DOI 10.1007/s10958-016-3034-4
Journal of Mathematical Sciences, Vol. 218, No. 3, October, 2016

ON QUASI-NONUNIFORM ESTIMATES FOR ASYMPTOTIC EXPANSIONS IN THE CENTRAL LIMIT THEOREM

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Improved asymptotic expansions are constructed in terms of the Chebyshev–Hermite polynomials in the local form of the central limit theorem for sums of independent identically distributed random variables under the condition of absolute integrability of some positive powers of the the characteristic function of a summand. The influence of the requirements to the order of existing moments on the accuracy of approximation is discussed. Theoretical results are illustrated by the example of a particular shifted exponential distribution.

Let $X_1, X_2, \ldots$ be independent random variables with zero means, unit variances, and common distribution P. Denote the distribution of the normalized sum $(X_1 + \ldots + X_n)/\sqrt{n}$ by P_n and the standard normal distribution with density $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ by Φ. The central limit theorem (CLT) states that for large n the distribution P_n is close to Φ. In probability theory there is a traditional problem of estimation of the proximity of P_n to Φ as well as of construction of asymptotic expansions which bring more accurate approximation of P_n than Φ. We are interested in approximations in the local form of CLT for densities. It is well known that in this problem it is necessary to impose additional restrictions to ensure the existence of these densities and the validity of the local form of the CLT. As such restriction we will use the condition

$$\int_{-\infty}^{\infty} |f(t)|^v dt < \infty, \tag{1}$$

where $f(t)$ is the characteristic function of the distribution P, v is a positive number. This condition guarantees the existence of continuous and bounded densities $p_n(x)$ for $n \geqslant v$ and the validity of the relation $p_n(x) \to \varphi(x)$, $n \to \infty$, $-\infty < x < \infty$, called the local form of CLT for densities. It should be noted that from the existence of the density $p_n(x)$ for some n it follows that $f(t) \to 0,\ t \to \infty$.

In order to obtain estimates of proximity of $p_n(x)$ and $\varphi(x)$ and to construct asymptotic expansions for densities $p_n(x)$ it is necessary to impose additional restrictions on the distribution P associated with the existence of moments $\alpha_k = \int_{-\infty}^{\infty} x^k P(dx)$, $k \geqslant 3$ is integer, or absolute moments $\beta_s = \int_{-\infty}^{\infty} |x|^s P(dx)$, $s > 2$. We will assume the existence of moments of the orders 4, 5, or 6.

We need the Chebyshev–Hermite polynomials $H_k(x) = (-1)^k \varphi^{(k)}(x)/\varphi(x)$, $k = 0, 1, \ldots$, in particular, $H_0(x) \equiv 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, $H_5(x) = x^5 - 10x^3 + 15x$, $H_6(x) = x^6 - 15x^4 + 45x^2 - 15$. From the results stated and proved below it follows that the estimate

$$|p_n(x) - \varphi(x)| \leqslant \frac{|A_1(x)|}{\sqrt{n}} \varphi(x) + O\left(\frac{1}{n}\right),\ n \to \infty, \tag{2}$$

is valid for the distributions P with finite fourth moment, where

$$A_1(x) = \frac{\theta_3}{3!} H_3(x).$$

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Proceedings of the XXXII International Seminar on Stability Problems for Stochastic Models, Trondheim, Norway, June 16–21, 2014.

For the distributions P with finite fifth moment we have

$$\left|p_n(x) - \varphi(x)\left(1 + \frac{A_1(x)}{\sqrt{n}}\right)\right| \leqslant \frac{|A_2(x)|}{n}\varphi(x) + O\left(\frac{1}{n^{3/2}}\right),\ n \to \infty, \tag{3}$$

where

$$A_2(x) = \frac{\theta_4}{4!}H_4(x) + \frac{1}{2}\left(\frac{\theta_3}{3!}\right)^2 H_6(x).$$

Whereas for the distributions P with finite sixth moment

$$\left|p_n(x) - \varphi(x)\left(1 + \frac{A_1(x)}{\sqrt{n}} + \frac{A_2(x)}{n}\right)\right| \leqslant \frac{|A_3(x)|}{n^{3/2}}\varphi(x) + O\left(\frac{1}{n^2}\right),\ n \to \infty, \tag{4}$$

where

$$A_3(x) = \frac{\theta_5}{5!}H_5(x) + \frac{\theta_3}{3!}\frac{\theta_4}{4!}H_7(x) + \frac{1}{6}\left(\frac{\theta_3}{3!}\right)^3 H_9(x).$$

In (2)–(4) the first terms on the right-hand sides are equivalent (for $n \to \infty$) to the left-hand sides for all x where these terms do not vanish, and their calculation is not difficult. We call these estimates quasinonuniform due to the fact that the term "nonuniform estimates" is occupied, and some terms on the right-hand sides of (2)–(4) are uniform over x. The left-hand sides of (2)–(4) change noticeably with x, and the right-hand sides of (2)–(4) capture these changes. The values θ_l can be calculated by formula (5). Explicit estimates for the values $O(\cdots)$ are given below. All these results were obtained using the Edgeworth–Cramér expansions of the density $p_n(x)$.

Here and below $\theta_l = \int\limits_{-\infty}^{\infty} H_l(x)P(dx)$ are the numbers that we call Chebyshev–Hermite moments of the distribution P; they are finite if and only if moments α_l are finite and can be calculated by the formulas

$$\frac{\theta_l}{l!} = \sum_{j=0}^{[l/2]} \frac{\alpha_{l-2j}}{(l-2j)!}\frac{(-1)^j}{2^j j!}, \qquad l = 0, 1, \dots. \tag{5}$$

In particular, $\theta_0 = 1$ for any distribution P; for distributions with zero mean and unit variance, we have $\theta_1 = \theta_2 = 0$, $\theta_3 = \alpha_3$, $\theta_4 = \alpha_4 - 3$, $\theta_5 = \alpha_5 - 10\alpha_3$.

Good estimates of the values $O(\cdots)$ from (2)–(4) are quite cumbersome, but for each distribution P and for any number n each of them can be brought to numerical values. Below we will present some numerical illustrations that use the centered exponential distribution with parameter 1 as the distribution P; its density is $p(x) = 0$, $x < -1$, and $p(x) = e^{-(x+1)}$, $x \geqslant -1$. For brevity we will call it just the exponential distribution (ED). It has zero mean, unit variance, and for it $\alpha_3 = 2$, $\alpha_4 = 9$, $\alpha_5 = 44$, $\alpha_6 = 265$. This distribution is one of the exclusive distributions for which the density $p_n(x)$ is very easy to calculate. For the ED we have

$$p_n(x) = \sqrt{n}\frac{n^n}{n!e^n}\left(1 + \frac{x}{\sqrt{n}}\right)^{n-1} e^{-x\sqrt{n}}, \qquad x \geqslant -\sqrt{n},$$

(these are the densities of the centered and normalized Erlang distributions). The explicit form of the densities allows us to get numerical and graphical illustrations of the results. In order to build graphics tools that are available on the site `ru.numberempire.com` can be used. For better understanding of the essence of matter one can build for $n = 100$ the graphs of the functions:

1) $p_n(x)$, $\varphi(x)$, and $p_n(x) - \varphi(x)$;
2) $p_n(x) - \varphi(x)$, $A_1(x)\varphi(x)/\sqrt{n}$, and $p_n(x) - \varphi(x)(1 + (A_1(x)\varphi(x))/\sqrt{n})$;
3) $p_n(x) - \varphi(x)(1 + (A_1(x)\varphi(x))/\sqrt{n})$, $(A_2(x)\varphi(x))/n$, and their difference.

It is useful to compare the last difference with the function $A_3(x)\varphi(x)/n^{3/2}$. In order to build these graphs the numbers $\theta_3/3!$, $\theta_4/4!$, $\theta_5/5!$ are needed, which are 1/3, 1/4, 1/5 for the ED, and the number $\sqrt{100}\frac{100^{100}}{100!e^{100}} = 0.39860996$. It is possible to obtain the formulas for $H_7(x)$ and $H_9(x)$ by differentiating the function $e^{-x^2/2}$; they can be found in [2] and [3].

Now we give the asymptotic values (see below) of the upper bounds of the values $O(\cdots)$ from (2)–(4) for the ED (the symbol $\lesssim$ should read "does not exceed the value that is equivalent to ..."):

$$O\left(\frac{1}{n}\right) \lesssim 0.03\frac{\beta_4}{n} + \frac{|A_2^*(x)|\varphi(x)}{n} + \frac{1.9}{n^{3/2}} + \frac{6}{n^2} + \frac{10.4}{n^{5/2}}, \qquad n \to \infty, \tag{6}$$

where

$$A_2^*(x) = H_4(x)/40 + H_6(x)/18;$$

$$O\left(\frac{1}{n^{3/2}}\right) \lesssim 0.0124\frac{\beta_5}{n^{3/2}} + \frac{|A_3^*(x)|\varphi(x)}{n^{3/2}} + \frac{11.4}{n^2} + \frac{57}{n^{5/2}} + \frac{72}{n^3} + \frac{1.5}{n^{7/2}}, \qquad n \to \infty, \tag{7}$$

where

$$A_3^*(x) = -H_5(x)/72 + H_7(x)/12 + H_9(x)/162;$$

$$O\left(\frac{1}{n^2}\right) \lesssim 0.0048\frac{\beta_6}{n^2} + \frac{|A_4^*(x)|\varphi(x)}{n^2} + \frac{71}{n^{5/2}} + \frac{434}{n^3} + \frac{736}{n^{7/2}} + \frac{13}{n^4} + \frac{13}{n^5}, \qquad n \to \infty, \tag{8}$$

where

$$A_4^*(x) = -0.043651H_6(x) + (1/15 + 1/32)H_8(x) + H_{10}(x)/72 + H_{12}(x)/1944.$$

To calculate the first terms on the right-hand sides of (6)–(8), we did not use the assumption that we are dealing with the ED. The polynomials $A_2^*(x)$ and $A_3^*(x)$ are the analogs of $A_2(x)$ and $A_3(x)$, but in these definitions the values $\theta_4^{(4,\lambda_4)} = 0.4\alpha_4 - 3$ and $\theta_5^{(5,\lambda_5)} = 5\alpha_5/12 - 10\alpha_3$ are used instead of $\theta_4 = \alpha_4 - 3$ and $\theta_5 = \alpha_5 - 10\alpha_3$ (the formal definitions $\theta_k^{(k,\lambda_k)}$ and λ_k are given below). The polynomial $A_4^*(x)$ is

$$\left(\frac{\theta_6^{(6,\lambda_6)}}{6!} - \frac{1}{2}\left(\frac{\theta_3}{3!}\right)^2\right)H_6(x) + \left(\frac{\theta_3}{3!}\frac{\theta_5}{5!} + \frac{1}{2}\left(\frac{\theta_4}{4!}\right)^2\right)H_8(x) + \frac{1}{2}\left(\frac{\theta_3}{3!}\right)^2\frac{\theta_4}{4!}H_{10}(x) + \frac{1}{24}\left(\frac{\theta_3}{3!}\right)^4 H_{12}(x),$$

where

$$\frac{\theta_6^{(6,\lambda_6)}}{6!} = \frac{3\alpha_6/7}{6!} - \frac{\alpha_4}{4!2} + \frac{1}{24}.$$

Inequalities (6)–(8) follow from Theorems 1 and 2 involving the values $B_{k,n,j}$, $j = 1, 2$, which are equivalent to $\int_{-\infty}^{\infty} |x|^k \varphi(x)dx/2\pi$. These limit values are used in the calculations of the right-hand sides of (6)–(8).

Let us make some comments about the estimates (2), (6) and give some modifications of (6). Looking at the graphs of the functions $p_n(x) - \varphi(x)$ and $A_1(x)\varphi(x)/\sqrt{n}$ for the ED for $n = 100$ it is easy to see that they are very close to each other (the maximum of the absolute value of the difference between these functions does not exceed 0.00115). From the well-known results it follows that for distributions P with the finite fourth moment under condition (1) the following expansion is valid:

$$p_n(x) - \varphi(x) = \frac{\alpha_3}{3!\sqrt{n}}H_3(x)\varphi(x) + \frac{\alpha_4 - 3}{4!n}H_4(x)\varphi(x) + \frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x) + o\left(\frac{1}{n}\right), \quad n \to \infty. \tag{9}$$

The order of decrease of the value $o(1/n)$ in the general case cannot be improved.

From (9) it follows that

$$\left|p_n(x)-\varphi(x)-\frac{\alpha_3}{3!\sqrt{n}}H_3(x)\varphi(x)\right| \leqslant \left|\frac{\alpha_4-3}{4!n}H_4(x)\varphi(x)+\frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x)\right| + o\left(\frac{1}{n}\right), \quad n\to\infty. \tag{10}$$

This estimate cannot be improved at all points x where the first term on the right side does not vanish, because for these x the first term on the right side is equivalent to

$$\left|p_n(x)-\varphi(x)-\frac{\alpha_3}{3!\sqrt{n}}H_3(x)\varphi(x)\right|$$

(note that for the ED for $n = 100$ the maximum value of the last function is close to 0.00115; the maximum value of the function $|p_n(x)-\varphi(x)|$ is equal to $0.0193\ldots$). The only drawback of (10) is the presence of the value $o\left(\frac{1}{n}\right)$, which makes it unusable for quantitative computation. Analogs of (10) that are devoid of this defect can be to obtained from Theorem 1 formulated and proved below. From this theorem it follows that for all x

$$p_n(x)-\varphi(x)=\frac{\alpha_3}{3!\sqrt{n}}H_3(x)\varphi(x)+\frac{\lambda\alpha_4-3}{4!n}H_4(x)\varphi(x)+\frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x)+R,$$

where $0\leqslant\lambda\leqslant 1$ is a parameter whose choice is at our disposal and

$$|R|\leqslant q_4(\lambda)\frac{\beta_4}{4!n}\frac{3}{\sqrt{2\pi}}+O\left(\frac{1}{n^{3/2}}\right), \quad n\to\infty,$$

where

$$q_4(\lambda)=\sup_{u>0}\frac{|e^{iu}-(1+iu+\frac{(iu)^2}{2!}+\frac{(iu)^3}{3!}+\lambda\frac{(iu)^4}{4!})|}{u^4/4!}.$$

For $0\leqslant\lambda\leqslant 0.4$ we have the equality $q_4(\lambda)=1-\lambda$; for $\lambda\geqslant 0.4$ the sum $\lambda+q_4(\lambda)$ is greater than 1 and always $q_4(\lambda)\leqslant 1$, $q_4(1)=1$. It is possible to obtain an explicit estimate of $O(\frac{1}{n^{3/2}})$ from Theorem 1. From the last expansion of $p_n(x)-\varphi(x)$ we obtain the estimate

$$\left|p_n(x)-\varphi(x)-\frac{\alpha_3}{3!\sqrt{n}}H_3(x)\varphi(x)-\frac{\lambda\alpha_4-3}{4!n}H_4(x)\varphi(x)-\frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x)\right|\leqslant$$

$$\leqslant(1-\lambda)\frac{\beta_4}{4!n}\frac{3}{\sqrt{2\pi}}+O\left(\frac{1}{n^{3/2}}\right), \qquad n\to\infty, \tag{11}$$

for any $0\leqslant\lambda\leqslant 0.4$. This estimate cannot be improved in the following sense: the multiplication of the first term on its right-hand side by any constant less than 1 makes the evaluation at $x=0$ incorrect for sufficiently large n.

To make this sure, note that (9) implies the relation

$$p_n(x)-\varphi(x)\sim\frac{\alpha_3}{3!\sqrt{n}}H_3(x)\varphi(x)+\frac{\alpha_4-3}{4!n}H_4(x)\varphi(x)+\frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x), \qquad n\to\infty,$$

for those x, where the right-hand side does not vanish or, which is the same,

$$p_n(x)-\varphi(x)-\frac{\alpha_3}{3!\sqrt{n}}H_3(x)\varphi(x)-\frac{\lambda\alpha_4-3}{4!n}H_4(x)\varphi(x)-\frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x)\sim$$

$$\sim\frac{(1-\lambda)\alpha_4}{4!n}H_4(x)\varphi(x), \qquad n\to\infty, \tag{12}$$

for those x where $H_4(x) \neq 0$. This is evident if we note that $\alpha_4 = \beta_4$ and $H_4(0)\varphi(0) = \frac{3}{\sqrt{2\pi}}$.

Thus, estimate (10) which cannot be brought to the numerical values cannot be improved at all points x, with the possible exception of six; estimate (11) which can be brought to the numerical values cannot be improved only at one point $x = 0$. For the ED the asymptotic expansion

$$\frac{\alpha_3}{3!\sqrt{n}} H_3(x)\varphi(x) + \frac{\lambda\alpha_4 - 3}{4!n} H_4(x)\varphi(x) + \frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x) \tag{13}$$

of the difference $p_n(x) - \varphi(x)$, which is used in (11), for $\lambda = 0.4$ (this is the maximal value at which (12) is valid) approximates this difference for a great number of points x worse than the first term of this expansion. This is easily seen by drawing the graphs of the function $p_n(x) - \varphi(x) - \frac{\alpha_3}{3!\sqrt{n}} H_3(x)\varphi(x)$ for the ED as well as the difference between $p_n(x) - \varphi(x)$ and function (13). In particular, for $x = 0$ and $n = 100$ we have $p_n(x) - \varphi(x) = p_n(x) - \varphi(x) - \frac{\alpha_3}{3!\sqrt{n}} H_3(x)\varphi(x) = -0.000332\ldots$, at the same time the difference between $p_n(x) - \varphi(x)$ and the function (13) is equal to $0.00269\ldots$. This disadvantage is partly offset by the fact that for expansion (13) we have an explicit estimate, but precisely because of it, we use the inequality

$$\left| p_n(x) - \varphi(x) - \frac{\alpha_3}{3!\sqrt{n}} H_3(x)\varphi(x) \right| \leqslant$$

$$\leqslant 0.6 \frac{\beta_4}{4!n} \frac{3}{\sqrt{2\pi}} + \left| \frac{0.4\alpha_4 - 3}{4!n} H_4(x)\varphi(x) + \frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x) \right| + O\left(\frac{1}{n^{3/2}}\right),$$

whence follow (2) and (6) ((6) is valid for the ED).

It is easy to understand that the drawback of expansion (13) mentioned above, although linked to its merits, is associated with the “irregularity” of the asymptotic expansions. All the terms of the asymptotic expansions of the difference $p_n(x) - \varphi(x)$, beginning with the second, consist of several summands. If the distribution P has many moments, then by adding new terms to the asymptotic expansion, in general, we usually improve the accuracy of the approximation of the difference $p_n(x) - \varphi(x)$. However, if we add terms that make up new terms of expansion, one by one, the picture may change dramatically. Consider a simple example. The second term of asymptotic expansion of the difference $p_n(x) - \varphi(x)$ is the sum of two summands

$$\frac{\alpha_4 - 3}{4!n} H_4(x)\varphi(x) + \frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x).$$

Sketching the graph of this sum and the individual terms for the ED it is easy to see that the maximum absolute value of the sum is considerably less than the maximum of the absolute values of each term; moreover, near the local extrema the terms have opposite signs and the summation annihilates each other. In particular, for the ED for $n = 100$ at the point $x = 0$ the first of these terms is equal to $0.002992\ldots$, the second is equal to -0.003324, and their sum is equal to -0.0003325.

However, if we have information only about the fourth moment of the distribution P, then in order to get expansions the accuracy of which cannot be improved at least at one point, we will be able to include only the value $\lambda\alpha_4$ in the main part of the expansion, $0 \leqslant \lambda \leqslant 0.4$.

Expansion (13) can be slightly transformed and written in the form

$$\frac{\alpha_3}{3!\sqrt{n}} H_3(x)\varphi(x) + \lambda \frac{\alpha_4 - 3}{4!n} H_4(x)\varphi(x) + \frac{\lambda}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x) +$$

$$+ (1 - \lambda)\left(\frac{-3}{4!} H_4(x)\varphi(x) + \frac{1}{2}\left(\frac{\alpha_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x)\right).$$

Now it is easy to see that when $0 \leqslant \lambda \leqslant 0.4$, we have

$$\left| p_n(x) - \varphi(x) - \left(\frac{A_1(x)}{\sqrt{n}} + \lambda \frac{A_2(x)}{n} \right) \varphi(x) \right| \leqslant$$

$$\leqslant (1-\lambda) \frac{\beta_4}{4!n} \frac{3}{\sqrt{2\pi}} + (1-\lambda) \left| \frac{-3}{4!n} H_4(x)\varphi(x) + \frac{1}{2} \left(\frac{\alpha_3}{3!\sqrt{n}} \right)^2 H_6(x)\varphi(x) \right| + O\left(\frac{1}{n^{3/2}} \right),$$

whence for $\lambda = 0.4$ we obtain

$$\left| p_n(x) - \varphi(x) - \left(\frac{A_1(x)}{\sqrt{n}} + 0.4 \frac{A_2(x)}{n} \right) \varphi(x) \right| \leqslant 0.03 \frac{\beta_4 + 3}{n} + 0.05 \left(\frac{\alpha_3}{\sqrt{n}} \right)^2 + O\left(\frac{1}{n^{3/2}} \right), \tag{14}$$

here we took into account the fact that $\max_x |H_{2k}(x)\varphi(x)| = |H_{2k}(0)|\varphi(0) = (2k-1)!!/\sqrt{2\pi} \leqslant 0.4(2k-1)!!$. The function

$$\left(\frac{A_1(x)}{\sqrt{n}} + 0.4 \frac{A_2(x)}{n} \right) \varphi(x)$$

approximates the difference $p_n(x) - \varphi(x)$ for the ED better than the function $A_1(x)\varphi(x)/\sqrt{n}$, estimate (14) is good enough, but we cannot speak of the unimprovability of (14), since, for example, for the ED with $n = 100$ the left-hand side of (14) does not exceed 0.00075, whereas $0.03\frac{\beta_4}{n} = 0.0027$.

It should be noted that the "irregularity" of the asymptotic expansions does not take place for all distributions. If the "irregular" does not take place, then the expansion from Theorem 1 is acceptable. Apparently, the second term of the asymptotic expansion of the difference $p_n(x) - \varphi(x)$ is regular for the distributions P with $\alpha_4 \leqslant 3$.

In order to avoid the hassles associated with the "irregularity" (if there is any), instead of approximation (13) of the difference $p_n(x) - \varphi(x)$ we can use the approximation

$$\frac{\alpha_3}{3!\sqrt{n}} H_3(x)\varphi(x) + \frac{\alpha_4 - 3}{4!n} H_4(x)\varphi(x) + \frac{1}{2} \left(\frac{\alpha_3}{3!\sqrt{n}} \right)^2 H_6(x)\varphi(x), \tag{15}$$

related to the case $\lambda = 1$; for this approximation estimate (11) is valid with $1 - \lambda$ replaced by 1, that is, the following inequality holds:

$$\left| p_n(x) - \varphi(x) - \frac{A_1(x)}{\sqrt{n}} \varphi(x) - \frac{A_2(x)}{n} \varphi(x) \right| \leqslant \frac{\beta_4}{4!n} \frac{3}{\sqrt{2\pi}} + O\left(\frac{1}{n^{3/2}} \right), \qquad n \to \infty. \tag{16}$$

Thus, for approximation (15) we can obtain an estimate that is somewhat less accurate than (11); at the same time approximation (15) is significantly more precise than all the approximations that were discussed above. This is easy to ascertain by drawing the graph of the difference between $p_n(x) - \varphi(x)$ and function (15) for the ED. In fact, estimate (16) is very rough because approximation (15) is very precise. In particular, the maximum value of the absolute value of the mentioned difference for the ED with $n = 100$ is equal to 0.0001555, the quantity $\frac{\beta_4}{4!n} \frac{3}{\sqrt{2\pi}}$ equals 0.004488, and the ratio of right-hand and left-hand sides of (16) is greater than 28. Such a large distinction of the right-hand and left-hand sides of (14) questions the quality of estimate (16). It is hardly possible to consider the quality of an estimate which cannot correctly reflect the order of the estimated quantity as high. The cause of roughness of estimate (16) is clear. The quantity under the absolute value on the left-hand side is equivalent to the third term of the asymptotic expansion of the difference $p_n(x) - \varphi(x)$, i.e., to the quantity $A_3(x)\varphi(x)/n^{3/2}$, the maximum of the absolute value of which does not exceed 0.000137 for the ED with $n = 100$, and on the right-hand side of (16) the moments of order no higher than 4 are used so that it cannot decrease faster than $1/n$. However, from estimate (16) it is possible to obtain fully informative results for less precise expansions. So, (16) implies the relation

$$\left|p_n(x) - \varphi(x) - \frac{A_1(x)}{\sqrt{n}}\varphi(x)\right| \leqslant \frac{|A_2(x)|}{n}\varphi(x) + \frac{\beta_4}{4!n}\frac{3}{\sqrt{2\pi}} + O\left(\frac{1}{n^{3/2}}\right), \qquad n \to \infty.$$

As it has been mentioned above, the maximum value of the right-hand side approximately equals 0.00115 for the ED in case $n = 100$. It is easy to check that the maximum value of the first addend on the right-hand side of this inequality is less than 0.00101, and the ratio of the sum of the first two addends on the right-hand side to the maximum of the left-hand side is less than 4.8, i.e., the right-hand side of the last inequality gives at least the right order of the maximum of the left-hand side.

Obviously, the following inequality follows from (16):

$$|p_n(x) - \varphi(x)| \leqslant \left|\frac{A_1(x)}{\sqrt{n}}\varphi(x) + \frac{A_2(x)}{n}\varphi(x)\right| + \frac{\beta_4}{4!n}\frac{3}{\sqrt{2\pi}} + O\left(\frac{1}{n^{3/2}}\right), \qquad n \to \infty.$$

It is easy to notice that the graphs of the left-hand side and of the first addend of the right-hand side of this inequality are very close to the ED when $n = 100$ just by plotting them. Moreover, if we add 0.00156 to the first addend on the right-hand side, then we get a function that estimates the left-hand side on the real axis. The second addend on the right-hand side of the last inequality for the ED for $n = 100$ is almost 30 times greater than 0.000156. However, this estimate is consistent because the second addend on the right-hand side is less than 25% of the maximum value of the first addend (and about 6% when $n = 1600$, it is necessary to take into account that $\sqrt{1600}\frac{1600^{1600}}{1600!e^{1600}} = 0.398921503$ when counting $p_n(x)$, $n = 1600$).

The same reasoning can be given for (3), (7) and (4), (8).

The technique used below is based on the application of the inversion formula for the Fourier transform

$$p_n(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-itx} f^n\left(\frac{t}{\sqrt{n}}\right) dt, \quad -\infty < x < \infty, \tag{17}$$

for all $n \geqslant v$, the repeated application of the equality

$$a^k - b^k = \sum_{j=0}^{k-1} a^{k-j-1} b^j (a-b), \tag{18}$$

for any complex a, b and natural k the expansion of the multiplication $e^{t^2/2} f(t)$ in the segment of a Taylor series, and modifications of Taylor expansions of characteristic functions, suggested in [7]. We will say a few words about this.

One of the key roles in application of the method of characteristic functions is played by the expansions of characteristic functions in the segments of a Taylor series. These expansions of the characteristic functions of distributions P with finite moments of order k usually look like

$$f(t) = \sum_{j=0}^{k} \frac{\alpha_j}{j!}(it)^j + o(t^k), \quad t \to 0, \tag{19}$$

or

$$f(t) = \sum_{j=0}^{k-1} \frac{\alpha_j}{j!}(it)^j + \gamma\frac{\beta_k}{k!}t^k, \quad -\infty < t < \infty,$$

where $\gamma = \gamma(t)$ is a complex function such that $|\gamma| \leqslant 1$. Further on, all such functions (even within one formula) will be designated as γ, the relation $h(t) = \gamma H(t)$ is equivalent to the inequality $|h(t)| \leqslant |H(t)|$, therefore, for example, the equalities $-\gamma = \gamma$, $\gamma\cdot\gamma = \gamma$ take place; if $A(t), B(t)$ are nonnegative functions, then $\gamma A(t) + \gamma B(t) = \gamma(A(t) + B(t))$ where the functions γ are different, etc. In [7] modifications of these expansions were suggested that allow us to write $f(t)$ in the form

$$f(t) = \sum_{j=0}^{k-1} \frac{\alpha_j}{j!}(it)^j + \lambda\frac{\alpha_k}{k!}(it)^k + \gamma q_k(\lambda)\frac{\beta_k}{k!}t^k, \quad -\infty < t < \infty. \tag{20}$$

Here λ is an arbitrary number from $[0,1]$, the definitions of the functions $q_k(\lambda)$ are the same as that of the function $q_4(\lambda)$ given above. For our purposes it is only important that the equalities $q_k(\lambda) = 1-\lambda$ and $q_k(1) = 1$ take place if $0 \leqslant \lambda \leqslant \lambda_k = \frac{k}{2(k+1)}$. The studies on these expansions were continued in [6]; see also [5].

The powers of the variable it in the expansions of the characteristic function $f(t)$ given above are associated with the moments of the distribution P. For the function $e^{t^2/2}f(t)$ (similarly to (19)), instead of the moments α_j, the Chebyshev–Hermite moments θ_j of the distribution P are used. The use of these moments is justified by the fact that in the expansions of the densities $p_n(x)$ by the system of functions $H_l(x)\varphi(x)$, $l = 0, 1, \ldots$, the coefficients of these functions are formed by means of the moments and the Chebyshev–Hermite quasimoments of the distribution P_n, and they are related to the moments and the Chebyshev–Hermite quasimoments of the distribution P in a quite simple way (see, e.g., [3, Chap. 4, Sections 1 and 4]). We will use this relationship in the proof of the lemmas given below. In this case it is not necessary to refer to [3]. The expansions of the densities $p_n(x)$ are the same as the expansions of the relation $p_n(x)/\varphi(x)$ in the Chebyshev–Hermite polynomials, and this system of polynomials is a complete orthogonal system of functions in the Hilbert space $L_2(\varphi)$.

We will use two types of the Chebyshev–Hermite quasimoments of the distribution P. The quasimoments $\theta_l^{(k)}$, $k \leqslant l$, of the distribution P can be calculated by formula (5) on the right-hand side of which it is required to omit terms related to the moments α_j, $j > k$, and the quasimoments $\theta_l^{(k,\lambda)}$, $k \leqslant l$, can be calculated as $\theta_l^{(k)}$ but instead of α_k, the product $\lambda\alpha_k$ is used, where λ is the one from (20). In the estimates of the residuals of the expansions, the Chebyshev–Hermite moments and quasimoments with the norm sign $||\cdot||$ will be used. The formulas for these quantities will be given at their appearance.

Let us make some comments about the technique of the construction of expansions and introduce some notation. When $n \geqslant v$, inversion formula (17) yields

$$p_n(x) - \varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\left(f^n\left(\frac{t}{\sqrt{n}}\right) - g^n\left(\frac{t}{\sqrt{n}}\right)\right)dt, \qquad -\infty < x < \infty,$$

where $g(t) = e^{-t^2/2}$ is the characteristic function of the standard normal law. The right-hand side of this equality can be written as

$$\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}} e^{-itx}\left(f^n\left(\frac{t}{\sqrt{n}}\right) - g^n\left(\frac{t}{\sqrt{n}}\right)\right)dt +$$

$$+\frac{1}{2\pi}\int_{|t|>T\sqrt{n}} e^{-itx}f^n\left(\frac{t}{\sqrt{n}}\right)dt - \frac{1}{2\pi}\int_{|t|>T\sqrt{n}} e^{-itx}g^n\left(\frac{t}{\sqrt{n}}\right)dt, \tag{21}$$

where $T > 0$ is the parameter, the choice of which is discussed below. It is almost obvious that the absolute value of the last element does not exceed

$$\frac{1}{\pi T\sqrt{n}}e^{-T^2n/2}, \tag{22}$$

and the absolute value of the last but one element does not exceed

$$\frac{\sqrt{n}}{\pi}\int\limits_T^\infty |f(t)|^n dt \leqslant \frac{\sqrt{n}\alpha^{n-v}(T)}{\pi}\int\limits_T^\infty |f(t)|^v dt, \tag{23}$$

where

$$\alpha(T) = \sup\{|f(t)| : t \geqslant T\} < 1,$$

and these terms decrease exponentially as n grows. To prove the last inequality we should notice that the assumption that the supremum is equal to 1 and the relation $f(t) \to 0$, $t \to \infty$, which was discussed after formula (1), result in that $|f(t_0)| = 1$ for some $t_0 \geqslant T$, which means that the distribution P is lattice, contradicting the fact that $f(t) \to 0$, $t \to \infty$.

We will consider the first element in (21). From (18) it follows that

$$f^n\left(\frac{t}{\sqrt{n}}\right) - g^n\left(\frac{t}{\sqrt{n}}\right) = \left(f\left(\frac{t}{\sqrt{n}}\right) - g\left(\frac{t}{\sqrt{n}}\right)\right)\sum_{j=0}^{n-1} f^{n-j-1}g^j = \left(e^{t^2/2n}f\left(\frac{t}{\sqrt{n}}\right) - 1\right)\sum_{j=0}^{n-1} f^{n-j-1}g^{j+1}.$$

The arguments of the powers of $f(\cdot)$ and $g(\cdot)$ are $t/\sqrt{n}$, and from now on we will omit them. Using the expansion of $e^{t^2/2n}f(\frac{t}{\sqrt{n}})$, following from Lemma 1, we can think of the first multiplier on the right-hand side of the last equality as a linear combination of

$$A(P)\left(\frac{it}{\sqrt{n}}\right)^l \quad \text{and} \quad \gamma C(P)\left(\frac{t}{\sqrt{n}}\right)^l e^{t^2/2n};$$

hence $f^n(\frac{t}{\sqrt{n}}) - g^n(\frac{t}{\sqrt{n}})$ is a linear combination of

$$A(P)\left(\frac{it}{\sqrt{n}}\right)^l \sum_{j=0}^{n-1} f^{n-j-1}g^{j+1} \quad \text{and} \quad \gamma C(P)\left(\frac{t}{\sqrt{n}}\right)^l \sum_{j=0}^{n-1} f^{n-j-1}g^j. \tag{24}$$

We need to find the integral of this difference multiplied by $e^{-itx}/2\pi$ over the interval $[-T\sqrt{n}, T\sqrt{n}]$. The integral of the second term in (24) is bounded by

$$\frac{|C(P)|}{n^{l/2}}\frac{1}{2\pi}\int\limits_{-T\sqrt{n}}^{T\sqrt{n}} |t|^l n\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)dt = \frac{|C(P)|}{n^{(l-2)/2}}B_{l,n,1},$$

where $\mu(t) = \max(|f(t)|, g(t))$,

$$B_{l,n,j} = \frac{1}{2\pi}\int\limits_{-T\sqrt{n}}^{T\sqrt{n}} |t|^l \mu^{n-j}\left(\frac{t}{\sqrt{n}}\right)dt, \quad 0 \leqslant j < n, \quad l \geqslant 0.$$

For any real t and any fixed j we have $\mu^{n-j}(\frac{t}{\sqrt{n}}) \to e^{-t^2/2}$ as $n \to \infty$. Moreover, with T chosen in an appropriate way,

$$B_{l,n,j} \to B_l = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |t|^l e^{-t^2/2}dt, \ n \to \infty.$$

However, in most cases, good bounds for $|f(t)|$ as well as for $\mu(t)$, can be obtained only for small t and this dictates the choice of the parameter T. Also note that for some distributions in the definition of $B_{l,n,j}$ we can consider the limit as $T \to \infty$, and the last relation remains valid.

Hence the second quantity in (24) is equivalent to

$$\frac{|C(P)|}{n^{(l-2)/2}}B_l.$$

The first quantity in (24) can be expressed as

$$A(P)\left(\frac{it}{\sqrt{n}}\right)^l S_1,$$

where

$$S_k=\sum_{j_1+\ldots+j_k\leqslant n-k} f^{n-j_1-\ldots-j_k-k}g^{j_1+\ldots+j_k+k},$$

$1\leqslant k\leqslant n$ (here the sum is over all sets of nonnegative integer numbers $j_1,\ldots,j_k$ such that $j_1+\ldots+j_k\leqslant$ $\leqslant n-k$; the total number of sets is C_n^k). It is easy to see that for any k, $1\leqslant k<n$, we have

$$S_k=C_n^k g^n+\left(e^{t^2/2n}f\left(\frac{t}{\sqrt{n}}\right)-1\right)S_{k+1}. \tag{25}$$

Indeed,

$$S_k=\sum_{j_1+\ldots+j_k\leqslant n-k-1}(f^{n-j_1-\ldots-j_k-k}-g^{n-j_1-\ldots-j_k-k})g^{j_1+\ldots+j_k+k}+C_n^k g^n.$$

By virtue of (18),

$$S_k=C_n^k g^n+\sum_{j_1+\ldots+j_k\leqslant n-k-1}\left(\sum_{j_{k+1}=0}^{n-j_1-\ldots-j_k-k-1} f^{n-j_1-\ldots-j_k-k-j_{k+1}-1}g^{j_{k+1}}(f-g)\right)g^{j_1+\ldots+j_k+k}=$$
$$=C_n^k g^n+\left(e^{t^2/2n}f\left(\frac{t}{\sqrt{n}}\right)-1\right)\times$$
$$\times\sum_{j_1+\ldots+j_k\leqslant n-k-1}\left(\sum_{j_{k+1}=0}^{n-j_1-\ldots-j_k-k-1} f^{n-j_1-\ldots-j_k-j_{k+1}-k-1}g^{j_1+\ldots+j_{k+1}+k+1}\right).$$

It is not difficult to verify that the last multiple sum coincides with S_{k+1} (more detailed arguments can be found in [3, Chap. 4, Paragraph 14], and in [4]). Thus, the first value in (24) can be written as

$$A(P)\left(\frac{it}{\sqrt{n}}\right)^l ng^n+A(P)\left(\frac{it}{\sqrt{n}}\right)^l\left(e^{t^2/2n}f\left(\frac{t}{\sqrt{n}}\right)-1\right)S_2.$$

The second addend can be converted more (these transformations depend on l), and the integration of the first gives the value

$$\frac{A(P)}{n^{(l-2)/2}}\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}} e^{-itx}(it)^l e^{-t^2/2}dt=\frac{A(P)}{n^{(l-2)/2}}H_l(x)\varphi(x)+K,$$

where

$$|K|\leqslant\frac{|A(P)|}{n^{(l-2)/2}}\frac{1}{\pi}\int_{T\sqrt{n}}^{\infty} t^l e^{-t^2/2}dt,$$

this value decreases exponentially as n grows. Here we used the well-known inversion formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}(it)^l e^{-t^2/2}dt = H_l(x)\varphi(x), \quad l = 0, 1, \dots .$$

Let us proceed to the exact statements and proofs. Below α_k and β_k are used to denote the moments of even orders.

Lemma 1. *For distributions P with a finite fourth moment, for any $0 \leqslant \lambda \leqslant 1$ the following representations are true $(-\infty < t < \infty)$:*

$$e^{t^2/2}f(t) = 1+\frac{\theta_3}{3!}(it)^3+\frac{\theta_4^{(4,\lambda)}}{4!}(it)^4+\frac{\theta_5^{(3)}}{5!}(it)^5+\gamma q_4(\lambda)\frac{\beta_4}{4!}t^4e^{t^2/2}+\gamma\frac{||\theta_6^{(4,\lambda)}||}{6!}t^6e^{t^2/2}+\gamma\frac{||\theta_7^{(3)}||}{7!}t^7e^{t^2/2}, \quad (26)$$

where

$$\frac{||\theta_6^{(4,\lambda)}||}{6!} = \frac{\lambda\alpha_4}{4!2} + \frac{1}{24}, \quad \frac{||\theta_7^{(3)}||}{7!} = \frac{|\alpha_3|}{3!2^2 2!};$$

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \frac{\theta_4^{(4,\lambda)}}{4!}(it)^4 + \gamma q_4(\lambda)\frac{\beta_4}{4!}t^4e^{t^2/2} + \gamma\frac{||\theta_5^{(3)}||}{5!}t^5e^{t^2/2} + \gamma\frac{||\theta_6^{(4,\lambda)}||}{6!}t^6e^{t^2/2}, \quad (27)$$

where

$$\frac{||\theta_5^{(3)}||}{5!} = \frac{|\alpha_3|}{3!2};$$

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \gamma\frac{||\theta_4||}{4!}t^4e^{t^2/2} + \gamma\frac{||\theta_5^{(3)}||}{5!}t^5e^{t^2/2}, \quad (28)$$

where

$$\frac{||\theta_4||}{4!} = \frac{\beta_4 + 3}{4!}.$$

Proof. From (20) it follows that

$$e^{t^2/2}f(t) = e^{t^2/2}\left(1 - \frac{t^2}{2} + \frac{\alpha_3}{3!}(it)^3 + \lambda\frac{\alpha_4}{4!}(it)^4\right) + \gamma q_4(\lambda)\frac{\beta_4}{4!}t^4e^{t^2/2}.$$

It is obvious that

$$e^{t^2/2}\left(1 - \frac{t^2}{2}\right) = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \sum_{j=3}^{\infty}\left(\frac{t^2}{2}\right)^j\frac{1}{j!} - \frac{t^2}{2}\left(1 + \frac{t^2}{2} + \sum_{j=2}^{\infty}\left(\frac{t^2}{2}\right)^j\frac{1}{j!}\right) =$$

$$= 1 - \frac{t^4}{8} + \left(\frac{t^2}{2}\right)^3\left(\sum_{j=0}^{\infty}\left(\frac{t^2}{2}\right)^j\left(\frac{1}{(j+3)!} - \frac{1}{(j+2)!}\right)\right) = 1 - \frac{t^4}{8} - \left(\frac{t^2}{2}\right)^3\sum_{j=0}^{\infty}\left(\frac{t^2}{2}\right)^j\frac{1}{(j+1)!(j+3)}.$$

Hence,

$$e^{t^2/2}\left(1 - \frac{t^2}{2}\right) = 1 - \frac{t^4}{8} + \gamma\frac{t^6}{24}e^{t^2/2}. \quad (29)$$

So,

$$e^{t^2/2}f(t) = 1 - \frac{t^4}{8} + \gamma\frac{t^6}{24}e^{t^2/2} + \frac{\alpha_3}{3!}(it)^3 + \frac{\alpha_3}{3!}(it)^3\frac{t^2}{2} + \frac{\alpha_3}{3!}(it)^3\sum_{j=2}^{\infty}\left(\frac{t^2}{2}\right)^j\frac{1}{j!} +$$

$$+\lambda\frac{\alpha_4}{4!}(it)^4 + \lambda\frac{\alpha_4}{4!}(it)^4\sum_{j=1}^{\infty}\left(\frac{t^2}{2}\right)^j\frac{1}{j!} + \gamma q_4(\lambda)\frac{\beta_4}{4!}t^4e^{t^2/2} = 1 + \frac{\alpha_3}{3!}(it)^3 + \frac{\lambda\alpha_4 - 3}{4!}(it)^4 - \frac{\alpha_3}{3!2}(it)^5+$$

$$+\gamma q_4(\lambda)\frac{\beta_4}{4!}t^4e^{t^2/2}+\gamma\lambda\frac{\alpha_4}{4!2}t^6e^{t^2/2}+\gamma\frac{t^6}{24}e^{t^2/2}+\gamma\frac{\alpha_3}{3!2^22!}t^7e^{t^2/2}.$$

The right-hand side of the last equality differs from the right-hand side of (26) only in the symbols used.

The proof of Eq. (27) differs from the previous one only in using the representation

$$e^{t^2/2}\frac{\alpha_3}{3!}(it)^3=\frac{\alpha_3}{3!}(it)^3+\frac{\alpha_3}{3!}(it)^3\sum_{j=1}^{\infty}\left(\frac{t^2}{2}\right)^j\frac{1}{j!}=$$

$$=\frac{\alpha_3}{3!}(it)^3-\frac{\alpha_3}{3!2}(it)^5\sum_{j=0}^{\infty}\left(\frac{t^2}{2}\right)^j\frac{1}{(j+1)!}=\frac{\theta_3}{3!}(it)^3+\gamma\frac{\alpha_3}{3!2}t^5e^{t^2/2}.$$

To prove (28) note that the following analogue of (29) holds:

$$e^{t^2/2}\left(1-\frac{t^2}{2}\right)=1+\gamma\frac{t^4}{8}e^{t^2/2},$$

so

$$e^{t^2/2}f(t)=e^{t^2/2}\left(1-\frac{t^2}{2}+\frac{\alpha_3}{3!}(it)^3\right)+\gamma\frac{\beta_4}{4!}t^4e^{t^2/2}=1+\gamma\frac{t^4}{8}e^{t^2/2}+\frac{\alpha_3}{3!}(it)^3e^{t^2/2}+\gamma\frac{\beta_4}{4!}t^4e^{t^2/2}=$$

$$=1+\frac{\alpha_3}{3!}(it)^3+\gamma\frac{\beta_4+3}{4!}t^4e^{t^2/2}+\gamma\frac{\alpha_3}{3!2}t^5e^{t^2/2}.$$

The right-hand side of the last equality differs from the right-hand side of (28) only in the symbols used.

Lemma 2. *For distributions P with a finite fourth moment, for any $0\leqslant\lambda\leqslant1$ and any $-\infty<t<\infty$ and $n\geqslant3$ the following representation is true*:

$$f^n\left(\frac{t}{\sqrt{n}}\right)-g^n\left(\frac{t}{\sqrt{n}}\right)=\frac{\theta_3}{3!\sqrt{n}}(it)^3g^n+\frac{\theta_4^{(4,\lambda)}}{4!n}(it)^4g^n+\frac{n-1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2(it)^6g^n+\gamma q_4(\lambda)\frac{\beta_4}{4!n}t^4\mu^{n-1}+$$

$$+\frac{\gamma}{2}q_4(\lambda)\frac{\theta_3}{3!\sqrt{n}}\frac{\beta_4}{4!n}t^7\mu^{n-1}+\frac{\gamma}{6}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_4||}{4!n}t^{10}\mu^{n-1}+\frac{\theta_5^{(3)}}{5!}\left(\frac{it}{\sqrt{n}}\right)^5S_1+2\frac{\theta_3}{3!}\frac{\theta_4^{(4,\lambda)}}{4!}\left(\frac{it}{\sqrt{n}}\right)^7S_2+$$

$$+\left(\frac{\theta_3}{3!}\right)^3\left(\frac{it}{\sqrt{n}}\right)^9S_3+\gamma\frac{||\theta_6^{(4,\lambda)}||}{6!n^2}t^6\mu^{n-1}+\frac{\gamma}{2}\frac{\theta_3}{3!\sqrt{n}}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}t^8\mu^{n-1}+\frac{\gamma}{2}\frac{\theta_4^{(4,\lambda)}}{4!n}\frac{||\theta_4||}{4!n}t^8\mu^{n-1}+\gamma\frac{||\theta_7^{(3)}||}{7!n^{5/2}}t^7\mu^{n-1}+$$

$$+\frac{\gamma}{2}\frac{\theta_3}{3!\sqrt{n}}\frac{||\theta_6^{(4,\lambda)}||}{7!n^2}t^9\mu^{n-1}+\frac{\gamma}{2}\frac{\theta_4^{(4,\lambda)}}{4!n}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}t^9\mu^{n-1}+\frac{\gamma}{6}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_5^{(3)}||}{5!n^{3/2}}t^{11}\mu^{n-1}.$$

Proof. The assertion stated before Lemma 1 entails the following:

$$f^n\left(\frac{t}{\sqrt{n}}\right)-g^n\left(\frac{t}{\sqrt{n}}\right)=\left(e^{t^2/2n}f\left(\frac{t}{\sqrt{n}}\right)-1\right)S_1. \tag{30}$$

Using (26) one can easily notice that the right-hand side of the last equation can be represented as

$$\frac{\theta_3}{3!}\left(\frac{it}{\sqrt{n}}\right)^3S_1+\frac{\theta_4^{(4,\lambda)}}{4!}\left(\frac{it}{\sqrt{n}}\right)^4S_1+\frac{\theta_5^{(3)}}{5!}\left(\frac{it}{\sqrt{n}}\right)^5S_1+$$

$$+\gamma q_4(\lambda)\frac{\beta_4}{4!}\left(\frac{t}{\sqrt{n}}\right)^4n\mu^{n-1}+\gamma\frac{||\theta_6^{(4,\lambda)}||}{6!}\left(\frac{t}{\sqrt{n}}\right)^6n\mu^{n-1}+\gamma\frac{||\theta_7^{(3)}||}{7!}\left(\frac{t}{\sqrt{n}}\right)^7n\mu^{n-1}$$

(hereafter we omit argument $t/\sqrt{n}$ of the function μ). Using Eq. (25) for $k = 1$, we see that the sum of the first two terms can be written as

$$\frac{\theta_3}{3!}\left(\frac{it}{\sqrt{n}}\right)^3 ng^n + \frac{\theta_4^{(4,\lambda)}}{4!}\left(\frac{it}{\sqrt{n}}\right)^4 ng^n + \frac{\theta_3}{3!}\left(\frac{it}{\sqrt{n}}\right)^3\left(e^{t^2/2n} f\left(\frac{t}{\sqrt{n}}\right) - 1\right) S_2 +$$
$$+\frac{\theta_4^{(4,\lambda)}}{3!}\left(\frac{it}{\sqrt{n}}\right)^4\left(e^{t^2/2n} f\left(\frac{t}{\sqrt{n}}\right) - 1\right) S_2. \tag{31}$$

Using Eq. (27), one can see that the third term in this expression may be represented as

$$\frac{\theta_3}{3!}\left(\frac{it}{\sqrt{n}}\right)^3\left(\frac{\theta_3}{3!}\left(\frac{it}{\sqrt{n}}\right)^3 + \frac{\theta_4^{(4,\lambda)}}{4!}\left(\frac{it}{\sqrt{n}}\right)^4\right) S_2 +$$
$$+\gamma q_4(\lambda)\frac{\theta_3}{3!}\frac{\beta_4}{4!}\left(\frac{t}{\sqrt{n}}\right)^7 C_n^2\mu^{n-1} + \gamma\frac{\theta_3}{3!}\frac{||\theta_5^{(3)}||}{5!}\left(\frac{t}{\sqrt{n}}\right)^8 C_n^2\mu^{n-1} + \gamma\frac{\theta_3}{3!}\frac{||\theta_6^{(4,\lambda)}||}{7!}\left(\frac{t}{\sqrt{n}}\right)^9 C_n^2\mu^{n-1}.$$

Using Eq. (28) one can see that the last term of (31) can be written as

$$\frac{\theta_3}{3!}\frac{\theta_4^{(4,\lambda)}}{4!}\left(\frac{it}{\sqrt{n}}\right)^7 S_2 + \gamma\frac{\theta_4^{(4,\lambda)}}{4!}\frac{||\theta_4||}{4!}\left(\frac{t}{\sqrt{n}}\right)^8 C_n^2\mu^{n-1} + \gamma\frac{\theta_4^{(4,\lambda)}}{4!}\frac{||\theta_5^{(3)}||}{5!}\left(\frac{t}{\sqrt{n}}\right)^9 C_n^2\mu^{n-1}.$$

Summing up the preliminary results, we obtain

$$f^n\left(\frac{t}{\sqrt{n}}\right) - g^n\left(\frac{t}{\sqrt{n}}\right) =$$
$$= \frac{\theta_3}{3!\sqrt{n}}(it)^3 g^n + \frac{\theta_4^{(4,\lambda)}}{4!n}(it)^4 g^n + \left(\frac{\theta_3}{3!}\right)^2\left(\frac{it}{\sqrt{n}}\right)^6 S_2 + \gamma q_4(\lambda)\frac{\beta_4}{4!n}t^4\mu^{n-1} + \frac{\gamma}{2}q_4(\lambda)\frac{\theta_3}{3!\sqrt{n}}\frac{\beta_4}{4!n}t^7\mu^{n-1} +$$
$$+\frac{\theta_5^{(3)}}{5!}\left(\frac{it}{\sqrt{n}}\right)^5 S_1 + 2\frac{\theta_3}{3!}\frac{\theta_4^{(4,\lambda)}}{4!}\left(\frac{it}{\sqrt{n}}\right)^7 S_2 + \gamma\frac{||\theta_6^{(4,\lambda)}||}{6!n^2}t^6\mu^{n-1} + \frac{\gamma}{2}\frac{\theta_3}{3!\sqrt{n}}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}t^8\mu^{n-1} +$$
$$+\frac{\gamma}{2}\frac{\theta_4^{(4,\lambda)}}{4!n}\frac{||\theta_4||}{4!n}t^8\mu^{n-1} + \gamma\frac{||\theta_7^{(3)}||}{7!n^{5/2}}t^7\mu^{n-1} + \frac{\gamma}{2}\frac{\theta_3}{3!\sqrt{n}}\frac{||\theta_6^{(4,\lambda)}||}{7!n^2}t^9\mu^{n-1} + \frac{\gamma}{2}\frac{\theta_4^{(4,\lambda)}}{4!n}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}t^9\mu^{n-1}. \tag{32}$$

Notice that the third term of the right-hand side of this equation can be written as

$$\left(\frac{\theta_3}{3!}\right)^2\left(\frac{it}{\sqrt{n}}\right)^6 C_n^2 g^n + \left(\frac{\theta_3}{3!}\right)^2\left(\frac{it}{\sqrt{n}}\right)^6\left(e^{t^2/2n} f\left(\frac{t}{\sqrt{n}}\right) - 1\right) S_3.$$

Then, using (28), we can easily verify the validity of the lemma.

Theorem 1. *For distributions P with a finite fourth moment and for which condition* (1) *is satisfied, for any $0 \leqslant \lambda \leqslant \lambda_4 = 0.4$ and any $-\infty < x < \infty$ for $n \geqslant \max(3, v)$ the following representation is true:*

$$p_n(x) - \varphi(x) = \frac{\theta_3}{3!\sqrt{n}}H_3(x)\varphi(x) + \frac{\theta_4^{(4,\lambda)}}{4!n}H_4(x)\varphi(x) + \frac{n-1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x) + R + K,$$

where

$$|R| \leqslant q_4(\lambda)\frac{\beta_4}{4!n}B_{4,n,1} + \frac{q_4(\lambda)}{2}\frac{|\theta_3|}{3!\sqrt{n}}\frac{\beta_4}{4!n}B_{7,n,1} + \frac{|\theta_5^{(3)}|}{5!n^{3/2}}B_{5,n,0} + \frac{|\theta_3|}{3!\sqrt{n}}\frac{|\theta_4^{(4,\lambda)}|}{4!n}B_{7,n,0} + \frac{1}{6}\left|\frac{\theta_3}{3!\sqrt{n}}\right|^3 B_{9,n,0} +$$
$$+\frac{||\theta_6^{(4,\lambda)}||}{6!n^2}B_{6,n,1} + \frac{1}{2}\frac{|\theta_3|}{3!\sqrt{n}}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}B_{8,n,1} + \frac{1}{2}\frac{|\theta_4^{(4,\lambda)}|}{4!n}\frac{||\theta_4||}{4!n}B_{8,n,1} + \frac{1}{6}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_4||}{4!n}B_{10,n,1} +$$

$$+\frac{||\theta_7^{(3)}||}{7!n^{5/2}}B_{7,n,1}+\frac{1}{2}\frac{|\theta_3|}{3!\sqrt{n}}\frac{||\theta_6^{(4,\lambda)}||}{6!n^2}B_{9,n,1}+\frac{1}{2}\frac{|\theta_4^{(4,\lambda)}|}{4!n}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}B_{9,n,1}+\frac{1}{6}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_5^{(3)}||}{5!n^{3/2}}B_{11,n,1},$$

and the value K is the sum of the terms that decrease exponentially as n grows; the absolute value of one of them is no greater than (22), the absolute value of the other is no greater than the right-hand side of (23), and the others can be bounded by the terms in the main part of the expansion for the difference $p_n(x)-\varphi(x)$ with the coefficients at $H_l(x)\varphi(x)$ replaced by their absolute values and these functions by

$$\frac{1}{\pi}\int_{T\sqrt{n}}^{\infty} t^l e^{-t^2/2}dt,$$

where T is an arbitrary parameter.

Proof. With account of the comments made before Lemma 1, it suffices to compare the statements of Lemma 2 and Theorem 1, if we notice that

$$|S_k| \leqslant C_n^k \mu^n.$$

After this note the statement of the theorem becomes obvious.

As has been already mentioned above, for some distributions, including the exponential one, it is allowed to pass to the limit as $T\to\infty$. If we perform the corresponding calculations (not too complicated, but quite cumbersome) with the numbers $B_{k,n,0}$ and $B_{k,n,1}$ being changed to their asymptotic equivalents B_k, we can be certain that for the ED, the bound of $|R|$ from Theorem 1 is equivalent to the sum no greater than

$$0.03\frac{\beta_4}{n}+0.032\frac{|\theta_3|}{\sqrt{n}}\frac{\beta_4}{n}+\frac{1.31}{n^{3/2}}+\frac{5.65}{n^2}+\frac{10.4}{n^{5/2}}<\frac{0.27}{n}+\frac{1.9}{n^{3/2}}+\frac{5.65}{n^2}+\frac{10.4}{n^{5/2}}$$

(here $\lambda=\lambda_4=0.4$).

Now let us notice that in the main part of the expansion from Theorem 1 we can separate the addend

$$-\frac{1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2 H_6(x)\varphi(x).$$

It appeared quite naturally but "too early." Its presence in the second expansion from Theorem 2 is necessary, but in the expansion from Theorem 1 it makes the approximation worse (one can easily make this sure by plotting), which is why we will "transfer" it to the residuary part of the expansion. Now it is easy to be certain about (6). The term, which decreases as n^{-2} as n grows in (6), increased a little as compared to the same addend on the right-hand side of the previous inequality. This is because we have added $\max_x \frac{|(\theta_3/3!)^2 H_6(x)\varphi(x)|}{2n^2} < \frac{1}{3n^2}$ to the last term mentioned.

Similarly, it is easy to ascertain the validity of the following representations.

Statement 1. *For distributions P with a finite fifth moment for all $0\leqslant\lambda\leqslant 1$ the following representation of the quantity $e^{t^2/2}f(t)$ takes place for all $-\infty<t<\infty$:*

$$e^{t^2/2}f(t)=1+\frac{\theta_3}{3!}(it)^3+\frac{\theta_4}{4!}(it)^4+\frac{\theta_5^{(5,\lambda)}}{5!}(it)^5+\frac{\theta_6^{(4)}}{6!}(it)^6+\gamma q_5(\lambda)\frac{\beta_5}{5!}t^5e^{t^2/2}+$$
$$+\gamma\frac{||\theta_7^{(5,\lambda)}||}{7!}t^7e^{t^2/2}+\gamma\frac{||\theta_8^{(4)}||}{8!}t^8e^{t^2/2},$$

where

$$\frac{||\theta_7^{(5,\lambda)}||}{7!}=\frac{\lambda|\alpha_5|}{5!2}+\frac{|\alpha_3|}{3!2^2 2!},\quad \frac{||\theta_8^{(4)}||}{8!}=\frac{\alpha_4}{4!2^2 2!}+\frac{1}{128};$$

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \frac{\theta_4}{4!}(it)^4 + \frac{\theta_5^{(5,\lambda)}}{5!}(it)^5 + \gamma q_5(\lambda)\frac{\beta_5}{5!}t^5 e^{t^2/2} + \gamma\frac{||\theta_6^{(4)}||}{6!}t^6 e^{t^2/2} + \gamma\frac{||\theta_7^{(5,\lambda)}||}{7!}t^7 e^{t^2/2},$$

where

$$\frac{||\theta_6^{(4)}||}{6!} = \frac{\alpha_4}{4!2} + \frac{1}{24}, \quad \frac{||\theta_7^{(5,\lambda)}||}{7!} = \frac{\lambda|\alpha_5|}{5!2} + \frac{|\alpha_3|}{3!2^2 2!};$$

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \frac{\theta_4}{4!}(it)^4 + \gamma\frac{||\theta_5||}{5!}t^5 e^{t^2/2} + \gamma\frac{||\theta_6^{(4)}||}{6!}t^6 e^{t^2/2},$$

where

$$\frac{||\theta_5||}{5!} = \frac{\beta_5}{5!} + \frac{|\alpha_3|}{3!2}, \quad \frac{||\theta_6^{(4)}||}{6!} = \frac{\alpha_4}{4!2} + \frac{1}{24}.$$

To construct the first of these representations we used the equality

$$e^{t^2/2}\left(1 - \frac{t^2}{2}\right) = 1 - \frac{(it)^4}{8} + \frac{(it)^6}{24} + \gamma\frac{t^8}{128}e^{t^2/2}.$$

Statement 2. *For distributions P with a finite sixth moment for all $0 \leqslant \lambda \leqslant 1$, the following representations of the quantity $e^{t^2/2}f(t)$ take place for all $-\infty < t < \infty$:*

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \frac{\theta_4}{4!}(it)^4 + \frac{\theta_5}{5!}(it)^5 + \frac{\theta_6^{(6,\lambda)}}{6!}(it)^6 + \frac{\theta_7^{(5)}}{7!}(it)^7 + \gamma q_6(\lambda)\frac{\beta_6}{6!}t^6 e^{t^2/2} + \\ + \gamma\frac{||\theta_8^{(6,\lambda)}||}{8!}t^8 e^{t^2/2} + \gamma\frac{||\theta_9^{(5)}||}{9!}t^9 e^{t^2/2},$$

where

$$\frac{||\theta_8^{(6,\lambda)}||}{8!} = \frac{\lambda\alpha_6}{6!2} + \frac{\alpha_4}{4!2^2 2!} + \frac{1}{128}, \quad \frac{||\theta_9^{(5)}||}{9!} = \frac{|\alpha_5|}{5!2^2 2!} + \frac{|\alpha_3|}{3!2^3 3!};$$

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \frac{\theta_4}{4!}(it)^4 + \frac{\theta_5}{5!}(it)^5 + \frac{\theta_6^{(6,\lambda)}}{6!}(it)^6 + \gamma q_6(\lambda)\frac{\beta_6}{6!}t^6 e^{t^2/2} + \gamma\frac{||\theta_7^{(5)}||}{7!}t^7 e^{t^2/2} + \gamma\frac{||\theta_8^{(6,\lambda)}||}{8!}t^8 e^{t^2/2},$$

where

$$\frac{||\theta_7^{(5)}||}{7!} = \frac{|\alpha_5|}{5!2} + \frac{|\alpha_3|}{3!2^2 2!}, \quad \frac{||\theta_8^{(6,\lambda)}||}{8!} = \frac{\lambda\alpha_6}{6!2} + \frac{\alpha_4}{4!2^2 2!} + \frac{1}{128};$$

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \frac{\theta_4}{4!}(it)^4 + \frac{\theta_5}{5!}(it)^5 + \gamma\frac{||\theta_6||}{6!}t^6 e^{t^2/2} + \gamma\frac{||\theta_7^{(5)}||}{7!}t^7 e^{t^2/2},$$

where

$$\frac{||\theta_6||}{6!} = \frac{\alpha_6}{6!} + \frac{\alpha_4}{4!2} + \frac{1}{24};$$

$$e^{t^2/2}f(t) = 1 + \frac{\theta_3}{3!}(it)^3 + \frac{\theta_4}{4!}(it)^4 + \gamma\frac{||\theta_5||}{5!}t^5 e^{t^2/2} + \gamma\frac{||\theta_6^{(4)}||}{6!}t^6 e^{t^2/2}.$$

These representations together with (28) yield the following analogs of Lemma 2.

Statement 3. *For distributions P with a finite fifth moment, the following representation takes place for all $0 \leqslant \lambda \leqslant 1$ and all $-\infty < t < \infty$ when $n \geqslant 4$:*

$$f^n\left(\frac{t}{\sqrt{n}}\right) - g^n\left(\frac{t}{\sqrt{n}}\right) = \frac{\theta_3}{3!\sqrt{n}}(it)^3 g^n + \frac{\theta_4}{4!n}(it)^4 g^n + \frac{n-1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2 (it)^6 g^n + \frac{\theta_5^{(5,\lambda)}}{5!n^{3/2}}(it)^5 g^n +$$

$$+\frac{n-1}{n}\frac{\theta_3}{3!\sqrt{n}}\frac{\theta_4}{4!n}(it)^7 g^n + \frac{(n-1)(n-2)}{6n^2}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^3 (it)^9 g^n + \frac{\theta_6^{(4)}}{6!}\left(\frac{it}{\sqrt{n}}\right)^6 S_1 +$$

$$+\left(2\frac{\theta_3}{3!}\frac{\theta_5^{(5,\lambda)}}{5!}+\left(\frac{\theta_4}{4!}\right)^2\right)\left(\frac{it}{\sqrt{n}}\right)^8 S_2+3\left(\frac{\theta_3}{3!}\right)^2\frac{\theta_4}{4!}\left(\frac{it}{\sqrt{n}}\right)^{10} S_3+\left(\frac{\theta_3}{3!}\right)^4\left(\frac{it}{\sqrt{n}}\right)^{12} S_4+$$

$$+\gamma q_5(\lambda)\frac{\beta_5}{5!n^{3/2}}t^5\mu^{n-1}+\frac{\gamma}{2}q_5(\lambda)\frac{\theta_3}{3!\sqrt{n}}\frac{\beta_5}{5!n^{3/2}}t^8\mu^{n-1}+\gamma\frac{||\theta_7^{(5,\lambda)}||}{7!n^{5/2}}t^7\mu^{n-1}+$$

$$+\frac{\gamma}{2}\left(\frac{|\theta_3|}{3!\sqrt{n}}\frac{||\theta_6^{(4)}||}{6!n^2}+\frac{|\theta_4|}{4!n}\frac{||\theta_5||}{5!n^{3/2}}+\frac{||\theta_4||}{4!n}\frac{||\theta_5^{(5,\lambda)}||}{5!n^{3/2}}\right)t^9\mu^{n-1}+$$

$$+\frac{\gamma}{6}\left(2\frac{|\theta_3|}{3!\sqrt{n}}\frac{|\theta_4|}{4!n}\frac{||\theta_4||}{4!n}+\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_5||}{5!n^{3/2}}\right)t^{11}\mu^{n-1}+$$

$$+\frac{\gamma}{24}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^3\frac{||\theta_4||}{4!n}t^{13}\mu^{n-1}+\gamma\frac{||\theta_8^{(4)}||}{8!n^3}t^8\mu^{n-1}+$$

$$+\frac{\gamma}{2}\left(\frac{|\theta_3|}{3!\sqrt{n}}\frac{||\theta_7^{(5,\lambda)}||}{7!n^{5/2}}+\frac{|\theta_4|}{4!n}\frac{||\theta_6^{(4)}||}{6!n^2}+\frac{|\theta_5^{(5,\lambda)}|}{5!n^{3/2}}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}\right)t^{10}\mu^{n-1}+$$

$$+\frac{\gamma}{6}\left(2\frac{|\theta_3|}{3!\sqrt{n}}\frac{|\theta_4|}{4!n}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}+\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_6^{(4)}||}{6!n^2}\right)t^{12}\mu^{n-1}+\frac{\gamma}{24}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^3\frac{||\theta_5^{(3)}||}{5!n^{3/2}}t^{14}\mu^{n-1}.$$

Statement 4. *For distribution P with a finite sixth moment, the following representation takes place for all $0\leqslant\lambda\leqslant 1$ and all $-\infty<t<\infty$ when $n\geqslant 5$:*

$$f^n\left(\frac{t}{\sqrt{n}}\right)-g^n\left(\frac{t}{\sqrt{n}}\right)=\frac{\theta_3}{3!\sqrt{n}}(it)^3g^n+\frac{\theta_4}{4!n}(it)^4g^n+\left(\frac{\theta_6^{(6,\lambda)}}{6!n^2}+\frac{n-1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\right)(it)^6g^n+\frac{\theta_5}{5!n^{3/2}}(it)^5g^n+$$

$$+\frac{n-1}{n}\frac{\theta_3}{3!\sqrt{n}}\frac{\theta_4}{4!n}(it)^7g^n+\frac{(n-1)(n-2)}{6n^2}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^3(it)^9g^n+\frac{n-1}{2n}\left(2\frac{\theta_3}{3!\sqrt{n}}\frac{\theta_5}{5!n^{3/2}}+\left(\frac{\theta_4}{4!n}\right)^2\right)(it)^8g^n+$$

$$\frac{(n-1)(n-2)}{2n^2}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{\theta_4}{4!n}(it)^{10}g^n+\frac{(n-1)\dots(n-3)}{24n^3}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^4(it)^{12}g^n+\frac{\theta_7^{(5)}}{7!}\left(\frac{it}{\sqrt{n}}\right)^7 S_1+$$

$$+2\left(\frac{\theta_3}{3!}\frac{\theta_6^{(6,\lambda)}}{6!}+\frac{\theta_4}{4!}\frac{\theta_5}{5!}\right)\left(\frac{it}{\sqrt{n}}\right)^9 S_2+3\left(\left(\frac{\theta_3}{3!}\right)^2\frac{\theta_5}{5!}+\frac{\theta_3}{3!}\left(\frac{\theta_4}{4!}\right)^2\right)\left(\frac{it}{\sqrt{n}}\right)^{11} S_3+4\left(\frac{\theta_3}{3!}\right)^3\frac{\theta_4}{4!}\left(\frac{it}{\sqrt{n}}\right)^{13} S_4+$$

$$+\left(\frac{\theta_3}{3!}\right)^5\left(\frac{it}{\sqrt{n}}\right)^{15} S_5+\gamma q_6(\lambda)\frac{\beta_6}{6!n^2}t^6\mu^{n-1}+\frac{\gamma}{2}q_6(\lambda)\frac{\theta_3}{3!\sqrt{n}}\frac{\beta_6}{6!n^2}t^9\mu^{n-1}+\gamma\frac{||\theta_8^{(6,\lambda)}||}{8!n^3}t^8\mu^{n-1}+$$

$$+\frac{\gamma}{2}\left(\frac{|\theta_3|}{3!\sqrt{n}}\frac{||\theta_7^{(5)}||}{7!n^{5/2}}+\frac{|\theta_4|}{4!n}\frac{||\theta_6||}{6!n^2}+\frac{||\theta_4||}{4!n}\frac{||\theta_6^{(6,\lambda)}||}{6!n^2}+\frac{|\theta_5|}{5!n^{3/2}}\frac{||\theta_5||}{5!n^{3/2}}\right)t^{10}\mu^{n-1}+$$

$$\frac{\gamma}{6}\left(\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_6||}{6!n^2}+2\frac{|\theta_3|}{3!\sqrt{n}}\frac{|\theta_4|}{4!n}\frac{||\theta_5||}{5!n^{3/2}}+2\frac{|\theta_3|}{3!\sqrt{n}}\frac{||\theta_4||}{4!n}\frac{|\theta_5|}{5!n^{3/2}}+\left(\frac{\theta_4}{4!n}\right)^2\frac{||\theta_4||}{4!n}\right)t^{12}\mu^{n-1}+$$

$$+\frac{\gamma}{24}\left(\left|\frac{\theta_3}{3!\sqrt{n}}\right|^3\frac{||\theta_5||}{5!n^{3/2}}+3\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{|\theta_4|}{4!n}\frac{||\theta_4||}{4!n}\right)t^{14}\mu^{n-1}+\frac{\gamma}{120}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^4\frac{||\theta_4||}{4!n}t^{16}\mu^{n-1}+\gamma\frac{||\theta_9^{(5)}||}{9!n^{7/2}}t^9\mu^{n-1}+$$

$$+\frac{\gamma}{2}\left(\frac{|\theta_3|}{3!\sqrt{n}}\frac{||\theta_8^{(6,\lambda)}||}{8!n^3}+\frac{|\theta_4|}{4!n}\frac{||\theta_7^{(5)}||}{7!n^{5/2}}+\frac{|\theta_5|}{5!n^{3/2}}\frac{||\theta_6^{(4)}||}{6!n^2}+\frac{||\theta_5^{(3)}||}{5!n^{3/2}}\frac{|\theta_6^{(6,\lambda)}|}{6!n^2}\right)t^{11}\mu^{n-1}+$$

$$+\frac{\gamma}{6}\left(\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{||\theta_7^{(5)}||}{7!n^{5/2}}+2\frac{|\theta_3|}{3!\sqrt{n}}\frac{|\theta_4|}{4!n}\frac{||\theta_6^{(4)}||}{6!n^2}+2\frac{|\theta_3|}{3!\sqrt{n}}\frac{|\theta_5|}{5!n^{3/2}}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}+\left(\frac{\theta_4}{4n}\right)^2\frac{||\theta_5^{(3)}||}{5!n^{3/2}}\right)t^{13}\mu^{n-1}+$$

$$\frac{\gamma}{24}\left(\left|\frac{\theta_3}{3!\sqrt{n}}\right|^3\frac{||\theta_6^{(4)}||}{6!n^2}+3\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{|\theta_4|}{4!n}\frac{||\theta_5^{(3)}||}{5!n^{3/2}}\right)t^{15}\mu^{n-1}+\frac{\gamma}{120}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^4\frac{||\theta_5^{(3)}||}{5!n^{3/2}}t^{17}\mu^{n-1}.$$

With the help of these representations it is easy to prove the validity of the following analogs of Theorem 1.

Theorem 2. *For distributions P with a finite fifth moment under condition* (1), *the following representation takes place for all $0\leqslant\lambda\leqslant\lambda_5=5/12$ and for all $-\infty<x<\infty$, when $n\geqslant\max(4,v)$:*

$$p_n(x)-\varphi(x)=\frac{\theta_3}{3!\sqrt{n}}H_3(x)\varphi(x)+\frac{\theta_4}{4!n}H_4(x)\varphi(x)+\frac{n-1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2H_6(x)\varphi(x)+\frac{\theta_5^{(5,\lambda)}}{5!n^{3/2}}H_5(x)\varphi(x)+$$

$$\frac{n-1}{n}\frac{\theta_3}{3!\sqrt{n}}\frac{\theta_4}{4!n}H_7(x)\varphi(x)+\frac{(n-1)(n-2)}{6n^2}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^3H_9(x)\varphi(x)+R+K,$$

where

$$|R|\leqslant(1-\lambda)\frac{\beta_5}{5!n^{3/2}}B_{5,n,1}+\frac{1-\lambda}{2}\frac{|\theta_3|}{3!\sqrt{n}}\frac{\beta_5}{5!n^{3/2}}B_{8,n,1}+$$

$$\frac{1}{n^2}\left(\frac{\theta_6^{(4)}}{6!}B_{6,n,0}+\left|2\frac{\theta_3}{3!}\frac{\theta_5^{(5,\lambda)}}{5!}+\left(\frac{\theta_4}{4!}\right)^2\right|\frac{B_{8,n,0}}{2}+\left(\frac{\theta_3}{3!}\right)^2\frac{|\theta_4|}{4!}\frac{B_{10,n,0}}{2}+\left(\frac{\theta_3}{3!}\right)^4\frac{B_{12,n,0}}{24}\right)+\frac{||\theta_7^{(5,\lambda)}||}{7!n^{5/2}}B_{7,n,1}+$$

$$+\frac{1}{n^{5/2}}\left(\frac{|\theta_3|}{3!}\frac{||\theta_6^{(4)}||}{6!}+\frac{|\theta_4|}{4!}\frac{||\theta_5||}{5!}+\frac{||\theta_4||}{4!}\frac{|\theta_5^{(5,\lambda)}|}{5!}\right)\frac{B_{9,n,1}}{2}+\frac{1}{n^{5/2}}\left(2\frac{|\theta_3|}{3!}\frac{|\theta_4|}{4!}\frac{||\theta_4||}{4!}+\left(\frac{\theta_3}{3!}\right)^2\frac{||\theta_5||}{5!}\right)\frac{B_{11,n,1}}{6}+$$

$$+\frac{1}{n^{5/2}}\left(\frac{\theta_3}{3!}\right)^3\frac{||\theta_4||}{4!}\frac{B_{13,n,1}}{24}+\frac{||\theta_8^{(4)}||}{8!n^3}B_{8,n,1}+\frac{1}{n^3}\left(\frac{|\theta_3|}{3!}\frac{||\theta_7^{(5,\lambda)}||}{7!}+\frac{|\theta_4|}{4!}\frac{||\theta_6^{(4)}||}{6!}+\frac{|\theta_5^{(5,\lambda)}|}{5!}\frac{||\theta_5^{(3)}||}{5!}\right)\frac{B_{10,n,1}}{2}+$$

$$+\frac{1}{n^3}\left(2\frac{|\theta_3|}{3!}\frac{|\theta_4|}{4!}\frac{||\theta_5^{(3)}||}{5!}+\left(\frac{\theta_3}{3!}\right)^2\frac{||\theta_6^{(4)}||}{6!}\right)\frac{B_{12,n,1}}{6}+\frac{1}{n^3}\left|\frac{\theta_3}{3!}\right|^3\frac{||\theta_5^{(3)}||}{5!}\frac{B_{14,n,1}}{24},$$

and the quantity K was defined as in Theorem 1.

For distributions P with a finite sixth moment under condition (1), *the following representation takes place for all $0\leqslant\lambda\leqslant\lambda_6=3/7$ and for all $-\infty<x<\infty$, when $n\geqslant\max(5,v)$:*

$$p_n(x)-\varphi(x)=\frac{\theta_3}{3!\sqrt{n}}H_3(x)\varphi(x)+\frac{\theta_4}{4!n}H_4(x)\varphi(x)+\left(\frac{\theta_6^{(6,\lambda)}}{6!n^2}+\frac{n-1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\right)H_6(x)\varphi(x)+$$

$$+\frac{\theta_5}{5!n^{3/2}}H_5(x)\varphi(x)+\frac{n-1}{n}\frac{\theta_3}{3!\sqrt{n}}\frac{\theta_4}{4!n}H_7(x)\varphi(x)+$$

$$+\frac{(n-1)(n-2)}{6n^2}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^3H_9(x)\varphi(x)+\frac{n-1}{2n}\left(2\frac{\theta_3}{3!\sqrt{n}}\frac{\theta_5}{5!n^{3/2}}+\left(\frac{\theta_4}{4!n}\right)^2\right)H_8(x)\varphi(x)+$$

$$+\frac{(n-1)(n-2)}{2n^2}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{\theta_4}{4!n}H_{10}(x)\varphi(x)+\frac{(n-1)\dots(n-3)}{24n^3}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^4H_{12}(x)\varphi(x)+R+K,$$

where

$$|R|\leqslant(1-\lambda)\frac{\beta_6}{6!n^2}B_{6,n,1}+\frac{1-\lambda}{2}\frac{|\theta_3|}{3!\sqrt{n}}\frac{\beta_6}{6!n^2}B_{9,n,1}+$$

$$+\frac{1}{n^{5/2}}\left(\frac{|\theta_7^{(5)}|}{7!}B_{7,n,0}+\left|\frac{\theta_3}{3!}\frac{\theta_6^{(6,\lambda)}}{6!}+\frac{\theta_4}{4!}\frac{\theta_5}{5!}\right|B_{9,n,0}+\left|\left(\frac{\theta_3}{3!}\right)^2\frac{\theta_5}{5!}+\frac{\theta_3}{3!}\left(\frac{\theta_4}{4!}\right)^2\right|\frac{B_{11,n,0}}{2}+\left|\frac{\theta_3}{3!}\right|^3\frac{|\theta_4|}{4!}\frac{B_{13,n,0}}{6}+\right.$$

$$\left.+\left|\frac{\theta_3}{3!}\right|^5\frac{B_{15,n,0}}{120}\right)+\frac{||\theta_8^{(6,\lambda)}||}{8!n^3}B_{8,n,1}+\frac{1}{n^3}\left(\frac{|\theta_3|}{3!}\frac{||\theta_7^{(5)}||}{7!}+\frac{|\theta_4|}{4!}\frac{||\theta_6||}{6!}+\frac{||\theta_4||}{4!}\frac{|\theta_6^{(6,\lambda)}|}{6!}+\frac{|\theta_5|}{5!}\frac{||\theta_5||}{5!}\right)\frac{B_{10,n,1}}{2}+$$

$$+\frac{1}{n^3}\left(\left(\frac{|\theta_3|}{3!}\right)^2\frac{||\theta_6||}{6!}+2\frac{|\theta_3|}{3!}\frac{|\theta_4|}{4!}\frac{||\theta_5||}{5!}+2\frac{|\theta_3|}{3!}\frac{||\theta_4||}{4!}\frac{|\theta_5|}{5!}+\left(\frac{\theta_4}{4!}\right)^2\frac{||\theta_4||}{4!}\right)\frac{B_{12,n,1}}{6}+$$

$$+\frac{1}{n^3}\left(\left|\frac{\theta_3}{3!}\right|^3\frac{||\theta_5||}{5!}+3\left(\frac{\theta_3}{3!}\right)^2\frac{|\theta_4|}{4!}\frac{||\theta_4||}{4!}\right)\frac{B_{14,n,1}}{24}+\frac{1}{n^3}\left(\frac{\theta_3}{3!}\right)^4\frac{||\theta_4||}{4!}\frac{B_{16,n,1}}{120}+\frac{||\theta_9^{(5)}||}{9!n^{7/2}}B_{9,n,1}+$$

$$+\frac{1}{n^{7/2}}\left(\frac{|\theta_3|}{3!}\frac{||\theta_8^{(6,\lambda)}||}{8!}+\frac{|\theta_4|}{4!}\frac{||\theta_7^{(5)}||}{7!}+\frac{|\theta_5|}{5!}\frac{||\theta_6^{(4)}||}{6!}+\frac{||\theta_5^{(3)}||}{5!}\frac{||\theta_6^{(6,\lambda)}||}{6!}\right)\frac{B_{11,n,1}}{2}+$$

$$+\frac{1}{n^{7/2}}\left(\left(\frac{\theta_3}{3!}\right)^2\frac{||\theta_7^{(5)}||}{7!}+2\frac{|\theta_3|}{3!}\frac{|\theta_4|}{4!}\frac{||\theta_6^{(4)}||}{6!}+2\frac{|\theta_3|}{3!}\frac{|\theta_5|}{5!}\frac{||\theta_5^{(3)}||}{5!}+\left(\frac{\theta_4}{4!}\right)^2\frac{||\theta_5^{(3)}||}{5!}\right)\frac{B_{13,n,1}}{6}+$$

$$+\frac{1}{n^{7/2}}\left(\left|\frac{\theta_3}{3!}\right|^3\frac{||\theta_6^{(4)}||}{6!}+3\left(\frac{\theta_3}{3!}\right)^2\frac{|\theta_4|}{4!}\frac{||\theta_5^{(3)}||}{5!}\right)\frac{B_{15,n,1}}{24}+\frac{1}{n^{7/2}}\left(\frac{\theta_3}{3!}\right)^4\frac{||\theta_5^{(3)}||}{5!}\frac{B_{17,n,1}}{120},$$

and the quantity K was defined as in Theorem 1.

Similarly to the proof of Theorem 1, it is easy to ascertain for ED that in the first expansion of Theorem 2 the quantity $|R|$ does not exceed the quantity equivalent to

$$0.0124\frac{\beta_5}{n^{3/2}}+0.017\frac{|\theta_3|}{\sqrt{n}}\frac{\beta_5}{n^{3/2}}+\frac{9.4}{n^2}+\frac{56}{n^{5/2}}+\frac{72}{n^3}<0.0124\frac{\beta_5}{n^{3/2}}+\frac{11}{n^2}+\frac{56}{n^{5/2}}+\frac{72}{n^3}$$

(here we used $\lambda=\lambda_5=5/12$, and $\beta_5<44.3$). The small increase of the term in (7) that decreases as n^{-2} with growth of n as compared to the same term on the right-hand side of the previous inequality is because the quantity $\max\limits_x\frac{|(\theta_3/3!)^2H_6(x)\varphi(x)|}{2n^2}$, which does not exceed $\frac{1}{3n^2}$, was added. The small increase of the term in (7) that decreases as $n^{-5/2}$ with growth of n as compared to the same term on the right-hand side of the previous inequality is related to the fact that

$$\max_x\frac{|(\theta_3/3!)(\theta_4/4!)H_7(x)+1/2(\theta_3/3!)^3H_9(x)|\varphi(x)}{n^{5/2}}<\frac{1}{n^{5/2}}$$

was added. Also, there is a term decreasing as $n^{-7/2}$, which estimates $\max\limits_x\frac{|(\theta_3/3!)^3H_9(x)\varphi(x)|}{3n^{7/2}}$.

Similarly to the proof of Theorem 1 it is easy to ascertain for ED in the second expansion of Theorem 2 that the quantity $|R|$ does not exceed the quantity equivalent to

$$0.0048\frac{\beta_6}{n^2}+0.0081\frac{|\theta_3|}{\sqrt{n}}\frac{\beta_6}{n^2}+\frac{65}{n^{5/2}}+\frac{432}{n^3}+\frac{734}{n^{7/2}}<0.0048\frac{\beta_6}{n^2}+\frac{70}{n^{5/2}}+\frac{431.3}{n^3}+\frac{734}{n^{7/2}}$$

(here we used $\lambda=\lambda_6=3/7$ and $\beta_6=265$). The following equality also takes place:

$$\left(\frac{\theta_6^{(6,\lambda)}}{6!n^2}-\frac{1}{2n}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\right)H_6(x)\varphi(x)+\frac{n-1}{2n}\left(2\frac{\theta_3}{3!\sqrt{n}}\frac{\theta_5}{5!n^{3/2}}+\left(\frac{\theta_4}{4!n}\right)^2\right)H_8(x)\varphi(x)+$$

$$+\frac{(n-1)(n-2)}{2n^2}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^2\frac{\theta_4}{4!n}H_{10}(x)\varphi(x)+\frac{(n-1)\dots(n-3)}{24n^3}\left(\frac{\theta_3}{3!\sqrt{n}}\right)^4H_{12}(x)\varphi(x)=$$

$$= \frac{1}{n^2}\left(\left(\frac{\theta_6^{(6,\lambda)}}{6!} - \frac{1}{2}\left(\frac{\theta_3}{3!}\right)^2\right) H_6(x) + \left(\frac{\theta_3}{3!}\frac{\theta_5}{5!} + \frac{1}{2}\left(\frac{\theta_4}{4!}\right)^2\right) H_8(x) + \frac{1}{2}\left(\frac{\theta_3}{3!}\right)^2 \frac{\theta_4}{4!} H_{10}(x) + \right.$$

$$\left. + \frac{1}{24}\left(\frac{\theta_3}{3!}\right)^4 H_{12}(x)\right)\varphi(x) + \frac{R_1(x)}{n^3} + \frac{R_2(x)}{n^4} + \frac{R_3(x)}{n^5},$$

where

$$R_1(x) = -\left(\left(\frac{\theta_3}{3!}\frac{\theta_5}{5!} + \frac{1}{2}\left(\frac{\theta_4}{4!}\right)^2\right) H_8(x) + \frac{3}{2}\left(\frac{\theta_3}{3!}\right)^2 \frac{\theta_4}{4!} H_{10}(x) + \frac{1}{4}\left(\frac{\theta_3}{3!}\right)^4 H_{12}(x)\right)\varphi(x),$$

$$R_2(x) = \left(\left(\frac{\theta_3}{3!}\right)^2 \frac{\theta_4}{4!n} H_{10}(x) + \frac{11}{24}\left(\frac{\theta_3}{3!}\right)^4 H_{12}(x)\right)\varphi(x), \quad R_3(x) = -\frac{1}{4}\left(\frac{\theta_3}{3!}\right)^4 H_{12}(x)\varphi(x).$$

For ED the absolute value of the maximum of the function $R_1(x)$ does not exceed 2.1, and the absolute value of the maximum of the functions $R_2(x)$ and $R_3(x)$ does not exceed 13. And now it is easy to ascertain the validity of (8).

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