DOI 10.1007/s10958-016-2946-3
Journal of Mathematical Sciences, Vol. 216, No. 6, August, 2016

NEW NONSINGULARITY CONDITIONS FOR GENERAL MATRICES AND THE ASSOCIATED EIGENVALUE INCLUSION SETS

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The paper suggests generalizations of some known sufficient nonsingularity conditions for matrices with constant principal diagonal and the corresponding eigenvalue inclusion sets to the cases of arbitrary matrices and matrices with nonzero diagonal entries. Bibliography: 11 *titles.*

1. In [9], for matrices with constant principal diagonal, Melman suggested new nonsingularity conditions and the associated eigenvalue inclusion sets, which are unions of specific ovals of Cassini. Melman's research has at least two motivations. First, for matrices $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with

$$a_{11} = \cdots = a_{nn} = \xi, \tag{1}$$

the simplest classical Gerschgorin inclusion set ([3]; also see, e.g., [11, Theorem 1.1])

$$\Gamma(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : \ |z - a_{ii}| \le r'_i(A)\}, \tag{2}$$

consisting of disks centered at the diagonal matrix entries, as well as the Ostrowski–Brauer inclusion set ([1, 10]; also see, e.g., [11, Theorem 2.2])

$$\Delta(A) = \bigcup_{\substack{i,j=1\\ i\ne j}}^{n} \left\{z \in \mathbb{C} : \ |z - a_{ii}|\,|z - a_{jj}| \le r'_i(A)\, r'_j(A)\right\}, \tag{3}$$

consisting of the Cassini ovals with foci at the diagonal matrix entries, both degenerate to the single disks

$$\{z \in \mathbb{C} : \ |z - \xi| \le \max_{1\le i\le n} r'_i(A)\} \tag{4}$$

and

$$\left\{z \in \mathbb{C} : \ |z - \xi| \le \max_{1\le i\ne j\le n} \{r'_i(A)\, r'_j(A)\}^{1/2}\right\}, \tag{5}$$

respectively.

Here and in what follows, we use the notation

$$r'_i(A) = \sum_{\substack{j=1\\ j\ne i}}^{n} |a_{ij}|, \quad r_i(A) = r'_i(A) + |a_{ii}|, \quad i = 1, \ldots, n,$$

i.e., $r_i(A)$ and $r'_i(A)$ are the ith complete and deleted absolute row sums of A; $r'(A) = (r'_i(A))$ and $r(A) = (r_i(A))$ are the corresponding vectors; $c'_i(A) = r'_i(A^T)$, $i = 1, \ldots, n$, are the deleted absolute column sums of A, and $c'(A) = (c'_i(A))$;

$$\operatorname{Spec} A = \bigcup_{i=1}^{n} \lambda_i(A)$$

is the eigenspectrum of A.

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 439, 2015, pp. 128–144. Original article submitted September 23, 2015.

The second motivation for considering matrices with constant principal diagonal is that this class of matrices contains the class of Toeplitz matrices. Accordingly, in [9], the general results were adapted to Toeplitz matrices, for which they are considerably less expensive to compute.

To be specific, we recall that in [9] the following two equivalent general results were established.

Theorem 1. *If a matrix* $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, *satisfies condition* (1), *then*

$$\operatorname{Spec} A \subseteq \Omega(A) \equiv \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : \ \left|(z-\xi)^2 - (B^2)_{ii}\right| \le r_i'(B^2) \right\}, \tag{6}$$

where we denote

$$A = \xi I_n - B; \tag{7}$$

in addition, the following inclusion holds:

$$\Omega(A) \subseteq \Delta(A).$$

Theorem 2. *If, under the assumptions of Theorem* 1,

$$|\xi^2 - (B^2)_{ii}| > r_i'(B^2), \quad i = 1, \ldots, n, \tag{8}$$

then A is nonsingular; in addition, the sufficient nonsingularity conditions (8) *are weaker than the Ostrowski–Brauer nonsingularity conditions*

$$|a_{ii}|\,|a_{jj}| > r_i'(A)\, r_j'(A), \quad 1 \le i \ne j \le n, \tag{9}$$

corresponding to the eigenvalue inclusion set $\Delta(A)$ *(see* (3)*), which reduce, for A satisfying* (1), *to the single inequality*

$$|\xi|^2 > \max_{i\ne j} \{ r_i'(A)\, r_j'(A)\}. \tag{10}$$

As was indicated in [9], Theorems 1 and 2 "can be adapted to cases where the diagonal elements are not constant but clustered around a certain value, provided that the clustering is tight enough compared to the magnitude of the nondiagonal elements of the matrix."

In this paper, we provide some generalizations of Theorems 1 and 2 to arbitrary matrices $A \in \mathbb{C}^{n\times n}$ and to matrices with nonzero diagonal entries. In general, no clustering of the diagonal matrix entries is assumed. However, we also propose general nonsingularity conditions and the corresponding eigenvalue inclusion sets, which are best suited for matrices with clustered diagonal entries.

The paper is organized as follows. In the next section, we generalize Theorems 1 and 2 to arbitrary matrices and, in particular, provide generalizations especially suited for matrices with clustered diagonal entries. In Sec. 3, a series of generalizations of Theorem 2 to matrices with nonzero diagonal entries is suggested, and it is shown that the conditions obtained are weaker than the known conditions of the so-called diagonal dominance of order k, $k \ge 0$, introduced in [5]. The generalizations established in Secs. 2 and 3 are based on imposing the condition of strict diagonal dominance on some auxiliary matrices, whose nonsingularity implies the nonsingularity of a given matrix. In Sec. 4, new nonsingularity conditions and eigenvalue inclusion sets are derived by applying known sufficient nonsingularity conditions, more complicated than the condition of strict diagonal dominance, to the auxiliary matrices considered in Secs. 2 and 3. Concluding remarks are presented in Sec. 5.

2. First of all, we will show that for a matrix A with constant principal diagonal, Melman's nonsingularity conditions (8) are actually weaker than not only the classical Ostrowski–Brauer conditions (9) but also their relaxed version, see (11) below, which takes into account the zero/nonzero pattern of the entries of A. Consequently, Melman's inclusion set $\Omega(A)$ is contained in a subset $\Delta'(A)$, see (17), of the Ostrowski–Brauer set $\Delta(A)$. To this end, we recall the following definition and theorem [5] (also see [4]).

A matrix $A \in \mathbb{C}^{n\times n}$, $n \geq 1$, is called *quasiirreducible* if all its diagonal entries that are irreducible components of order 1 of the matrix A are nonzero. In particular, every matrix with nonzero diagonal entries obviously is quasiirreducible.

Theorem 3 ([5]). *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 1$, is quasiirreducible and satisfies the conditions*

$$|a_{ii}|^{\alpha}\,|a_{jj}|^{1-\alpha} > r_i'(A)^{\alpha}\, r_j'(A)^{1-\alpha} \quad \text{for all} \quad i \neq j \quad \text{such that} \quad a_{ij} \neq 0 \tag{11}$$

for a certain $0 \leq \alpha \leq 1$, then A is nonsingular.

Now we are ready to show that for a matrix satisfying (1), the nonsingularity conditions (8) are weaker than conditions (11) with $\alpha = 1/2$, which, in the case under consideration, read as

$$|\xi|^2 > \max_{i\neq j:\, a_{ij}\neq 0} \{r_i'(A)\, r_j'(A)\}, \tag{12}$$

i.e., conditions (8) stem from condition (12).

Indeed, since

$$r_i(B^2) = \{|B^2|e\}_i \leq \{|B|r'(A)\}_i = \sum_{j\neq i:\, a_{ij}\neq 0} |a_{ij}| r_j'(A)$$
$$\leq r_i'(A) \max_{j\neq i:\, a_{ij}\neq 0} \left\{r_j'(A)\right\} \leq \max_{j\neq i:\, a_{ij}\neq 0} \left\{r_i'(A)\, r_j'(A)\right\},$$

from (12) it follows that

$$|\xi|^2 > r_i(B^2) = |(B^2)_{ii}| + r_i'(B^2), \quad i = 1, \ldots, n,$$

implying that

$$r_i'(B^2) < |\xi|^2 - |(B^2)_{ii}| \leq \left|\xi^2 - (B^2)_{ii}\right|, \quad i = 1, \ldots, n.$$

Thus, condition (12) is stronger than conditions (8).

Equivalently, Melman's inclusion set $\Omega(A)$ is contained in the disk

$$\left\{ z \in \mathbb{C} :\ |z - \xi| \leq \max_{i\neq j:\, a_{ij}\neq 0} \left\{r_i'(A)\, r_j'(A)\right\}^{1/2} \right\}, \tag{13}$$

corresponding to the nonsingularity condition (12).

Remark 1. If condition (12) is fulfilled, then (see [6]) the matrix $A = \xi I_n - B$ is a nonsingular H-matrix, whereas under the weaker conditions (8), as the following simple example demonstrates, the matrix A is not necessarily an H-matrix. Thus, conditions (8) differ from the majority of known nonsingularity conditions, which ensure that A is a nonsingular H-matrix.

Example 1. Let

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad \text{whence} \quad \xi = 0, \quad B^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Obviously, B^2 is a strictly diagonally dominant (sdd) matrix, and conditions (8) are fulfilled. However, since $D_A = 0$, the matrix A cannot be an H-matrix.

Now let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, be an arbitrary matrix. We write it as

$$A = D_A - B, \quad \text{where} \quad D_A = \operatorname{diag}(a_{11}, \ldots, a_{nn}).$$

Observe that for A to be nonsingular, it is obviously sufficient that the matrix

$$C(A) \equiv A(D_A + B) = D_A^2 - B^2 + (D_A B - B D_A) \tag{14}$$

be nonsingular. (Observe that the matrix $D_AB - BD_A$ has zero diagonal entries.) Thus, any condition sufficient for $C(A)$ to be nonsingular is a fortiori sufficient for A to be nonsingular. In particular, if $C(A)$ is an sdd matrix, i.e.,

$$|a_{ii}^2 - (B^2)_{ii}| > r_i'(B^2 + (BD_A - D_AB)), \quad i = 1, \dots, n, \tag{15}$$

then A is nonsingular.

Obviously, in the case where $D_A = \xi I_n$, we have $BD_A - D_AB = 0$, and conditions (15) reduce to the Melman nonsingularity conditions (8).

Thus, we have obtained the following generalization of Theorem 2 to arbitrary matrices.

Theorem 4. *If a matrix* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$, *satisfies conditions* (15), *then it is nonsingular.*

It is also worth mentioning that conditions (15), involving the additional matrix $BD_A - D_AB$, are not essentially more expensive to check than conditions (8), because, in the general case, the most expensive part in computing both (8) and (15) is the computation of the entries of B^2.

In terms of the eigenvalue inclusion sets, Theorem 4 can equivalently be stated as follows.

Theorem 5. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$. *Then*

$$\operatorname{Spec} A \subseteq \Omega'(A) \equiv \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : \ |(z - a_{ii})^2 - (B^2)_{ii}| \leq r_i'(B^2 + BD_A - D_AB) \right\}. \tag{16}$$

Proof. If $\lambda \in \operatorname{Spec} A$, then the matrix $A - \lambda I_n$ cannot satisfy (15), whence, for a certain i, $1 \leq i \leq n$,

$$|(a_{ii} - \lambda)^2 - (B^2)_{ii}| \leq r_i'(B^2 + B(D_A - \lambda I_n) - (D_A - \lambda I_n)B),$$

and, in order to complete the proof, it only remains to observe that

$$r_i'(B^2 + B(D_A - \lambda I_n) - (D_A - \lambda I_n)B) = r_i'(B^2 + BD_A - D_AB). \qquad \square$$

Note that the set $\Omega'(A)$, as well as the Melman set $\Omega(A)$, is a union of Cassini ovals.

Unfortunately, in the general case, the inclusion

$$\Omega'(A) \subseteq \Delta'(A),$$

where

$$\Delta'(A) \equiv \bigcup_{\substack{i,j=1 \\ i\neq j:\ a_{ij}\neq 0}}^{n} \left\{ z \in \mathbb{C} : \ |z - a_{ii}|\, |z - a_{jj}| \leq r_i'(A)\, r_j'(A) \right\}, \tag{17}$$

which holds, as we have seen, for matrices with constant principal diagonal, is not always valid, i.e., the conditions (see (11) with $\alpha = 1/2$)

$$|a_{ii}|\, |a_{jj}| > r_i'(A)\, r_j'(A) \quad \text{for all} \quad i \neq j \quad \text{such that} \quad a_{ij} \neq 0 \tag{18}$$

do not necessarily imply that $C(A)$ is an sdd matrix. Moreover, the classical Ostrowski–Brauer conditions (9) neither imply (15). Indeed, consider the following simplest example, for which conditions (9) and (18) coincide.

Example 2. Let

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}.$$

Then conditions (9) and (18) are obviously fulfilled, but the matrix

$$C(A) = \begin{pmatrix} -1 & -4 \\ 2 & 7 \end{pmatrix}$$

is not diagonally dominant.

Vice versa, as the following example demonstrates, conditions (15) neither imply (18). Thus, in general, conditions (15) are incomparable with (9) and (18).

Example 3. Let

$$A = \begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}.$$

Then the matrix

$$C(A) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

is strictly diagonally dominant, but A satisfies neither (18) nor (9).

Consider a modification of the above approach, which is as well applicable to arbitrary matrices, but appears best suited for matrices with clustered diagonal entries, for which

$$|a_{ii} - \xi| \le \varepsilon, \quad i \in S, \tag{19}$$

where $\varepsilon \ge 0$ and S is a subset of the index set $\{1, \dots, n\}$.

Now, instead of the matrix $C(A)$, defined in (14), we consider the matrix

$$C_\xi(A) \equiv (D_A - B)(\xi I_n + B) = \xi D_A - B^2 + (D_A - \xi I_n)B, \tag{20}$$

which depends on the scalar parameter ξ. Then, requiring that $C_\xi(A)$ be an sdd matrix, we arrive at the following sufficient nonsingularity conditions for A.

Theorem 6. *Let $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \ge 2$, and let $\xi \in \mathbb{C}$. If the conditions*

$$|\xi a_{ii} - (B^2)_{ii}| > r_i'(B^2 - (a_{ii} - \xi)B), \quad i = 1, \dots, n, \tag{21}$$

then A is nonsingular.

Obviously, as $\varepsilon \to 0$ in (19), the inequalities in (21) corresponding to $i \in S$ tend to the corresponding inequalities in (8), and Theorem 6 provides an extension of Theorem 2 to arbitrary matrices, which takes into consideration the clustering of the diagonal entries.

By applying Theorem 6 to the matrix $A - \lambda I_n$, where $\lambda \in \operatorname{Spec} A$, with ξ changed for $\xi - \lambda$, we obtain the corresponding generalization of Theorem 1.

Theorem 7. *Let $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \ge 2$, and let $\xi \in \mathbb{C}$. Then*

$$\operatorname{Spec} A \subseteq \Omega_\xi''(A) \equiv \bigcup_{i=1}^{n} \left\{z \in \mathbb{C} : \left|(z - a_{ii})\,(z - \xi) - (B^2)_{ii}\right| \le r_i'(B^2 - (a_{ii} - \xi)B)\right\}. \tag{22}$$

Proof. We have

$$C_{\xi-\lambda}(A - \lambda I_n) = (\xi - \lambda)(D_A - \lambda I_n) + (D_A - \xi I_n)B - B^2.$$

Since the matrix $C_{\xi-\lambda}(A - \lambda I_n)$ is singular, from Theorem 6 it follows that for a certain i, $1 \le i \le n$, we necessarily have

$$\left|(\xi - \lambda)(a_{ii} - \lambda) - (B^2)_{ii}\right| \le r_i'(B^2 - (a_{ii} - \xi)B).$$

This completes the proof. □

It is clear that the set $\Omega_\xi''(A)$ is a union of Cassini ovals with foci at the roots

$$z_\pm = \frac{1}{2}\left\{a_{ii} + \xi \pm \sqrt{(a_{ii} - \xi)^2 + 4(B^2)_{ii}}\right\}$$

of the quadratic equation

$$z^2 - (a_{ii} + \xi)z + a_{ii}\xi - (B^2)_{ii} = 0.$$

Furthermore, the set $\Omega''_\xi(A)$ is obviously contained in the following unions of Cassini ovals with foci at the diagonal entries of A and the point ξ:

$$\Omega''_\xi(A) \subseteq \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : \; |(z-a_{ii})\,(z-\xi)| \le (B^2)_{ii} + r'_i(B^2 - (a_{ii}-\xi)B) \right\}$$
$$\subseteq \bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : \; |(z-a_{ii})\,(z-\xi)| \le r_i(B^2) + |a_{ii}-\xi|\, r'_i(B) \right\}.$$

Obviously, Theorem 7 can be strengthened as follows.

Corollary 1. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \ge 2$. *Then*

$$\operatorname{Spec} A \subseteq \Omega''(A) \equiv \bigcap_{\xi\in\mathbb{C}} \Omega''_\xi(A).$$

Corollary 1 is of theoretical interest. In practice, for instance, in the case of matrices whose diagonal entries are clustered around p values $\xi_1,\dots,\xi_p$, $p \ge 1$, one can apply the following larger but computable eigenvalue inclusion set:

$$\operatorname{Spec} A \subseteq \bigcap_{i=1}^{p} \Omega''_{\xi_i}(A).$$

3. Now we present a series of alternative generalizations of Melman's nonsingularity conditions (8), but this time only to matrices $A = (a_{ij}) = D_A - B$ with nonzero diagonal entries,

$$a_{ii} \ne 0, \quad i = 1,\dots,n. \tag{23}$$

For such matrices, the nonsingularity of A obviously amounts to the nonsingularity of the Jacobi scaled matrix

$$\bar{A} \equiv D_A^{-1}A = I_n - D_A^{-1}B \equiv I_n - \bar{B}.$$

Observe that the latter matrix has constant principal diagonal, $D_{\bar{A}} = I_n$. By applying Theorem 2 to $\bar{A}$, we obtain that if

$$|1-(\bar{B}^2)_{ii}| > r'_i(\bar{B}^2), \quad i = 1,\dots,n, \tag{24}$$

then both matrices A and $\bar{A}$ are nonsingular.

Obviously, if $D_A = \xi I_n$, $\xi \ne 0$, then conditions (24) are equivalent to Melman's conditions (8).

Unfortunately, in the case where $D_A \ne \xi I_n$, the inequalities describing the eigenvalue inclusion sets associated with the nonsingularity conditions (24) involve λ in a rather complicated way, which makes these sets practically useless.

As is readily seen, conditions (24) can be replaced by the single stronger sufficient condition

$$1 > \max_{1\le i\le n} r_i(\bar{B}^2), \tag{25}$$

from which conditions (24) trivially follow.

Condition (25), in its turn, can be strengthened to the condition

$$1 > \max_{1\le i\le n} r_i(|\bar{B}|^2), \tag{26}$$

which is less expensive to compute (because $r(|\bar{B}|^2)$ can be computed by multiplying $|\bar{B}|$ by vectors twice) and still weaker than the relaxed Ostrowski–Brauer conditions (18).

Indeed,

$$r_i(|\bar{B}|^2) = (|\bar{B}|\, r(\bar{B}))_i = \sum_{j\neq i:\, a_{ij}\neq 0} \frac{|a_{ij}|}{|a_{ii}|} r_j(\bar{B}) \leq \frac{r'_i(A)}{|a_{ii}|} \max_{j\neq i:\, a_{ij}\neq 0} \left\{ \frac{r'_j(A)}{|a_{jj}|} \right\},$$

whence

$$\max_{1\leq i\leq n} r_i(|\bar{B}|^2) \leq \max_{1\leq i\neq j\leq n:\, a_{ij}\neq 0} \frac{r'_i(A)\, r'_j(A)}{|a_{ii}|\, |a_{jj}|}.$$

Therefore, conditions (18) imply (26), and we have the implications

$$(18) \Longrightarrow (26) \Longrightarrow (25) \Longrightarrow (24). \tag{27}$$

The following theorem extends conditions (24) to the case of an arbitrary $k \geq 1$.

Theorem 8. *If, for a certain $k \geq 1$, a matrix $A = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$, with nonzero diagonal entries satisfies the conditions*

$$|1 - (\bar{B}^k)_{ii}| > r'_i(\bar{B}^k), \quad i = 1, \ldots, n, \tag{28}$$

then it is nonsingular.

Proof. Define the matrix

$$\bar{C}^{(k)} = \bar{A}(I_n + \bar{B} + \cdots + \bar{B}^{k-1}) = I_n - \bar{B}^k, \quad k \geq 1.$$

Obviously, for $\bar{A}$ (and also A) to be nonsingular, suffice it that the matrix $\bar{C}^{(k)}$ be nonsingular. It remains to observe that conditions (28) exactly mean that $\bar{C}^{(k)}$ is an sdd matrix, implying that it is nonsingular. □

Note that for $k = 1$ conditions (28) simply require that A be an sdd matrix, whereas for $k = 2$ they reduce to (24).

For matrices with constant principal diagonal, $D_A = \xi I_n$, Theorem 8 implies the following generalization of Theorem 2.

Corollary 2. *If a matrix $A = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$, with $D_A = \xi I_n$, $\xi \in \mathbb{C}$, satisfies the conditions*

$$|\xi^k - (B^k)_{ii}| > r'_i(B^k), \quad i = 1, \ldots, n, \tag{29}$$

where $k \geq 1$, then A is nonsingular.

Proof. Indeed, if $\xi \neq 0$, then A is nonsingular by Theorem 8. Otherwise, we have $A = B$, and conditions (29), meaning that $A^k = B^k$ is an sdd matrix, guarantee that A is nonsingular. □

Obviously, conditions (28) can be replaced by the stronger sufficient conditions

$$1 > \max_{1\leq i\leq n} r_i(\bar{B}^k) \tag{30}$$

and

$$1 > \max_{1\leq i\leq n} r_i(|\bar{B}|^k), \tag{31}$$

which generalize conditions (25) and (26), respectively, to $k \geq 1$.

Now we show that condition (31) is weaker than the so-called conditions of strict diagonal dominance of order k, $k \geq 1$, with weight $\alpha = 1$ (see [5]) for the matrix $\bar{A} = I_n - \bar{B}$, which read as

$$\prod_{j=1}^{k} \frac{r'_{i_j}(A)}{|a_{i_j i_j}|} < 1 \tag{32}$$

for all paths $(i_1, \ldots, i_k)$ in the digraph G_{A-D_A} associated with the matrix $A - D_A$.

Indeed, for $k = 1$, we have the equalities

$$r_i(|\bar{B}|) = \frac{r_i'(A)}{|a_{ii}|}, \quad i = 1, \ldots, n,$$

whereas for $k \geq 2$,

$$r_i(|\bar{B}|^k) = (|\bar{B}| \cdot r(|\bar{B}|^{k-1}))_i = \sum_{i_2 \neq i:\, a_{ii_2} \neq 0} \frac{|a_{ii_2}|}{|a_{ii}|} r_{i_2}(|\bar{B}|^{k-1})$$

$$\leq \frac{r_i'(A)}{|a_{ii}|} \max_{i_2 \neq i:\, a_{ii_2} \neq 0} \left\{ r_{i_2}(|\bar{B}|^{k-1}) \right\}.$$

By induction, we readily obtain that

$$\max_{1 \leq i \leq n} r_i(|\bar{B}|^k) \leq \max_{i_1, \ldots, i_k} \prod_{j=1}^{k} \frac{r_{i_j}'(A)}{|a_{i_j i_j}|}, \tag{33}$$

where the maximum on the right-hand side is taken over all paths $(i_1, \ldots, i_k)$ in the digraph G_{A-D_A}.

In view of (33), condition (31) is implied by conditions (32). Furthermore, the row sums $r_i(|\bar{B}|^k)$, $i = 1, \ldots, n$, occurring in (31), can be rather inexpensively computed by multiplying the matrix $|\bar{B}|$ by vectors k times.

Note also that under the strict conditions (32), A is a nonsingular H-matrix (see [6, Theorem 3.2]).

Thus, for $k \geq 1$ we have the implication string

$$(32) \Longrightarrow (31) \Longrightarrow (30) \Longrightarrow (28),$$

similar to (27).

4. All the nonsingularity conditions presented in the previous sections are based on the requirement that a certain auxiliary matrix, associated with a given matrix A, must be strictly diagonally dominant. However, the nonsingularity of an auxiliary matrix can also be ensured by imposing other sufficient nonsingularity conditions on it, which lead to other eigenvalue inclusion sets.

Below, we illustrate this approach by applying the classical "mixed" Ostrowski conditions [10] (also see, e.g., [11, Theorem 1.16]), which are recalled below, and the relaxed Ostrowski–Brauer conditions (18) to the matrices $C(A)$ and $C_\xi(A)$ (see (14) and (20)). This permits us to generalize the corresponding results obtained in [7] and [8] for matrices with constant principal diagonal to arbitrary matrices.

Theorem 9. *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $n \geq 2$, and let $0 \leq \alpha \leq 1$. If*

$$|a_{ii}| > [r_i'(A)]^\alpha \, [c_i'(A)]^{1-\alpha}, \quad i = 1, \ldots, n, \tag{34}$$

then A is nonsingular.

By applying Theorem 9 to the matrix $C(A)$, we immediately obtain the following nonsingularity result, generalizing Theorem 4, which corresponds to the case $\alpha = 1$ in (34).

Theorem 10. *Let $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n \times n}$, $n \geq 2$, and let $0 \leq \alpha \leq 1$. If*

$$|a_{ii}^2 - (B^2)_{ii}| > \left[r_i'(B^2 + BD_A - D_A B)\right]^\alpha \, \left[c_i'(B^2 + BD_A - D_A B\right]^{1-\alpha}, \quad i = 1, \ldots, n, \tag{35}$$

then A is nonsingular.

Note that Theorem 10 generalizes Theorem 2.2 in [7] to arbitrary matrices and reduces to the latter for matrices with constant principal diagonal.

The counterpart of Theorem 10 in terms of the eigenvalue inclusion set is the following generalization of Theorem 5 in this paper. Simultaneously, it generalizes Theorem 2.5 in [7], which is stated for matrices with constant principal diagonal, to the general case.

Theorem 11. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$. *Then*

$$\operatorname{Spec} A \subseteq \bigcap_{0\leq\alpha\leq 1} \bigcup_{i=1}^{n} \Big\{ z \in \mathbb{C} : \ |(z - a_{ii})^2 - (B^2)_{ii}| \leq \left[r_i'(B^2 + BD_A - D_A B)\right]^{\alpha} \left[c_i'(B^2 + BD_A - D_A B)\right]^{1-\alpha} \Big\}. \tag{36}$$

By applying Theorem 9 to the matrix $C_\xi(A)$, we obtain the following generalization of Theorem 6.

Theorem 12. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$, *let* $0 \leq \alpha \leq 1$, *and let* $\xi \in \mathbb{C}$. *If*

$$|\xi a_{ii} - (B^2)_{ii}| > \left[r_i'(B^2 - (a_{ii} - \xi)B)\right]^{\alpha} \left[c_i'(B^2 - (a_{ii} - \xi)B)\right]^{1-\alpha}, \ i = 1, \dots, n, \tag{37}$$

then A *is nonsingular.*

The corresponding generalizations of Theorem 7 and Corollary 1 are as follows.

Theorem 13. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$, *and let* $\xi \in \mathbb{C}$. *Then*

$$\operatorname{Spec} A \subseteq \bigcap_{0\leq\alpha\leq 1} \bigcup_{i=1}^{n} \Big\{ z \in \mathbb{C} : \ |(z - a_{ii})(z - \xi) - (B^2)_{ii}| \leq \left[r_i'(B^2 - (a_{ii} - \xi)B)\right]^{\alpha} \left[c_i'(B^2 - (a_{ii} - \xi)B)\right]^{1-\alpha} \Big\}. \tag{38}$$

Corollary 3. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$. *Then*

$$\operatorname{Spec} A \subseteq \bigcap_{\xi\in\mathbb{C}} \bigcap_{0\leq\alpha\leq 1} \bigcup_{i=1}^{n} \Big\{ z \in \mathbb{C} : \ |(z - a_{ii})(z - \xi) - (B^2)_{ii}| \leq \left[r_i'(B^2 - (a_{ii} - \xi)B)\right]^{\alpha} \left[c_i'(B^2 - (a_{ii} - \xi)B)\right]^{1-\alpha} \Big\}. \tag{39}$$

It should be indicated that the eigenvalue inclusion sets for the matrix A occurring in Theorems 11 and 13, whose definitions involve the intersection over α, are not of theoretical interest only because they can alternatively be defined in such a way that they become practically computable, see [2, 7].

In a similar way, applying the nonsingularity conditions (18) to the matrices $C(A)$ and $C_\xi(A)$, we come to the following strengthenings of Theorems 4–7.

Theorem 14. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 1$. *If*

$$|a_{ii}^2 - (B^2)_{ii}|\,|a_{jj}^2 - (B^2)_{jj}| > r_i'(B^2 + BD_A - D_A B)\, r_j'(B^2 + BD_A - D_A B) \quad \text{for all} \quad i \neq j \quad \text{such that} \quad (B^2)_{ij} + a_{ij}a_{jj} - a_{ii}a_{ij} \neq 0, \tag{40}$$

then A *is nonsingular.*

Theorem 15. *Let* $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 1$, *and let* $\xi \in \mathbb{C}$. *If*

$$|\xi a_{ii} - (B^2)_{ii}|\,|\xi a_{jj} - (B^2)_{jj}| > r_i'(B^2 - (a_{ii} - \xi)B)\, r_j'(B^2 - (a_{jj} - \xi)B) \quad \text{for all} \quad i \neq j \quad \text{such that} \quad (B^2)_{ij} + (a_{ii} - \xi)a_{ij} \neq 0, \tag{41}$$

then A *is nonsingular.*

The eigenvalue inclusion counterparts of Theorems 14 and 15 are presented below.

Theorem 16. *Let $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then*

$$\operatorname{Spec} A \subseteq \bigcup_{\substack{i\neq j:\\ (B^2)_{ij}+a_{ij}a_{jj}-a_{ii}a_{ij}\neq 0}} \big\{z\in\mathbb{C}:\ |(z-a_{ii})^2-(B^2)_{ii}|\,|(z-a_{jj})^2-(B^2)_{jj}| \leq r_i'(B^2+BD_A-D_AB)\, r_j'(B^2+BD_A-D_AB)\big\}. \tag{42}$$

Theorem 17. *Let $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$, and let $\xi \in \mathbb{C}$. Then*

$$\operatorname{Spec} A \subseteq \bigcup_{\substack{i\neq j:\\ (B^2)_{ij}+a_{ij}a_{jj}-a_{ii}a_{ij}\neq 0}} \big\{z\in\mathbb{C}:\ |(z-a_{ii})(z-\xi)-(B^2)_{ii}|\,|(z-a_{jj})(z-\xi)-(B^2)_{jj}| \leq r_i'(B^2-(a_{ii}-\xi)B)\, r_j'(B^2-(a_{jj}-\xi)B)\big\}. \tag{43}$$

Corollary 4. *Let $A = (a_{ij}) = D_A - B \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then*

$$\operatorname{Spec} A \subseteq \bigcap_{\xi\in\mathbb{C}} \bigcup_{\substack{i\neq j:\\ (B^2)_{ij}+(a_{ii}-\xi)a_{ij}\neq 0}} \big\{z\in\mathbb{C}:\ |(z-a_{ii})(z-\xi)-(B^2)_{ii}|\,|(z-a_{jj})(z-\xi)-(B^2)_{jj}| \leq r_i'(B^2-(a_{ii}-\xi)B)\, r_j'(B^2-(a_{jj}-\xi)B)\big\}. \tag{44}$$

Note that in the case of matrices with constant principal diagonal, Theorems 14, 15 and 16, 17 improve (by taking into account the matrix sparsity pattern) Lemma 2.1 and Theorem 2.2 in [8].

Statement of the nonsingularity conditions resulting from application of Theorem 9 and conditions (18) to the matrices $\bar{C}^{(k)} = I_n - \bar{B}^k$, $k \geq 1$, is left to the reader.

5. In this paper, we have suggested a general approach, which enables one to obtain nonsingularity conditions and eigenvalue inclusion sets for a given matrix A. This approach is based on imposing known nonsingularity conditions (in particular, the simplest condition of strict diagonal dominance) on some auxiliary matrices, whose nonsingularity implies the nonsingularity of A. In this way, a number of new sufficient nonsingularity conditions and the associated eigenvalue inclusion sets are obtained.

The results presented generalize recent results on the nonsingularity and eigenvalue inclusion sets for matrices with constant principal diagonal, established in [7–9], to arbitrary matrices and matrices with nonzero diagonal entries.

Obviously, the approach developed can readily be used in conjunction with other known nonsingularity conditions, which will result in new nonsingularity conditions and eigenvalue inclusion sets for matrices.

Translated by L. Yu. Kolotilina.

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