mapsto x^\wedge$ is called *canonical embedding.*

A formula is *bounded* or *restricted* provided that each bound variable in it is restricted by a bounded quantifier, i.e., a quantifier ranging over a particular set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall\, x \in y)$ or $(\exists\, x \in y)$.

1.2.2. The restricted transfer principle. *Let* $\varphi(u_1, \dots, u_n)$ *be a bounded formula of* ZFC. *Then the following is also a theorem of* ZFC: *for every collection* $x_1, \dots, x_n \in \mathbb{V}$ *the equivalence*

$$\varphi(x_1, \dots, x_n) \iff \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge)$$

holds.

1.2.3. Given an arbitrary element x of the Boolean-valued universe $\mathbb{V}^{(\mathbb{B})}$, define the class $x{\downarrow}$ by

$$x{\downarrow} := \{y \in \mathbb{V}^{(\mathbb{B})} : [\![y \in x]\!] = \mathbb{1}\}.$$

This class is called the *descent* of x. Moreover, $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbb{V}$ for every element $x \in \mathbb{V}^{(\mathbb{B})}$. If $[\![x \neq \varnothing]\!] = \mathbb{1}$, then $x{\downarrow}$ is a nonempty set.

1.2.4. Suppose that f is a map from X to Y within $\mathbb{V}^{(\mathbb{B})}$. More precisely, f, X, and Y are in $\mathbb{V}^{(\mathbb{B})}$ and $[\![f\colon X \to Y]\!] = \mathbb{1}$. There exist a unique map $f{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ (in the sense of the von Neumann universe $\mathbb{V}$) such that

$$[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1} \quad (x \in X{\downarrow}).$$

Moreover, for a nonempty subset A of X within $\mathbb{V}^{(\mathbb{B})}$ (i.e., $[\![\varnothing \neq A \subset X]\!] = \mathbb{1}$) we have $f{\downarrow}(A{\downarrow}) = f(A){\downarrow}$. The map $f{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ is called the *descent* of f from $\mathbb{V}^{(\mathbb{B})}$. The descent $f{\downarrow}$ of every internal map f is *extensional*:

$$[\![x = x']\!] \leq [\![f{\downarrow}(x) = f{\downarrow}(x')]\!] \quad (x, x' \in X{\downarrow}).$$

For the descents of composite, inverse, and identity map we have

$$(g \circ f){\downarrow} = g{\downarrow} \circ f{\downarrow}, \quad (f^{-1}){\downarrow} = (f{\downarrow})^{-1}, \quad (I_X){\downarrow} = I_{X{\downarrow}}.$$

By virtue of these rules, we can consider the descent operation as a functor from the category of $\mathbb{B}$-valued sets and mappings to the category of the standard sets and mappings (i.e., those in the sense of $\mathbb{V}$).

1.2.5. Given $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$, denote by $(x_1, \dots, x_n)^{\mathbb{B}}$ the corresponding ordered n-tuple inside $\mathbb{V}^{(\mathbb{B})}$. Assume that P is an n-ary relation on X inside $\mathbb{V}^{(\mathbb{B})}$, i.e., $X, P \in \mathbb{V}^{(\mathbb{B})}$ and $[\![P \subset X^{n^{\wedge}}]\!] = \mathbb{1}$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that $(x_1, \dots, x_n) \in P' \iff [\![(x_1, \dots, x_n)^{\mathbb{B}} \in P]\!] = \mathbb{1}$. Slightly abusing notation, we denote P' by the occupied symbol $P{\downarrow}$ and call $P{\downarrow}$ the *descent* of P.

1.2.6. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$, i.e., let x be some set composed of $\mathbb{B}$-valued sets or, symbolically, $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\varnothing{\uparrow} := \varnothing$ and $\operatorname{dom}(x{\uparrow}) := x$, $\operatorname{Im}(x{\uparrow}) := \{\mathbb{1}\}$ if $x \neq \varnothing$. The element $x{\uparrow}$ (of the nonseparated universe $\bar{\mathbb{V}}^{(\mathbb{B})}$, i.e., the distinguished representative of the class $\{y \in \bar{\mathbb{V}}^{(\mathbb{B})} : [\![y = x{\uparrow}]\!] = \mathbb{1}\}$) is the *ascent* of x. For the corresponding element in the separated universe $\mathbb{V}^{(\mathbb{B})}$ the same name and notation are preserved.

1.2.7. Let $X, Y, f \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$ and let f be a mapping from X to Y. There exists a mapping $f{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ within $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![f{\uparrow}(x) = f(x)]\!] = \mathbb{1} \quad (x \in X)$$

if and only if f is *extensional*, i.e., the relation

$$[\![x = x']\!] \leq [\![f(x) = f(x')]\!] \quad (x, x' \in X)$$

holds. The map $f{\uparrow}$ with the above property is unique and satisfies the relation $f{\uparrow}(A{\uparrow}) = f(A){\uparrow}$ $(A \subset X)$. The composite of extensional maps is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents inside $\mathbb{V}^{(\mathbb{B})}$:

$$\mathbb{V}^{(\mathbb{B})} \models (g \circ f){\uparrow} = g{\uparrow} \circ f{\uparrow}.$$

Observe also that if f and f^{-1} are extensional then $(f{\uparrow})^{-1} = (f^{-1}){\uparrow}$.

1.2.8. Suppose that $X \in \mathbb{V}$, $X \neq \varnothing$, i.e., X is a nonempty set. Let $\iota := \iota_X$ denote the standard name embedding $x \mapsto x^{\wedge}$ $(x \in X)$. Then $\iota(X){\uparrow} = X^{\wedge}$ and $X = \iota^{-1}(X^{\wedge}{\downarrow})$. Take $Y \in \mathbb{V}^{(\mathbb{B})}$ with $[\![Y \neq \varnothing]\!] = \mathbb{1}$. Using the above relations, we can extend the ascent operation to the case of a map f from X to $Y{\downarrow}$ and descent operation to the case of an internal map g from $X^{\wedge}$ to Y, i.e., $[\![g\colon X^{\wedge} \to Y]\!] = \mathbb{1}$.

The maps $f{\upharpoonleft} := (f \circ \iota^{-1}){\uparrow}$ and $g{\downharpoonright} := (g{\downarrow}) \circ \iota$ are called *modified ascent* of f and *modified descent* of g, respectively. (Sometimes, when there is no ambiguity, we speak of ascents and descents, using simple arrows.) It is easy to see that $g{\downharpoonright}$ is the unique map from X to $Y{\downarrow}$ satisfying

$$[\![g{\downharpoonright}(x) = g(x^{\wedge})]\!] = \mathbb{1} \quad (x \in X)$$

and $f{\upharpoonleft}$ is the unique map from $X^{\wedge}$ to Y within $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![f{\upharpoonleft}(x^{\wedge}) = f(x)]\!] = \mathbb{1} \quad (x \in X).$$

1.2.9. Given $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote by $\operatorname{mix}(X)$ the set of all mixtures of the form $\operatorname{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following assertions are referred to as the *rules for canceling arrows* or the *Escher rules.* Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f\colon X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![Y, Y' \neq \varnothing]\!] = [\![g\colon Y \to Y']\!] = \mathbb{1}$. Then

$$X{\uparrow}{\downarrow} = \operatorname{mix}(X), \quad Y{\downarrow}{\uparrow} = Y;$$
$$f{\uparrow}{\downarrow} = f, \quad g{\downarrow}{\uparrow} = g.$$

There are some other cancellation rules.

1.3. Boolean-Valued Reals and Vector Lattices. The main results of the section tells us that the Boolean-valued interpretation of the field of reals (complex numbers) is a real (complex) universally complete vector lattice. Everywhere below, $\mathbb{B}$ is a complete Boolean algebra and $\mathbb{V}^{(\mathbb{B})}$ is the corresponding Boolean-valued universe.

1.3.1. By virtue of the transfer and maximum principles, there exists an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\mathscr{R}$ is a field of reals$]\!] = \mathbb{1}$. Note also that $\varphi(x)$, formally presenting the expressions of the axioms of an Archimedean ordered field x, is bounded; therefore, by the restricted transfer principle $[\![\varphi(\mathbb{R}^\wedge)]\!] = \mathbb{1}$, i.e., $[\![\mathbb{R}^\wedge$ is an Archimedean ordered field$]\!] = \mathbb{1}$. Thus, we will assume that $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$, while the elements $0 := 0^\wedge$ and $1 := 1^\wedge$ are the zero and unity of $\mathscr{R}$ within the model $\mathbb{V}^{(\mathbb{B})}$.

1.3.2. Let Ⓡ be the underlying set of the field $\mathscr{R}$, on which the addition $\oplus$, multiplication $\otimes$, and ordering $\circledleq$ are given. Then $\mathscr{R}$ is a 6-tuple $(Ⓡ, \oplus, \otimes, \circledleq, 0^\wedge, 1^\wedge)$ within $\mathbb{V}^{(\mathbb{B})}$; in symbols, $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = (Ⓡ, \oplus, \otimes, \circledleq, 0^\wedge, 1^\wedge)$.

The descent $\mathscr{R}{\downarrow}$ of the field $\mathscr{R}$ is the descent of the underlying set $\mathbf{R} := Ⓡ{\downarrow}$ together with the descended operations $+ := \oplus{\downarrow}$, $\cdot := \odot{\downarrow}$, order relation $\leqslant := \circledleq{\downarrow}$, and distinguished elements $0 := 0^\wedge$ and $\mathbb{1} := 1^\wedge$; in symbols, $\mathscr{R}{\downarrow} = (\mathbf{R}, +, \cdot, \leqslant, 0, \mathbb{1})$. Also, we can introduce multiplication by the standard reals in $\mathscr{R}{\downarrow}$ by the rule

$$y = \lambda x \iff [\![y = \lambda^\wedge \odot x]\!] = \mathbb{1} \quad (\lambda \in \mathbb{R},\ x, y \in \mathbf{R}).$$

1.3.3. The Gordon theorem. *Let $\mathscr{R}$ be the reals within $\mathbb{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$ (with the descended operations and order) is a universally complete vector lattice with a weak order unit $\mathbb{1}$. Moreover, there exists a Boolean isomorphism χ of $\mathbb{B}$ onto the Boolean algebra of projection $\mathbb{P}(\mathscr{R}{\downarrow})$ such that for all $x, y \in \mathscr{R}{\downarrow}$ and $b \in \mathbb{B}$ we have*

$$\chi(b)x = \chi(b)y \iff b \leq [\![x = y]\!], \quad \chi(b)x \leq \chi(b)y \iff b \leq [\![x \leq y]\!]. \tag{G}$$

1.3.4. A vector lattice is an *f-algebra* if it is simultaneously a real algebra and satisfies, for all $a, x, y \in X_+$, the following conditions:

(1) $x \geq 0$ and $y \geq 0$ imply $xy \geq 0$;
(2) $x \perp y = 0$ implies that $(ax) \perp y$ and $(xa) \perp y$.

The multiplication in every (Archimedean) f-algebra is commutative and associative. An f-algebra is called *semi-prime* if $xy = 0$ implies $x \perp y$ for all x and y. The universally complete vector lattice $\mathscr{R}{\downarrow}$ with the descended multiplications is a semiprime f-algebra with ring unit $\mathbb{1} = 1^\wedge$.

1.3.5. By the maximum principle, there is an element $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ for which

$$[\![\mathscr{C} \text{ is the set of complex numbers}]\!] = \mathbb{1}.$$

Since the equality $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ is expressed by a bounded set-theoretic formula, from the restricted transfer principle we obtain

$$[\![\mathbb{C}^\wedge = \mathbb{R}^\wedge \oplus i^\wedge \mathbb{R}^\wedge]\!] = \mathbb{1}.$$

Moreover, $\mathbb{R}^\wedge$ is assumed to be a dense subfield of $\mathscr{R}$; therefore, we can also assume that $\mathbb{C}^\wedge$ is a dense subfield of $\mathscr{C}$. If 1 is the unity of $\mathbb{C}$, then $1^\wedge$ is the unity of $\mathscr{C}$ inside $\mathbb{V}^{(\mathbb{B})}$. We write i instead of $i^\wedge$ and

$\mathbb{1}$ instead of $1^\wedge$. By the Gordon theorem $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$; consequently, $\mathscr{C}{\downarrow}$ is a universally complete complex vector lattice, i.e., the complexification of a vector lattice $\mathscr{R}{\downarrow}$. Moreover, $\mathscr{C}{\downarrow}$ is a complex f-algebra defined as the complexification of a real f-algebra with a ring unit $\mathbb{1} := 1^\wedge$.

1.3.6. Let A be an f-algebra. A vector lattice X is said to be an *f-module over A* if the following holds:

(1) X is a module over A (with respect to a multiplication $A \times X \ni (a, x) \mapsto ax \in X$);
(2) $ax \geq 0$ for all $a \in A_+$ and $x \in X_+$;
(3) $x \perp y$ implies $ax \perp y$ for all $a \in A_+$ and $x, y \in X$.

A vector lattice X has a natural f-module structure over $\mathrm{Orth}(X)$, i.e., $\pi x := \pi(x)$ for all $x \in X$ and $\pi \in \mathrm{Orth}(X)$. Clearly, X is an f-module over an arbitrary f-submodule $A \subset \mathrm{Orth}(X)$ and, in particular, over $\mathscr{Z}(X)$.

1.3.7. Theorem. *Let X be an f-module over $\mathscr{Z}(Y)$ with Y a Dedekind complete vector lattice and $\mathbb{B} = \mathbb{P}(Y)$. Then there exists $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ such that* $[\![\mathscr{X}$ is a vector lattice over $\mathscr{R}]\!] = \mathbb{1}$, *$\mathscr{X}{\downarrow}$ is an f-module over A^{u}, and there is an f-module isomorphism h from X to $\mathscr{X}{\downarrow}$ satisfying* $\mathscr{X}{\downarrow} = \mathrm{mix}\big(h(X)\big)$.

1.3.8. The Gordon theorem was established in [22]. The concept of an f-module was introduced in [57].

1.4. Boolean-Valued Functionals. We will demonstrate in this section how Boolean-valued analysis works by transferring some results from order-bounded functionals to operators. Below, X and Y stand for vector lattices, where Y is an order dense sublattice in $\mathscr{R}{\downarrow}$.

1.4.1. Let $\mathbb{B}$ be a complete Boolean algebra and let $\mathscr{R}$ be the field of reals in $\mathbb{V}^{(\mathbb{B})}$. The fact that X is a vector lattice over the ordered field $\mathbb{R}$ can be rewritten as a restricted formula, say, $\varphi(X, \mathbb{R})$. Hence, recalling the restricted transfer principle, we come to the identity $[\![\varphi(X^\wedge, \mathbb{R}^\wedge)]\!] = \mathbb{1}$, which amounts to saying that $X^\wedge$ is a vector lattice over the ordered field $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$.

Let $X^{\wedge\sim} := L^\sim(X^\wedge, \mathscr{R})$ be the space of order-bounded $\mathbb{R}^\wedge$-linear functionals from $X^\wedge$ to $\mathscr{R}$. More precisely, $\mathscr{R}$ is considered as a vector space over the field $\mathbb{R}^\wedge$ and by the maximum principle there exists $X^{\wedge\sim} \in \mathbb{V}^{(\mathbb{B})}$ such that

$$[\![X^{\wedge\sim} \text{ is a vector space over } \mathscr{R} \text{ of } \mathbb{R}^\wedge\text{-linear order-bounded functionals from } X^\wedge \text{ to } \mathscr{R} \text{ ordered by the cone of positive functionals}]\!] = \mathbb{1}.$$

A functional $\tau \in X^{\wedge\sim}$ is positive if $[\![(\forall\, x \in X^\wedge)\ \tau(x) \geq 0]\!] = \mathbb{1}$.

1.4.2. It can easily be seen that the Riesz–Kantorovich theorem remains true if X is a vector lattice over a dense subfield $\mathbb{P} \subset \mathbb{R}$ and Y is a Dedekind complete vector lattices (over $\mathbb{R}$) and $L^\sim(X, Y)$ is replaced by $L^\sim_{\mathbb{P}}(X, Y)$, the vector spaces over $\mathbb{R}$ of all $\mathbb{P}$-linear order-bounded operators from X to Y, ordered by the cone of positive operators: $L^\sim_{\mathbb{P}}(X, Y)$ is a Dedekind complete vector lattice. The Boolean-valued interpretation of this fact yields that $X^{\wedge\sim} := L^\sim_{\mathbb{R}^\wedge}(X^\wedge, \mathbb{R})$ is a Dedekind complete vector lattice within $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} := \mathbb{P}(Y)$. In particular, the descent $X^{\wedge\sim}{\downarrow}$ of the space $X^{\wedge\sim}$ is a Dedekind complete vector lattice. Let $L^\sim_{dp}(X, Y)$ and $\mathrm{Hom}(X, T)$ stand respectively for the space of disjointness preserving order-bounded operators and the set of all lattice homomorphisms from X to Y. The following is based on the construction from 1.2.8.

1.4.3. Theorem. *Let X and Y be vector lattices with Y universally complete and presented as $Y = \mathscr{R}{\downarrow}$. Given $T \in L^\sim(X, Y)$, the modified ascent $T{\uparrow}$ is an order-bounded $\mathbb{R}^\wedge$-linear functional on $X^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$, i.e., $[\![T{\uparrow} \in X^{\wedge\sim}]\!] = \mathbb{1}$. The mapping $T \mapsto T{\uparrow}$ is a lattice isomorphism between the Dedekind complete vector lattices $L^\sim(X, Y)$ and $X^{\wedge\sim}{\downarrow}$.*

1.4.4. Corollary. *Given operators $R, S \in L^\sim(X, Y)$, put $\sigma := S{\uparrow}$ and $\tau := T{\uparrow}$. The following are true:*

(1) $S \leq T \iff [\![\sigma \leq \tau]\!] = \mathbb{1}$;
(2) $S = |T| \iff [\![\sigma = |\tau|]\!] = \mathbb{1}$;
(3) $S \perp T \iff [\![\sigma \perp \tau]\!] = \mathbb{1}$;

(4) $[\![T \in \mathrm{Hom}(X,Y)]\!] \iff [\![\tau \in \mathrm{Hom}(X^{\wedge},\mathscr{R})]\!] = \mathbb{1}$;
(5) $T \in L^{\sim}_{dp}(X,Y) \iff [\![\tau \in (X^{\wedge\sim})_{dp}]\!] = \mathbb{1}$.

1.4.5. Consider a vector lattice X, and let D be an order ideal in X. A linear operator T from D into X is *band preserving* provided that $x \perp y$ implies $Tx \perp y$ for all $x \in D$ and $y \in X$, or, equivalently, $Tx \in \{x\}^{\perp\perp}$ for all $x \in D$ (the disjoint complements are taken in X). If X is a vector lattice with the principal projection property and $D \subset X$ is an order dense ideal, then a linear operator $T\colon D \to X$ is band preserving if and only if T commutes with band projections: $\pi Tx = T\pi x$ for all $\pi \in \mathbb{P}(X)$ and $x \in D$.

1.4.6. Let $\mathrm{End}_N(X_{\mathbb{C}})$ be the set of all band preserving endomorphisms of $X_{\mathbb{C}}$ with $X := \mathscr{R}{\downarrow}$. Clearly, $\mathrm{End}_N(X_{\mathbb{C}})$ is a complex vector space. Moreover, $\mathrm{End}_N(X_{\mathbb{C}})$ becomes a faithful unitary module over the ring $X_{\mathbb{C}}$ on letting gT be equal to $gT\colon x \mapsto g{\cdot}Tx$ for all $x \in X_{\mathbb{C}}$. This is immediate since the multiplication by an element of $X_{\mathbb{C}}$ is band preserving and the composite of band preserving operators is band preserving too.

1.4.7. By $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C})$ we denote the element of $\mathbb{V}^{(\mathbb{B})}$ that represents the space of all $\mathbb{C}^{\wedge}$-linear operators from $\mathscr{C}$ into $\mathscr{C}$. Then $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C})$ is a vector space over $\mathscr{C}$ inside $\mathbb{V}^{(\mathbb{B})}$, and $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$ is a faithful unitary module over a complex f-algebra $X_{\mathbb{C}}$.

1.4.8. Proposition. *A linear operator T on a universally complete vector lattice X or $X_{\mathbb{C}}$ is band preserving if and only if T is extensional.*

Proof. Take a linear operator $T\colon X \to X$. By the Gordon theorem the extensionality condition $[\![x = y]\!] \le [\![Tx = Ty]\!]$ $(x, y \in X = \mathscr{R}{\downarrow})$ amounts to saying that the identity $\pi x = \pi y$ implies $\pi Tx = \pi Ty$ for all $x, y \in X$ and $\pi \in \mathbb{P}(X)$. By linearity of T the latter is equivalent to $\pi x = 0 \implies \pi Tx = 0$ $(x \in X, \pi \in \mathbb{P}(X))$. Substituting $y := \pi^{\perp} y$ yields $\pi T \pi^{\perp} = 0$ or, which is the same, $\pi T = \pi T \pi$. According to 1.4.5, T is band preserving. The complex case is treated by complexification. □

1.4.9. Theorem. *The modules $\mathrm{End}_N(X_{\mathbb{C}})$ and $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$ are isomorphic. The isomorphy can be established by sending a band preserving operator to its ascent. The same remains true when $\mathscr{C}$ and $\mathbb{C}$ are replaced by $\mathscr{R}$ and $\mathbb{R}$, respectively.*

Proof. By virtue of Proposition 1.4.8, we can apply the constructions of 1.2.4 and 1.2.7, as well as the cancellation rules 1.2.9. □

2. Band Preserving Operators

2.1. Wickstead's Problem and Cauchy's Functional Equation. In this section, we demonstrate that the band preserving operators in universally complete vector lattices are solutions in disguise of the Cauchy functional equation and the Wickstead problem amounts to that of regularity of all solutions to the equation.

2.1.1. The Wickstead problem. When are we so happy in a vector lattice that all band preserving linear operators turn out to be order-bounded?

This question was raised by Wickstead in [79]. Further progress is presented in [4, 5, 28, 41, 42, 62]. The approach combining logical, algebraical, and analytical tools was presented in [41–43]. See a survey of the main ideas and results on the problem and its modifications in [30].

The answer depends on the vector lattice in which the operator in question acts. Therefore, the problem can be reformulated as follows: Characterize the vector lattices in which every band preserving linear operators is order-bounded.

Let X be a universally complete vector lattice, and let T be a band preserving linear operator in X. By the Gordon theorem we can assume that $X = \mathscr{R}{\downarrow}$, where $\mathscr{R}$ is the field of reals within $\mathbb{V}^{(\mathbb{B})}$ and

$\mathbb{B} = \mathbb{P}(X)$. Moreover, according to Theorem 1.4.9, we can assume further that $T = \tau{\downarrow}$, where $\tau \in \mathbb{V}^{(\mathbb{B})}$ is an internal $\mathbb{R}^{\wedge}$-linear function from $\mathscr{R}$ to $\mathscr{R}$. It can easily be seen that T is order-bounded if and only if

$$[\![\tau \text{ is order-bounded (i.e., } \tau \text{ is bounded on every intervals } [a,b] \subset \mathscr{R})]\!] = \mathbb{1}.$$

2.1.2. By $\mathbb{F}$ we denote either $\mathbb{R}$ or $\mathbb{C}$. The *Cauchy functional equation* with unknown function $f\colon \mathbb{F} \to \mathbb{F}$ has the form

$$f(x+y) = f(x) + f(y) \quad (x, y \in \mathbb{F}).$$

It is easy that a solution to the equation is automatically $\mathbb{Q}$-homogeneous, i.e., it satisfies another functional equation:

$$f(qx) = qf(x) \quad (q \in \mathbb{Q},\ x \in \mathbb{F}).$$

In the sequel we will be interested in a more general situation. Namely, we will consider the simultaneous functional equations

$$\begin{cases} f(x+y) = f(x) + f(y) & (x, y \in \mathbb{F}), \\ f(px) = pf(x) & (p \in \mathbb{P},\ x \in \mathbb{F}), \end{cases} \tag{L}$$

where $\mathbb{P}$ is a subfield of $\mathbb{F}$ that includes $\mathbb{Q}$. Denote by $\mathbb{F}_{\mathbb{P}}$ the field $\mathbb{F}$, which is considered as a vector space over $\mathbb{P}$. Clearly, solutions to the simultaneous equations (L) are precisely $\mathbb{P}$-linear functions from $\mathbb{F}_{\mathbb{P}}$ to $\mathbb{F}_{\mathbb{P}}$.

2.1.3. Let $\mathscr{E}$ be a Hamel basis for a vector space $\mathbb{F}_{\mathbb{P}}$, and let $\mathscr{F}(\mathscr{E}, \mathbb{F})$ be the space of all functions from $\mathscr{E}$ to $\mathbb{F}$. The solution set of (L) is a vector space over $\mathbb{F}$ isomorphic with $\mathscr{F}(\mathscr{E}, \mathbb{F})$. Such an isomorphism can be implemented by sending a solution f to the restriction $f|_{\mathscr{E}}$ of f to $\mathscr{E}$. The inverse isomorphism $\varphi \mapsto f_\varphi$ ($\varphi \in \mathscr{F}(\mathscr{E}, \mathbb{F})$) is defined by

$$f_\varphi(x) := \sum_{e \in \mathscr{E}} \varphi(e)\psi(e) \quad (x \in \mathbb{F}_{\mathbb{P}}),$$

where $x = \sum_{e \in \mathscr{E}} \psi(e)e$ is the expansion of x with respect to Hamel basis $\mathscr{E}$.

2.1.4. Theorem. *Each solution of* (L) *is either* $\mathbb{F}$*-linear or everywhere dense in* $\mathbb{F}^2 := \mathbb{F} \times \mathbb{F}$. *In particular,* f_φ *is continuous if and only if* $\varphi(e)/e = \text{const}$ $(e \in \mathscr{E})$.

2.1.5. Now assume that $\mathbb{F} = \mathbb{C}$ and $\mathbb{P} := \mathbb{P}_0 + i\mathbb{P}_0$ with $\mathbb{P}_0$ a subfield in $\mathbb{R}$. Then the space of solutions of the system (L) is a complexification of the space of solution of the same system with $\mathbb{P} := \mathbb{P}_0$. In more detail, if $g\colon \mathbb{R} \to \mathbb{R}$ is a $\mathbb{P}_0$-linear function, then we have the unique $\mathbb{P}$-linear function $\tilde{g}\colon \mathbb{C} \to \mathbb{C}$ defined as

$$\tilde{g}(z) = g(x) + ig(y) \quad (z = x + iy \in \mathbb{C}).$$

Conversely, if $f\colon \mathbb{C} \to \mathbb{C}$ is a $\mathbb{P}$-linear function, then there is a unique pair of $\mathbb{P}_0$-linear functions $g_1, g_2\colon \mathbb{R} \to \mathbb{R}$ such that $f(z) = \tilde{g}_1(z) + i\tilde{g}_2(z)$ $(z \in \mathbb{C})$. Thus, every solution f of (L) can be represented in the form $f = f_1 + if_2$, where $f_1, f_2\colon \mathbb{C} \to \mathbb{C}$ are $\mathbb{P}_0$-linear and $f_i(\mathbb{R}) \subset \mathbb{R}$ $(i = 1, 2)$. We say that f is *monotone* or *bounded* if so are f_1 and f_2.

2.1.6. Proposition. *Let* $\mathbb{P}$ *be a subfield of* $\mathbb{F}$*, while* $\mathbb{P} := \mathbb{P}_0 + i\mathbb{P}_0$ *for some dense subfield* $\mathbb{P}_0 \subset \mathbb{R}$*, in case* $\mathbb{F} = \mathbb{C}$*. The following are equivalent*:

(1) $\mathbb{F} = \mathbb{P}$;

(2) *every solution to* (L) *is order-bounded.*

Proof. The implication (1) $\Longrightarrow$ (2) is trivial. Prove the converse by way of contradiction. The assumption that $\mathbb{F} \neq \mathbb{P}$ implies that each Hamel basis $\mathscr{E}$ for the vector space $\mathbb{F}_{\mathbb{P}}$ contains at least two nonzero distinct elements $e_1, e_2 \in \mathscr{E}$. Define the function $\psi\colon \mathscr{E} \to \mathbb{F}$ so that $\psi(e_1)/e_1 \neq \psi(e_2)/e_2$. Then the $\mathbb{P}$-linear function $f = f_\psi\colon \mathbb{F} \to \mathbb{F}$, coinciding with ψ on $\mathscr{E}$, would exist by 2.1.2 and be discontinuous by Theorem 2.1.4. $\square$

2.1.7. Add to the system (L) the equation $f(xy) = f(x)f(y)$ (or $f(xy) = f(x)y + xf(y)$) $(x, y \in \mathbb{F})$. A solution of the resulting system is called $\mathbb{P}$-*endomorphism* ($\mathbb{P}$-*derivation*). The existence of the nontrivial $\mathbb{P}$-endomorphism and $\mathbb{P}$-derivation can be obtained similarly, but using a transcendental basis instead of a Hamel basis (see [6, 35]). Interpreting such existence results in a Boolean-valued model yields the existence of band preserving endomorphism and derivations of a universally complete f-algebra (see [42, 43], as well as [47]).

2.2. Locally One-Dimensional Vector Lattices. Boolean-valued representation of a vector lattice is a vector sublattice in $\mathscr{R}$ considered as a vector lattice over $\mathbb{R}^\wedge$. It stands to reason to find out what construction in a vector lattice corresponds to a Hamel basis for its Boolean-valued representation.

2.2.1. A vector lattice X is said to have a *cofinal family of band projections* if for each nonzero band B in X there exists a nonzero band projection π on X such that $\pi(X) \subset B$ (see [2]).

Let X be a vector lattice with a cofinal family of band projections. We will say that $x, y \in X$ *differ* at $\pi \in \mathbb{P}(X)$ provided that $\pi|x - y|$ is a weak order unit in $\pi(X)$ or, equivalently, if $\pi(X) \subset |x - y|^{\perp\perp}$. Clearly, x and y differ at π whenever $\rho x = \rho y$ implies $\pi\rho = 0$ for all $\rho \in \mathbb{P}(X)$. A subset $\mathscr{E}$ of X is said to be *locally linearly independent* provided that, for an arbitrary nonzero band projection π in X and each collection of the elements $e_1, \dots, e_n \in \mathscr{E}$ that are pairwise different at π, and each collection of reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, the condition $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$ implies that $\lambda_k = 0$ for all $k := 1, \dots, n$. In other words, $\mathscr{E}$ is locally linearly independent if for every band projection $\pi \in \mathbb{P}(X)$ any subset of $\pi(\mathscr{E})$ consisting of nonzero members pairwise different at π is linearly independent.

An inclusion-maximal locally linearly independent subset of X is called a *local Hamel basis* for X.

2.2.2. Proposition. *Each vector lattice X with a cofinal family of band projections has a local Hamel basis for X.*

2.2.3. A locally linearly independent set $\mathscr{E}$ in G is a local Hamel basis if and only if for every $x \in G$ there exist a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{P}(G)$ and a family of reals $(\lambda_{\xi,e})_{\xi\in\Xi,\, e\in\mathscr{E}}$ such that

$$x = \text{o-}\sum_{\xi\in\Xi}\Bigg(\sum_{e\in\mathscr{E}} \lambda_{\xi,e}\pi_\xi e\Bigg)$$

and for every $\xi \in \Xi$ the set $\{e \in \mathscr{E} : \lambda_{\xi,e} \neq 0\}$ is finite and consists of nonzero elements pairwise different at π_ξ. Moreover, the representation is unique up to refinements of the partition of unity (see [2, Sec. 6; 39, Sec. 5.1]). The following result of [37, Proposition 4.6(1)] explains why and how the concept of local Hamel basis is such a useful technical tool (see [2]).

2.2.4. Proposition. *Assume that $\mathscr{E}, \mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$, $[\![\mathscr{E} \subset \mathscr{X}]\!] = \mathbb{1}$, $[\![\mathscr{X}$ is a vector subspace of $\mathscr{R}_\mathbb{R}]\!] = \mathbb{1}$, and $X := \mathscr{X}{\downarrow}$. Then*

$$[\![\mathscr{E} \text{ is a Hamel basis for the vector space } \mathscr{X} \text{ (over } \mathbb{R}^\wedge)]\!] = \mathbb{1}$$

if and only if $\mathscr{E}{\downarrow}$ is a local Hamel basis for X.

2.2.5. A vector lattice X is said to be *locally one-dimensional* if for any two nondisjoint $x_1, x_2 \in X$ there exist nonzero components u_1 and u_2 of x_1 and x_2, respectively, such that u_1 and u_2 are proportional (see [2, Definition 11.1]). Equivalent definitions see in [39, Proposition 5.1.2].

2.2.6. Proposition. *Let X be a laterally complete vector lattice, and let $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ be its Boolean-valued representation with $\mathbb{B} := \mathbb{P}(X)$. Then X is locally one-dimensional if and only if $\mathscr{X}$ is one-dimensional vector lattice over $\mathbb{R}^\wedge$ in $\mathbb{V}^{(\mathbb{B})}$, i.e., $[\![\mathscr{R} = \mathbb{R}^\wedge]\!] = \mathbb{1}$.*

2.2.7. Proposition. *A universally complete vector lattice is locally one-dimensional if and only if every band preserving linear operator in it is order-bounded.*

Proof. By the Gordon theorem we can assume that $X = \mathscr{R}{\downarrow}$ with $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} \simeq \mathbb{P}(X)$. Thus, the problem reduces to existence of a discontinuous solution to the Cauchy functional equation (L) and the claim follows from Proposition 2.1.6. □

2.2.8. Proposition. *Let $\mathbb{R}$ is a transcendental extension of a subfield $\mathbb{P} \subset \mathbb{R}$. There exists an $\mathbb{P}$-linear subspace $\mathscr{X}$ in $\mathbb{R}$ such that $\mathscr{X}$ and $\mathbb{R}$ are isomorphic vector spaces over $\mathbb{P}$ but they are not isomorphic as ordered vector spaces over $\mathbb{P}$.*

Proof. Let $\mathscr{E}$ be a Hamel basis of a $\mathbb{P}$-vector space $\mathbb{R}$. Since $\mathscr{E}$ is infinite, we can choose a proper subset $\mathscr{E}_0 \subsetneq \mathscr{E}$ of the same cardinality: $|\mathscr{E}_0| = |\mathscr{E}|$. If $\mathscr{X}$ denotes the $\mathbb{P}$-subspace of $\mathbb{R}$ generated by $\mathscr{E}_0$, then $\mathscr{X}_0 \subsetneq \mathbb{R}$ and $\mathscr{X}$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{P}$. If $\mathscr{X}$ and $\mathbb{R}$ were isomorphic as ordered vector spaces over $\mathbb{P}$, then $\mathscr{X}$ would be order complete and, in consequence, we would have $\mathscr{X} = \mathbb{R}$, a contradiction. □

2.2.9. Theorem. *Let X be a nonlocally one-dimensional universally complete vector lattice. Then there exist a vector sublattice $X_0 \subset X$ and a band preserving linear bijection $T\colon X_0 \to X$ such that T^{-1} is also band preserving but X_0 and X are not lattice isomorphic.*

Proof. We can assume, without loss of generality, that $X = \mathscr{R}{\downarrow}$ and $[\![\mathscr{R} \neq \mathbb{R}^{\wedge}]\!] = \mathbb{1}$. By Proposition 2.2.8 there exist an $\mathbb{R}^{\wedge}$-linear subspace $\mathscr{X}$ in $\mathscr{R}$ and $\mathbb{R}^{\wedge}$-linear isomorphism τ from $\mathscr{X}$ onto $\mathscr{R}$, while $\mathscr{X}$ and $\mathscr{R}$ are not isomorphic as ordered vector spaces over $\mathbb{R}^{\wedge}$. Put $X_0 := \mathscr{X}{\downarrow}$, $T := \tau{\downarrow}$, and $S := \tau^{-1}{\downarrow}$. The maps S and T are band preserving and linear. Moreover, $S = (\tau{\downarrow})^{-1} = T^{-1}$. It remains to observe that X_0 and X are lattice isomorphic if and only if $\mathscr{X}$ and $\mathscr{R}$ are isomorphic as ordered vector spaces. □

2.2.10. Let γ be a cardinal. A vector lattice X is said to be *Hamel γ-homogeneous* whenever there exists a local Hamel basis of cardinality γ in X consisting of strongly distinct weak order units. Two elements $x, y \in X$ are said to be *strongly distinct* if $|x - y|$ is a weak order unit in X.

2.2.11. Proposition. *Let X be a universally complete vector lattice. There is a band X_0 in X such that $X_0^{\perp}$ is locally one-dimensional and there exists a partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(X_0)$ with Γ a set of infinite cardinals such that $\pi_\gamma X_0$ is Hamel γ-homogeneous for all $\gamma \in \Gamma$.*

2.2.12. A local Hamel basis is also called a *d-basis*. This concept stems from [17], but for the first time in the context of disjointness preserving operators in [4, 5]. Various aspects of the concept can be found in [2, 3]. Theorem 2.2.6 was established in [28], while Proposition 2.2.7 in [5, 62]. Theorem 2.2.9 was proved in [3] not involving Boolean-valued approach. Theorem 2.2.11 was never published.

2.3. Algebraic Band Preserving Operators. In this section, some description of algebraic orthomorphisms on a vector lattice is given and the Wickstead problem for algebraic operators is examined.

2.3.1. Let $\mathbb{P}[x]$ be a ring of polynomials in variable x over a field $\mathbb{P}$. An operator T on a vector space X over a field $\mathbb{P}$ is said to be *algebraic* if there exists a nonzero $\varphi \in \mathbb{P}[x]$, a polynomial with coefficients in $\mathbb{P}$, for which $\varphi(T) = 0$.

For an algebraic operator T, there exists a unique polynomial φ_T such that $\varphi_T(T) = 0$, the leading coefficient of φ_T equals to 1, and φ_T divides each polynomial ψ with $\psi(T) = 0$. The polynomial φ_T is called the *minimal polynomial* of T. The simple examples of algebraic operators yield a projection P (an idempotent operator, $P^2 = P$) in X with $\varphi_P(\lambda) = \lambda^2 - \lambda$ whenever $P \neq 0, I_X$, and a nilpotent operator S ($S^m = 0$ for some $m \in \mathbb{N}$) in X with $\varphi_S(\lambda) = \lambda^k$, $k \leq m$.

For an operator T on X, the set of all eigenvalues of T will be denoted throughout by $\sigma_p(T)$. A real number λ is a root of φ_T if and only if $\lambda \in \sigma_p(T)$. In particular, $\sigma_p(T)$ is finite. If $b - a^2 > 0$ for some $a, b \in \mathbb{R}$, then $T^2 + 2aT + bI$ is a weak order unit in $\mathrm{Orth}(X)$ for every $T \in \mathrm{Orth}(X)$ (see [14]).

2.3.2. Proposition. *Let X be a vector lattice, and let T in $\mathrm{Orth}(X)$ be algebraic. Then*

$$\varphi_T(x) = \prod_{\lambda\in\sigma_p(T)} (x - \lambda).$$

2.3.3. Proposition. *Consider the universally complete vector lattice $X = \mathscr{R}{\downarrow}$. Let T be a band preserving linear operator on X and τ an $\mathbb{R}^{\wedge}$-linear function on $\mathscr{R}$. For $\varphi \in \mathbb{R}[x]$, $\varphi(x) = a_0 + a_1x + \dots + a_nx^n$, define $\hat{\varphi} \in \mathbb{R}^{\wedge}[x]$ by $\hat{\varphi}(x) = a_0^{\wedge} + a_1^{\wedge}x + \dots + a_n^{\wedge}x^{n^{\wedge}}$. Then*

$$\hat{\varphi}(\tau){\downarrow} = \varphi(\tau{\downarrow}), \quad \varphi(T){\uparrow} = \hat{\varphi}(T{\uparrow}).$$

Proof. It follows from 1.2.4 and 1.2.7 that $(\tau^{n^\wedge})\!\downarrow = (\tau\!\downarrow)^n$ and $(T^n)\!\uparrow = (T\!\uparrow)^{n^\wedge}$. Thus, it remains to apply Theorem 1.4.9. □

2.3.4. A linear operator T on a vector lattice X is said to be *diagonal* if $T = \lambda_1 P_1 + \dots + \lambda_m P_m$ for some collections of reals $\lambda_1, \dots, \lambda_m$ and projection operators $P_1, \dots, P_m$ on X with $P_i \circ P_j = 0$ $(i \neq j)$. In the equality above, we can and will assume that $P_1 + \dots + P_n = I_X$ and that $\lambda_1, \dots, \lambda_m$ are pairwise different. An algebraic operator T is diagonal if and only if the minimal polynomial of T has the form $\varphi_T(x) = (x - \lambda_1) \cdot \ldots \cdot (x - \lambda_m)$ with pairwise different $\lambda_1, \dots, \lambda_m \in \mathbb{R}$.

We call an operator T on X *strongly diagonal* if there exist pairwise disjoint band projections $P_1, \dots, P_m$ and real numbers $\lambda_1, \dots, \lambda_m$ such that $T = \lambda_1 P_1 + \dots + \lambda_m P_m$. In particular, every strongly diagonal operator on X is an orthomorphism.

2.3.5. Proposition. *Let $T = \lambda_1 P_1 + \dots + \lambda_m P_m$ be a diagonal operator on a vector lattice X. Then T is band preserving if and only if the projection operators $P_1, \dots, P_m$ are band preserving.*

Proof. The sufficiency is obvious. To prove the necessity, observe first that if T is band preserving, then so is T^n for all $n \in \mathbb{N}$ and thus $\varphi(T)$ is band preserving for every polynomial $\varphi \in \mathbb{R}[x]$. Next, make use of the representation $P_j = \varphi_j(T)$ $(j := 1, \dots, m)$, where $\varphi_j \in \mathbb{R}[x]$ is an interpolation polynomial defined by $\varphi_j(\lambda_k) = \delta_{jk}$ with δ_{jk} the Kronecker symbol. □

2.3.6. Theorem. *Let X be a universally complete vector lattice. The following are equivalent*:

(1) *the Boolean algebra $\mathbb{P}(X)$ is σ-distributive*;
(2) *every algebraic band preserving operator in X is order-bounded*;
(3) *every algebraic band preserving operator in X is strongly diagonal*;
(4) *every band preserving diagonal operator in X is strongly diagonal*;
(5) *every band preserving nilpotent operator in X is order-bounded*;
(6) *every band preserving nilpotent operator in X is trivial.*

Proof. The only nontrivial implications are (2) $\Longrightarrow$ (3) and (6) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) We have to prove that an algebraic orthomorphism on X is strongly diagonal. Let T be an orthomorphism in X and $\varphi(T) = 0$, where φ is a minimal polynomial of T, so that $\varphi(\lambda) = (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_m)$ with $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Since T admits a unique extension to an orthomorphism on X^{u}, we can assume without loss of generality that $X = X^{\mathrm{u}} = \mathscr{R}\!\downarrow$ and $\tau = T\!\uparrow$. Then $[\![\tau(x) = \lambda_0 x \ (x \in \mathscr{R})]\!] = \mathbb{1}$ for some $\lambda_0 \in \mathscr{R}$. It is seen from Proposition 2.3.3 that $\hat{\varphi}(\lambda_0) = 0$ and thus $(\lambda_0 - \lambda_1^\wedge) \cdot \ldots \cdot (\lambda_0 - \lambda_m^\wedge) = 0$ or $\lambda_0 \in \{\lambda_1^\wedge, \dots, \lambda_m^\wedge\}$ within $\mathbb{V}^{(\mathbb{B})}$. Put $P_l := \chi(b_l)$ with $b_l := [\![\lambda_0 = \lambda_l^\wedge]\!]$ and observe that $\{P_1, \dots, P_m\}$ is a partition of unity in $\mathbb{P}(X)$. Moreover, given $x \in X$, we see that $b_l \leq [\![Tx = \tau x = \lambda_0 x]\!] \wedge [\![\lambda_0 = \lambda_l^\wedge]\!] \leq [\![Tx = \lambda_l^\wedge x]\!]$, so that $P_l T x = P_l(\lambda_l x) = \lambda_l P_l(x)$. Summing up over $l = 1, \dots, m$, we get $Tx = \lambda_1 P_1 x + \dots + \lambda_m P_m$.

(6) $\Longrightarrow$ (1) Arguing for a contradiction, assume that assertion (2) of Theorem 2.3.6 is fulfilled and construct a nonzero band preserving nilpotent operator in X. By Propositions 2.2.7 and 2.1.6 we have that $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} \neq \mathbb{R}^\wedge$ and thus $\mathscr{R}$ is an infinite-dimensional vector space over $\mathbb{R}^\wedge$ within $\mathbb{V}^{(\mathbb{B})}$. Let $\mathscr{E} \subset \mathscr{R}$ be a Hamel basis and choose an infinite sequence $(e_n)_{n\in\mathbb{N}}$ of pairwise distinct elements in $\mathscr{E}$. Fix a natural number $m > 1$ and define an $\mathbb{R}^\wedge$-linear function $\tau \colon \mathscr{R} \to \mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$ by letting $\tau(e_{km+i}) = e_{km+i-1}$ if $2 \leq i \leq m$, $\tau(e_{km+1}) = 0$ for all $k := 0, 1, \dots$, and $\tau(e) = 0$ if $e \neq e_n$ for all $n \in \mathbb{N}$. In other words, if $\mathscr{R}_0$ is the $\mathbb{R}^\wedge$-linear subspace of $\mathscr{R}$ generated by the sequence $(e_n)_{n\in\mathbb{N}}$, then $\mathscr{R}_0$ is an invariant subspace for τ and τ is the linear operator associated with the infinite block matrix $\operatorname{diag}(A, \dots, A, \dots)$ with equal blocks in the principal diagonal and A the Jordan block of size m with eigenvalue 0. It follows that τ is discontinuous and $\tau^m = 0$ by construction. Consequently, $T := \tau\!\downarrow$ is a band preserving linear operator in X and $T^m = 0$ by Proposition 2.3.3, but T is not order-bounded; a contradiction. □

2.3.7. Algebraic order-bounded disjointness preserving operators in vector lattices were treated in [14], where, in particular, the Propositions 2.3.2 and 2.3.5 were proved. Theorem 2.3.6 was obtained in [48].

2.4. Involutions and Complex Structures. The main result of this section tells us that in a real non-locally one-dimensional universally complete vector lattice there are band preserving complex structures and nontrivial band preserving involutions.

2.4.1. A linear operator T on a vector lattice X is called *involutory* or an *involution* if $T \circ T = I_X$ (or, equivalently, $T^{-1} = T$) and is called a *complex structure* if $T \circ T = -I_X$ (or, equivalently, $T^{-1} = -T$). The operator $P - P^{\perp}$, where P is a projection operator on X and $P^{\perp} = I_X - P$, is an involution. The involution $P - P^{\perp}$ with band projections P is referred to as *trivial*.

2.4.2. Proposition. *Let X be a Dedekind complete vector lattice. Then there is no order-bounded band preserving the complex structure in X and there is no nontrivial order-bounded band preserving involution in X.*

Proof. An order-bounded band preserving operator T on a universally complete vector lattice X with weak unit $\mathbb{1}$ is a multiplication operator: $Tx = ax$ $(x \in X)$ for some $a \in X$. It follows that T is an involution if and only if $a^2 = \mathbb{1}$ and hence there is a band projection P on E with $a = P\mathbb{1} - P^{\perp}\mathbb{1}$ or $T = P - P^{\perp}$. If T is a complex structure on E, then the corresponding equation $a^2 = -\mathbb{1}$ has no solution. □

2.4.3. Theorem. *Let $\mathbb{F}$ be a dense subfield of $\mathbb{R}$, and let $B \subset \mathbb{R}$ be a nonempty finite or countable set. Then there exists a discontinuous $\mathbb{F}$-linear function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ f = I_{\mathbb{R}}$ and $f(x) = x$ for all $x \in B$.*

Proof. Let $\mathscr{E} \subset \mathbb{R}$ be a Hamel basis of $\mathscr{R}$ over $\mathbb{R}^{\wedge}$. Every $x \in B$ can be written as $x = \sum\limits_{e \in \mathscr{E}} \lambda_e(x)e$, where $\lambda_e(x) \in \mathbb{F}$ for all $e \in \mathscr{E}$. Put $\mathscr{E}(x) := \{e \in \mathscr{E} \colon \lambda_e(x) \neq 0\}$ and $\mathscr{E}_0 = \bigcup\limits_{x \in B} \mathscr{E}(x)$. Since B is finite or countable, so is also $\mathscr{E}_0$. Hence $\mathscr{E} \setminus \mathscr{E}_0$ has the cardinality of continuum. There exists a decomposition $\mathscr{E} \setminus \mathscr{E}_0 := \mathscr{E}_1 \cup \mathscr{E}_2$, where $\mathscr{E}_1$ and $\mathscr{E}_2$ are disjoint sets both having the same cardinality. Hence there exists a one-to-one mapping g_0 from $\mathscr{E}_1$ onto $\mathscr{E}_2$ with the inverse $g_0^{-1} \colon \mathscr{E}_2 \to \mathscr{E}_1$.

Now we define the function $g\colon \mathscr{E} \to \mathscr{E}$ as follows:

$$g(h) = \begin{cases} g_0(h) & \text{for } h \in \mathscr{E}_1, \\ g_0^{-1}(h) & \text{for } h \in \mathscr{E}_2, \\ h & \text{for } h \in \mathscr{E}_0. \end{cases} \tag{1}$$

It can easily be checked that the $\mathbb{F}$-linear extension $f\colon \mathbb{R} \to \mathbb{R}$ of a function g is the sought involution. □

2.4.4. Theorem. *Let $\mathbb{F}$ be a dense subfield of $\mathbb{R}$. Then there exists a discontinuous $\mathbb{F}$-linear function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ f = -I_{\mathbb{R}}$.*

Proof. The proof is similar to that of Theorem 2.4.3 with the minor modifications: put $\mathscr{E}_0 = \varnothing$ and define

$$g(h) = \begin{cases} -g_0(h) & \text{for } h \in \mathscr{E}_1, \\ g_0^{-1}(h) & \text{for } h \in \mathscr{E}_2. \end{cases}$$ □

Interpreting Theorems 2.4.3 and 2.4.4 in a Boolean-valued model yields the result.

2.4.5. Theorem. *Let X be a universally complete real vector lattice that is not locally one-dimensional. Then*

(1) *for every nonempty finite or countable set $B \subset X$ there exists a band preserving involution T on X with $T(x) = x$ for all $x \in B$;*
(2) *there exists a band preserving the complex structure on X.*

Proof. Assume that $X = \mathscr{R}{\downarrow}$. Take a one-to-one function $\nu\colon \mathbb{N} \to X$ with $B = \operatorname{Im}(\nu)$. The function $\nu{\uparrow}\colon \mathbb{N}^{\wedge} \to X$ can fail to be one-to-one within $\mathbb{V}^{(\mathbb{B})}$ but $B{\uparrow}$ is again finite or countable, as $B{\uparrow} = \operatorname{Im}(\nu{\uparrow})$

by 1.2.7. By Theorem 2.4.3 there exists an $\mathbb{R}^{\wedge}$-linear function $\tau\colon \mathscr{R} \to \mathscr{R}$ such that $[\![\tau \circ \tau = I_{\mathscr{R}}]\!] = \mathbb{1}$ and

$$\mathbb{1} = [\![(\forall\, x \in B{\uparrow})\ \tau(x) = x]\!] = \big[\!\big[(\forall\, n \in \mathbb{N}^{\wedge})\ \tau\big(\nu{\uparrow}(n)\big) = \nu{\uparrow}(n)\big]\!\big]$$
$$= \bigwedge_{n\in\mathbb{N}} \big[\!\big[\tau\big(\nu{\uparrow}(n^{\wedge})\big) = \nu{\uparrow}(n^{\wedge})\big]\!\big] = \bigwedge_{n\in\mathbb{N}} \big[\!\big[\tau\big(\nu(n)\big) = \nu(n)\big]\!\big] = \bigwedge_{n\in\mathbb{N}} \big[\!\big[\tau{\downarrow}\big(\nu(n)\big) = \nu(n)\big]\!\big].$$

It follows that if $T := \tau{\downarrow}$, then $T \circ T = I_X$ by 1.2.4 and $T\big(\nu(n)\big) = \nu(n)$ for all $n \in \mathbb{N}$, as required. The second claim is proved in a similar way, using Theorem 2.4.4. □

2.4.6. Corollary. *Let X be a universally complete vector lattice. Then the following are equivalent*:

(1) *X is locally one-dimensional*;
(2) *there is no nontrivial band preserving involution on X*;
(3) *there is no band preserving the complex structure on X.*

2.4.7. Corollary. *Let X be a universally complete real vector lattice that is not locally one-dimensional. Then X admits a structure of complex vector space with a band preserving complex multiplication.*

Proof. A complex structure T on X allows us to define on X a structure of a vector space over the complex numbers $\mathbb{C}$, by setting $(\alpha + i\beta)x = \alpha x + \beta T(x)$ for all $z = \alpha + i\beta \in \mathbb{C}$ and $x \in X$. If T is band preserving, then the map $x \mapsto zx$ $(x \in X)$ is evidently band preserving for all $z \in \mathbb{C}$. □

2.4.8. The main results of this section were obtained in [49]. In connection with Corollary 2.4.7, we should mention the problem of existence of a complex structure and spaces with few operators (see [18, 25, 26, 70, 71]).

3. Disjointness Preserving Operators

3.1. Characterization and Representation. Now we will demonstrate that some properties of disjointness preserving operators are just Boolean-valued interpretations of elementary properties of disjointness preserving functionals.

3.1.1. Theorem. *Assume that Y has the projection property. An order-bounded linear operator $T\colon X \to Y$ is disjointness preserving if and only if* $\ker(bT)$ *is an order ideal in X for every projection $b \in \mathbb{P}(Y)$.*

Proof. The necessity is obvious, and so only the sufficiency will be proved. Suppose that $\ker(bT)$ is an order ideal in X for every $b \in \mathbb{P}(Y)$. We can assume that $Y \subset \mathscr{R}{\downarrow}$ by the Gordon theorem. Take $|y| \le |x|$ and put $b := [\![Tx = 0]\!]$. Then $bTx = 0$ by (G) and $bTy = 0$ by hypothesis. Again, using (G) we have that $b \le [\![Ty = 0]\!]$. Thus, $[\![Tx = 0]\!] \le [\![Ty = 0]\!]$ or, what is the same, $[\![Tx = 0]\!] \Longrightarrow [\![Ty = 0]\!] = \mathbb{1}$. Now, put $\tau := T{\uparrow}$ and ensure that $\ker(\tau)$ is an order ideal in $X^{\wedge}$ within $\mathbb{V}^{(\mathbb{B})}$. Making use of the fact that $|x| \le |y|$ if and only if $[\![x^{\wedge} \le y^{\wedge}]\!] = \mathbb{1}$, we deduce that

$$[\![\ker(\tau) \text{ is an order ideal in } X^{\wedge}]\!]$$
$$= [\![(\forall\, x, y \in X^{\wedge})\ (\tau(x) = 0 \wedge |y| \le |x| \to \tau(y) = 0)]\!]$$
$$= \bigwedge_{x,y\in X} [\![(\tau(x^{\wedge}) = 0]\!] \wedge [\![|y^{\wedge}| \le |x^{\wedge}|]\!] \Longrightarrow [\![\tau(y^{\wedge}) = 0]\!]$$
$$= \bigwedge \{[\![T(x) = 0]\!] \Longrightarrow [\![T(y) = 0]\!] : x, y \in X,\ |y| \le |x|\} = \mathbb{1}.$$

Apply within $\mathbb{V}^{(\mathbb{B})}$ the fact that a functional τ is disjointness preserving if and only if $\ker(\tau)$ is an order ideal in $X^{\wedge}$. It follows that T is disjointness preserving by Corollary 1.4.4(5). □

3.1.2. A similar reasoning shows that if Y has the projection property, then for an order-bounded disjointness preserving linear operator $T \in L^{\sim}(X, Y)$ there exists a band projection $\pi \in \mathbb{P}(Y)$ such that $T^{+} = \pi|T|$ and $T^{-} = \pi^{\perp}|T|$. In particular, $T = (\pi - \pi^{\perp})|T|$ and $|T| = (\pi - \pi^{\perp})T$. To ensure this, observe that the functional $\tau := T\!\uparrow$ is disjointness preserving if and only if either τ or $-\tau$ is a lattice homomorphism.

From this fact it follows that $T \in L^{\sim}(X,Y)$ is disjointness preserving if and only if $(Tx)^+ \perp (Ty)^-$ for all $x, y \in X_+$. Indeed, given $x, y \in X_+$ we can write $(Tx)^+ = (Tx) \vee 0 \le T^+x = \pi|T|x$ and, similarly, $(Ty)^- \le \pi^{\perp}|T|y$. Hence $(Tx)^+ \wedge (Ty)^- = 0$.

3.1.3. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. For a pair of disjointness preserving operators T_1 and T_2 from X to Y, there exist a band projection $\pi \in \mathbb{P}(Y)$, a lattice homomorphism $T \in \mathrm{Hom}(X,Y)$, and orthomorphisms $S_1, S_2 \in \mathrm{Orth}(Y)$ such that*

$$|S_1| + |S_2| = \pi, \quad \pi T_1 = S_1 T, \quad \pi T_2 = S_2 T,$$
$$\mathrm{Im}(\pi^{\perp}T_1)^{\perp\perp} = \mathrm{Im}(\pi^{\perp}T_2)^{\perp\perp} = \pi(Y), \quad \pi^{\perp}T_1 \perp \pi^{\perp}T_2.$$

Proof. As usual, there is no loss of generality in assuming that $Y = \mathscr{R}\!\downarrow$. Put $\tau_1 := T_1\!\uparrow$ and $\tau_2 := T_2\!\uparrow$. The desired result is a Boolean-valued interpretation of the following fact: If the disjointness preserving functionals τ_1 and τ_2 are not proportional, then they are nonzero and disjoint. Put $b := [\![\tau_1 \text{ and } \tau_2 \text{ are proportional}]\!]$ and $\pi := \chi(b)$. Then within $\mathbb{V}^{([0,b])}$ there exist a lattice homomorphism $\tau\colon X^{\wedge} \to \mathscr{R}$ and reals $\sigma_1, \sigma_2 \in \mathscr{R}$ such that $\tau_i = \sigma_i \tau$. If the function $\bar{\sigma}_i$ is defined as $\bar{\sigma}_i\colon \lambda \mapsto \sigma_i\lambda$ $(\lambda \in \mathscr{R})$, then the operators $S_1 := \bar{\sigma}_1\!\downarrow$, $S_2 := \bar{\sigma}_2\!\downarrow$, and $T := \tau\!\downarrow$ (with the modified descents taken from $\mathbb{V}^{([0,b])}$) satisfy the first line of the required conditions. Moreover, $\pi^{\perp} = \chi(b^*)$ and by transfer we have

$$b^* = [\![\tau_1 \ne 0]\!] \wedge [\![\tau_2 \ne 0]\!] \wedge [\![|\tau_1| \wedge |\tau_2| = 0]\!],$$

so that the second line of required conditions is also satisfied. □

3.1.4. Corollary. *Let X and Y be vector lattices with Y Dedekind complete. The sum $T_1 + T_2$ of two disjointness preserving operators $T_1, T_2\colon X \to Y$ is disjointness preserving if and only if there exist pairwise disjoint band projections $\pi, \pi_1, \pi_2 \in \mathbb{P}(Y)$, orthomorphisms $S_1, S_2 \in \mathrm{Orth}(Y)$, and a lattice homomorphism $T \in \mathrm{Hom}(X,Y)$ such that*

$$\pi + \pi_1 + \pi_2 = I_Y, \quad |S_1| + |S_2| = \pi,$$
$$T(X)^{\perp\perp} = \pi(Y), \quad \pi_1 T_2 = \pi_2 T_1 = 0, \quad \pi T_1 = S_1 T, \quad \pi T_2 = S_2 T.$$

Consequently, in this case $T_1 + T_2 = \pi_1 T_1 + \pi_2 T_2 + (S_1 + S_2)T$.

3.1.5. Corollary. *The sum $T_1 + T_2$ of two disjointness preserving operators $T_1, T_2\colon X \to Y$ is disjointness preserving if and only if $T_1(x_1) \perp T_2(x_2)$ for all $x_1, x_2 \in X$ with $x_1 \perp x_2$.*

Proof. The necessity is immediate from Theorem 3.1.4, since $T_1 = \pi_1 T_1 + S_1 T$ and $T_2 = \pi_2 T_2 + S_2 T$. To see the sufficiency, observe that if T_1 and T_2 meet the above condition, then $T_k x_1 \perp T_l x_2$ $(k, l := 1, 2)$ and so $(T_1 + T_2)(x_1) \perp (T_1 + T_2)(x_2)$ for every pair of disjoint elements $x_1, x_2 \in X$. □

3.1.6. Aspects of the theory of disjointness preserving operators are presented in [29, 39, 40]. Recent results on disjointness preserving operators are surveyed in [13]. In particular, the concept of disjointness preserving set of operators is discussed in the survey. In terms of this concept, Corollary 3.1.5 can be reformulated as follows: *$T_1 + T_2$ is disjointness preserving if and only if $\{T_1, T_2\}$ is a disjointness preserving set of operators* [13, Lemma 5.2].

3.2. Polydisjoint Operators. The aim of the present section is to describe the order ideal in the space of order-bounded operators that is generated by the order bounded disjointness preserving operators (= d-homomorphisms) in terms of n-disjoint operators.

3.2.1. Let X and Y be vector lattices, and let n be a positive integer. A linear operator $T\colon X \to Y$ is said to be *n-disjoint* if, for every collection of $n+1$ pairwise disjoint elements $x_0, \dots, x_n \in X$, the infimum of $\{|Tx_k|\colon k := 0, 1, \dots, n\}$ equals zero; symbolically:

$$(\forall\, x_0, x_1 \dots, x_n \in X)\ x_k \perp x_l\ (k \ne l) \Longrightarrow |Tx_0| \wedge \dots \wedge |Tx_n| = 0.$$

An operator is called *polydisjoint* if it is n-disjoint for some $n \in \mathbb{N}$. A 1-disjoint operator is just a disjointness preserving operator.

3.2.2. Consider some simple properties of n-disjoint operators. Let X and Y be vector lattices with Y Dedekind complete.

(1) $T \in L^{\sim}(X, Y)$ is n-disjoint if and only if $|T|$ is n-disjoint.

(2) Let $T_1, \ldots, T_n$ be order-bounded and disjointness preserving operators from X to Y. Then $T := T_1 + \cdots + T_n$ is n-disjoint.

3.2.3. Proposition. *An order-bounded functional on a vector lattice is n-disjoint if and only if it is representable as a disjoint sum of n order-bounded disjointness preserving functionals. Such representation is unique up to permutation.*

Proof. Assume that f is a positive n-disjoint functional on a vector lattice $C(Q)$. Prove that the corresponding Radon measure μ is the sum of n Dirac measures. This is equivalent to saying that the support of μ consists of n points. If there are $n+1$ points $q_0, q_1, \ldots, q_n \in Q$ in the support of μ, then we can choose pairwise disjoint compact neighborhoods $U_0, U_1, \ldots, U_n \in Q$ of these points and next take pairwise disjoint open sets $V_k \subset Q$ with $\mu(U_k) > 0$ and $U_k \subset V_k$ $(k = 0, 1, \ldots, n)$. Using the Tietze–Urysohn theorem, construct a continuous function x_k on Q that vanishes on $Q \setminus V_k$ and is identically one on U_k. Then $x_0 \wedge x_1 \wedge \cdots \wedge x_n = 0$ but none of the reals $f(x_0), f(x_1), \ldots, f(x_n)$ is equal to zero, since $f(x_k) \geq \mu(U_k) > 0$ for all $k := 0, 1, \ldots, n$. This contradiction shows that the support of μ consists of n points. The general case is reduced to what was proved by using the Kreĭns–Kakutani representation theorem. □

3.2.4. Theorem. *An order-bounded operator from a vector lattice to a Dedekind complete vector lattice is n-disjoint for some $n \in \mathbb{N}$ if and only if it is representable as a disjoint sum of n order-bounded disjointness preserving operators.*

Proof. Assume that an operator $T \in L^{\sim}(X, Y)$ is n-disjoint and denote $\tau := T{\uparrow} \in \mathbb{V}^{(\mathbb{B})}$. It is deduced by direct calculation of Boolean truth values that $\tau \colon X^{\wedge} \to \mathscr{R}$ is an order-bounded n-disjoint functional within $\mathbb{V}^{(\mathbb{B})}$. Using the transfer principle and applying Proposition 3.2.3 to τ yields pairwise disjoint order-bounded disjointness preserving functionals $\tau_1, \ldots, \tau_n$ on $X^{\wedge}$ with $\tau = \tau_1 + \cdots + \tau_n$. It remains to observe that the linear operators $T_1 := \tau_1{\downarrow}, \ldots, T_n := \tau_n{\downarrow}$ from X to Y are order-bounded, disjointness preserving, and $T_1 + \cdots + T_n = T$. Moreover, if $k \neq j$, then

$$0 = (\tau_k \wedge \tau_l){\downarrow} = \tau_k{\downarrow} \wedge \tau_l{\downarrow} = T_k \wedge T_l,$$

so that T_k and T_l are disjoint. □

3.2.5. It can easily be seen that the representation of an order-bounded n-disjoint operator in Theorem 3.1.4 is unique up to mixing: if $T = T_1 + \cdots + T_n = S_1 + \cdots + S_m$ for two pairwise disjoint collections $\{T_1, \ldots, T_n\}$ and $\{S_1, \ldots, T_n\}$ of order-bounded disjointness preserving operators, then for every $j = 1, \ldots, m$ there exists a disjoint family of projections $\pi_{1j}, \ldots, \pi_{nj} \in \mathbb{P}(Y)$ such that $S_j = \pi_{1j} T_1 + \cdots + \pi_{nj} T_n$ for all $j := 1, \ldots, m$.

3.2.6. Corollary. *A positive operator from a vector lattice to a Dedekind complete vector lattice is n-disjoint if and only if it is the sum of n pairwise disjoint lattice homomorphisms.*

3.2.7. Corollary. *The set of polydisjoint operators from a vector lattices to a Dedekind complete vector lattices coincides with the order ideal in the vector lattice of order-bounded operators generated by lattice homomorphisms.*

3.2.8. The characterizations of sums of disjointness preserving operators (Theorem 3.2.4), sums of lattice homomorphisms (Corollary 3.2.6), and the ideal of order-bounded operators generated by lattice homomorphism (Corollary 3.2.7) were proved in [12] using standard tools. For an algebraic approach to the problem, see [39].

3.3. Differences of Lattice Homomorphisms. This section presents a characterization of order-bounded operators representable as a difference of two lattice homomorphisms. The starting point of this question is the celebrated Stone theorem about the structure of vector sublattices in the Banach lattice $C(Q,\mathbb{R})$ of continuous real functions on a compact space Q. This theorem can be rephrased in the above terms as follows.

3.3.1. The Stone theorem. *Each closed vector sublattice of $C(Q,\mathbb{R})$ is the intersection of the kernels of some differences of lattice homomorphisms on $C(Q,\mathbb{R})$.*

3.3.2. In view of the Stone theorem it is reasonable to refer to a difference of lattice homomorphisms on a vector lattice X as a *two-point relation* on X. We are not obliged to assume here that the lattice homomorphisms under study act into the reals $\mathbb{R}$. Thus, a linear operator $T\colon X \to Y$ between vector lattices is said to be a *two-point relation* on X whenever it is written as a difference of two lattice homomorphisms. An operator $bT := b \circ T$ with $b \in \mathbb{B} := \mathbb{P}(Y)$ is called a *stratum* of T.

3.3.3. The kernel $\ker(bT)$ of each stratum of a two point relation T is evidently a sublattice of X, since it is determined by an equation $bT_1x = bT_2x$. Thus, each stratum bT of an order-bounded disjointness preserving operator $T\colon X \to Y$ is a two-point relation on X and so its kernel is a vector sublattice of X. The main result of this section says that the converse is valid too. To handle the corresponding scalar problem, a formula of subdifferential calculus is used (see [44, 51]). In the following form of this auxiliary fact, the *positive decomposition* of a functional f means a representation $f = f_1 + \dots + f_N$ with positive functionals $f_1, \dots, f_N$.

3.3.4. The decomposition theorem. *Assume that $H_1, \dots, H_N$ are cones in a vector lattice X and f and g are positive functionals on X. The inequality*

$$f(h_1 \vee \dots \vee h_N) \geq g(h_1 \vee \dots \vee h_N)$$

holds for all $h_k \in H_k$ $(k := 1, \dots, N)$ if and only if to each positive decomposition $(g_1, \dots, g_N)$ of g there is a positive decomposition $(f_1, \dots, f_N)$ of f such that

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k,\ k := 1, \dots, N).$$

3.3.5. Proposition. *Let $\mathbb{F}$ be a dense subfield in $\mathbb{R}$ and let X be a vector lattice over $\mathbb{F}$. An order-bounded $\mathbb{F}$-linear functional from X to $\mathbb{R}$ is a two-point relation if and only if its kernel is an $\mathbb{F}$-linear sublattice of the ambient vector lattice.*

Proof. Let l be an order-bounded functional on a vector lattice X. Put $f := l^+$, $g := l^-$, and $H := \ker(l)$. It suffices to demonstrate only that g is a lattice homomorphism, i.e., $[0,g] = [0,1]g$ (see [39]). So, we take $0 \leq g_1 \leq g$ and put $g_2 := g - g_1$. We can assume that $g_1 \neq 0$ and $g_1 \neq g$. By hypothesis, for all $h_1, h_2 \in \ker(l)$ we have the $f(h_1 \vee h_2) \geq g(h_1 \vee h_2)$. By the decomposition theorem there is a positive decomposition $f = f_1 + f_2$ such that $f_1(h) - g_1(h) = 0$ and $f_2(h) - g_2(h) = 0$ for all $h \in H$. Since $H = \ker(f - g)$, we see that there are reals α and β satisfying $f_1 - g_1 = \alpha(f - g)$ and $f_2 - g_2 = \beta(f - g)$. Clearly, $\alpha + \beta = 1$ (for otherwise $f = g$ and $l = 0$). Therefore, one of the reals α and β is strictly positive. If $\alpha > 0$, then we have $g_1 = \alpha g$ for f and g disjoint. If $\beta > 0$, then, arguing similarly, we see that $g_2 = \beta g$. Hence, $0 \leq \beta \leq 1$ and we again see that $g_1 \in [0,1]g$. □

3.3.6. Theorem. *An order-bounded operator from a vector lattice to a Dedekind complete vector lattice is a two-point relation if and only if the kernel of its every stratum is a vector sublattice of the ambient vector lattice.*

Proof. The necessity is obvious, so only the sufficiency will be proved. Take $T \in L^{\sim}(X,Y)$, and let $\ker(bT) := (bT)^{-1}(0)$ be a vector sublattice in X for all $b \in \mathbb{P}(Y)$. We apply the Boolean-valued "scalarization" on putting $Y = \mathscr{R}{\downarrow}$.

Put $\tau := T{\uparrow}$ and observe that the validity of the identities $T^+{\uparrow} = \tau^+$ and $T^-{\uparrow} = \tau^-$ within $\mathbb{V}^{(\mathbb{B})}$ follows from Corollary 1.4.4(2) (and is proved by easy calculation of Boolean truth values). Moreover,

$$[\![\ker(\tau) \text{ is a vector sublattice of } X^{\wedge}]\!] = \mathbb{1}.$$

Indeed, given $x, y \in X$, put
$$b := [\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!].$$
This means that $x, y \in \ker(bT)$. Hence, we see by hypothesis that $bT(x \vee y) = 0$, whence $b \leq [\![T(x \vee y) = 0^\wedge]\!]$. Replacing T by τ yields
$$[\![\tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge]\!] \leq [\![\tau(x \vee y)^\wedge = 0^\wedge]\!].$$
A straightforward calculation of Boolean truth values completes the proof:
$$\begin{aligned}&[\![\ker(\tau) \text{ is a Riesz subspace of } X^\wedge]\!]\\ &\quad= [\![(\forall\, x, y \in X^\wedge)(\tau(x) = 0^\wedge \wedge \tau(y) = 0^\wedge \to \tau(x \vee y) = 0^\wedge)]\!]\\ &\quad= \bigwedge_{x,y\in X} \Big[\!\Big[\tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge \to \tau\big((x \vee y)^\wedge\big) = 0^\wedge\Big]\!\Big] = \mathbb{1}.\end{aligned}$$
□

3.3.7. Theorems 3.3.4 and 3.3.6 were obtained in [50] and [52], respectively. On using of the above terminology, the Meyer theorem (see [39, 3.3.1(5)] and [21, 63]) reads as follows: *Each order-bounded disjointness preserving operator between vector lattices is a two-point relation.* This fact can easily be deduced from Theorem 3.3.6, since $\ker(bT)$ is a vector sublattice, whenever T is disjointness preserving.

3.4. Sums of Lattice Homomorphisms. In this section, we will give a description for an order bounded operator T whose modulus can be presented as the sum of two lattice homomorphisms in terms of the properties of the kernels of the strata of T. Thus, we reveal the connection between the 2-disjoint operators and Grothendieck subspaces.

3.4.1. Recall that a subspace H of a vector lattice is a *G-space* or *Grothendieck subspace* provided that H enjoys the following property:
$$(\forall\, x, y \in H)\ (x \vee y \vee 0 + x \wedge y \wedge 0 \in H). \tag{2}$$

3.4.2. This condition appears as follows. In 1955, Grothendieck [27] pointed out the subspaces with the above condition in the vector lattice $C(Q, \mathbb{R})$ of continuous functions on a compact space Q defining them by means of a family of relations A with each relation $\alpha \in \mathrm{A}$ having the form
$$f(q^1_\alpha) = \lambda_\alpha f(q^2_\alpha) \quad (q^1_\alpha, q^2_\alpha \in Q,\ \lambda_\alpha \in \mathbb{R},\ \alpha \in \mathrm{A}).$$
These spaces yield examples of L^1-predual Banach spaces that are not AM-spaces. In 1969, Lindenstrauss and Wulpert gave a characterization of such subspaces by means of the property 3.4.1 and introduced the term G-space (see [55]). Some related properties of Grothendieck spaces are presented also in [54, 67].

3.4.3. Theorem. *Let $\mathbb{F}$ be a dense subfield in $\mathbb{R}$, and let X be a vector lattice over $\mathbb{F}$. The modulus of an order-bounded $\mathbb{F}$-linear functional from X to $\mathbb{R}$ is the sum of two lattice homomorphisms if and only if the kernel of this functional is a Grothendieck subspace of X.*

Proof. The proof relied on the decomposition theorem (Theorem 3.3.4) (see [53]). □

3.4.4. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. The modulus of an order-bounded operator $T\colon X \to Y$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B} := \mathbb{P}(Y)$ is a Grothendieck subspace of the ambient vector lattice X.*

Proof. The proof runs along the lines of Sec. 3.3. We apply the technique of Boolean-valued "scalarization" reducing the operator problems to the case of functionals, which is handled in Theorem 3.4.3. Put $Y = \mathscr{R}{\downarrow}$ and $\tau := T{\uparrow}$ and proceed with τ within $\mathbb{V}^{(\mathbb{B})}$. First, we observe the useful calculation:
$$\begin{aligned}&[\![\ker(l) \text{ is a Grothendieck subspace of } X^\wedge]\!]\\ &\quad= [\![(\forall\, x, y \in X^\wedge)\ (\tau(x) = 0^\wedge \wedge \tau(y) = 0^\wedge \to \tau(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge)]\!]\\ &\quad= \bigwedge_{x,y\in X} \Big[\!\Big[\tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge \to \tau\big((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge\big) = 0^\wedge\Big]\!\Big].\end{aligned} \tag{$*$}$$

Sufficiency. Take $x, y \in X$ and put

$$b := [\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!].$$

It follows from (G) (see the Gordon theorem) that $x, y \in \ker(bT)$. By hypothesis $\ker(bT)$ is a Grothendieck subspace and so $bT(x \vee y \vee 0 + x \wedge y \wedge 0) = 0$. By using (G) again, we get

$$[\![Tx = 0^\wedge]\!] \wedge [\![Ty = 0^\wedge]\!] = b \leq [\![T(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge]\!].$$

Now, it follows from $(*)$ that $[\![\ker(l)$ is a Grothendieck subspace of $X^\wedge]\!] = \mathbb{1}$. By the transfer principle we can apply Theorem 3.4.3 to τ within $\mathbb{V}^{(\mathbb{B})}$, consequently, $|\tau| = \tau_1 + \tau_2$ with τ_1 and τ_2 lattice homomorphisms within $\mathbb{V}^{(\mathbb{B})}$. It can easily be seen that the operators $T_1 := \tau_1{\downarrow}$ and $T_2 := \tau_2{\downarrow}$ from X to $\mathscr{R}{\downarrow}$ are lattice homomorphisms and $|T| = T_1 + T_2$.

Necessity. Assume that $|T| = T_1 + T_2$ for some lattice homomorphisms $T_1, T_2 \colon X \to Y$ and denote $\tau := T{\uparrow}$, $\tau_1 := T_1{\uparrow}$, and $\tau_2 := T_2{\uparrow}$. It can easily be checked that inside $\mathbb{V}^{(\mathbb{B})}$ we have $\tau, \tau_1, \tau_2 \colon X^\wedge \to \mathscr{R}$ and $|\tau| = \tau_1 + \tau_2$; moreover, τ_1 and τ_2 are lattice homomorphisms. By Theorem 3.4.3 and the transfer principle $[\![\ker(l)$ is a Grothendieck subspace of $X^\wedge]\!] = \mathbb{1}$. Making use of $(*)$, we infer

$$[\![\tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge]\!] \leq \Big[\!\!\Big[\tau\big((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge\big) = 0^\wedge\Big]\!\!\Big].$$

Now, if $b \in \mathbb{B}$ and $bTx = bTy = 0$, then

$$\Big[\!\!\Big[l\big((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge\big) = 0^\wedge\Big]\!\!\Big] \geq b,$$

whence by the Gordon theorem we get $bT(x \vee y \vee 0 + x \wedge y \wedge 0) = 0$. $\square$

3.4.5. The main result of the section (Theorem 3.4.4) was obtained in [53]. The sums of Riesz homomorphisms were first described in [12] in terms of n-disjoint operators (see Sec. 3.3). A survey of some conceptually close results on n-disjoint operators is given in [1, Sec. 5.6].

4. Order Continuous Operators

4.1. Maharam Operators. Now we examine some class of order continuous positive operators that behave in many instances like functionals. In fact, such operators are representable as Boolean-valued order continuous functionals.

4.1.1. Throughout this section, X and Y are vector lattices with Y Dedekind complete. A linear operator $T \colon X \to Y$ is said to have the *Maharam property* or is said to be *order interval preserving* whenever $T[0, x] = [0, Tx]$ for every $0 \leq x \in X$, i.e., if for arbitrary $0 \leq x \in X$ and $0 \leq y \leq Tx$ there is some $0 \leq u \in X$ such that $Tu = y$ and $0 \leq u \leq x$. A *Maharam operator* is an order-bounded order continuous operator whose modulus enjoys the Maharam property.

We say that a linear operator $S \colon X \to Y$ is *absolutely continuous* with respect to T and write $S \preccurlyeq T$ if $|S|x \in \{|T|x\}^{\perp\perp}$ for all $x \in X_+$. It can be easily seen that if $S \in \{T\}^{\perp\perp}$, then $S \preccurlyeq T$, but the converse can be false.

4.1.2. The null ideal $\mathscr{N}_T$ of an order bounded operator $T \colon X \to Y$ is defined by $\mathscr{N}_T := \{x \in X : |T|(|x|) = 0\}$. Observe that $\mathscr{N}_T$ is indeed an ideal in X. The disjoint complement of $\mathscr{N}_T$ is referred to as the *carrier* of T and is denoted by $\mathscr{C}_T$, so that $\mathscr{C}_T := \mathscr{N}_T^{\perp}$. An operator T is called *strictly positive* whenever $0 < x \in X$ implies $0 < |T|(x)$. Clearly, $|T|$ is strictly positive on $\mathscr{C}_T$. Sometimes we find it convenient to denote $X_T := \mathscr{C}_T$ and $Y_T := (\operatorname{Im} T)^{\perp\perp}$.

4.1.3. As an examples of Maharam operators, we consider conditional expectation and Bochner integration. Take a probability space (Q, Σ, μ), and let Σ_0 and μ_0 be a σ-subalgebra of Σ and the restriction of μ to Σ_0. The conditional expectation operator $\mathscr{E}(\cdot, \Sigma_0)$ is a Maharam operator from $L^1(Q, \Sigma, \mu)$ onto $L^1(Q, \Sigma_0, \mu_0)$. The restriction of $\mathscr{E}(\cdot, \Sigma_0)$ to $L^p(Q, \Sigma, \mu)$ is also a Maharam operator from $L^p(Q, \Sigma, \mu)$ to $L^p(Q, \Sigma_0, \mu_0)$. These facts are immediate in view of the simple properties of conditional expectation.

Let (Q, Σ, μ) be a probability space, and let Y be a Banach lattice. Consider the space $X := L^1(Q, \Sigma, \mu, F)$ of Bochner integrable Y-valued functions, and let $T \colon E \to F$ denote the Bochner integral

$Tf := \int\limits_Q f\,d\mu$. If the Banach lattice Y has order continuous norm (in this case, Y is order complete), then X is a Dedekind complete vector lattice under the natural order

$$f \geq 0 \Longleftrightarrow f(t) \geq 0$$

for almost all $t \in Q$ and T is a Maharam operator. See more examples in [39, 40].

4.1.4. A positive operator $T\colon X \to Y$ is said to have the *Levi property* if $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order-bounded in Y. Given an order-bounded order continuous operator T from X to Y, denote by $\mathscr{D}_m(T)$ the largest ideal of the universal completion X^{u} onto which we can extend the operator T by order continuity. For a positive order continuous operator T we have $X = \mathscr{D}_m(T)$ if and only if T has the Levi property.

The following theorem describes an important property of Maharam operators, enabling us to embed them into an appropriate Boolean-valued universe as order continuous functionals.

4.1.5. Theorem. *Let X and Y be some vector lattices with Y having the projection property, and let T be a Maharam operator from X to Y. Then there exist an order closed subalgebra $\mathscr{B}$ of $\mathbb{B}(X_T)$ consisting of projection bands and a Boolean isomorphism h from $\mathbb{B}(Y_T)$ onto $\mathscr{B}$ such that $T\big(h(L)\big) \subset L$ for all $L \in \mathbb{B}(Y_T)$.*

The Boolean algebra of projections $\mathscr{B}$ in Theorem 4.1.5 as well as the corresponding Boolean algebra of bands admits a simple description. For $L \in \mathbb{B}(Y_T)$ denote by $h(L)$ the band in $\mathbb{B}(X_T)$ corresponding to the band projection $h([K])$.

4.1.6. Proposition. *For a band $K \in \mathbb{B}(X_T)$ the following are equivalent:*

(1) *$Tu = Tv$ and $u \in K$ imply $v \in K$ for all $u, v \in X_+$;*
(2) *$T(K'_+) \subset T(K_+)^{\perp\perp}$ implies $K' \subset K$ for all $K' \in \mathbb{B}(X_T)$;*
(3) *$K = h(L)$ for some $L \in \mathbb{B}(Y)$.*

A band $K \in \mathbb{B}(X_T)$ (as well as the corresponding band projection $[K] \in \mathbb{P}(X_T)$) is said to be *T-saturated* if one of (and then all) the conditions 4.1.6(1)–(3) is fulfilled.

The following can be deduced from Proposition 4.1.6 by the Freudenthal spectral theorem.

4.1.7. Proposition. *If X and Y are Dedekind complete vector lattices and T is a Maharam operator from X to Y, then there exists an f-module structure on X over an f-algebra $\mathscr{Z}(Y)$ such that an order-bounded operator $S\colon X \to Y$ is absolutely continuous with respect to T if and only if S is $\mathscr{Z}(Y)$-linear.*

We now state the main result of the section.

4.1.8. Theorem. *Let X be a Dedekind complete vector lattice, $Y := \mathscr{R}{\downarrow}$, and let $T\colon X \to Y$ be a strictly positive Maharam operator with $Y = Y_T$. Then there are $\mathscr{X}, \tau \in \mathbb{V}^{(\mathbb{B})}$ satisfying the following:*

(1) $\mathbb{V}^{(\mathbb{B})} \models$ "$\mathscr{X}$ is a Dedekind complete vector lattice and $\tau\colon \mathscr{X} \to \mathscr{R}$ is an strictly positive order continuous functional with the Levi property";
(2) *$\mathscr{X}{\downarrow}$ is a Dedekind complete vector lattice and a unitary f-module over the f-algebra $\mathscr{R}{\downarrow}$;*
(3) *$\tau{\downarrow}\colon \mathscr{X}{\downarrow} \to \mathscr{R}{\downarrow}$ is a strictly positive Maharam operator with the Levi property and an $\mathscr{R}{\downarrow}$-module homomorphism;*
(4) *there exists a lattice isomorphism φ from X into $\mathscr{X}{\downarrow}$ such that $\varphi(X)$ is an order dense ideal of $\mathscr{X}{\downarrow}$ and $T = \tau{\downarrow} \circ \varphi$.*

4.1.9. The Maharam operator stems from the theory of Maharam's "full-valued" integrals (see [59–61]). Theorem 4.1.8 was established in [36,37]. More results, applications, and references on Maharam operators are in [39, 40]. See [44] for some extension of this theory to sublinear and convex operators.

4.2. Representation of Order Continuous Operators. Theorem 4.1.8 enables us to state that every fact on order continuous functionals ought to have a parallel variant for Maharam operators that can be proved by the Boolean-valued machinery. The aim of this section is to prove an operator version of the following result.

4.2.1. Theorem. *Let X be a vector lattice and assume that $X_n^\sim$ separates the points of X. Then there exist an order dense ideals L and X' in X^{u} and a linear functional $\tau\colon L \to \mathbb{R}$ such that the following assertions hold*:

(1) $X' = \{x' \in X' \colon xx' \in L \text{ for all } x \in X\}$;

(2) *τ is strictly positive, o-continuous, and has the Levi property*;

(3) *for every $\sigma \in X_n^\sim$ there exists a unique $x' \in X'$ such that*

$$\sigma(x) = \tau(x \cdot x') \quad (x \in X);$$

(4) *the map $\sigma \mapsto x'$ is a lattice isomorphism of $X_n^\sim$ onto X'.*

4.2.2. To translate Theorem 4.2.1 into a result on operators we need some preparation. Let X and Y be f-modules over an f-algebra A. A linear operator $T\colon X \to Y$ is called *A-linear* if $T(ax) = aTx$ for all $x \in X$ and $a \in A$. Denote by $L^A(X,Y)$ the set of all order-bounded A-linear operators from X to Y and put $L_n^A(X,Y) := L^A(X,Y) \cap L_n^\sim(X,Y)$.

We say that a set $\mathscr{T} \subset L^\sim(X,Y)$ *separates the points* of X whenever, given nonzero $x \in X$, there exists $T \in \mathscr{T}$ such that $Tx \neq 0$. In the case of a Dedekind complete Y and the sublattice $\mathscr{T} \subset L^\sim(X,Y)$ this is equivalent to saying that for every nonzero $x \in X_+$ there is a positive operator $T \in \mathscr{T}$ with $Tx \neq 0$.

4.2.3. Given a real vector lattice $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$, denote by $\mathscr{X}^\sim$ and $\mathscr{X}_n^\sim$ the internal vector lattices of order-bounded and order continuous functionals on $\mathscr{X}$, respectively. More precisely, $[\![\sigma \in \mathscr{X}^\sim]\!] = \mathbb{1}$ and $[\![\sigma \in \mathscr{X}_n^\sim]\!] = \mathbb{1}$ mean that

$$[\![\sigma\colon \mathscr{X} \to \mathscr{R} \text{ is an order-bounded functional}]\!] = \mathbb{1}$$

and

$$[\![\sigma\colon \mathscr{X} \to \mathscr{R} \text{ is an order continuous functional}]\!] = \mathbb{1},$$

respectively. Put $X := \mathscr{X}{\downarrow}$ and $A := \mathscr{R}{\downarrow}$.

4.2.4. Theorem. *The mapping assigning to each $\sigma \in \mathscr{X}^\sim{\downarrow}$ its descent $S := \sigma{\downarrow}$ is a lattice isomorphism of $\mathscr{X}^\sim{\downarrow}$ and $\mathscr{X}_n^\sim{\downarrow}$ onto $L^A(X,\mathscr{R}{\downarrow})$ and $L_n^A(X,\mathscr{R}{\downarrow})$, respectively. Moreover,* $[\![\mathscr{X}^\sim$ separates the points of $\mathscr{X}]\!] = \mathbb{1}$ ($[\![\mathscr{X}_n^\sim)$ separates the points of $\mathscr{X}]\!] = \mathbb{1}$) *if and only if $L^A(X,\mathscr{R}{\downarrow})$ (respectively, $L_n^A(X,\mathscr{R}{\downarrow})$) separates the points of X.*

4.2.5. By the Gordon theorem we can also assume that $Y^{\mathrm{u}} = \mathscr{R}{\downarrow}$. Of course, in this case we can identify A^{u} with Y^{u}. In view of Theorem 1.3.7 there exists a real Dedekind complete vector lattice $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} = \mathbb{P}(Y)$ such that $\mathscr{X}{\downarrow}$ is an f-module over A^{u}, and there is an f-module isomorphism h from X to $\mathscr{X}{\downarrow}$ satisfying $\mathscr{X}{\downarrow} = \mathrm{mix}\big(h(X)\big)$. In virtue of Theorem 4.2.4 $\mathscr{X}_n^\sim$ separates the points of $\mathscr{X}$. The transfer principle tells us that Theorem 4.2.1 is true within $\mathbb{V}^{(\mathbb{B})}$, so that there exist an order dense ideal $\mathscr{L}$ in $\mathscr{X}^{\mathrm{u}}$ and a strictly positive linear functional $\tau\colon \mathscr{L} \to \mathscr{R}$ with the Levi property such that the order ideal $\mathscr{X}' = \{x' \in \mathscr{X}^{\mathrm{u}} \colon x'\mathscr{X} \subset \mathscr{L}\}$ is lattice isomorphic to $\mathscr{X}_n^\sim$; moreover, the isomorphism is implemented by assigning the functional $\sigma_{x'} \in \mathscr{X}_n^\sim$ to $x' \in \mathscr{X}'$ by $\sigma_{x'}(x) = \tau(xx')$ $(x \in \mathscr{X})$.

4.2.6. Put $\hat{X} := \mathscr{X}{\downarrow}$, $\hat{L} := \mathscr{L}{\downarrow}$, $\hat{T} := \tau{\downarrow}$, and $\hat{X}' := \mathscr{X}'{\downarrow}$. By Theorem 4.2.4 we can identify the universally complete vector lattices X^{u}, $\hat{X}^{\mathrm{u}}$, and $\mathscr{X}^{\mathrm{u}}{\downarrow}$ as well as X with a laterally dense sublattice in $\hat{X}$. Then $\hat{L}$ is an order dense ideal in $\hat{X}^{\mathrm{u}}$ and an f-module over A^{u}, while $\hat{T}\colon \hat{L} \to Y^{\mathrm{u}}$ is a strictly positive Maharam operator with the Levi property. Since the multiplication in X^{u} is the descent of the internal multiplication in $\mathscr{X}^{\mathrm{u}}$, we have the representation $\hat{X}' = \{x' \in X^{\mathrm{u}} \colon x'\hat{X} \subset \hat{L}\}$. Moreover, $\hat{X}'$ is f-module isomorphic to

$L^A_n(\hat{X}, Y^{\mathrm{u}})$ by assigning to $x' \in \hat{X}$ the operator $\hat{S}_{x'} \in L^A_n(\hat{X}, Y^{\mathrm{u}})$ defined as $\hat{S}_{x'}(x) = \hat{T}(xx')$ $(x \in \hat{X})$. Now, defining

$$L := \{x \in \hat{L} \colon \hat{T}x \in Y\}, \quad T := \hat{T}|_L, \quad X' := \{x' \in \hat{X}' \colon x'X \subset L\}$$

yields that if $x' \in X'$, then $S_{x'} := \hat{S}_{x'}|_X$ is contained in $L^A_n(X, Y)$. Conversely, an arbitrary $S \in L^A_n(X, Y)$ has a representation $Sx = \hat{T}(xx')$ $(x \in X)$ with some $x' \in \hat{X}'$, so that $\hat{T}(xx') \in Y$ for all $x \in X$ and hence $x' \in X'$, $xx' \in L$ for all $x \in X$, and $Sx = T(xx')$ $(x \in X)$ by the above definitions.

4.2.7. Theorem. *Let X be an f-module over $A := \mathscr{Z}(Y)$ with Y a Dedekind complete vector lattice, and let $L^A_n(X, Y)$ separates the points of X. Then there exist an order dense ideal L in X^{u} and a strictly positive Maharam operator $T \colon L \to Y$ such that the order ideal $X' = \{x' \in X' \colon (\forall\, x \in X)\ xx' \in L\} \subset X^{\mathrm{u}}$ is lattice isomorphic to $L^A_n(X, Y)$. The isomorphism is implemented by assigning the operator $S_{x'} \in L^A_n(X, Y)$ to an element $x' \in X'$ by the formula*

$$S_{x'}(x) = \Phi(xx') \quad (x \in X).$$

If there exists a strictly positive $T_0 \in L^A_n(X, Y)$, then we can choose L and T such that $X \subset L$ and $T|_X = T_0$.

Below, in Theorems 4.2.8–4.2.10, X and Y are Dedekind complete vector lattices.

4.2.8. The Hahn decomposition theorem. *Let $S \colon X \to Y$ be a Maharam operator. Then there is a band projection $\pi \in \mathbb{P}(X)$ such that $S^+ = S \circ \pi$ and $S^- = -S \circ \pi^\perp$. In particular, $|S| = S \circ (\pi - \pi^\perp)$.*

4.2.9. The Nakano theorem. *Let $T_1, T_2 \colon X \to Y$ be order-bounded operators such that $T := |T_1| + |T_2|$ is a Maharam operator. Then T_1 and T_2 are disjoint if and only if so are their carriers*; *symbolically, $T_1 \perp T_2 \iff \mathscr{C}_{T_1} \perp \mathscr{C}_{T_2}$.*

4.2.10. The Radon–Nikodým theorem. *Assume that $T \colon X \to Y$ is a positive Maharam operator. A positive operator $S \colon X \to Y$ belongs to $\{T\}^{\perp\perp}$ if and only if there exists an orthomorphism $0 \leq \rho \in \mathrm{Orth}^\infty(X)$ with $Sx = T(\rho x)$ for all $x \in \mathscr{D}(\rho)$.*

4.2.11. Theorem 4.2.1 is proved in [78, Theorem 2.1]. It can be also extracted from [77, Theorem IX.3.1] or [39, Theorem 3.4.8]. Theorem 4.2.7 was proved in [36] (also see [39]). Theorems 4.2.8–4.2.10, first obtained in [57], can easily be deduced from Theorem 4.2.7, or can be proved by the general scheme of “Boolean-valued scalarization.”

4.3. Conditional Expectation Type Operators. The conditional expectation operators have many remarkable properties related to the order structure of the underlying function space. Boolean-valued analysis enables us to demonstrate that some much more general class of operators shares these properties.

4.3.1. Let Z be a universally complete vector lattice with unity $\mathbb{1}$. Recall that Z is an f-algebra with multiplicative unit $\mathbb{1}$. Assume that $\Phi \colon L^1(\Phi) \to Y$ is a Maharam operator with the Levi property. We will write $L^0(\Phi) := Z$ whenever $L^1(\Phi)$ is an order dense ideal in Z. Denote also by $L^\infty(\Phi)$ the order ideal in Z generated by $\mathbb{1}$. Considering an order ideal $X \subset Z$, we will always assume that $L^\infty(\Phi) \subset X \subset L^1(\Phi)$. The *associate space* X' is defined as the set of all $x' \in L^0(\Phi)$ for which $xx' \in L^1(\Phi)$ for all $x \in X$. Clearly, X' is an order ideal in Z.

If (Ω, Σ, μ) is a probability space and $\mathscr{X}_0$ is an order closed vector sublattice of $L^\infty(\Omega, \Sigma, \mu)$ containing 1_Ω, then there exists a σ-subalgebra Σ_0 of Σ such that $\mathscr{X}_0 = L^\infty(\Omega, \Sigma_0, \mu_0)$, with $\mu_0 = \mu|_{\mathscr{X}_0}$ (see [19, Lemma 2.2]). Interpreting this fact and the properties of conditional expectation in a Boolean-valued model yields the following result.

4.3.2. Theorem. *Let $\Phi \colon L^1(\Phi) \to Y$ be a strictly positive Maharam operator with $Y = Y_\Phi$ and let Z_0 be an order closed sublattice in $L^0(\Phi)$. If $\mathbb{1} \in X_0 := L^1(\Phi) \cap Z_0$ and the restriction $\Phi_0 := \Phi|_{X_0}$ has the Maharam property, then $X_0 = L^1(\Phi_0)$ and there exists an operator $\mathrm{E}(\cdot|Z_0)$ from $L^1(\Phi)$ onto $L^1(\Phi_0)$ such that*

(1) $\mathrm{E}(\cdot|Z_0)$ *is an order continuous positive linear projection*;
(2) $\mathrm{E}(\cdot|Z_0)$ *commutes with all saturated projections, i.e.,* $\mathrm{E}(h(\pi)x|Z_0) = h(\pi)\mathrm{E}(x|Z_0)$ *for all* $\pi \in \mathbb{P}_\Phi(X)$ *and* $x \in L^1(\Phi)$;
(3) $\Phi(xy) = \Phi\big(y\mathrm{E}(x|Z_0)\big)$ *for all* $x \in L^1(\Phi)$ *and* $y \in L^\infty(\Phi_0)$;
(4) $\Phi_0(|\mathrm{E}(x|Z_0)|) \leq \Phi(|x|)$ *for all* $x \in L^1(\Phi)$;
(5) $\mathrm{E}(\cdot|Z_0)$ *satisfies the averaging identity, i.e.,* $\mathrm{E}(v\mathrm{E}(x|Z_0)|Z_0) = \mathrm{E}(v|Z_0)\mathrm{E}(x|Z_0)$ *for all* $x \in L^1(\Phi)$ *and* $v \in L^\infty(\Phi)$.

4.3.3. We will call the operator $\mathrm{E}(\cdot|Z_0)$ defined by Theorem 4.3.2 the *conditional expectation operator with respect to* Z_0. Take $w \in X'$ and observe that $\mathrm{E}(wx|Z_0) \in L^1(\Phi_0)$ is well defined for all $x \in X$. If, moreover, $\mathrm{E}(wx|Z_0) \in X$ for every $x \in X$, then we can define a linear operator $T\colon X \to X$ by putting $Tx := \mathrm{E}(wx|Z_0)$ $(x \in X)$. Clearly, T is order-bounded and order continuous. Furthermore, for all $x \in X_+$ we have

$$T^+x = \mathscr{E}(w^+x|Z_0), \quad T^-x = \mathscr{E}(w^-x|Z_0), \quad |T|x = \mathscr{E}(|w|x|Z_0).$$

In particular, T is positive if and only if so is w. Putting $x := wx$ and $y := \mathbb{1}$ in Theorem 4.3.2(3), we get

$$\Phi(wx) = \Phi(wx\mathbb{1}) = \Phi\big(\mathscr{E}(wx|Z_0)\big) = \Phi(Tx)$$

for all $x \in X$. Now, x can be chosen to be a component of $\mathbb{1}$ with $wx = w^+$ or $wx = w^-$, so that $T = 0$ implies $\Phi(w^+) = 0$ and $\Phi(w^-) = 0$, since Φ is strictly positive. Thus, $w \in X'$ is uniquely determined by T.

We say that T satisfies the *averaging identity* if $T(y \cdot Tx) = Ty \cdot Tx$ for all $x \in X$ and $y \in L^\infty(\Phi)$. Now we present two well-known results. By $\mathscr{E}(\cdot|\Sigma_0)$ we denote the conditional expectation operator with respect to a σ-algebra Σ_0.

4.3.4. Theorem. *Let* (Ω, Σ, μ) *be probability space, and let* $\mathscr{X}$ *be an order ideal in* $L^1(\Omega, \Sigma, \mu)$ *containing* $L^\infty(\Omega, \Sigma, \mu)$. *For a linear operator* $\mathscr{T}$ *on* $\mathscr{X}$ *the following are equivalent*:
(1) $\mathscr{T}$ *is order continuous, satisfies the averaging identity, and keeps* $L^\infty(\Omega, \Sigma, \mu)$ *invariant*;
(2) *there exist* $w \in \mathscr{X}'$ *and a sub-*σ*-algebra* Σ_0 *of* Σ *such that* $\mathscr{T}x = \mathscr{E}(wx|\Sigma_0)$ *for all* $x \in \mathscr{X}$.

4.3.5. Theorem. *For a subspace* $\mathscr{X}$ *of* $L^1(\Omega, \Sigma, \mu)$ *the following are equivalent*:
(1) $\mathscr{X}$ *is the range of a positive contractive projection*;
(2) $\mathscr{X}$ *is a closed vector sublattice of* $L^1(\Omega, \Sigma, \mu)$;
(3) *there exists a lattice isometry from some* $L^1(\Omega', \Sigma', \mu')$ *space onto* $\mathscr{X}$.

The following two results can be proved by interpreting Theorems 4.3.4 and 4.3.5 in a Boolean-valued model.

4.3.6. Theorem. *Let* $\Phi\colon L^1(\Phi) \to Y$ *be a strictly positive Maharam operator, and let* X *be an order dense ideal in* $L^1(\Phi)$ *including* $L^\infty(\Phi)$. *For a linear operator* T *on* X *the following are equivalent*:
(1) T *is order continuous, satisfies the averaging identity, leaves invariant the subspace* $L^\infty(\Phi)$, *and commutes with all* Φ*-saturated projections*;
(2) *there exist* $w \in X'$ *and an order closed sublattice* Z_0 *in* $L^0(\Phi)$ *containing a unit element* $\mathbb{1}$ *of* $L^1(\Phi)$ *such that the restriction of* Φ *onto* $L^1(\Phi) \cap Z_0$ *has the Maharam property and* $Tx = \mathscr{E}(wx|Z_0)$ *for all* $x \in X$.

4.3.7. Theorem. *For each subspace* X_0 *of* $L^1(\Phi)$ *the following statements are equivalent*:
(1) X *is the range of a positive* Φ*-contractive projection*;
(2) X *is a closed vector sublattice of* $L^1(\Phi)$ *invariant under all* Φ*-saturated projections*;
(3) *there exists a Maharam operator* $\Psi\colon L^1(\Psi) \to Y$ *and a lattice isomorphism* h *from* $L^1(\Psi)$ *onto* X *such that* $\Phi(|Tx|) = \Psi(|x|)$ *for all* $x \in L^1(\Psi)$.

4.3.8. Theorems 4.3.4 and 4.3.5 can be found in [19, Proposition 3.1] and [20, Lemma 1], respectively. Theorems 4.3.6 and 4.3.7 are published for the first time.

4.4. Maharam Extension. The general properties of Maharam operators can be deduced from the corresponding facts about functionals with the help of Theorem 4.1.8. Nevertheless, these methods can be also useful in studying arbitrary regular operators.

4.4.1. Proposition. *Suppose that X is a vector lattice over a dense subfield $\mathbb{F} \subset \mathbb{R}$ and $\varphi\colon X \to \mathbb{R}$ is a strictly positive $\mathbb{F}$-linear functional. There exist a Dedekind complete vector lattice X^φ containing X and a strictly positive order continuous linear functional $\bar{\varphi}\colon X^\varphi \to \mathbb{R}$ with the Levi property extending φ such that for every $x \in X^\varphi$ there is a sequence (x_n) in X with $\lim\limits_{n\to\infty} \bar{\varphi}(|x - x_n|) = 0$.*

Proof. Put $d(x, y) := \varphi(|x - y|)$ and note that (X, d) is a metric space. Let X^φ be the completion of the metric space (X, d), and let $\bar{\varphi}$ be the extension of φ to X^φ by continuity. It is not difficult to ensure that X^φ is a Banach lattice with additive norm $\|\cdot\|^\varphi := \bar{\varphi}(|\cdot|)$ that contains X as a norm dense $\mathbb{F}$-linear sublattice. Thus, $\bar{\varphi}$ is a strictly positive order continuous linear functional on X^φ with the Levi property. □

4.4.2. Put $L^1(\varphi) := X^\varphi$, and let $\bar{X}$ stand for the order ideal in $L^1(\varphi)$ generated by X. Then $(L^1(\varphi), \|\cdot\|^\varphi)$ is an AL-space and $\bar{X}$ is a Dedekind complete vector lattice. Moreover, X is norm dense in $L^1(\varphi)$ and, hence, in $\bar{X}$.

Given a nonempty subset U of a lattice L, we denote by $U^\uparrow$ ($U^\downarrow$) the set of elements $x \in L$ representable in the form $x = \sup(A)$ (respectively, $x = \inf(A)$), where A is an upward (respectively, downward) directed subset of U. Moreover, we set $U^{\uparrow\downarrow} := (U^\uparrow)^\downarrow$, etc. If in the above definition A is countable, then we write $U^{\upharpoonleft}$, $U^{\downharpoonleft}$, and $U^{\upharpoonleft\downharpoonleft}$ instead of $U^\uparrow$, $U^\downarrow$, and $U^{\uparrow\downarrow}$. Recall that for the Dedekind completion X^δ we have $X^\delta = X^\uparrow = X^\downarrow$.

4.4.3. Proposition. *$\bar{X} = X^{\downharpoonleft\upharpoonleft} = X^{\upharpoonleft\downharpoonleft}$ and $L^1(\varphi) = X^{\downharpoonleft\upharpoonleft} = X^{\upharpoonleft\downharpoonleft}$ with both $(\cdot)^{\downharpoonleft\upharpoonleft}$ and $(\cdot)^{\upharpoonleft\downharpoonleft}$ taken in $\bar{X}$ and $L^1(\varphi)$, respectively.*

Translating Propositions 4.4.1 and 4.4.3 by means of Boolean-valued "scalarization" leads to the following result.

4.4.4. Theorem. *Let X and Y be vector lattices with Y Dedekind complete and T a strictly positive linear operator from X to Y. There exist a Dedekind complete vector lattice $\bar{X}$ and a strictly positive Maharam operator $\bar{T}\colon \bar{X} \to Y$ satisfying the following conditions*:

(1) *there exist a lattice homomorphism $\iota\colon X \to \bar{X}$ and an f-algebra homomorphism $\theta\colon \mathscr{Z}(Y) \to \mathscr{Z}(\bar{X})$ such that*
$$\alpha T x = \bar{T}\big(\theta(\alpha)\iota(x)\big) \quad \big(x \in X,\ \alpha \in \mathscr{Z}(Y)\big);$$

(2) *$\iota(X)$ is a majorizing sublattice in $\bar{X}$ and $\theta\big(\mathscr{Z}(Y)\big)$ is an order closed sublattice and subring of $\mathscr{Z}(\bar{X})$;*

(3) *$\bar{X} = \big(X \odot \mathscr{Z}(Y)\big)^{\downarrow\uparrow}$, where $X \odot \mathscr{Z}(Y)$ is a subspace of $\bar{X}$ consisting of all finite sums $\sum\limits_{k=1}^{n} \theta(\alpha_k)\iota(x_k)$ with $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n \in \mathscr{Z}(Y)$.*

4.4.5. The pair $(\bar{X}, \bar{T})$ (or $\bar{T}$ for short) is called a *Maharam extension* of T if it satisfies conditions (1)–(3) of Theorem 4.4.4. The pair $(\bar{X}, \iota)$ is also called a *Maharam extension space* for T. Two Maharam extensions T_1 and T_2 of T with the respective Maharam extension spaces (X_1, ι_1) and (X_2, ι_2) are said to be *isomorphic* if there exists a lattice isomorphism h of X_1 onto X_2 such that $T_1 = T_2 \circ h$ and $\iota_2 = h \circ \iota_1$. It is not difficult to ensure that the Maharam extension is unique up to isomorphism.

4.4.6. Let X and Y be vector lattices with Y Dedekind complete, $T\colon X \to Y$ a strictly positive operator, and let $(\bar{X}, \bar{T})$ be the Maharam extension of T. Consider the universal completion $\bar{X}^{\mathrm{u}}$ of $\bar{X}$ with a fixed f-algebra structure. Let $L_1(\Phi)$ be the greatest order dense ideal in $\bar{X}^{\mathrm{u}}$ onto which $\bar{T}$ can be extended by order continuity. In more detail,
$$L^1(T) := \{x \in \bar{X}^{\mathrm{u}}\colon \bar{T}([0, |x|] \cap \bar{X}) \text{ is order-bounded in } Y\},$$
$$\hat{T}x := \sup\{\bar{T}u\colon u \in \bar{X},\ 0 \le u \le x\} \ \ (x \in L^1(T)_+), \quad \hat{T}x = \hat{T}x^+ - \hat{T}x^- \ \ \big(x \in L^1(T)\big).$$

Define the Y-valued norm $\lbrack\!\lbrack\cdot\rbrack\!\rbrack$ on $L^1(T)$ by $\lbrack\!\lbrack u\rbrack\!\rbrack := \hat{T}(|u|)$. In terms of the theory of lattice normed spaces, $(L^1(T), \lbrack\!\lbrack\cdot\rbrack\!\rbrack)$ is a *Banach–Kantorovich lattice* (see [39, Chap. 2]). In particular, $\lbrack\!\lbrack au\rbrack\!\rbrack = |a|\lbrack\!\lbrack u\rbrack\!\rbrack$ $(a \in \mathscr{Z}(Y)$, $u \in L^1(T))$.

4.4.7. Theorem. *For every operator $S \in \{T\}^{\perp\perp}$, there is a unique element $z = z_T \in \bar{X}^{\mathrm{u}}$ satisfying*

$$Sx = \hat{T}\big(z \cdot \imath(x)\big) \quad (x \in X).$$

The correspondence $T \mapsto z_T$ establishes a lattice isomorphism between the band $\{T\}^{\perp\perp}$ and the order dense ideal in $\bar{X}^{\mathrm{u}}$ defined by

$$\{z \in \bar{X}^{\mathrm{u}} \colon z \cdot \imath(X) \subset L_1(T)\}.$$

Proof. This result is a variant of the Radon–Nikodým theorem for positive operators and can be obtained as a combination of Theorems 4.2.10 and 4.4.4 or proved by means of Boolean-valued "scalarization". □

4.4.8. The Maharam extension stems from the corresponding extension result given by D. Maharam for F-integrals [59–61]. For operators in Dedekind complete vector lattices, this construction was performed in [7, 8] in three different ways. One of them, based upon the embedding $x \mapsto \bar{x}$ of a vector lattice X into $L^\sim(L^\sim(X,Y),Y)$ defined as $\bar{x}(T) := Tx$ $(T \in L^\sim(X,Y))$, was independently discovered in [56]. The main difference is that in [56] the Maharam extension was constructed for an arbitrary collection of order-bounded operators. For some further properties of the Maharam extension, see [39, 56].

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