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ON THE EXTENDABILITY OF LOCALLY DEFINED ISOMETRIES OF A PSEUDO-RIEMANNIAN MANIFOLD

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ABSTRACT. Let η be a stationary subalgebra of the Lie algebra ζ of all Killing vector fields on a pseudo-Riemannian analytic manifold, G be a simply connected Lie group generated by the algebra ζ, and H be its subgroup generated by the subalgebra η. Then the subgroup H is closed in G.

An analytic mapping of an open subset U of an analytic manifold M to an analytic manifold N may admit an analytic extension to the whole manifold M. The study of analytic extensions of local isometries of Riemannian or pseudo-Riemannian manifolds is of particular interest. It is easy to prove that an isometry $f : U \to V$ between two open subsets of a complete, simply connected Riemannian manifold M can be analytically extended to an isometry $f : M \to M$. However, it is not always possible to construct an analytic extension of a locally defined Riemannian analytic metric to a metric of a complete manifold. A natural generalization of the notion of a complete manifold is the notion of nonextendability introduced by Helgason in [1]. Nonextendable properly Riemannian manifolds and the possibility of extension of isometries locally defined on them are examined in [3, 4]. We present a generalization of results of [3] to the case of pseudo-Riemannian manifolds. In particular, we prove that if a pseudo-Riemannian metric is defined on a ball and the Lie algebra of infinitesimal isometries of this metric has zero center and the dimension of this Lie algebra at a fixed point coincides with the dimension of the ball, then this metric can be extended to a metric of a complete pseudo-Riemannian manifold.

Consider a pseudo-Riemannian analytic manifold M and an isometry $\varphi : U \to V$ between open subsets of it. The following question appears: Under which conditions the mapping $\varphi : U \to V$ can be analytically extended to an isometry $\varphi : M \to M$ of the whole manifold? The existence of an extension along a continuous curve for complete manifolds was proved in the classical monograph by Helgason [1]. However, even for complete manifolds, an extension along curves can be ambiguous. For non-complete manifolds, an analytic extension of a local isometry along an arbitrary curve does not exist. However, an extension of an infinitesimal isometry always exists.

Infinitesimal isometries on a pseudo-Riemannian manifold are vector fields $\xi^k(x)$ satisfying the following system of differential equations:

$$\sum_{k=1}^{n}\left(\frac{\partial g_{ij}}{\partial x^k}\xi^k - g_{kj}\frac{\partial \xi^k}{\partial x^i} - g_{ik}\frac{\partial \xi^k}{\partial x^j}\right) = 0.$$

Such vector fields are called Killing vector fields.

Theorem 1. *Let M be an analytic pseudo-Riemannian manifold, X be a Killing vector field defined in a domain $U \subset M$, and let $\gamma(t)$, $0 \le t \le 1$, be a continuous curve in M such that $\gamma(0) \in U$. Then the vector field X can be analytically extended along γ.*

Proof. Assume that X can be analytically extended to a neighborhood of each point $\gamma(t)$ for $t < t_1 \le 1$. We prove that X can also be extended to a neighborhood of the point $q = \gamma(t_1)$. Let V be a normal

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neighborhood of the point q, which is also a normal neighborhood of each of its points (see [1]). Consider $t < t_1$ such that $p = \gamma(t) \in V$.

A vector field X generates a local one-parameter isometry group φ_s in a neighborhood of each point $\gamma(t)$, $t < t_1$. We prove that for all sufficiently small values of s, local isometries φ_s can also be analytically extended to a neighborhood of the point $q = \gamma(t_1)$. Then the vector field of velocities of this local isometry group is an analytic extension of the vector field X to a neighborhood of the point q.

Consider a connected open set $V_0 \subset V$ that contains the points p and q, whose closure also lies in V, $\overline{V_0} \subset V$, $p, q \in V_0$. Consider a small neighborhood $V' \subset V_0$ of the point q and connect the point p with an arbitrary point $q' \in V'$ by a segment of a geodesic $\alpha(t)$, $0 \le t \le 1$. Let

$$Y = \frac{d\alpha}{dt}(0) \in T_pM, \quad p_s = \varphi_s(p), \quad Y_s = \varphi_s(Y).$$

From the point p_s, we eject a geodesic $\beta(t)$, $0 \le t \le 1$, such that

$$\frac{d\beta}{dt}(0) = Y_s.$$

For sufficiently small values of s we have $\beta(t) \in V_0$, $0 \le t \le 1$. We set $\varphi_s(q') = \beta(1)$. The mapping obtained is an analytic extension of the isometry φ_s. □

Remark. This proof allows one to generalize Theorem 1 to the case of spaces with affine connection. Namely, an infinitesimal affine transform X of an analytic space with affine connection M defined on an open set $U \subset M$ can be analytically extended along an arbitrary continuous curve $\gamma(t)$ on M.

In the general case, the basic impossibility of extension of an infinitesimal isometry to an isometry of the manifold M in the whole is caused by the fact that the pair consisting of the Lie algebra ζ and a stationary subalgebra $\eta \subset \zeta$ of it may not generate a homogeneous manifold. More precisely, under the condition that each local one-parameter group generated by a vector field $X \in \zeta$ is extendable, the isometry group G of the manifold M and an orbit $K \subset M$ of this group appear, but this is not always possible.

Theorem 1 allows one to identify the Lie algebra ζ of all Killing vector fields defined in some neighborhood of a point $p \in M$ with the Lie algebra of all Killing vector fields on the pseudo-Riemannian analytic manifold M. Let $\eta \subset \zeta$ be a stationary subalgebra. $X \in \eta$ if and only if $X(p) = 0$. Let G be a simply connected Lie group with the Lie algebra ζ and $H \subset G$ be the subgroup generated by the subalgebra $\eta \subset \zeta$. The exponential mapping defined an isometric action of the group G in some neighborhood U of the point p determined for elements $g \in W$ from some neighborhood of 0 in G. An extension of these isometries to the whole manifold M defines the orbit $K = G(p)$ as a differentiable submanifold of the manifold M diffeomorphic to the factor-group G/H. However, the factor-group G/H is a differentiable manifold if and only if the subgroup H is closed in G. For pseudo-Riemannian manifolds, one can find the following sufficiently general condition for the metric under which the subgroup H is closed in G: the algebra ζ has zero center.

Theorem 2. *Let ζ be the Lie algebra of all Killing vector fields on a pseudo-Riemannian analytic manifold M, $p \in M$ be a fixed point, $\eta \subset \zeta$ be a stationary subalgebra consisting of vector field $X \in \zeta$ such that $X(p) = 0$, G be a simply connected Lie group with the Lie algebra ζ, and $H \in G$ be the subgroup generated by the subalgebra $\eta \subset \zeta$. If ζ has zero center, then H is closed in G.*

The proof of Theorem 2 in the general form for the case of a Riemannian manifold follows from the description of quasi-complete manifolds whose definition and properties are stated below. Here we present an algebraic proof for a pseudo-Riemannian manifold in the case of a locally homogeneous manifold, i.e., under the condition

$$\dim M = \dim \zeta - \dim \eta.$$

Assume that the subgroup H is not closed in G. Let the group $\overline{H}$ be the closure of H in G and $\overline{\eta} \subset \zeta$ be the Lie algebra of the group $\overline{H} \subset G$. As was proved in the classical work of Mal'cev [2], the subalgebra η is a normal subgroup of the Lie algebra $\overline{\eta}$. Therefore, the adjoint representation Ad of the group G in the algebra ζ,

$$\operatorname{Ad} g(X) = g^{-1} X g \quad \forall X \in \zeta,\ g \in G,$$

defines a linear mapping in the vector space $V = \zeta/\eta$ if $g \in \overline{H}$.

Consider a normal neighborhood $U \subset M$ of the point p and a closed ball of radius r (in the normal coordinates) $B_r \subset U$. Due to the compactness of B_r, there exists a neighborhood of the identity $W \subset G$ such that all elements $g \in W$ define an isometry $g : B_r \to U$ (g belongs to a local one-parameter transformation group generated by some Killing vector field on M). In the tangent space T_pM identified with the vector space $V = \zeta/\eta$ we choose a basis and consider transformations $\operatorname{Ad} h$, $h \in H \cap W$, and $\operatorname{Ad} g$, $g \in \overline{H}$, as matrices.

Consider a vector field $Z \subset \overline{\eta}$, $Z \notin \eta$, and a local one-parameter subgroup $H_t \in \overline{H}$ generated by the vector field Z. Each element $h_t \in H_t \cap W$ is the limit of some sequence $h_n \in H$ such that $h_n \in W$ starting from some number. For $t \leq 1$, the norms of the matrices of linear transforms $\operatorname{Ad} h_t$ and $\operatorname{Ad} h_n = h_n$ of the vector space V are bounded by some constant C. Therefore, the mappings $\operatorname{Ad} h_t$ and $\operatorname{Ad} h_n$ map the ball B_δ to the ball B_r, $B_\delta \subset B_r \subset V$, $\delta \leq r/C^2 n$. The bounded sequence of matrices $\operatorname{Ad} h_n$ defined the sequence of isometric mappings defined on the neighborhood $B_\delta \subset M$ of the point $p \in M$. Then the mapping $\operatorname{Ad} h_t$ also defines (in the normal coordinates) an isometric mapping defined on B_δ.

The right multiplication by an element h_t defines a linear mapping on the tangent space $T_pM = \zeta/\eta$, which, in its turn, determines, for all $t \leq 1$, an isometric mapping defined on some neighborhood of the point $p \in M$, since the right multiplication by h_t is the superposition of isometric mappings $\operatorname{Ad} h_t : X \to h_t^{-1} X h_t$ and $h_t : X \to h_t X$. Therefore, a one-parameter isometry group $X \to X h_t$ commutes with the action of each sufficiently small element of the group G on U. Hence the Killing vector field corresponding to this one-parameter group of right multiplications commutes with the whole Lie algebra ζ, which contradicts the theorem.

Except for the nonclosedness of the subgroup H generated by a stationary subalgebra, there exist other obstructions for the extension of a local isometry of a Riemannian analytic manifold. For example, a local isometry of a complete manifold M can be analytically extended in a neighborhood of each point of this manifold (perhaps, ambiguously). If we remove from M a closed subset, the extendability property of local isometries loses.

Consider a class of Riemannian analytic manifolds that are locally isometric to each other, i.e., that have isometric open subsets. This class of manifolds is defined only by local properties of the metric. The following question appears: For which metrics does there exist a manifold M possessing the property of analytic extendability of local isometries $\varphi : U \to V$ to isometries $\varphi : M \to M$ and which properties does this manifold possess? Primarily, such a manifold must be nonextendable.

Definition 1. A pseudo-Riemannian analytic manifold M is said to be *nonextendable* if it cannot be analytically embedded as an open subset in a pseudo-Riemannian analytic manifold N different from M.

As an example of a nontrivial punctured covering Euclidean plane, the nonextendability property is not sufficient for the extendability of local isometries to isometries of the whole manifold. One can define a manifold possessing the property of extendability of all local isometries to isometries of the whole manifold, for example, for metrics whose Lie algebra if Killing vector fields has zero center.

Definition 2. An oriented Riemannian analytic manifold M is said to be *quasi-complete* if it is nonextendable and does not admit nontrivial, orientation-preserving Killing vector fields of isometries $\varphi : U \to V$ between open subsets.

Theorem 3. *Each Riemannian ball with an analytic metric such that the Lie algebra of all Killing vector fields on this ball has zero center is contained in a quasi-complete manifold.*

The idea of the proof of Theorem 3 is an analytic extension of the initial Riemannian manifold U to a nonextendable manifold M_1 and the subsequent factorization with respect to all local isometries that preserve orientation and Killing vector fields. A detailed study of quasi-complete manifolds and the proof of Theorem 3 can be found in [3].

Theorem 4. *Each isometry $\varphi : U \to V$ between open subsets of quasi-complete manifolds M and N can be analytically extended to an isometry $\varphi : M \to N$.*

The idea of the proof of Theorem 4 is as follows: the nonextendability of the isometry φ contradicts the nonextendability of the manifold N.

Theorem 4 implies that a quasi-complete manifold is unique in the class if locally isometric manifolds. The term "quasi-complete manifold" can be explained by the fact that a complete manifold whose Lie algebra of Killing vector fields has no center is also quasi-complete. The proof of Theorem 4 and other properties of quasi-complete manifolds can be found in [3].

Remark. In the case of a properly Riemannian manifold, Theorem 2 is a consequence of Theorem 4.

This work is a natural continuation of [4].

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