DOI 10.1007/s10958-016-3025-5
Journal of Mathematical Sciences, Vol. 218, No. 2, October, 2016

ESTIMATION OF PROBABILITIES FOR MULTIDIMENSIONAL BIRTH-DEATH PROCESSES*

A. I. Zeifman[1], A. S. Sipin[2], A. V. Korotysheva[2], T. L. Panfilova[2], Ya. A. Satin[2], G. N. Shilova[2], and V. Yu. Korolev[3]

We consider a multidimensional inhomogeneous birth-death process and obtain bounds for the probabilities of the corresponding one-dimensional processes.

1. Introduction and preliminaries

Multidimensional birth-death processes are objects of a number of studies in queueing theory. Firstly, the problem of the product form solutions for such models was considered; see, for instance, [6] and references therein. Some specific examples of multidimensional birth-death processes as queueing models can be found in [4,5]. If the process is inhomogeneous and the transition intensities are of a more general form, then the problem of computation of any probabilistic characteristics of the queueing model becomes considerably more complicated. In this note we try to obtain a number of such bounds. Our approach is based on the method of investigation of inhomogeneous birth-death processes; see the detailed discussion and some preliminary results in [3,7–10]. Main results of the present paper were briefly formulated in [14].

Let $\mathbf{X}(t) = (X_1(t), \dots, X_d(t))$ be a d-dimensional birth-death process such that in the interval $(t, t+h)$ the following transitions are possible with order h: the birth of a particle of type j, the death of a particle of type j. Let $\lambda_{j,\mathbf{m}}(t)$ be the corresponding birth rate (of the transition from the state $\mathbf{m} = (m_1, \dots, m_d) = \sum_{i=1}^d m_i \mathbf{e}_i$ to the state $\mathbf{m} + \mathbf{e}_j$), and $\mu_{j,\mathbf{m}}(t)$ be the corresponding death intensity (of the transition from the state $\mathbf{m} = (m_1, \dots, m_d) = \sum_{i=1}^d m_i \mathbf{e}_i$ to the state $\mathbf{m} - \mathbf{e}_j$). Denote $p_{\mathbf{m}}(t) = \Pr(\mathbf{X}(t) = \mathbf{m})$.

Let now the (countable) state space of the vector process under consideration be ordered in a special way, say $0, 1, \dots$. By $p_i(t)$ denote the corresponding state probabilities, and by $\mathbf{p}(t)$ denote the corresponding column vector of state probabilities. Applying our standard approach (see details in [3,8–10]), we assume in addition, that all intensity functions are linear combinations of a finite number of locally integrable on $[0, \infty)$ nonegative functions, and, moreover, that, in new enumeration,

$$\Pr\left(X\left(t+h\right) = j / X\left(t\right) = i\right) = \begin{cases} q_{ij}\left(t\right) h + \alpha_{ij}\left(t, h\right), & j \neq i, \\ 1 - \sum\limits_{k \neq i} q_{ik}\left(t\right) h + \alpha_i\left(t, h\right), & j = i, \end{cases} \tag{1}$$

where all $\alpha_i(t, h)$ are $o(h)$ uniformly in i, i.e., $\sup_i |\alpha_i(t, h)| = o(h)$. We also assume that the boundedness condition

$$\lambda_{j,\mathbf{m}}(t) \leqslant L < \infty, \qquad \mu_{j,\mathbf{m}}(t) \leqslant M < \infty, \tag{2}$$

holds for any j, $\mathbf{m}$ and almost all $t \geqslant 0$. Then the probabilistic dynamics of the process is represented by the forward Kolmogorov system:

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}(t), \tag{3}$$

[1] Vologda State University, Institute of Informatics Problems, FRC CSC RAS; ISEDT, RAS, Vologda, Russia, e-mail: a_zeifman@mail.ru

[2] Vologda State University, Vologda, Russia

[3] Lomonosov Moscow State University, Institute of Informatics Problems, FRC CSC RAS, Moscow, Russia

* This work was supported by the Russian Foundation for Basic Research, projects No. 13–07–00223, 14–07–00041, 15–01–01698, 15–07–02341; and by Ministry of Education and Science.

Proceedings of the XXXII International Seminar on Stability Problems for Stochastic Models, Trondheim, Norway, June 16–21, 2014.

 1072-3374/16/2182-0238

where $A(t)$ is the corresponding infinitesimal (intensity) matrix. Throughout the paper by $\|\cdot\|$ we denote the l_1-norm, i.e., $\|\mathbf{x}\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{i,j=0}^{\infty}$. Let Ω be the set all stochastic vectors, i.e., l_1-vectors with nonegative coordinates and unit norm. Hence assumption (2) implies the inequality

$$\|A(t)\| \leqslant H = 2d\,(L + M) < \infty, \tag{4}$$

for any j, $\mathbf{m}$ and almost all $t \geqslant 0$. Hence the operator function $A(t)$ from l_1 into itself is bounded for almost all $t \geqslant 0$ and is locally integrable on $[0;\infty)$. Therefore we can consider (3) as a differential equation in the space l_1 with a bounded operator. It is well known [1] that the Cauchy problem for differential Eq. (3) has a unique solution for an arbitrary initial condition, and $\mathbf{p}(s) \in \Omega$ implies $\mathbf{p}(t) \in \Omega$ for $t \geqslant s \geqslant 0$.

2. Bounds for birth-death process

Here we briefly consider the general approach and the corresponding bounds for an inhomogeneous one-dimensional birth-death process (BDP); see details in [3, 7–9]. Let $X(t)$, $t \geqslant 0$, be a BDP with birth and death rates $\lambda_n(t)$ and $\mu_n(t)$, respectively. Let $p_{ij}(s,t) = \Pr\{X(t) = j\,|X(s) = i\}$ for $i, j \geqslant 0$, $0 \leqslant s \leqslant t$, be the transition probability functions of the process $X = X(t)$ and $p_i(t) = \Pr\{X(t) = i\}$ be the state probabilities. By $\mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T$, $t \geqslant 0$, we denote the column vector of state probabilities. We suppose that the boundedness assumptions (2) hold and, hence, $\|A(t)\| \leqslant H < \infty$.

1. Weak ergodicity

We recall that a Markov chain $X(t)$ is called weakly ergodic if $\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \to 0$ as $t \to \infty$ for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$. Put $E_k(t) = E\{X(t)\,|X(0) = k\}$. Consider an increasing sequence of positive numbers $\{d_i\}$, $i = 1, 2, \dots$, $d_1 = 1$, and the corresponding triangular matrix D:

$$D = \begin{pmatrix} d_1 & d_1 & d_1 & \cdots \\ 0 & d_2 & d_2 & \cdots \\ 0 & 0 & d_3 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{5}$$

Let l_{1D} be the space of sequences:

$$l_{1D} = \left\{\mathbf{z} = (p_1, p_2, \cdots)^T : \|\mathbf{z}\|_{1D} \equiv \|D\mathbf{z}\| < \infty\right\}.$$

Put

$$d = \inf_{i \geqslant 1} d_i = 1, \quad W = \inf_{i \geqslant 1} \frac{d_i}{i}, \quad g_i = \sum_{n=1}^{i} d_n.$$

Consider the following expressions:

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+1}}{d_k}\lambda_{k+1}(t) - \frac{d_{k-1}}{d_k}\mu_k(t), \quad k \geqslant 0, \tag{6}$$

and

$$\alpha(t) = \inf_{k \geqslant 0} \alpha_k(t). \tag{7}$$

The property $\mathbf{p}(t) \in \Omega$ for any $t \geqslant 0$ allows us to set $p_0(t) = 1 - \sum_{i \geqslant 1} p_i(t)$ and obtain the following system from (3) for corresponding BDP:

$$\frac{d\mathbf{z}(t)}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \tag{8}$$

where $\mathbf{z(t)} = (p_1(t), p_2(t), \dots)^T$, $\mathbf{f(t)} = (\lambda_0(t), 0, 0, \dots)^T$, $B(t) = (b_{ij}(t))_{i,j=1}^{\infty}$ and

$$b_{ij} = \begin{cases} -(\lambda_0 + \lambda_1 + \mu_1), & \text{if} \quad i = j = 1, \\ \mu_2 - \lambda_0, & \text{if} \quad i = 1, \ j = 2, \\ -\lambda_0, & \text{if} \quad i = 1, \ j > 2, \\ -(\lambda_j + \mu_j), & \text{if} \quad i = j > 1, \\ \mu_j, & \text{if} \quad i = j - 1 > 1, \\ \lambda_j, & \text{if} \quad i = j + 1 > 1, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

This is a linear non-homogeneous differential system, the solution of which can be written as

$$\mathbf{z}(t) \quad = \quad V(t,0)\mathbf{z}(0) + \int_0^t V(t,\tau)\mathbf{f}(\tau)\, d\tau, \tag{10}$$

where $V(t,\tau) = V(t)V^{-1}(\tau)$ is the Cauchy operator of (8). Consider Eq. (8) in the space l_{1D}. We have

$$DBD^{-1} = \begin{pmatrix} -(\lambda_0 + \mu_1) & \frac{d_1}{d_2}\mu_2 & 0 & \ddots & \\ \frac{d_2}{d_1}\lambda_1 & -(\lambda_1 + \mu_2) & \frac{d_2}{d_3}\mu_3 & 0 & \ddots \\ 0 & \frac{d_3}{d_2}\lambda_2 & -(\lambda_2 + \mu_3) & \frac{d_3}{d_4}\mu_4 & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{11}$$

Next, $\mathbf{f}(t)$ and $B(t)$ are bounded and locally integrable on $[0,\infty)$ as a vector function and an operator function in l_{1D}, respectively. Now we have the following bound for the logarithmic norm $\gamma\left(B(t)\right)$ in l_{1D}:

$$\gamma\left(B\right)_{1D} = \gamma\left(DB(t)D^{-1}\right)_1 = \sup_{i\geqslant 0}\left(\frac{d_{i+1}}{d_i}\lambda_{i+1}(t) + \frac{d_{i-1}}{d_i}\mu_i(t) - (\lambda_i(t) + \mu_{i+1}(t))\right) =$$
$$= -\inf_{k\geqslant 0}\left(\alpha_k\left(t\right)\right) = -\alpha(t), \tag{12}$$

in accordance with (7). Hence

$$\|V(t,s)\|_{1D} \leqslant e^{-\int\limits_s^t \alpha(\tau)\, d\tau}. \tag{13}$$

Therefore,

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} = \|\mathbf{z}^*(t) - \mathbf{z}^{**}(t)\|_{1D} \leqslant e^{-\int\limits_s^t \alpha(\tau)\, d\tau}\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}, \tag{14}$$

for any $t \geqslant s \geqslant 0$ and any initial conditions $\mathbf{p}^*(s), \mathbf{p}^{**}(s)$.

Moreover, inequality $\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\| \leqslant 2\|\mathbf{z}(s)\| \leqslant 4\|\mathbf{p}^*(s) - \mathbf{p}^{**}(s)\|_{1D}$ implies the bound

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leqslant 4e^{-\int\limits_s^t \alpha(\tau)\, d\tau}\sum_{i\geqslant 1} g_i |p_i^*(s) - p_i^{**}(s)|. \tag{15}$$

Let now

$$\alpha(t) \geqslant \alpha^* > 0 \tag{16}$$

for almost all $t \geqslant 0$. Then we obtain the inequality

$$\|\mathbf{z}(t)\|_{1D} \leqslant \|V(t)\|_{1D}\|\mathbf{z}(0)\|_{1D} + \int\limits_0^t \|V(t,\tau)\|_{1D}\|\mathbf{f}(\tau)\|_{1D}\, d\tau \leqslant e^{-\alpha^* t}\|\mathbf{z}(0)\|_{1D} + \frac{H}{2\alpha^*}, \tag{17}$$

because $\lambda_0(t) \leqslant \frac{H}{2}$ for almost all $t \geqslant 0$. On the other hand, all $p_i(t) \geqslant 0$, and therefore

$$\|\mathbf{z}(t)\|_{1D} = \sum_{i\geqslant 1} p_i(t) \sum_{k=1}^{i} d_k \geqslant \sum_{i\geqslant N} d_i p_i(t). \tag{18}$$

Hence

$$\sum_{i=N}^{\infty} d_i p_i(t) \leqslant e^{-\alpha^* t}\|\mathbf{z}(0)\|_{1D} + \frac{H}{2\alpha^*}, \tag{19}$$

and

$$\sum_{i=N}^{\infty} p_i(t) \leqslant d_N^{-1} e^{-\alpha^* t}\|\mathbf{z}(0)\|_{1D} + \frac{H}{2\alpha^* d_N}, \tag{20}$$

for any N and any $t \geqslant 0$.

Theorem 1. *Let a BDP with rates $\lambda_k(t)$ and $\mu_k(t)$ be given. Assume that*

$$\int_0^{\infty} \alpha(t)\, dt = +\infty. \tag{21}$$

Then $X(t)$ is weakly ergodic, and bounds (14), (15) *hold for any $t \geqslant s \geqslant 0$ and any initial conditions $\mathbf{p}^*(s), \mathbf{p}^{**}(s)$. If, instead of* (21) *we have*(16), *then $X(t)$ is exponentially weakly ergodic, and bounds* (19) *and* (20) *hold.*

Corollary 1. *Let, in addition, the numbers d_i grow sufficiently fast so that $W > 0$. Then $X(t)$ has the limiting mean, say $\phi(t)$, and the following bound holds*:

$$|\phi(t) - E_k(t)| \leqslant \frac{4}{W} e^{-\int_0^t \alpha(\tau)\, d\tau} \|\mathbf{p}(0) - \mathbf{e_k}\|_{1D}. \tag{22}$$

2. Null ergodicity

We recall that a Markov chain $X(t)$ is called null-ergodic, if $p_k(t) \to 0$ as $t \to \infty$ for any initial condition $\mathbf{p}(0)$ and any k.

Consider a decreasing sequence of positive numbers $\{\delta_i\}$, $i = 0, 1, \ldots$, $\delta_0 = 1$, and the corresponding diagonal matrix Δ with diagonal entries $\{\delta_k\}$. Let $l_{1\Delta}$ be the space of sequences:

$$l_{1\Delta} = \left\{ (p_0, p_1, \ldots)^T : \|\mathbf{z}\|_{1\Delta} \equiv \|\Delta \mathbf{p}\| < \infty \right\}.$$

Consider the following expressions:

$$\zeta_k(t) = \lambda_k(t) + \mu_k(t) - \frac{\delta_{k+1}}{\delta_k}\lambda_k(t) - \frac{\delta_{k-1}}{\delta_k}\mu_k(t), \quad k \geqslant 0, \tag{23}$$

and

$$\zeta(t) = \inf_{k\geqslant 0} \zeta_k(t). \tag{24}$$

Then we have the following bound for the logarithmic norm $\gamma(A(t))$ in $l_{1\Delta}$:

$$\gamma(A)_{1\Delta} = \gamma\left(\Delta A(t) \Delta^{-1}\right)_1 = \sup_{i\geqslant 0} \left(\frac{\delta_{i+1}}{\delta_i}\lambda_i(t) + \frac{\delta_{i-1}}{\delta_i}\mu_i(t) - (\lambda_i(t) + \mu_i(t)) \right) =$$
$$= -\inf_{k\geqslant 0} (\zeta_k(t)) = -\zeta(t), \tag{25}$$

in accordance with (23). Hence

$$\|U(t,s)\|_{1\Delta} \leqslant e^{-\int\limits_s^t \zeta(\tau)\,d\tau}, \tag{26}$$

where $U(t,s) = U(t)U^{-1}(s)$ is the Cauchy operator of (3) for the corresponding BDP. Therefore,

$$\sum_{k=0}^{\infty} \delta_k p_k(t) = \|\mathbf{p}(t)\|_{1\Delta} \leqslant e^{-\int\limits_s^t \zeta(\tau)\,d\tau} \|\mathbf{p}(s)\|_{1\Delta} \leqslant e^{-\int\limits_s^t \zeta(\tau)\,d\tau}, \tag{27}$$

for any $t \geqslant s \geqslant 0$ and any initial condition $\mathbf{p}(s)$.

Theorem 2. *Let a BDP with rates $\lambda_k(t)$ and $\mu_k(t)$ be given. Assume that*

$$\int\limits_0^{\infty} \zeta(t)\,dt = +\infty. \tag{28}$$

Then $X(t)$ is null-ergodic, and bound (27) holds for any $t \geqslant s \geqslant 0$ and any initial condition $\mathbf{p}(s)$. If, instead of (28) we have

$$\zeta(t) \geqslant \zeta^* > 0, \tag{29}$$

then $X(t)$ is exponentially null-ergodic, and the bound

$$\sum_{k=0}^{N} p_k(t) \leqslant \frac{\delta_k}{\delta_N} e^{-\zeta^* t}, \tag{30}$$

holds for any $t \geqslant 0$, any initial condition $X(0) = k$, and any natural N.

3. Bounds for probabilities in multidimensional BDP

Consider the one-dimensional process $X_j(t)$. Then $X_j(t)$ is a (generally non-Markovian) birth and death process with birth intensities

$$\lambda_k = \frac{\sum_{\mathbf{m},m_j=k} \lambda_{j,\mathbf{m}}(t) p_{\mathbf{m}}(t)}{\sum_{\mathbf{m},m_j=k} p_{\mathbf{m}}(t)} \tag{31}$$

and death intensities

$$\mu_k = \frac{\sum_{\mathbf{m},m_j=k} \mu_{j,\mathbf{m}}(t) p_{\mathbf{m}}(t)}{\sum_{\mathbf{m},m_j=k} p_{\mathbf{m}(t)}}. \tag{32}$$

For any fixed initial distribution $\mathbf{p}(0)$ and any $t > 0$ the probability distribution $\mathbf{p}(t)$ is unique. Hence $\lambda_k = \lambda_k(\mathbf{p}(0), t)$ and $\mu_k = \mu_k(\mathbf{p}(0), t)$ uniquely define the corresponding birth-death system (3) for the state probabilities of the process $X_j(t)$ with a given initial condition, and Theorems 1 and 2 allow one to obtain their bounds. Let for all $\mathbf{m}$ and any $t \geqslant 0$

$$l_j \leqslant \lambda_{j,\mathbf{m}}(t) \leqslant L_j, \quad m_j \leqslant \mu_{j,\mathbf{m}}(t) \leqslant M_j. \tag{33}$$

Then $H = 2\sum_{j=1}^{d}(L_j + M_j)$.

Theorem 3. *Let*

$$L_j < m_j, \quad \alpha_* = l_j + m_j - 2\sqrt{L_j M_j} > 0, \tag{34}$$

for some j. Then the following bound holds:

$$\Pr\left(X_j(t) \leqslant n / X_j(0) = k\right) \geqslant 1 - \beta^{1-n}\left(\frac{H}{2\alpha^*} + e^{-\alpha^* t} \cdot \sum_{i=0}^{k-1} \beta^i\right), \tag{35}$$

for any natural n, k and any $t \geqslant 0$, where $\beta = \sqrt{M_j/L_j} > 1$.

Proof. Put $d_n = \beta^{n-1}, \quad n \geqslant 1$. Then

$$\lambda_k + \mu_{k+1} - \frac{d_{k+1}}{d_k}\lambda_{k+1} - \frac{d_{k-1}}{d_k}\mu_k \geqslant l_j + m_j - \beta L_j - M_j/\beta = \alpha_*, \tag{36}$$

hence, in accordance with (20), we have

$$\Pr\left(X_j(t) \leqslant n / X_j(0) = k\right) \geqslant 1 - \beta^{1-n}\left(\frac{H}{2\alpha^*} + e^{-\alpha^* t}\|\mathbf{z}(0)\|_{1D}\right), \tag{37}$$

and (35), for any n, k and any $t \geqslant 0$.

Theorem 4. *Let*

$$M_j < l_j \tag{38}$$

for some j. Then

$$\Pr\left(X_j(t) \leqslant n / X_j(0) = k\right) \leqslant \sigma^{k-n} \cdot e^{-\zeta^* t}, \tag{39}$$

where $\sigma = \sqrt{M_j/l_j} < 1$, $\zeta^* = \left(\sqrt{l_j} - \sqrt{M_j}\right)^2$.

Proof. Put $\delta_n = \sigma^n, \quad n \geqslant 0$. Then

$$\lambda_k + \mu_k - \frac{\delta_{k+1}}{\delta_k}\lambda_k - \frac{\delta_{k-1}}{\delta_k}\mu_k \geqslant \lambda_k(1-\sigma) - \mu_k(1/\sigma - 1) \geqslant l_j(1-\sigma) - M_j(1/\sigma - 1) = \zeta^*, \tag{40}$$

and we can apply Theorem 2.

Remark 1. Instead of $X_j(t)$ we can obtain the same results for the one-dimensional process $Z(t) = |X(t)|$, that is, the number of all particles at the moment t.

Remark 2. There are a number of situations in which a complete study is possible. Namely, if all birth and death intensities for some $X_j(t)$ depend only on j and t (or all birth and death intensities for $Z(t)$ depend only on t) then we obtain the corresponding ordinary one-dimensional inhomogeneous BDP. Hence all results of the previous section can be applied.

Remark 3. In a weakly ergodic case our bounds for the state probabilities can be used to study the truncations and perturbation bounds for the corresponding processes; see the results for inhomogeneous BDPs obtained in [11–13].

Remark 4. All of our results can be extended for the case of so-called birth-death-transformation process, i.e., for the process $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))$ such that in the interval $(t, t+h)$ the following transitions are possible with order h: the birth of a particle of type j, the death of a particle of type j, and the transformation of a particle of type j_1 to type j_2.

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