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CHARACTERIZATION OF γ-SUBGAUSSIAN RANDOM ELEMENTS IN A BANACH SPACE

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Abstract. We give a characterization of weakly subgaussian random elements that are γ-subgaussian in infinite-dimensional Banach and Hilbert spaces.

CONTENTS

1. Introduction

To formulate our result, we need a brief preparation.

Lemma 1.1. *For a real valued random variable ξ, the following statements are equivalent*:

(i) *ξ is a centered Gaussian variable*;

(ii) *there exists $a \geq 0$ such that*

$$\mathbf{E}\, e^{t\xi} = e^{\frac{1}{2}\, t^2 a^2} \quad \forall t \in \mathbb{R}. \tag{1.1}$$

Moreover, if (1.1) *holds for some $a \geq 0$, then $\mathbf{E}\,\xi = 0$, $(\mathbf{E}\,\xi^2)^{\frac{1}{2}} = a$, and*

$$\mathbf{E}\, e^{\varepsilon\xi^2} = \frac{1}{\sqrt{1 - 2a^2\varepsilon}} < \infty \quad \forall \varepsilon \in \left]0, 1/2a^2\right[.$$

In what follows, we consider Banach and Hilbert spaces over the field $\mathbb{R}$ of real numbers. For a Banach space X, we write X^* for the dual space of X.

Every considered random element with values in a Banach space is assumed to have a separable range.

A random element ξ with values in a Banach space X is said to be *centered Gaussian* if for every $x^* \in X^*$ the random variable $\langle x^*, \xi\rangle$ is centered Gaussian.

A random element ξ with values in a Banach space X is said to be *γ-subgaussian* (cf. [6] and [3, Remark 1.4]; see also [8]) if there exists a centered Gaussian random element η with values in X such that

$$\mathbf{E}\, e^{\langle x^*, \xi\rangle} \leq \mathbf{E}\, e^{\langle x^*, \eta\rangle} \quad \forall x^* \in X^*.$$

In the case where $X = \mathbb{R}$, the notion of a γ-subgaussian random element in $\mathbb{R}$ coincides with the notion of a subgaussian random variable introduced in [5].

Lemma 1.2 (see [1])**.** *For a real valued random variable ξ, the following statements are equivalent*:

(i) *ξ is subgaussian*;

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(ii) *there exists* $a \ge 0$ *such that*

$$\mathbf{E}\, e^{t\xi} \le e^{\frac{1}{2}t^2a^2} \quad \forall t \in \mathbb{R}. \tag{1.2}$$

Moreover, if (1.2) *holds for some* $a \ge 0$*, then necessarily*

$$\mathbf{E}\,\xi = 0, \quad (\mathbf{E}\,\xi^2)^{\frac{1}{2}} \le a.$$

Taking into account Lemma 1.2, to each real-valued subgaussian random variable ξ we can associated a quantity $\tau(\xi)$ defined by the equality

$$\tau(\xi) := \inf\Big\{a \ge 0 :\ \mathbf{E}\, e^{t\xi} \le e^{\frac{1}{2}t^2a^2}\ \forall t \in \mathbb{R}\Big\},$$

which is called the *Gaussian deviation* ("écart de Gauss," see [5]) or the *Gaussian standard* (see [1]) of ξ.

Lemma 1.3 (see [2, 5]; see also [8, Proposition 2.1 and Corollary 2.1])**.** *For a real-valued random variable* ξ *the following statements are equivalent*:

(i) ξ *is subgaussian*;

(ii) *there exists* $\varepsilon > 0$ *such that*

$$\mathbf{E}\, e^{\varepsilon\xi^2} < \infty, \quad \mathbf{E}\,\xi = 0.$$

Moreover, if (i) *holds, then*

$$\mathbf{E}\, e^{\varepsilon\xi^2} \le \frac{1}{\sqrt{1 - 2\varepsilon\tau^2(\xi)}} < \infty \quad \forall \varepsilon \in \Big]0, \frac{1}{2\tau^2(\xi)}\Big[$$

and

$$(\mathbf{E}\,\xi^p)^{\frac{1}{p}} \le \beta_p \tau(\xi) \quad \forall p \in \,]0,\infty[\,,$$

where

$$\beta_p = \begin{cases} 1 & \text{if } p \in \,]0,2], \\ 2^{1/p}\left(\dfrac{p}{e}\right)^{1/2} & \text{if } p \in \,]2,\infty[. \end{cases}$$

A random element ξ with values in a Banach space X is said to be *weakly subgaussian* (see [8]) if for every $x^* \in X^*$ the random variable $\langle x^*, \xi\rangle$ is subgaussian.

In what follows, we denote by H an infinite-dimensional separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$.

Definition 1.4 (see [3, Definition 2.1])**.** Let $\boldsymbol{e} := \{e_n,\ n \in \mathbb{N}\}$ be an orthonormal basis of H. A random element ξ with values in H is *subgaussian with respect to* $\boldsymbol{e}$ if the following conditions hold:

(i) for every $x \in H$, the real-valued random variable $\langle x, \xi\rangle$ is subgaussian (i.e., if ξ is weakly subgaussian);

(ii) $\sum\limits_{n=1}^{\infty} \tau^2(\langle e_n, \xi\rangle) < \infty$.

We have the following complement to Definition 1.4:

Proposition 1.5. *Let* $\boldsymbol{e} := \{e_n,\ n \in \mathbb{N}\}$ *be an orthonormal basis of* H*. For a random element* ξ *with values in* H *the following statements are equivalent*:

(i) ξ *is subgaussian with respect to* $\boldsymbol{e}$;

(ii) *for every* $n \in \mathbb{N}$*, the real-valued random variable* $\langle e_n, \xi\rangle$ *is subgaussian and*

$$\sum_{n=1}^{\infty} \tau^2\big(\langle e_n, \xi\rangle\big) < \infty.$$

If X is a finite-dimensional Banach space, then weakly subgaussian random elements are γ-subgaussian (see [8, Proposition 4.4]). In each infinite-dimensional Banach space, there exists a weakly subgaussian random element that is not γ-subgaussian (see [8, Theorem 4.4]). By using the terminology of Definition 1.4, we give the following characterization of weakly subgaussian random elements in a separable Hilbert space that are γ-subgaussian.

Theorem 1.6. *For a random element ξ with values in H the following statements are equivalent*:

(i) *ξ is γ-subgaussian*;
(ii) *for every orthonormal basis $\boldsymbol{e} := \{e_n,\ n \in \mathbb{N}\}$ of H, the random element ξ is subgaussian with respect to $\boldsymbol{e}$.*

Let $\mathcal{SG}$ be the set of all real-valued subgaussian random variables defined on a fixed probability space. It is known that $\mathcal{SG}$ with respect to the natural pointwise operations is a vector space over $\mathbb{R}$ and if we identify the random variables that coincide a.s., then $(\mathcal{SG}, \tau(\cdot))$ is a Banach space (see [1, 2]).

For a weakly subgaussian random element ξ in a Banach space X, let $T_\xi : X^* \to \mathcal{SG}$ be the *induced operator*, which sends each $x^* \in X^*$ to the element $\langle x^*, \xi\rangle \in \mathcal{SG}$ (see [8, Proposition 4.2]).

Theorem 1.6 can be deduced from the following statement containing a characterization of weakly subgaussian random elements that are γ-subgaussian in the case of a reflexive type-2 Banach space.

Theorem 1.7. *For a random element ξ with values in a Banach space X consider the assertions*:

(i) *ξ is γ-subgaussian*;
(ii) *$T_\xi : X^* \to \mathcal{SG}$ is a 2-summing operator.*

Then (i) $\Longrightarrow$ (ii). *The implication* (ii) $\Longrightarrow$ (i) *is also valid under the condition that X is a reflexive type-2 space.*

Remark 1.8. We do not know whether the implication (ii) $\Longrightarrow$ (i) of Theorem 1.7 remains valid for a reflexive separable Banach space X that is not a type-2 space.

2. Proofs

Proof of Proposition 1.5. The implication (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i). Fix $x \in H$. We need to show only that the real valued random variable $\langle x, \xi\rangle$ is subgaussian. Clearly,

$$\langle x, \xi\rangle = \sum_{k=1}^{\infty} \langle x, e_k\rangle\langle e_k, \xi\rangle. \tag{2.1}$$

We also have

$$\sum_{k=1}^{\infty} \tau\big(\langle x, e_k\rangle\langle e_k, \xi\rangle\big) = \sum_{k=1}^{\infty} \big|\langle x, e_k\rangle\big| \tau\big(\langle e_k, \xi\rangle\big) \le \Big(\sum_{k=1}^{\infty} \langle x, e_k\rangle^2\Big)^{\frac12} \Big(\sum_{k=1}^{\infty} \tau^2\big(\langle e_k, \xi\rangle\big)\Big)^{\frac12} < \infty.$$

So,

$$\sum_{k=1}^{\infty} \tau\big(\langle x, e_k\rangle\langle e_k, \xi\rangle\big) < \infty. \tag{2.2}$$

From (2.2), since $(\mathcal{SG}, \tau(\cdot))$ is a Banach space, it follows that the series $\sum_k \langle x, e_k\rangle\langle e_k, \xi\rangle$ converges in $(\mathcal{SG}, \tau(\cdot))$ to the sum $\xi_x \in \mathcal{SG}$. From this and (2.1) we see that $\langle x, \xi\rangle = \xi_x$ a.s. Hence, $\langle x, \xi\rangle \in \mathcal{SG}$. ☐

Proof of Theorem 1.7. (i) $\Longrightarrow$ (ii). Since ξ is γ-subgaussian, there exists a centered Gaussian random element η with values in X such that

$$\mathbf{E}\, e^{\langle x^*, \xi\rangle} \le \mathbf{E}\, e^{\langle x^*, \eta\rangle} \quad \forall x^* \in X^*.$$

From this inequality we obtain

$$\tau(T_\xi x^*) \le \tau(T_\eta x^*) = \|T_\eta x^*\|_2 \quad \forall x^* \in X^*. \tag{2.3}$$

Since η is a centered Gaussian random element, it follows that T_η as an operator from X^* into L_2 is 2-summing. From this and (2.3) we get that T_ξ as an operator from X^* into $\mathcal{SG}$ is also 2-summing.

Assume that X is a reflexive type-2 space and show that under this assumption (ii) $\Longrightarrow$ (i). Since $T_\xi : X^* \to \mathcal{SG}$ is a 2-summing operator and X is reflexive, by Pietsch's domination theorem (see [9, Theorem 2.2.2] and [9, Exercise 5]) on the closed unit ball B_X of X we can find a finite positive Radon measure ν such that

$$\tau^2(T_\xi x^*) \le \int\limits_{B_X} \langle x^*, x\rangle^2 \, d\nu(x) \quad \forall x^* \in X^*. \tag{2.4}$$

Since X is of type 2 space, by [4, Theorem 3.5] we can find a centered Gaussian random element η with values in X such that

$$\int\limits_{B_X} \langle x^*, x\rangle^2 \, d\nu(x) = \mathbf{E}\langle x^*, \eta\rangle^2 \quad \forall x^* \in X^*. \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$\mathbf{E}\, e^{\langle x^*, \xi\rangle} \le e^{\frac{1}{2}\tau^2(\langle x^*, \xi\rangle)} = e^{\frac{1}{2}\tau^2(T_\xi x^*)} \le e^{\frac{1}{2}\mathbf{E}\langle x^*, \eta\rangle^2} = \mathbf{E}\, e^{\langle x^*, \eta\rangle} \quad \forall x^* \in X^*.$$

Consequently ξ is a γ-subgaussian random element in X. □

Proof of Theorem 1.6. The implication (i) $\Longrightarrow$ (ii) follows easily from the similar implication of Theorem 1.7.

(ii) $\Longrightarrow$ (i). The condition (ii) implies that the induced operator $T_\xi : H \to \mathcal{SG}$ has the following property:

$$\sum_{n=1}^{\infty} \tau^2(T_\xi e_n) < \infty$$

for every orthonormal basis $\boldsymbol{e} := \{e_n,\ n \in \mathbb{N}\}$ of H. This property by Slowikowski's theorem (see [7]) implies that $T_\xi : H \to \mathcal{SG}$ is a 2-summing operator. From this, since H is of type 2 and reflexive, by the implication (ii) $\Longrightarrow$ (i) of Theorem 1.7, we see that ξ is a γ-subgaussian random element in H. □

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REFERENCES

1. V. V. Buldygin and Yu. V. Kozachenko, "Sub-Gaussian random variables," *Ukr. Mat. Zh.*, **32**, 723–730 (1980).
2. V. V. Buldygin and Yu. V. Kozachenko, "Metric characteristics of random variables and processes," in: *Trans. Math. Monogr.*, **188**, Am. Math. Soc., Providence, Rhode Island (2000).
3. R. Giuliano Antonini, "Sub-Gaussian random variables in Hilbert spaces," *Rend. Sem. Mat. Univ. Padova*, **98**, 89–99 (1997).
4. J. Hoffmann-Jørgensen and G. Pisier, "The law of large numbers and the central limit theorem in Banach spaces," *Ann. Probab.*, **4**, No. 4, 587–599 (1976).
5. J.-P. Kahane, "Propriétés locales des fonctions à séries de Fourier aléatoires," *Stud. Math.*, **19**, 1–25 (1960).
6. M. Talagrand, "Regularity of Gaussian processes," *Acta Math.*, **159**, Nos. 1-2, 99–149 (1987).
7. N. D. Tien, V. I. Tarieladze, and R. Vidal, "On summing and related operators acting from a Hilbert space," *Bull. Polish Acad. Sci. Math.*, **46**, No. 4, 365–375 (1998).

8. N. N. Vakhania, V. V. Kvaratskhelia, and V. I. Tarieladze, “Weakly sub-Gaussian random elements in Banach spaces,” *Ukr. Mat. Zh.*, **57**, No. 9, 1187–1208 (2005).
9. N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, *Probability Distributions on Banach Spaces*, Math. Appl., Soviet Ser., **14**, Dordrecht (1987).

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