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FUNCTIONAL DIFFERENTIAL PARABOLIC EQUATIONS: INTEGRAL TRANSFORMATIONS AND QUALITATIVE PROPERTIES OF SOLUTIONS OF THE CAUCHY PROBLEM

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Abstract. In this monograph, we examine the Cauchy problem for second-order parabolic functional differential equations containing, in addition to differential operators, translation (generalized translation) operators acting with respect to spatial variables. The specified problems have important applications, such as the multilayer plates and envelopes theory, the diffusion processes theory, including biomathematical applications, models of nonlinear optics, etc. The main concern of the present work is the long-time behavior of solutions of studied problems.

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INTRODUCTION

We examine the Cauchy problem for second-order parabolic functional differential equations containing, in addition to differential operators, translation (generalized translation) operators acting with respect to spatial variables. The investigation of such nonlocal problems was started in the classical works of Tamarkin, Picone, and Carleman. Further development of the theory of functional differential (in particular, differential-difference) equations refers to Myshkis. Nowadays, this theory is deeply and actively developed by various mathematicians (see monographs [1, 28, 102] and the references therein as well as the series of papers [111–116, 119] devoted to functional differential equations in Banach spaces). The general theory of elliptic and parabolic functional differential equations (solvability, smoothness of generalized solutions, spectral properties of operators) was developed in [11, 12, 17, 18, 30, 39, 82–84, 97–107, 118, 123].

The specified problems have important applications, such as the multilayer plates and envelopes theory (see [81, 102]), the diffusion processes theory, including biomathematical applications (see [100, 117, 123]), models of nonlinear optics (see [91, 103, 104, 109, 120–122]), etc.

The main concern of the present work is the long-time behavior of solutions of studied problems. Recall that a stabilization of solutions frequently takes place for parabolic problems. This phenomenon (found by Petrovskii and Tikhonov in the first half of the 20 century) is the existence of a finite limit (in any sense) of the solution as $t \to \infty$. A well-known example is the necessary and sufficient condition of the (pointwise) stabilization of the Cauchy problem solution for the heat equation with a bounded initial-value function: the specified solution tends to a constant if and only if the limit

$$\lim_{r\to\infty} \frac{1}{\operatorname{mes}\{|x| < r\}} \int\limits_{|x|<r} u_0(x)dx$$

exists and is equal to the same constant. This condition is obtained in [95] (see also [96]). Further, the stabilization theory for parabolic equations was developed in [4–6, 9, 10, 19–27, 49, 69, 85, 86, 92–94, 124–129] and many other papers of various authors.

The stabilization of solutions also occurs in the elliptic theory. In particular, it takes place for the Dirichlet problem in subspaces (see [7, 8, 61, 77]): the direction of the stabilization is orthogonal to the boundary hyperplane, and the necessary and sufficient condition of the stabilization coincides with the classical condition from [95]. Thus, the behavior of the solution of the specified elliptic equation is similar to the behavior of solutions of parabolic equations. However, the complete coincidence does not take place: unlike the parabolic case, the fundamental solution decreases as a power.

At the moment, the classical stabilization theory can be regarded, in general, as complete: the research interest transits to nonclassical parabolic problems. This refers to the present work as well: it is devoted to functional differential parabolic equations.

Apart from regular equations (i.e., equations such that their coefficients have no singularities), we study singular functional differential parabolic equations containing the Bessel operator

$$\frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial y^2} + \frac{k}{y}\frac{\partial}{\partial y}$$

with positive parameter k acting with respect to one or several spatial variables.

Singularities of the above type arise models of mathematical physics such that the characteristic of the media (e.g., diffusion characteristics or heat–conductivity characteristics) have degenerate power-like heterogeneities.

Function-theory methods necessary for the investigation of such singularities and the general theory of the specified singular equations are developed in [34] (see also [31–33, 35]). A thorough investigation of parabolic equations containing the Bessel operator is given in [36–38, 42–45, 47] (see also references therein). Necessary and sufficient conditions of the stabilization of solutions of the specified singular parabolic equations are found in [48, 60, 64].

In the present work, we examine (apart from the Bessel operator) the general translation operator introduced and investigated in [41]. Thus, the functional differential equations studied are not only differential-difference ones, but are integrodifferential as well.

The work consists of the current introduction and four chapters.

In the first chapter, we use equations such that only low-order (more exactly, zero-order) terms are nonlocal. It is known that such terms characterize dissipation properties of the described process, and they become nonlocal once the dissipation delays. The case of the anisotropic media is the most interesting: the diffusion process is multidimensional and the delay is different for different directions (see, e.g., [100, 123]). Also, nonlocal terms of the above type arise in mathematical models of nonlinear optical systems with two-dimensional feedback, used, e.g., in contemporary computer technologies and in the study of laser bundles (see, e.g., [91, 103, 104, 120–122]).

The main result of Chap. 1 is Theorem 1.5.1 on the classical unique solvability of the Cauchy problem and Theorem 1.6.1 on the generalized weight asymptotic closeness of the investigated solution and the Cauchy problem solution with a transformed initial-value function for the heat equation; the latter theorem implies corollaries about the (pointwise) stabilization.

Note that the existence of generalized (in various senses) solutions of the specified problem was proved much earlier (see, e.g., [15, 16, 89, 90]), but stabilization theorems treat the solution behavior on low-dimensional manifolds (including one-dimensional ones), while the existence of a trace on such a manifold is not guaranteed even for strong solutions. Classical solutions, i.e., solutions possessing all derivatives (included to the equation) in the classical sense, satisfying the equation at any point of the half-space $\mathbb{R}^n \times (0, +\infty)$, and satisfying the initial-value condition (in the sense of one-sided limits as $t \to +0$) for any x from $\mathbb{R}^n$, possess the required properties; that is why its existence and integral representation are considered quite thoroughly (Secs. 1.1–1.4).

The proven weight asymptotic closeness of solutions is understood as follows: the difference between the solution of the studied functional differential equation, multiplied by the corresponding weight function, and the solution of the "standard" differential equation (more exactly, the heat equation) tends to zero if the independent variable of the studied solution tends to infinity along the ray rotated to a certain angle with respect to the initial-value hyperplane; this angle is uniquely determined by the coefficients of the low-order (i.e., nonlocal) part of the functional differential equation:

$$\lim_{t\to+\infty}\left[e^{-t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j}a_{jk}}u(x,t)-w\left(\frac{x_1+q_1t}{p_1},\dots,\frac{x_n+q_nt}{p_n},t\right)\right]=0,$$

where $w(x,t)$ is the bounded solution of the Cauchy problem for the heat equation with the initial-value function $u_0(p_1x_1,\dots,p_nx_n)$ and the constants p_j and q_j are determined by the coefficients of the nonlocal part of the original functional differential equation (here a_{jk} are the coefficients at translation operators).

This behavior of the solution is a qualitatively new effect compared with the classical case of differential equations: this phenomenon also occurs in the classical case, but this takes place only if the equation includes first-order terms. It turns out that zero-order terms can cause the same effect though their physical interpretation is principally else. Thus, qualitatively new effects caused by nonlocal terms of equations arise even in the case where the principal part of the equation is still classical one. Note that this entirely corresponds to the general parabolic theory (see [29]): low-order terms of a parabolic equation (unlike, e.g., the elliptic case) might have a principal impact to the qualitative properties of its solution.

The second chapter is devoted to regular equations with nonlocal principal terms, i.e., equations containing superpositions of second derivatives (including mixed ones) and translation operators with respect to any (spatial) coordinate directions. We consider the Cauchy problem solution in the sense of distributions (more exactly, in the sense of [15, 16]), and we prove (see Theorem 2.7.3) that the solution is classical in the subspace $\mathbb{R}^n \times (0, +\infty)$; this allows us to consider the behavior of solutions on one-dimensional manifolds and obtain theorems on the asymptotic closeness of solutions and on their stabilization. The main result (obtained in Theorem 2.7.4) is as follows: if the right-hand part of the equation is a homogeneous strong elliptic operator, then the asymptotic closeness takes place for the investigated solution and the Cauchy problem solution (with the same initial-value function) for the differential parabolic equation obtained from the original functional differential equation by means of setting all translations equal zero:

$$\lim_{t\to\infty}[u(x,t) - v(x,t)] = 0,$$

where $u(x,t)$ is the solution of the (functional differential) equation

$$\frac{\partial u}{\partial t} = \sum_{k,j,m=1}^{n} a_{kjm} \frac{\partial^2 u}{\partial x_k \partial x_j}(x_1, \ldots, x_{m-1}, x_m + h_{kjm}, x_{m+1}, \ldots, x_n, t)$$

and $v(x,t)$ is the solution of the (differential) equation

$$\frac{\partial v}{\partial t} = \sum_{k,j,m=1}^{n} a_{kjm} \frac{\partial^2 v}{\partial x_k \partial x_j}$$

(with the same initial-value function).

If the equation includes low-order terms (apart from the nonlocal principal part), then we obtain a weight asymptotic closeness; if the specified low-order terms are nonlocal ones, then effects specific for the nonlocal case described in Chap. 1 arise.

Note that the strong ellipticity assumption is quite important for the second chapter. Similarly to the case of bounded domains (see [102, §9]), there is an essential distinction between the strong ellipticity for differential and differential-difference operators.

In the third chapter, we study a parabolic integrodifferential equation with one spatial variable such that the Bessel operator and a linear combination of generalized translation operators act with respect to that variable; this is treated as a prototype case for singular functional differential equations. Similarly to the case of differential singular equations (see, e.g., [34]), we add the following condition to ensure the uniqueness of the solution: the solution is assumed to be even with respect to spatial variable. The unique solvability of such a problem is proved in Theorem 3.5.1. The properties of the one-dimensional fundamental solution constructed in this chapter are applied in the next chapter to construct the fundamental solution for the general singular case.

The fourth chapter is devoted to the general singular case: there are spatial variables such that second derivatives and translation operators act with respect to them (nonspecial variables) and spatial variables such that Bessel operators and generalized translation operators act with respect to them (special variables). Apart from the initial-value condition, we impose the evenness (with respect to special variables) condition for the solution. For the above problem, we prove the unique solvability (see Theorems 4.5.1 and 4.6.1) and the weight asymptotic closeness of its solution and the solution of a similar problem for certain differential singular equation (see Theorem 4.7.1).

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Chapter 1

EQUATIONS WITH NONLOCAL LOW-ORDER TERMS

In this chapter, we consider equations of the type

$$\frac{\partial u}{\partial t}=\Delta u+\sum_{h\in\mathcal{M}}a_h u(x-h,t), \tag{1.1}$$

where $\mathcal{M}$ is a finite set of vectors of $\mathbb{R}^n$ parallel to the coordinate axes (or any other orthogonal vector system). The motivation to study such equations relates both to the pure theory (nonclassical low-order terms are added to a parabolic equation) and applications, e.g., to problems of nonlinear optics: it is known (see, e.g., [121]) that a nonlinear optical system with so-called multipetal waves is described by the equation

$$\frac{\partial u}{\partial t}+u=D\Delta u+K(1+\gamma\cos u_g),$$

where $u(x,t)$ is the phase of the light wave, $u_g=u(g(x),t)$, g is a one-to-one transformation of spatial variables, different from the identity, the positive coefficients D and γ are the diffusion coefficient and the visibility of the interference picture respectively, and K (different from zero) is the nonlinearity coefficient depending on the intensity of the input field.

In [103, 104], this quasilinear equation is linearized to the form

$$\frac{\partial v}{\partial t}=D\Delta v-v-K\gamma\sin\omega v_g,$$

where the constant ω (the so-called spatially homogeneous stationary solution) is the root of the transcendental equation $\omega=K(1+\gamma\cos\omega)$.

If $g(x)$ is a translation operator, then the linearized equation coincides with Eq. (1.1) such that the set $\mathcal{M}$ consists of two elements (one of them is the zero vector).

1.1. Fundamental Solution (Single Spatial Variable)

Let $a,h\in\mathbb{R}^m$. In $\mathbb{R}^1\times(0,+\infty)$, consider the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\sum_{k=1}^{m}a_k u(x-h_k,t). \tag{1.2}$$

Define the following function on $\mathbb{R}^1\times(0,+\infty)$:

$$\mathcal{E}(x,t)\stackrel{\text{def}}{=}\mathcal{E}_{a,h}(x,t)\stackrel{\text{def}}{=}\int\limits_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^{m}a_k\cos h_k\xi)}\cos(x\xi-t\sum_{k=1}^{m}a_k\sin h_k\xi)d\xi. \tag{1.3}$$

It is easy to see that

$$|\mathcal{E}(x,t)|\le\int\limits_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^{m}a_k\cos h_k\xi)}d\xi\le\int\limits_0^\infty e^{(1-\xi^2)t}d\xi=e^t\int\limits_0^\infty e^{-\eta^2}\frac{d\eta}{\sqrt{t}}=\frac{e^t}{\sqrt{t}}\frac{\sqrt{\pi}}{2}.$$

Thus, for any $t_0,T\in(0,+\infty)$ integral (1.3) converges absolutely and uniformly with respect to $(x,t)\in\mathbb{R}^1\times[t_0,T]$. Therefore, $\mathcal{E}(x,t)$ is well defined on $\mathbb{R}^1\times(0,+\infty)$. Formally differentiate $\mathcal{E}$ under the integral sign:

$$\left|\frac{\partial^m}{\partial t^l\partial x^{m-l}}\left(e^{-t(\xi^2-\sum\limits_{k=1}^{m}a_k\cos h_k\xi)}\cos(x\xi-t\sum_{k=1}^{m}a_k\sin h_k\xi)\right)\right|P(\xi)e^{-t(\xi^2-\sum\limits_{k=1}^{m}a_k\cos h_k\xi)},$$

where $P(\xi)$ is a polynomial of power not exceeding $m+2l$. Hence,

$$\left|\frac{\partial^m}{\partial t^l \partial x^{m-l}}\left(e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\cos(x\xi - t\sum_{k=1}^m a_k \sin h_k\xi)\right)\right| \le A\xi^{m+2l}e^{(\sum\limits_{k=1}^m |a_k|-\xi^2)t}.$$

Further,

$$\int_0^\infty \xi^{m+2l}e^{(\sum\limits_{k=1}^m |a_k|-\xi^2)t}d\xi = e^{\sum\limits_{k=1}^m |a_k|t}\int_0^\infty \xi^{m+2l}e^{-\xi^2 t}d\xi = e^{\sum\limits_{k=1}^m |a_k|t}\int_0^\infty \frac{\eta^{m+2l}}{t^{\frac{m+2l}{2}}}e^{-\eta^2}\frac{d\eta}{\sqrt{t}}$$

$$= \frac{e^{\sum\limits_{k=1}^m |a_k|t}}{t^{\frac{m+2l+1}{2}}}\int_0^\infty \eta^{m+2l}e^{-\eta^2}d\eta = \frac{\Gamma(\frac{m+2l+1}{2})e^{\sum\limits_{k=1}^m |a_k|t}}{2t^{\frac{m+2l+1}{2}}}.$$

Therefore, the integral obtained by the formal differentiating of the integrated function converges absolutely and uniformly with respect to $(x,t)\in\mathbb{R}^1\times[t_0,T]$ for any $t_0,T\in(0,+\infty)$. Hence, the function $\mathcal{E}$ defined by relation (1.3) is infinitely differentiable on $\mathbb{R}^1\times(0,+\infty)$ and integration under the integral is valid. This implies that

$$\frac{\partial\mathcal{E}}{\partial t} = \int_0^\infty\Bigg[(\sum_{k=1}^m a_k\cos h_k\xi - \xi^2)e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\cos(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi)$$

$$+\ e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\sin(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi)\sum_{k=1}^m a_k\sin h_k\xi\Bigg]d\xi$$

$$= \int_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\Bigg[(\sum_{k=1}^m a_k\cos h_k\xi - \xi^2)\cos(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi)$$

$$+\sin(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi)\sum_{k=1}^m a_k\sin h_k\xi\Bigg]d\xi,$$

$$\frac{\partial\mathcal{E}}{\partial x} = -\int_0^\infty \xi e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\sin(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi)d\xi,$$

and

$$\frac{\partial^2\mathcal{E}}{\partial x^2} = -\int_0^\infty \xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\cos(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi)d\xi.$$

Therefore,

$$\frac{\partial\mathcal{E}}{\partial t} - \frac{\partial^2\mathcal{E}}{\partial x^2} = \int_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\Bigg[\sum_{k=1}^m a_k\cos h_k\xi\cos\left(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi\right)$$

$$+\sum_{k=1}^m a_k\sin h_k\xi\sin\left(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi\right)\Bigg]d\xi.$$

The latter relation is equal to

$$\sum_{k=1}^m a_k\int_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi)}\cos h_k\xi\cos\left(x\xi - t\sum_{k=1}^m a_k\sin h_k\xi\right)d\xi$$

$$+\sum_{k=1}^{m} a_k \int_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin h_k\xi \sin\left(x\xi - t\sum_{k=1}^{m} a_k \sin h_k\xi\right) d\xi$$

$$=\sum_{k=1}^{m} a_k \int_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos\left[(x-h_k)\xi t\sum_{k=1}^{m} a_k \sin h_k\xi\right] d\xi = \sum_{k=1}^{m} a_k \mathcal{E}(x-h_k,t).$$

Thus, the function $\mathcal{E}(x,t)$ satisfies (in the classical sense) Eq. (1.2) in the domain $\mathbb{R}^1 \times (0,+\infty)$.

We call $\mathcal{E}(x,t)$ the *fundamental solution* of Eq. (1.2). To show the reasonability of this term, we prove below that the convolution of $\mathcal{E}_{a,h}$ with any bounded initial-value function coincides with that initial-value function on the initial axis.

1.2. Fundamental Solution: Convolutions with Bounded Functions

Assuming that a positive t is fixed, estimate the behavior of $\mathcal{E}(x,t)$ as $x \to \infty$.

Let us prove the following assertion:

Lemma 1.2.1. *Let $t > 0$ and $a, h \in \mathbb{R}^m$. Then*

$$\lim_{x\to\infty} x^2 \mathcal{E}(x,t) = 0.$$

Proof. Decompose the function $\mathcal{E}(x,t)$ into its even and odd (with respect to x) terms $\mathcal{E}_1(x,t)$ and $\mathcal{E}_2(x,t)$:

$$\mathcal{E}_1(x,t) = \int_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos x\xi \, \cos\left(t\sum_{k=1}^{m} a_k \sin h_k\xi\right) d\xi$$

and

$$\mathcal{E}_2(x,t) = \int_0^\infty e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin x\xi \, \sin\left(t\sum_{k=1}^{m} a_k \sin h_k\xi\right) d\xi.$$

Change the variable: $\eta = x\xi$. This yields the relation

$$\mathcal{E}_1(x,t) = \frac{1}{x}\int_0^\infty e^{-t\left(\frac{\eta}{x}\right)^2} e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k\eta}{x}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k\eta}{x}\right) \cos\eta \, d\eta = \frac{1}{x}\int_0^\infty \psi\left(\frac{\eta}{x}\right) f(\eta) d\eta,$$

where

$$f(\tau) = \cos\tau \in L_\infty(\mathbb{R}^1_+),$$

$$\psi(\tau) = e^{-t\tau^2} e^{t\sum\limits_{k=1}^{m} a_k \cos\tau} \cos\left(t\sum_{k=1}^{m} a_k \sin\tau\right) \in L_1(\mathbb{R}^1_+).$$

Denoting $e^{-t\tau^2}$ by $\psi_0(\tau)$, we see that $\psi_0(\tau) \in L_1(\mathbb{R}^1_+)$. Further, the Mellin transform of the function $\psi_0(\tau)$ is defined on the real axis and it has no real zeros; indeed,

$$\int_0^\infty \tau^{ix}\psi_0(\tau)d\tau = \frac{1}{2t^{\frac{1+ix}{2}}}\int_0^\infty z^{\frac{ix-1}{2}} e^{-z} dz = \frac{\Gamma(\frac{1+ix}{2})}{2t^{\frac{1+ix}{2}}}.$$

Further,

$$\frac{1}{r}\int_0^\infty \psi_0\left(\frac{\tau}{r}\right) f(\tau)d\tau = \frac{\sqrt{\pi}}{2\sqrt{t}} e^{-\frac{r^2}{4t}} \overset{r\to\infty}{\longrightarrow} 0\,.$$

Then

$$\frac{1}{r}\int_0^\infty \psi\left(\frac{\tau}{r}\right) f(\tau)d\tau \overset{r\to\infty}{\longrightarrow} 0$$

due to the Wiener Tauberian theorem (see [13, p. 163]), i.e., $\mathcal{E}_1(x,t)$ tends to zero as $x\to\infty$ for all fixed $t>0$ and $a,h\in\mathbb{R}^m$.

Now, consider $\mathcal{E}_2(x,t)$.

Denote the function $e^{-t\tau^2}e^{t\sum\limits_{k=1}^m a_k\cos h_k\tau}\sin\left(t\sum\limits_{k=1}^m a_k\sin h_k\tau\right)$ by $\psi(\tau)\in L_1(\mathbb{R}^1_+)$. Denote the function $\sin\tau$ by $f(\tau)\in L_\infty(\mathbb{R}^1_+)$. Then

$$\frac{1}{r}\int_0^\infty \psi_0\left(\frac{\tau}{r}\right) f(\tau)d\tau = \frac{r}{2t}F\left(1,\frac{3}{2},-\frac{r^2}{4t}\right) \overset{r\to\infty}{\longrightarrow} 0,$$

where F denotes the confluent hypergeometric function of second type.

Thus, the assumptions of the Wiener Tauberian theorem are satisfied. Hence, for all fixed $t>0$ and $a,h\in\mathbb{R}^m$, we have

$$\mathcal{E}_2(x,t) = \frac{1}{x}\int_0^\infty \psi\left(\frac{\tau}{x}\right) f(\tau)d\tau \overset{r\to\infty}{\longrightarrow} 0\,.$$

Thus,

$$\lim_{x\to\infty}\mathcal{E}(x,t)=0$$

for any positive t and any $a,h\in\mathbb{R}^m$.

However, the obtained limit relation is not sufficient to prove the convergence of the convolution of the fundamental solution with bounded initial-value functions. We must estimate the rate of the proved decay. To do that, we integrate the term $\mathcal{E}_1(x,t)$ by parts:

$$\int_0^\infty e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\cos(t\sum_{k=1}^m a_k\sin h_k\xi)\cos x\xi d\xi$$

$$=\frac{1}{x}\Bigg[e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\cos(t\sum_{k=1}^m a_k\sin h_k\xi)\sin x\xi\Big|_{\xi=0}^{\xi=\infty}+t\int_0^\infty e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}$$

$$\times\left((2\xi+\sum_{k=1}^m h_ka_k\sin h_k\xi)\cos(t\sum_{k=1}^m a_k\sin h_k\xi)+\sin(t\sum_{k=1}^m a_k\sin h_k\xi)\sum_{k=1}^m h_ka_k\cos h_k\xi\right)\sin x\xi d\xi\Bigg]$$

$$=\frac{t}{x}\int_0^\infty e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\left(2\xi\cos(t\sum_{k=1}^m a_k\sin h_k\xi)+\sum_{k=1}^m a_kh_k\sin(h_k\xi+t\sum_{k=1}^m a_k\sin h_k\xi)\right)\sin x\xi d\xi.$$

Denote the derivative (with respect to ξ) of

$$e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\left(2\xi\cos(t\sum_{k=1}^m a_k\sin h_k\xi)+\sum_{k=1}^m a_kh_k\sin(h_k\xi+t\sum_{k=1}^m a_k\sin h_k\xi)\right)$$

by $\psi(\xi)$ and integrate by parts again. We see that

$$\mathcal{E}_1(x,t)=\frac{t}{x^2}\Bigg[e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi-\xi^2)}\Bigg(2\xi\cos(t\sum_{k=1}^m a_k\sin h_k\xi)$$

$$+\sum_{k=1}^m a_k h_k \sin(h_k\xi + t\sum_{k=1}^m a_k \sin h_k\xi)\Big) \cos x\xi\Big|_{\xi=\infty}^{\xi=0} + \int_0^\infty \psi(\xi)\cos x\xi d\xi\Bigg] = \frac{t}{x^2}\int_0^\infty \psi(\xi)\cos x\xi d\xi,$$

i.e., $x^2\mathcal{E}_1(x,t) = \dfrac{t}{x}\displaystyle\int_0^\infty \psi\left(\frac{\eta}{x}\right)\cos\eta d\eta.$

Since $\psi(\xi) \in L_1(\mathbb{R}^1_+)$, it follows that the assumptions of the Wiener Tauberian theorem are satisfied. Hence, $x^2\mathcal{E}_1(x,t) \overset{x\to\infty}{\longrightarrow} 0$ for all fixed $t > 0$ and $a, h \in \mathbb{R}^m$.

In the same way, consider the second term of the fundamental solution.

$$\mathcal{E}_2(x,t) = \frac{1}{x}\Bigg[e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi - \xi^2)} \sin(t\sum_{k=1}^m a_k \sin h_k\xi)\cos x\xi\Big|_{\xi=\infty}^{\xi=0}$$
$$-\int_0^\infty e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi - \xi^2)}\Big(t(\sum_{k=1}^m a_k h_k \sin h_k\xi + 2\xi)\sin(t\sum_{k=1}^m a_k\sin h_k\xi)$$
$$- t\cos(t\sum_{k=1}^m a_k \sin h_k\xi)\sum_{k=1}^m a_k h_k\cos h_k\xi\Big)\cos x\xi d\xi\Bigg].$$

Thus, the second term of the fundamental solution is equal to

$$-\frac{t}{x}\int_0^\infty e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi - \xi^2)}\Big(2\xi\sin(t\sum_{k=1}^m a_k\sin h_k\xi) - \sum_{k=1}^m a_k h_k\cos(h_k\xi + t\sum_{k=1}^m a_k\sin h_k\xi)\Big)\cos x\xi d\xi$$
$$= -\frac{t}{x^2}\Bigg[\sin x\xi\, e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi - \xi^2)}\Big(2\xi\sin(t\sum_{k=1}^m a_k\sin h_k\xi) - \sum_{k=1}^m a_k h_k\cos(h_k\xi + t\sum_{k=1}^m a_k\sin h_k\xi)\Big)\Big|_{\xi=0}^{\xi=\infty}$$
$$-\int_0^\infty \psi(\xi)\sin x\xi d\xi\Bigg] = \frac{t}{x^3}\int_0^\infty \psi(\frac{\eta}{x})\sin\eta d\eta,$$

where

$$\psi(\xi) = \Bigg[e^{t(\sum\limits_{k=1}^m a_k\cos h_k\xi - \xi^2)}\Big(2\xi\sin(t\sum_{k=1}^m a_k\sin h_k\xi) - \sum_{k=1}^m a_k h_k\cos(h_k\xi + t\sum_{k=1}^m a_k\sin h_k\xi)\Big)\Bigg]' \in L_1(\mathbb{R}^1_+).$$

By virtue of the Wiener Tauberian theorem, this implies that $x^2\mathcal{E}_2(x,t) \overset{x\to\infty}{\longrightarrow} 0$ for all fixed $t > 0$ and $a, h \in \mathbb{R}^m$, which completes the proof of Lemma 1.2.1. □

Estimate the behavior of derivatives of the fundamental solutions as $x \to \infty$.

The following assertion is valid.

Lemma 1.2.2. *Let $t > 0$ and $a, h \in \mathbb{R}^m$. Then*

$$\lim_{x\to\infty} x^2\frac{\partial^2\mathcal{E}}{\partial x^2}(x,t) = 0.$$

Proof. Consider the function

$$\frac{\partial^2\mathcal{E}_1(x,t)}{\partial x^2} = -\int_0^\infty \xi^2 e^{-t(\xi^2 - \sum\limits_{k=1}^m a_k\cos h_k\xi)}\cos(t\sum_{k=1}^m a_k\sin h_k\xi)\cos x\xi\, d\xi.$$

Integrating by parts, we see that

$$\frac{\partial^2 \mathcal{E}_1(x,t)}{\partial x^2} = \frac{1}{x}\left[\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos(t\sum_{k=1}^{m} a_k \sin h_k\xi)\sin x\xi\Big|_{\xi=\infty}^{\xi=0}\right.$$

$$\left. + \int_0^\infty \left(\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right)' \sin x\xi \, d\xi\right]$$

$$= \frac{1}{x}\int_0^\infty \left(\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right)' \sin x\xi \, d\xi.$$

Integrating by parts again, we obtain the relation

$$\frac{\partial^2 \mathcal{E}_1(x,t)}{\partial x^2} = \frac{1}{x^2}\left(\left[\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right]' \cos x\xi\Big|_{\xi=\infty}^{\xi=0}\right.$$

$$\left. + \int_0^\infty \left[\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right]'' \cos x\xi \, d\xi\right) = \frac{1}{x^3}\int_0^\infty \psi\left(\frac{\eta}{x}\right)\cos\eta \, d\eta,$$

where

$$\psi(\xi) = \left[\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right]'' \in L_1(\mathbb{R}^1_+).$$

Thus, the assumptions of the Wiener Tauberian theorem are satisfied. Hence, $x^2 \dfrac{\partial^2 \mathcal{E}_1(x,t)}{\partial x^2} \overset{x\to\infty}{\longrightarrow} 0$ for all fixed $t > 0$ and $a, h \in \mathbb{R}^m$.

In the same way, we have

$$\frac{\partial^2 \mathcal{E}_2(x,t)}{\partial x^2} = -\int_0^\infty \xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin(t\sum_{k=1}^{m} a_k \sin h_k\xi)\sin x\xi \, d\xi.$$

Integrating by parts, we see that the last expression is equal to

$$\frac{1}{x}\left[\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin(t\sum_{k=1}^{m} a_k \sin h_k\xi)\cos x\xi\Big|_{\xi=0}^{\xi=\infty}\right.$$

$$\left. - \int_0^\infty \left(\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right)' \cos x\xi \, d\xi\right]$$

$$= -\frac{1}{x}\int_0^\infty \left(\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right)' \cos x\xi \, d\xi.$$

Integrating by parts again, we obtain the relation

$$\frac{\partial^2 \mathcal{E}_2(x,t)}{\partial x^2} = -\frac{1}{x^2}\left(\left[\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right]' \sin x\xi\Big|_{\xi=0}^{\xi=\infty} - \int_0^\infty \psi(\xi)\sin x\xi d\xi\right),$$

where

$$\psi(\xi) = \left[\xi^2 e^{-t(\xi^2-\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \sin(t\sum_{k=1}^{m} a_k \sin h_k\xi)\right]'' \in L_1(\mathbb{R}^1_+).$$

Thus, the assumptions of the Wiener Tauberian theorem are satisfied. Therefore,

$$x^2 \frac{\partial^2 \mathcal{E}_2(x,t)}{\partial x^2} = \frac{1}{x} \int\limits_0^{\infty} \psi(\frac{\eta}{x}) \sin \eta \, d\eta \overset{x \to \infty}{\longrightarrow} 0,$$

which completes the proof of Lemma 1.2.2. □

Since $\mathcal{E}(x,t)$ satisfies Eq. (1.2) in $\mathbb{R}^1 \times (0,+\infty)$ (see the previous section), Lemmas 1.2.1 and 1.2.2 imply the following assertion.

Lemma 1.2.3. *Let* $t > 0$ *and* $a, h \in \mathbb{R}^m$. *Then*

$$\lim_{x \to \infty} x^2 \frac{\partial \mathcal{E}}{\partial t}(x,t) = 0.$$

This lemma can be proved directly as well (the proof is the same as the one for Lemma 1.2.2).

From Lemmas 1.2.1–1.2.3 and the fact that $\mathcal{E}(x,t)$ satisfies Eq. (1.2) in $\mathbb{R}^1 \times (0,+\infty)$, we deduce the following assertion:

Theorem 1.2.1. *Let* $u_0(x)$ *be continuous and bounded in* $\mathbb{R}^1$. *Then the function*

$$\int\limits_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t) u_0(\xi) d\xi$$

satisfies (*in the classical sense*) *Eq.* (1.2) *in* $\mathbb{R}^1 \times (0,+\infty)$.

Remark 1.2.1. The assumption of the continuity and boundedness of the function u_0 can be replaced by the assumption of its belonging to $L_\infty(\mathbb{R}^1)$. Under this assumption, in general, the specified convolution is not a classical solution of Eq. (1.2) anymore; it is its a. e. solution.

Remark 1.2.2. Continuing to integrate by parts in Lemmas 1.2.1-1.2.2, we see that

$$\lim_{x \to \infty} x^m \frac{\partial^{k+l} \mathcal{E}}{\partial t^k \partial x^l}(x,t) = 0$$

for all positive integers m, k, and l and any positive t.

1.3. Solutions of Cauchy Problems

Introduce the notation

$$u(x,t) = \int\limits_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t) u_0(\xi) d\xi$$

and impose to Eq. (1.2) the initial-value condition

$$u\big|_{t=0} = u_0(x), \tag{1.4}$$

where $u_0(x)$ is continuous and bounded in $\mathbb{R}^1$.

The function $u(x,t)$ is defined on $\mathbb{R}^1 \times (0,+\infty)$. Take an arbitrary real x_0 and investigate the behavior of $u(x_0,t)$ as $t \longrightarrow 0$.

Change the variable by the formula

$$\eta = \frac{x_0 - \xi}{2\sqrt{t}}.$$

This yields the relation

$$u(x_0,t) = 2\sqrt{t} \int\limits_{-\infty}^{+\infty} \mathcal{E}(2\sqrt{t}\eta, t) u_0(x_0 - 2\sqrt{t}\eta) d\eta.$$

Further, we have

$$\sqrt{t}\,\mathcal{E}(2\sqrt{t}\eta,t)=\sqrt{t}\int\limits_0^\infty e^{-t\left(\xi^2-\sum\limits_{k=1}^m a_k\cos h_k\xi\right)}\cos\left(2\sqrt{t}\eta\xi-t\sum_{k=1}^m a_k\sin h_k\xi\right)d\xi$$

$$=\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)dz.$$

Hence,

$$u(x_0,t)=2\int\limits_{-\infty}^{+\infty}u_0(x_0-2\sqrt{t}\eta)\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)dzd\eta. \tag{1.5}$$

On the other hand, the following relation holds: $u_0(x_0)=\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}u_0(x_0)e^{-\eta^2}d\eta.$

Consider the following difference:

$$\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\mathcal{E}(x_0-\xi,t)u_0(\xi)d\xi-u_0(x_0)$$

$$=\int\limits_{-\infty}^{+\infty}\left[\frac{2}{\pi}u_0(x_0-2\sqrt{t}\eta)\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)dz\right.$$

$$\left.-\frac{1}{\sqrt{\pi}}u_0(x_0)e^{-\eta^2}\right]d\eta=\int\limits_{-\infty}^{-A}+\int\limits_{-A}^{A}+\int\limits_{A}^{+\infty}\stackrel{\text{def}}{=}I_1+I_2+I_3. \tag{1.6}$$

Let us prove the following assertion.

Lemma 1.3.1.

$$\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)dz\stackrel{t\to+0}{\longrightarrow}\frac{\sqrt{\pi}}{2}e^{-\eta^2}$$

uniformly with respect to $\eta\in\mathbb{R}^1$.

Proof. We must prove that for any positive ε, there exists a positive δ such that for any $t\in(0,\delta)$ and any real η the following inequality holds:

$$\left|\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)dz-\int\limits_0^\infty e^{-z^2}\cos 2z\eta\,dz\right|<\varepsilon.$$

Let $\varepsilon>0$. Consider

$$\int\limits_0^\infty e^{-z^2}\left[e^{t\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)-\cos 2z\eta\right]dz$$

$$=\int\limits_0^\infty e^{-z^2}\left[e^{t\sum\limits_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\left(\cos 2z\eta\cos\left(t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)\right.\right.$$

$$- \sin 2z\eta \sin\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\Bigg) - \cos 2z\eta\Bigg]\, dz.$$

Select a small δ from $(0,1]$ such that the sine is monotone on $\left(-\delta\sum\limits_{k=1}^{m}|a_k|,\, \delta\sum\limits_{k=1}^{m}|a_k|\right)$ and the inequality

$$\sin\left(\delta\sum_{k=1}^{m}|a_k|\right) < \frac{\varepsilon}{2}\left(e^{t\sum\limits_{k=1}^{m}|a_k|}\int\limits_0^{\infty} e^{-z^2}\, dz\right)^{-1}$$

holds for any $t \in (-\delta, \delta)$. Then the following inequality holds provided that $t \in (0, \delta)$:

$$\left|\int\limits_0^{\infty} e^{-z^2} e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \sin 2z\eta \sin\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) dz\right|$$
$$\le \frac{\varepsilon}{2}\left(e^{t\sum\limits_{k=1}^{m}|a_k|}\int\limits_0^{\infty} e^{-z^2}\, dz\right)^{-1}\int\limits_0^{\infty} e^{-z^2} e^{t\sum\limits_{k=1}^{m}|a_k|}\, dz = \frac{\varepsilon}{2}.$$

It suffices to show that the specified (sufficiently small) δ can be chosen to satisfy the following inequality for any real z and η:

$$|\cos 2z\eta|\left|e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}}\cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) - 1\right| \le \frac{\varepsilon}{2}\left(\int\limits_0^{\infty} e^{-z^2}\, dz\right)^{-1}.$$

Thus, it suffices to show that

$$e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}}\cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) \xrightarrow{t\to+0} 1$$

uniformly with respect to $z \in \mathbb{R}^1$.

We have

$$e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}}\cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) - 1$$
$$= e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}}\left[1 - 2\sin^2\left(\frac{t}{2}\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right] - 1$$
$$= e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} - 1 - 2e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}}\sin^2\left(\frac{t}{2}\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right).$$

Select a small t such that $\sin^2\left(\frac{t}{2}\sum\limits_{k=1}^{m}|a_k|\right) \le \frac{\varepsilon}{8}e^{-\sum\limits_{k=1}^{m}|a_k|}$. Since there exists a (sufficiently small) semineighborhood of the origin such that the sine is monotone in it, it follows that the inequality

$$2\sin^2\left(\frac{t}{2}\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) \le \frac{\varepsilon}{4}e^{-\sum\limits_{k=1}^{m}|a_k|}$$

is valid for the selected t. Without loss of generality, we assume that the selected t belongs to $(0,1)$. Therefore, the following inequality holds for any real z:

$$2\sin^2\left(\frac{t}{2}\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \le \frac{\varepsilon}{4}.$$

Thus, it suffices to estimate $e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} - 1$.

Without loss of generality, we assume that t is sufficiently small to satisfy the inequality

$$1-\frac{\varepsilon}{4} < e^{-\sum\limits_{k=1}^{m}|a_k|t} < e^{\sum\limits_{k=1}^{m}|a_k|t} < 1+\frac{\varepsilon}{4}.$$

Hence, the inequality

$$1-\frac{\varepsilon}{4} < e^{-\sum\limits_{k=1}^{m}|a_k|t} < e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} < e^{\sum\limits_{k=1}^{m}|a_k|t} < 1+\frac{\varepsilon}{4}$$

is valid for any $z \in \mathbb{R}^1$.

This implies that

$$\left| e^{t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} - 1\right| < \frac{\varepsilon}{4}$$

for any $z \in \mathbb{R}^1$, which completes the proof of Lemma 1.3.1. □

Now, we can estimate the integrals of sum (1.6).

First, estimate $|I_3|$:

$$\int_0^\infty e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\cos 2\eta z dz$$
$$= e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\frac{\sin 2\eta z}{2\eta}\Big|_{z=0}^{z=\infty}$$
$$-\frac{1}{2\eta}\int_0^\infty \left[e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right]_z' \sin 2\eta z\, dz$$
$$= -\frac{1}{2\eta}\int_0^\infty \left[e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right]_z' \sin 2\eta z dz$$

(since $\eta > A > 0$).

The last expression can be estimated as follows:

$$-\frac{1}{2\eta}\left(\left[e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right]_z' \frac{\cos 2\eta z}{2\eta}\Big|_{z=\infty}^{z=0}\right.$$
$$\left.+\frac{1}{2\eta}\int_0^\infty \left[e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right]_z'' \cos 2\eta z dz\right)$$
$$= \frac{1}{4\eta^2}\left[e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\right]_z' \cos 2\eta z\Big|_{z=0}^{z=\infty}$$

$$-\frac{1}{4\eta^2}\int_0^\infty \left[e^{-z^2+t\sum_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)\right]''_z \cos 2\eta z\,dz.$$

Taking into account that $\eta > A > 0$, we see that the last expression is equal to

$$-\frac{1}{4\eta^2}\int_0^\infty \left[e^{-z^2+t\sum_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)\right]''_z \cos 2\eta z dz.$$

The last integral converges uniformly with respect to $\eta \in \mathbb{R}^1$ and if $t < 1$, then its absolute value is estimated from above by a constant depending only the vectors $a, h \in \mathbb{R}^m$; denote that constant by M. The same estimate holds for

$$\int_0^\infty e^{-z^2+t\sum_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\sin\left(t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)\sin 2\eta z\,dz,$$

which is the second term of the internal integral in I_3.

Thus,

$$|I_3| \le \frac{4M\sup|u_0|}{\pi}\int_A^\infty \frac{d\eta}{\eta^2} + \frac{\sup|u_0|}{\sqrt{\pi}}\int_A^\infty e^{-\eta^2}d\eta = \sup|u_0|\left(\frac{4M}{\pi A} + \frac{1}{\sqrt{\pi}}\int_A^\infty e^{-\eta^2}d\eta\right).$$

It is obvious that $|I_1|$ satisfies the same estimate.

Let ε be an arbitrary positive number. Without loss of generality, we assume that $t \le 1$. Select a (sufficiently large) positive A such that $|I_1| < \frac{\varepsilon}{3}$ and $|I_3| < \frac{\varepsilon}{3}$ and fix that A. Consider

$$I_2 = \int_{-A}^{A}\left[\frac{2}{\pi}u_0(x_0 - 2\sqrt{t}\eta)\int_0^\infty e^{-z^2+t\sum_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\right.$$
$$\left.\times\cos\left(2z\eta - t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)dz - \frac{1}{\sqrt{\pi}}u_0(x_0)e^{-\eta^2}\right]d\eta.$$

By virtue of Lemma 1.3.1 and the continuity of the function $u_0(x)$ at the point x_0, there exists a positive δ such that the inequalities $t < \delta$ and $|2\sqrt{t}\eta| < \delta$ imply the inequality

$$\left|\frac{2}{\pi}u_0(x_0 - 2\sqrt{t}\eta)\int_0^\infty e^{-z^2+t\sum_{k=1}^m a_k\cos\frac{h_kz}{\sqrt{t}}}\cos\left(2z\eta - t\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt{t}}\right)dz - \frac{u_0(x_0)}{\sqrt{\pi}}e^{-\eta^2}\right| < \frac{\varepsilon}{6A}.$$

Denote $\min\left(\delta, \frac{\delta^2}{4A^2}\right)$ by t_0; then $|I_2| < \frac{\varepsilon}{3}$ once $t < t_0$. Since ε is chosen arbitrarily, it follows that

$$\lim_{t\to+0}\left[\frac{1}{\pi}\int_{-\infty}^{+\infty}\mathcal{E}(x_0-\xi,t)u_0(\xi)d\xi - u_0(x_0)\right] = 0.$$

Taking into account the real x_0 is taken arbitrarily, we prove the following assertion.

Theorem 1.3.1. *Let $u_0(x)$ be continuous and bounded in $\mathbb{R}^1$. Then the function*

$$u(x,t) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_{-\infty}^{+\infty}\mathcal{E}(x-\xi,t)u_0(\xi)d\xi$$

is a classical solution of problem (1.2), (1.4).

In particular, using this theorem, one can compute the integral of the fundamental solution over the real axis:

Lemma 1.3.2.

$$\int_{-\infty}^{\infty} \mathcal{E}(x,t)dx = \pi e^{\sum\limits_{k=1}^{m} a_k t}.$$

Proof. Assign $u_0(x) \equiv 1$. This function is bounded. Hence, by virtue of Theorem 1.3.1, the function

$$y(x,t) \stackrel{\text{def}}{=} \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t)d\xi$$

satisfies Eq. (1.2) in $\mathbb{R}^1 \times (0,+\infty)$ and the initial-value condition $y(x,0) \equiv 1$ on $\mathbb{R}^1$. However, the function $y(x,t)$ does not depend on x:

$$\int_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t)d\xi = \int_{-\infty}^{+\infty} \mathcal{E}(\xi,t)d\xi = y(t),$$

i.e., $y(t)$ satisfies an ordinary differential equation

$$y' - \sum_{k=1}^{m} a_k y = 0$$

and the initial-value condition $y(0) = 1$. Thus, $y(t) = e^{\sum\limits_{k=1}^{m} a_k t}$, which completes the proof. ◻

1.4. Multidimensional Case

Let $x \in \mathbb{R}^n$. In $\mathbb{R}^n \times (0,+\infty)$, consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{k=1}^{m} a_k u(x - b_k h, t), \tag{1.7}$$

where a and b are arbitrary parameters from $\mathbb{R}^m$ and h is a fixed vector of length 1 in $\mathbb{R}^n$.

In $\mathbb{R}^n \times (0,+\infty)$, define the function

$$\mathcal{E}_{a,b,h,n}(x,t) \stackrel{\text{def}}{=} \mathcal{E}_{(n)}(x,t) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} e^{-t(|\xi|^2 - \sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi)} \cos\left(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi\right) d\xi. \tag{1.8}$$

Then

$$|\mathcal{E}_{(n)}(x,t)| \le \int_{\mathbb{R}^n} e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)} d\xi \le \int_{\mathbb{R}^n} e^{(\sum\limits_{k=1}^{m} |a_k| - |\xi|^2)t} d\xi$$

$$\le e^{\sum\limits_{k=1}^{m} |a_k| t} \int_{\mathbb{R}^n} e^{-t|\xi|^2} d\xi = e^{\sum\limits_{k=1}^{m} |a_k| t} \int_0^{\infty} \int_{|\xi|=r} e^{-t|\xi|^2} dS_\xi dr$$

$$= C_n e^{\sum\limits_{k=1}^{m} |a_k| t} \int_0^{\infty} e^{-tr^2} r^{n-1} dr = \frac{C_n}{2} e^{\sum\limits_{k=1}^{m} |a_k| t} \Gamma\left(\frac{n}{2}\right) t^{-\frac{n}{2}}$$

(here dS_ξ denotes the surface measure in $\mathbb{R}^n$ and C_n is the area of the unit sphere in $\mathbb{R}^n$).

Hence, integral (1.8) converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^n \times [t_0,T]$ for all $t_0, T \in (0,+\infty)$, i.e., $\mathcal{E}_{(n)}(x,t)$ is well defined in $\mathbb{R}^n \times (0,+\infty)$.

Formally differentiate $\mathcal{E}_{(n)}(x,t)$:

$$\frac{\partial \mathcal{E}_{(n)}}{\partial t} = \int\limits_{\mathbb{R}^n} (\sum_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2) e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)}$$

$$\times \cos(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)d\xi + \int\limits_{\mathbb{R}^n} e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)}$$

$$\times \sin(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi d\xi.$$

This expression is equal to

$$\int\limits_{\mathbb{R}^n} e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)} \Bigg[(\sum_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)\cos(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)$$

$$+\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi \sin(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)\Bigg] d\xi.$$

Therefore,

$$\frac{\partial \mathcal{E}_{(n)}}{\partial t} = \int\limits_{\mathbb{R}^n} \Bigg[\sum_{k=1}^{m} a_k \cos\left((x-b_k h)\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi\right)$$

$$-|\xi|^2\cos(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)\Bigg] e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)} d\xi.$$

Further, we have

$$\frac{\partial^2 \mathcal{E}_{(n)}}{\partial x_j^2} = -\int\limits_{\mathbb{R}^n} \xi_j^2\, e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)} \cos(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)d\xi, \quad j=\overline{1,n},$$

which implies that

$$\Delta\mathcal{E}_{(n)} = -\int\limits_{\mathbb{R}^n} |\xi|^2\, e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)} \cos(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)d\xi.$$

Therefore, we have

$$\frac{\partial \mathcal{E}_{(n)}}{\partial t} - \Delta\mathcal{E}_{(n)} = \sum_{k=1}^{m} a_k \int\limits_{\mathbb{R}^n} e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)} \cos\left[(x-b_k h)\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi\right] d\xi.$$

Thus, function $\mathcal{E}_{(n)}$ formally satisfies Eq. (1.7).

Let us check whether the above formal differentiating is legible:

$$\left|\frac{\partial^{l+|m|}}{\partial t^l \partial x_1^{m_1}\dots\partial x_n^{m_n}}\left[e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)}\cos(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)\right]\right| \leq P(\xi) e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)},$$

where $P(\xi)$ is a polynomial of power not exceeding $|m|+2l$ (here $|m| = m_1+\dots+m_n+2l$ is the multi-index length). Hence,

$$\left|\frac{\partial^{l+|m|}}{\partial t^l \partial x_1^{m_1}\dots\partial x_n^{m_n}}\left[e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k h\cdot\xi - |\xi|^2)}\cos(x\cdot\xi - t\sum_{k=1}^{m} a_k \sin b_k h\cdot\xi)\right]\right| \leq A|\xi|^{|m|+2l} e^{(\sum\limits_{k=1}^{m}|a_k| - |\xi|^2)t}.$$

Further, we have

$$\int\limits_{\mathbb{R}^n} |\xi|^{|m|+2l} e^{(\sum\limits_{k=1}^{m} |a_k|-|\xi|^2)t} d\xi = C e^{\sum\limits_{k=1}^{m} |a_k|t} \int\limits_0^{\infty} r^{|m|+2l+n-1} e^{-tr^2} dr = \frac{C\Gamma\left(\frac{m+n}{2}+l\right)}{2t^{\frac{m+n}{2}+l}} e^{\sum\limits_{k=1}^{m} |a_k|t}.$$

Therefore, the integral obtained by the formal differentiating of the function $\mathcal{E}_{(n)}$, for all $t_0, T \in (0,+\infty)$, converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^n \times [t_0, T]$. Therefore, the function $\mathcal{E}_{(n)}(x,t)$ defined by relation (1.8) is infinitely differentiable in $\mathbb{R}^n \times (0,+\infty)$ and satisfies (in the classical sense) Eq. (1.7).

Now, investigate the behavior of the function $\mathcal{E}_{(n)}(x,t)$ and its derivatives as $|x| \to \infty$. Without loss of generality, assume that $m = 1$ and $a_1 = 1$; redenote the vector $b_1 h$ by h. Further, rotate the coordinate system $\xi_1, \dots, \xi_n$ to an angle such that $x \cdot \xi = |x|\xi_1$ (the Jacobian of this change of variable is equal to unity). Then

$$\mathcal{E}_{(n)}(x,t) = \int\limits_{\mathbb{R}^n} e^{t(\cos \tilde{h}\cdot\xi - |\xi|^2)} \cos(|x|\xi_1 - t\sin \tilde{h}\cdot\xi) d\xi,$$

where $\tilde{h}$ is, in general, different from the vector from the vector h (moreover, $\tilde{h}$ depends on x; precisely, it depends on the ray containing the point x), but $|\tilde{h}| = |h|$. Assuming that $\tilde{h} = (|\tilde{h}|, 0, \dots, 0)$, we obtain that

$$\begin{aligned}\mathcal{E}_{(n)}(x,t) &= \int\limits_{\mathbb{R}^n} e^{t(\cos|\tilde{h}|\xi_1 - |\xi|^2)} \cos(|x|\xi_1 - t\sin|\tilde{h}|\xi_1) d\xi \\ &= \int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi'|^2} d\xi' \int\limits_{-\infty}^{+\infty} e^{t(\cos|\tilde{h}|\xi_1 - \xi_1^2)} \cos(|x|\xi_1 - t\sin|\tilde{h}|\xi_1) d\xi_1 \\ &= C_{n-1}\Gamma\left(\frac{n-1}{2}\right) t^{\frac{1-n}{2}} \mathcal{E}_{1,|\tilde{h}|}(|x|,t) = C_{n-1}\Gamma\left(\frac{n-1}{2}\right) t^{\frac{1-n}{2}} \mathcal{E}_{1,|h|}(|x|,t),\end{aligned} \tag{1.9}$$

where $\mathcal{E}_{1,|h|} = \mathcal{E}$ is still defined by (1.3).

However, the assumption that $\tilde{h} = (|\tilde{h}|, 0, \dots, 0)$ does not restrict the generality because the Laplace operator is invariant with respect to rotations. This means that $\tilde{h}$ in relation (1.9) varies from one ray to another, but the function $\mathcal{E}_{1,|h|}$ *is the same* for all $x \in \mathbb{R}^n$. Therefore, by virtue of Remark 1.3, the limit relation

$$\lim_{|x|\to\infty} |x|^{n+1} \mathcal{E}_{(n)}(x,t) = 0$$

holds for all positive t and $|h|$.

In the same way, we have

$$\Delta\mathcal{E}_{1,n} = -\int\limits_{\mathbb{R}^n} |\xi|^2 e^{t(\cos|\tilde{h}|\xi_1 - |\xi|^2)} \cos|x|\xi_1 \, \cos(t\sin|\tilde{h}|\xi_1) d\xi.$$

The last integral is equal to

$$\begin{aligned}&-\sum_{j=1}^{n} \int\limits_{\mathbb{R}^n} \xi_j^2 e^{t(\cos|\tilde{h}|\xi_1 - \xi_1^2 - \dots - \xi_n^2)} \cos|x|\xi_1 \, \cos(t\sin|\tilde{h}|\xi_1) d\xi \\ &= -\int\limits_{\mathbb{R}^n} \xi_1^2 e^{t(\cos|\tilde{h}|\xi_1 - \xi_1^2 - \dots - \xi_n^2)} \cos|x|\xi_1 \, \cos(t\sin|\tilde{h}|\xi_1) d\xi \\ &\quad -\sum_{j=2}^{n} \int\limits_{\mathbb{R}^n} \xi_j^2 e^{t(\cos|\tilde{h}|\xi_1 - \xi_1^2 - \dots - \xi_n^2)} \cos|x|\xi_1 \, \cos(t\sin|\tilde{h}|\xi_1) d\xi.\end{aligned}$$

The last expression is equal to

$$-\int\limits_{\mathbb{R}^{n-1}} e^{-t(\xi_2^2+\dots+\xi_n^2)}d\xi' \int\limits_{-\infty}^{+\infty} \xi_1^2 e^{t(\cos|\tilde{h}|\xi_1-\xi_1^2)} \cos|x|\xi_1 \, \cos(t\sin|\tilde{h}|\xi_1)d\xi_1$$

$$-\sum_{j=2}^{n}\int\limits_{\mathbb{R}^{n-1}} \xi_j^2 e^{-t(\xi_2^2+\dots+\xi_n^2)}d\xi' \int\limits_{-\infty}^{+\infty} e^{t(\cos|\tilde{h}|\xi_1-\xi_1^2)} \cos|x|\xi_1 \, \cos(t\sin|\tilde{h}|\xi_1)d\xi_1$$

$$= 2\frac{\partial^2\mathcal{E}_1}{\partial x^2}(|x|,t)\int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi'|^2}d\xi' - 2\mathcal{E}_1(|x|,t)\sum_{j=2}^{n}\int\limits_{\mathbb{R}^{n-1}} \xi_j^2 e^{-t|\xi'|^2}d\xi'.$$

Compute the integral in the second term:

$$\int\limits_{\mathbb{R}^{n-1}} \xi_j^2 e^{-t|\xi'|^2}d\xi' = \int\limits_{\mathbb{R}^{n-2}} e^{-t|\eta|^2}d\eta \int\limits_{-\infty}^{+\infty} \xi_j^2 e^{-t\xi_j^2}d\xi_j$$

$$= C_{n-2}\Gamma\left(\frac{n}{2}-1\right)t^{1-\frac{n}{2}}\int\limits_0^{\infty}\tau^2 e^{-t\tau^2}d\tau = \frac{1}{2}C_{n-2}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n}{2}-1\right)t^{-\frac{n+1}{2}}.$$

This implies that

$$\Delta\mathcal{E}_{1,n} = C_{n-1}\Gamma\left(\frac{n-1}{2}\right)t^{\frac{1-n}{2}}\frac{\partial^2\mathcal{E}_1}{\partial x^2}(|x|,t) - nC_{n-2}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n}{2}-1\right)t^{-\frac{n+1}{2}}\mathcal{E}_1(|x|,t).$$

Computing $\Delta\mathcal{E}_{2,n}$ in the same way and taking into account Remark 1.2.2, we see that the limit relation

$$\lim_{|x|\to\infty}|x|^{n+1}\Delta\mathcal{E}_{(n)}(x,t) = 0$$

holds for all positive t and $|h|$. Then, arguing as in the proof of Lemma 1.2.3, we obtain that

$$\lim_{|x|\to\infty}|x|^{n+1}\frac{\partial\mathcal{E}_{(n)}}{\partial t}(x,t) = 0$$

for all positive t and $|h|$. Thus, the following assertion is proved:

Theorem 1.4.1. *Let a function $u_0(x)$ be continuous and bounded in $\mathbb{R}^n$ (belong to the space $L_\infty(\mathbb{R}^n)$). Then the function $\int\limits_{\mathbb{R}^n}\mathcal{E}_{(n)}(x-\xi,t)u_0(\xi)d\xi$ satisfies Eq. (1.7) in the classical sense (a. e. respectively).*

To prove that the constructed solution satisfies the corresponding initial-value condition as well (apart from Eq. (1.7)), we represent the fundamental solution as follows:

$$\int\limits_{\mathbb{R}^n} e^{t(\sum_{k=1}^{m} a_k\cos b_k\xi_1-|\xi|^2)}\cos(x_1\xi_1+\dots+x_n\xi_n - t\sum_{k=1}^{m}a_k\sin b_k\xi_1)d\xi$$

$$= \int\limits_{\mathbb{R}^n} e^{t(\sum_{k=1}^{m} a_k\cos b_k\xi_1-\xi_1^2)}e^{-t|\xi'|^2}\cos(x_1\xi_1 - t\sum_{k=1}^{m}a_k\sin b_k\xi_1)\cos x'\cdot\xi'd\xi$$

$$-\int\limits_{\mathbb{R}^n} e^{t(\sum_{k=1}^{m} a_k\cos b_k\xi_1-\xi_1^2)}e^{-t|\xi'|^2}\sin(x_1\xi_1 - t\sum_{k=1}^{m}a_k\sin b_k\xi_1)\sin x'\cdot\xi'd\xi.$$

The last expression is equal to

$$\int\limits_{-\infty}^{\infty} e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k\xi_1 - \xi_1^2)} \cos(x_1\xi_1 - t\sum_{k=1}^{m} a_k \sin b_k\xi_1)d\xi_1 \int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi'|^2} \cos x'\cdot\xi' d\xi'$$

$$- \int\limits_{-\infty}^{\infty} e^{t(\sum\limits_{k=1}^{m} a_k \cos b_k\xi_1 - \xi_1^2)} \sin(x_1\xi_1 - t\sum_{k=1}^{m} a_k \sin b_k\xi_1)d\xi_1 \int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi'|^2} \sin x'\cdot\xi' d\xi'$$

$$= 2\mathcal{E}_{a,b}(x_1,t) \int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi'|^2} \cos x'\cdot\xi' d\xi'$$

(the second term vanishes because the integrand of its first factor is even).

Compute the last integral.

Without loss of generality (more exactly, up to a rotation of the coordinate system $\xi_1, \ldots, \xi_n$), we have $x'\cdot\xi' = |x|\xi_2$. Therefore,

$$\int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi'|^2} \cos x'\cdot\xi' d\xi' = \int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi'|^2} \cos|x'|\xi_2 d\xi = \int\limits_{-\infty}^{+\infty} e^{-t\xi_2^2} \cos|x'|\xi_2 d\xi_2 \int\limits_{\mathbb{R}^{n-2}} e^{-t|\xi''|^2} d\xi''$$

$$= \frac{\sqrt{\pi}}{\sqrt{t}} e^{-\frac{|x'|^2}{4t}} \int\limits_{\mathbb{R}^{n-2}} e^{-t(\xi_3^2+\cdots+\xi_n^2)} d\xi'' = \left(\frac{\pi}{t}\right)^{\frac{n-1}{2}} e^{-\frac{|x'|^2}{4t}}.$$

Thus, $\mathcal{E}_{(n)}(x,t) = 2\left(\frac{\pi}{t}\right)^{\frac{n-1}{2}} \mathcal{E}(x_1,t)e^{-\frac{|x'|^2}{4t}}$, where $x' = (x_2, \ldots, x_n)$ is a vector from $\mathbb{R}^{n-1}$.

Let $(y_0, x_1^0, \ldots x_{n-1}^0) \overset{\text{def}}{=} (y_0, x^0)$ be an arbitrary element of $\mathbb{R}^n$. Introduce the following notation:

$$u(x,t) \overset{\text{def}}{=} \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \mathcal{E}_{(n)}(x-\xi,t)u_0(\xi)d\xi. \tag{1.10}$$

Consider the difference $u(y_0, x^0, t) - u_0(y_0, x^0)$.

Change the variables:

$$\frac{y_0 - \xi_1}{2\sqrt{t}} = \eta, \; \frac{x_j^0 - \xi_{j+1}}{2\sqrt{t}} = z_j, \; j = \overline{1, n-1}.$$

This yields the relations

$$u(y_0, x^0, t) = \frac{2}{\pi^{\frac{n+1}{2}}} \int\limits_{\mathbb{R}^n} \sqrt{t}\mathcal{E}(2\sqrt{t}\eta, t)e^{-|z|^2} u_0\left(y_0 - 2\sqrt{t}\eta, x_1^0 - 2\sqrt{t}z_1, \ldots, x_{n-1}^0 - 2\sqrt{t}z_{n-1}\right)d\eta dz$$

$$= \frac{2}{\pi^{\frac{n+1}{2}}} \int\limits_{-\infty}^{+\infty} \int\limits_{\mathbb{R}^{n-1}} \sqrt{t}\mathcal{E}(2\sqrt{t}\eta, t)e^{-|\xi|^2} u_0\left(y_0 - 2\sqrt{t}\eta, x^0 - 2\sqrt{t}\xi\right) d\xi d\eta$$

and

$$u_0(y_0, x^0) = \frac{1}{\pi^{\frac{n}{2}}} \int\limits_{-\infty}^{+\infty} \int\limits_{\mathbb{R}^{n-1}} e^{-\eta^2 - |\xi|^2} u_0(y_0, x^0) d\xi d\eta.$$

Thus, we have

$$u(y_0,x^0,t)-u_0(y_0,x^0)$$
$$=\frac{2}{\pi^{\frac{n+1}{2}}}\int\limits_{-\infty}^{+\infty}\int\limits_{\mathbb{R}^{n-1}} e^{-|\xi|^2}\left[\sqrt{t}\mathcal{E}(2\sqrt{t}\eta,t)u_0\left(y_0-2\sqrt{t}\eta,x^0-2\sqrt{t}\xi\right)-\frac{\sqrt{\pi}}{2}u_0(y_0,x^0)e^{-\eta^2}\right]d\xi d\eta.$$

The last difference can be represented as

$$\frac{2}{\pi^{\frac{n+1}{2}}}\int\limits_{-\infty}^{+\infty}\int\limits_{\mathbb{R}^{n-1}} e^{-|\xi|^2}\left[u_0\left(y_0-2\sqrt{t}\eta,x^0-2\sqrt{t}\xi\right)\right.$$
$$\left.\times\int\limits_0^{\infty} e^{-z^2+t\sum\limits_{k=1}^{m}a_k\cos\frac{b_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^{m}a_k\sin\frac{b_kz}{\sqrt{t}}\right)dz-\frac{\sqrt{\pi}}{2}u_0(y_0,x^0)e^{-\eta^2}\right]d\xi d\eta$$

(see the deduction of representation (1.5) in the previous section).

Now, let $A>0$, $G_1=\left\{(\eta,\xi)\Big|\eta\in(-A,A),|\xi|<A\right\}$, and $G_2=\mathbb{R}^n\backslash G_1$; then $u(y_0,x^0,t)-u_0(y_0,x^0)=J_1+J_2$, where

$$J_j=\frac{2}{\pi^{\frac{n+1}{2}}}\int\limits_{G_j}\left[\int\limits_0^{\infty}u_0\left(y_0-2\sqrt{t}\eta,x^0-2\sqrt{t}\xi\right)e^{-z^2+t\sum\limits_{k=1}^{m}a_k\cos\frac{b_kz}{\sqrt{t}}}\right.$$
$$\left.\times\cos\left(2z\eta-t\sum_{k=1}^{m}a_k\sin\frac{b_kz}{\sqrt{t}}\right)dz-\frac{\sqrt{\pi}}{2}u_0(y_0,x^0)e^{-\eta^2}\right]e^{-|\xi|^2}d\eta d\xi,$$

$j=1,2$.

First, we estimate the integral J_2:

$$\left|\int\limits_{G_2}u_0(y_0,x^0)e^{-|\xi|^2-\eta^2}d\eta d\xi\right|\le\sup|u_0|\int\limits_{|\xi|^2+\eta^2\ge A^2}e^{-|\xi|^2-\eta^2}d\xi d\eta=C_n\sup|u_0|\int\limits_A^{\infty}r^{n-1}e^{-r^2}dr\overset{A\to\infty}{\longrightarrow}0$$

(due to the convergence of the last integral and boundedness of u_0).

To estimate the remaining term of the integral J_2, we decompose the integrating domain as follows:

$$G_2=\left\{\eta>A,\ \xi\in\mathbb{R}^{n-1}\right\}\cup\left\{\eta<-A,\ \xi\in\mathbb{R}^{n-1}\right\}\cup\{\eta\in[-A,A],|\xi|\ge A\}\overset{\text{def}}{=}G_{2,1}\cup G_{2,2}\cup G_{2,3};$$

correspondingly, for $j=1,2,3$, denote

$$\int\limits_{G_{2,j}}e^{-|\xi|^2}\int\limits_0^{\infty}u_0\left(y_0-2\sqrt{t}\eta,x^0-2\sqrt{t}\xi\right)e^{-z^2+t\sum\limits_{k=1}^{m}a_k\cos\frac{b_kz}{\sqrt{t}}}\cos\left(2z\eta-t\sum_{k=1}^{m}a_k\sin\frac{b_kz}{\sqrt{t}}\right)dzd\eta d\xi$$

by $\frac{\pi^{\frac{n+1}{2}}}{2}J_{2,j}$.

Then

$$J_{2,1}=\frac{2}{\pi^{\frac{n+1}{2}}}\int\limits_{G_{2,1}}u_0\left(y_0-2\sqrt{t}\eta,x^0-2\sqrt{t}\xi\right)$$
$$\times\int\limits_0^{\infty}e^{-z^2+t\sum\limits_{k=1}^{m}a_k\cos\frac{b_kz}{\sqrt{t}}}\cos 2z\eta\cos\left(t\sum_{k=1}^{m}a_k\sin\frac{b_kz}{\sqrt{t}}\right)dzd\eta d\xi$$

$$+\frac{2}{\pi^{\frac{n+1}{2}}}\int\limits_{G_{2,1}} u_0\left(y_0-2\sqrt{t}\eta, x^0-2\sqrt{t}\xi\right)$$

$$\times\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{b_kz}{\sqrt{t}}}\sin 2z\eta\sin\left(t\sum_{k=1}^m a_k\sin\frac{b_kz}{\sqrt{t}}\right)dzd\eta d\xi \stackrel{\text{def}}{=} J_{2,1,1}+J_{2,1,2}.$$

Earlier, estimating the integral I_3 in (1.6), we obtained that

$$\Big|\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{b_kz}{\sqrt{t}}}\cos 2z\eta\cos\left(t\sum_{k=1}^m a_k\sin\frac{b_kz}{\sqrt{t}}\right)dzd\eta d\xi\Big|\leq\frac{M}{\eta^2}$$

provided that $\eta\geq A$ and $t\in(0,1]$. Therefore,

$$|J_{2,1,1}|\leq\frac{2M\sup|u_0|}{\pi^{\frac{n+1}{2}}}\int\limits_{\mathbb{R}^{n-1}}e^{-|\xi|^2}d\xi\int\limits_A^\infty\frac{d\eta}{\eta^2}=\frac{2M\sup|u_0|}{\pi A}.$$

Estimating $J_{2,1,2}$ in the same way, we see that $|J_{2,1}|\leq\dfrac{4M\sup|u_0|}{\pi A}$.

In the same way, we obtain the inequality $|J_{2,2}|\leq\dfrac{4M\sup|u_0|}{\pi A}$.

Let us estimate

$$J_{2,3}=\frac{2}{\pi^{\frac{n+1}{2}}}\int\limits_{|\xi|\geq A}e^{-|\xi|^2}\int\limits_{-A}^{A}u_0\left(y_0-2\sqrt{t}\eta, x^0-2\sqrt{t}\xi\right)$$

$$\times\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{b_kz}{\sqrt{t}}}\cos 2z\eta\cos\left(t\sum_{k=1}^m a_k\sin\frac{b_kz}{\sqrt{t}}\right)dzd\eta d\xi.$$

For $t\in(0,1]$, we have

$$\Big|\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^m a_k\cos\frac{b_kz}{\sqrt{t}}}\cos 2z\eta\cos\left(t\sum_{k=1}^m a_k\sin\frac{b_kz}{\sqrt{t}}\right)dz\Big|\leq e^t\int\limits_0^\infty e^{-z^2}dz\leq\frac{\sqrt{\pi}}{2}e.$$

Therefore,

$$|J_{2,3}|\leq\frac{2\sup|u_0|e}{\pi^{\frac{n}{2}}}A\int\limits_{|\xi|\geq A}e^{-|\xi|^2}d\xi=\frac{2C_n\sup|u_0|e}{\pi^{\frac{n}{2}}}A\int\limits_A^\infty r^{n-2}e^{-r^2}dr.$$

The last expression tends to zero as $A\to\infty$. Indeed, we have

$$x\int\limits_x^\infty r^{n-2}e^{-r^2}dr=x\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}}-\int\limits_0^x r^{n-2}e^{-r^2}dr\right)=\frac{\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}}-\int\limits_0^x r^{n-2}e^{-r^2}dr}{\frac{1}{x}}.$$

Computing the limit

$$\lim_{x\to\infty}\frac{\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}}-\int\limits_0^x r^{n-2}e^{-r^2}dr}{\frac{1}{x}}$$

by means of the L'Hospital rule, we obtain the it is equal to $\lim\limits_{x\to\infty}x^ne^{-x^2}=0$.

Thus, for an arbitrary positive ε, one can select a positive A such that $|J_2| < \frac{\varepsilon}{2}$; fix such A and consider the integral J_1.

By virtue of the continuity of u_0 at the point (y_0, x^0) and Lemma 1.3.1, one can select a small positive t_0 such that for any t from $(0, t_0)$ and any (η, ξ) from G_1, we have

$$\left| u_0\left(y_0 - 2\sqrt{t}\eta, x^0 - 2\sqrt{t}\xi\right) \int_0^\infty e^{-z^2 + t\sum\limits_{k=1}^m a_k \cos\frac{b_k z}{\sqrt{t}}} \cos\left(2z\eta - t\sum_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}}\right) dz \right.$$
$$\left. - u_0(y_0, x^0)\frac{\sqrt{\pi}}{2}e^{-\eta^2} \right| < \frac{\varepsilon}{2}\frac{\pi^{\frac{n+1}{2}}}{2}\frac{1}{(2A)^n}, \quad \text{i.e.,} \quad |J_1| < \frac{\varepsilon}{2}.$$

Since the positive ε is selected arbitrarily, this implies that

$$\lim_{t\to+0} u(y_0, x_1^0, \ldots, x_{n-1}^0, t) = u_0(y_0, x_1^0, \ldots, x_{n-1}^0).$$

Since the real $x_1^0, \ldots, x_{n-1}^0$, and y_0 are selected arbitrarily, this proves the following assertion:

Theorem 1.4.2. *Let $u_0(x)$ be continuous and bounded in $\mathbb{R}^n$. Then the function defined by* (1.10) *is a classical solution of problem* (1.7), (1.4).

1.5. Uniqueness of Solutions

Take an arbitrary positive T and consider the function $u(x,t) = u(x', x_n, t)$ defined as

$$\frac{2}{\pi^{\frac{n+1}{2}}} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^{n-1}} u_0\left(x' - 2\sqrt{t}\xi, x_n - 2\sqrt{t}\eta\right) e^{-|\xi|^2} \int_0^\infty e^{-z^2 + t\sum\limits_{k=1}^m a_k \cos\frac{b_k z}{\sqrt{t}}} \cos\left(2z\eta - t\sum_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}}\right) dz d\xi d\eta \tag{1.11}$$

and satisfying (by virtue of Theorem 1.4.2) problem (1.7), (1.4).

Let us prove the following assertion:

Theorem 1.5.1. *Let $u_0(x)$ be continuous and bounded in $\mathbb{R}^n$. Then function* (1.11) *is a unique bounded solution of problem* (1.7), (1.4) *in the domain $\mathbb{R}^n \times (0,T)$.*

Proof. First, we prove that function (1.11) is bounded.

Treating $\cos\left(2z\eta - t\sum\limits_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}}\right)$ as the cosine of a difference, decompose $\frac{\pi^{\frac{n+1}{2}}}{2}u(x,t)$ into two terms; it suffices to estimate one of them. Estimate

$$I_1 \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^{n-1}} u_0\left(x' - 2\sqrt{t}\xi, x_n - 2\sqrt{t}\eta\right) e^{-|\xi|^2}$$
$$\times \int_0^\infty e^{-z^2 + t\sum\limits_{k=1}^m a_k \cos\frac{b_k z}{\sqrt{t}}} \cos 2z\eta \cos\left(t\sum_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}}\right) dz d\xi d\eta.$$

In I_1, represent the domain of integrating with respect to the variable (ξ, η) as $G_1 \cup G_2 \cup G_3 \cup G_4$, where

$$G_1 = \left\{(\xi,\eta)\Big|\eta \in (-1,1), |\xi| < 1\right\}, \; G_2 = \left\{(\xi,\eta)\Big|\eta > 1, \, \xi \in \mathbb{R}^{n-1}\right\},$$

$$G_3 = \left\{(\xi,\eta)\Big|\eta < -1, \, \xi \in \mathbb{R}^{n-1}\right\}, \text{ and } G_4 = \left\{(\xi,\eta)\Big|\eta \in [-1,1], |\xi| \geq 1\right\}.$$

The terms of the integral I_1 are denoted by J_1, J_2, J_3, and J_4 respectively. Estimate those terms.

$$|J_1| \le |G_1| \sup |u_0| e^{t\sum\limits_{k=1}^{m}|a_k|} \int\limits_0^\infty e^{-z^2} dz \le 2^{n-1}\sqrt{\pi} \sup |u_0| e^{T\sum\limits_{k=1}^{m}|a_k|}.$$

The absolute value of the internal integral in J_4 is estimated from above by

$$e^{t\sum\limits_{k=1}^{m}|a_k|} \int\limits_0^\infty e^{-z^2} dz \le e^{T\sum\limits_{k=1}^{m}|a_k|} \frac{\sqrt{\pi}}{2}.$$

Therefore,

$$|J_4| \le \sqrt{\pi} \sup |u_0| e^{T\sum\limits_{k=1}^{m}|a_k|} \int\limits_{\mathbb{R}^{n-1}} e^{-|\xi|^2} d\xi = \pi^{\frac{n}{2}} \sup |u_0| e^{T\sum\limits_{k=1}^{m}|a_k|}.$$

To estimate the term J_2, represent its internal integral as follows:

$$-\frac{1}{4\eta^2} \int\limits_0^\infty \left[e^{-z^2+t\sum\limits_{k=1}^{m} a_k \cos\frac{b_k z}{\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin\frac{b_k z}{\sqrt{t}} \right) \right]_z'' \cos 2\eta z \, dz. \tag{1.12}$$

Denote $t\sum\limits_{k=1}^{m} a_k \cos\dfrac{b_k z}{\sqrt{t}}$ and $t\sum\limits_{k=1}^{m} a_k \sin\dfrac{b_k z}{\sqrt{t}}$ by $f_1(z)$ and $f_2(z)$ respectively. Then

$$f_1'(z) = -\sqrt{t}\sum_{k=1}^{m} a_k b_k \sin\frac{b_k z}{\sqrt{t}}, \;\; f_1''(z) = -\sum_{k=1}^{m} a_k b_k^2 \cos\frac{b_k z}{\sqrt{t}},$$

$$f_2'(z) = \sqrt{t}\sum_{k=1}^{m} a_k b_k \cos\frac{b_k z}{\sqrt{t}}, \text{ and } f_2''(z) = -\sum_{k=1}^{m} a_k b_k^2 \sin\frac{b_k z}{\sqrt{t}}.$$

This implies the inequalities

$$|f_j| \le t\sum_{k=1}^{m} |a_k|, |f_j'| \le \sqrt{t}\sum_{k=1}^{m} |a_k||b_k|, \text{ and } |f_j''| \le \sum_{k=1}^{m} |a_k| b_k^2 \quad (j=1,2). \tag{1.13}$$

Further, we have

$$\begin{aligned}
&[e^{-z^2+f_1(z)}]' = (f_1'(z) - 2z) e^{-z^2+f_1(z)}, \\
&[e^{-z^2+f_1(z)}]'' = [(f_1'(z) - 2z)^2 + (f_1''(z) - 2)] e^{-z^2+f_1(z)}, \\
&[\cos f_2(z)]' = -f_2'(z) \sin f_2(z),
\end{aligned}$$

and

$$[\cos f_2(z)]'' = -f_2''(z)\sin f_2(z) - [f_2'(z)]^2 \cos f_2(z).$$

Hence,

$$\begin{aligned}
&\left[e^{-z^2+f_1(z)} \cos g(z) \right]'' = -e^{-z^2+f_1(z)} \left(f_2''(z)\sin f_2(z) + [f_2'(z)]^2 \cos f_2(z) \right) \\
&- 2f_2'(z)\sin f_2(z)[f_1'(z) - 2z] e^{-z^2+f_1(z)} + \cos f_2(z) e^{-z^2+f_1(z)} \left([f_1'(z)]^2 - 4z f_1'(z) + 4z^2 + f_1''(z) - 2 \right).
\end{aligned}$$

The absolute value of the last expression does not exceed

$$e^{t\sum\limits_{k=1}^{m}|a_k|} e^{-z^2} \left(|f_1'|^2 + 4z|f_1'| + |f_1''| + 4z^2 + 2 + 4z|f_2'| + 2|f_1'||f_2'| + |f_2''| + |f_2'|^2 \right).$$

This and estimates (1.13) imply that the absolute value of (1.12) does not exceed

$$\frac{e^{t\sum\limits_{k=1}^{m}|a_k|}}{2\eta^2}\int\limits_0^\infty e^{-z^2}\left[2t\left(\sum_{k=1}^{m}|a_k||b_k|\right)^2+4z\sqrt{t}\sum_{k=1}^{m}|a_k||b_k|+\sum_{k=1}^{m}|a_k|b_k^2+2z^2+1\right]dz$$

$$\leq\frac{M(1+T)e^{T\sum\limits_{k=1}^{m}|a_k|}}{\eta^2},$$

where M depends only on (complex) parameters a and b.

Thus,

$$|J_2|\leq M(1+T)e^{T\sum\limits_{k=1}^{m}|a_k|}\sup|u_0|\int\limits_{\mathbb{R}^{n-1}}e^{-|\xi|^2}d\xi\int\limits_1^\infty\frac{d\eta}{\eta^2}=M(1+T)e^{T\sum\limits_{k=1}^{m}|a_k|}\sup|u_0|\pi^{\frac{n-1}{2}}.$$

In the same way, we estimate $|J_3|$.

Thus, the boundedness of I_1 is proved. Therefore, the boundedness of $u(x,t)$ is proved as well.

Further, Eq. (1.7) can be represented as

$$\frac{\partial u}{\partial t}=\sum_{r=1}^{m+1}L_rP_ru,$$

where $P_{m+1}=\Delta$, $L_{m+1}=I$, $P_r=a_rI$, $r=\overline{1,m}$, and the operators L_r, $r=\overline{1,m}$, act as follows: $L_rg(x)=g(x-b_rh)$. It is known from [2] that no nontrivial bounded solutions of the Cauchy problem for such an equation exist for $u_0(x)\equiv 0$.

Since Eq. (1.7) is linear, it follows that Theorem 1.5.1 is proved. □

Now, consider problem (1.7), (1.4) in the half-space $\mathbb{R}^n\times(0,\infty)$. The function $u(x,t)$ defined by relation (1.11) is a classical solution of the specified problem such that for any t_0 from $(0,+\infty)$ the function $u(x,t)$ is bounded in the layer $\mathbb{R}^n\times[0,t_0]$. Let us show that Theorem 1.5.1 implies the following assertion.

Corollary 1.5.1. *Function* (1.11) *is the unique solution of problem* (1.7)*,* (1.4) *in the half-space* $\mathbb{R}^n\times(0,\infty)$ *bounded in* $\mathbb{R}^n\times[0,t_0]$ *for any positive* t_0.

Proof. Assume the converse: there exist two different solutions $u_1(x,t)$ and $u_2(x,t)$ possessing the above property. Then the function $v(x,t)\stackrel{\text{def}}{=}u_1(x,t)-u_2(x,t)$ is different from the identical zero, is bounded in $\mathbb{R}^n\times[0,t_0]$ for any positive t_0, satisfies Eq. (1.7), and satisfies Eq. (1.4) with the trivial initial-value function. There exists (x^*,t^*) from $\mathbb{R}^n\times(0,+\infty)$ such that $v(x^*,t^*)\neq 0$. Then, denoting t^*+1 by T, we see that for any finite T there exists a bounded solution of problem (1.7), (1.4) with the trivial initial-value function. This contradicts Theorem 1.5.1.

Corollary 1.5.1 is proved. □

Remark 1.5.1. We used the uniqueness of the bounded solution of problem (1.7), (1.4), but the assertion of [2, Theorem 2] is stronger: it defines a wider uniqueness class. Therefore, solution (1.11) is unique in a wider class as well. More exactly, it is the class of functions satisfying the following estimate for any positive T:

$$\sup_{t\in[0,T]}|u(x,t)|\leq Ce^{q|x|\log(|x|+1)}$$

if $q<\dfrac{1}{\max\limits_{1\leq k\leq m}|b_k|}$.

1.6. Asymptotic Properties of Solutions

In this section, we consider Eq. (1.7) in the following form:

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{j=1}^{n}\sum_{k=1}^{m_j} a_{jk} u(x + b_{jk}h_j, t), \tag{1.14}$$

where $h_j \overset{\text{def}}{=} (h_{j1}, \dots, h_{jn})$ are vectors of unit length pairwise orthogonal (for $j = \overline{1,n}$) in $\mathbb{R}^n$ and $a_{jk}, b_{jk} \in \mathbb{R}^1$ for $k = \overline{1, m_j}, j = \overline{1,n}$.

Then function (1.11), which is (by Theorem 1.5.1) the unique solution of problem (1.14), (1.4) in the half-space $\mathbb{R}^n \times (0,T)$ bounded in $\mathbb{R}^n \times [0, t_0]$ for any positive t_0, is represented as follows:

$$\left(\frac{2}{\pi}\right)^n \int_{\mathbb{R}^n} u_0(x - 2\sqrt{t}\eta) \prod_{j=1}^{n} \int_0^\infty e^{-z^2 + t\sum\limits_{k=1}^{m_j} a_{jk}\cos\frac{b_{jk}z}{\sqrt{t}}} \cos\left(2\eta_j z + t\sum_{k=1}^{m_j} a_{jk}\sin\frac{b_{jk}z}{\sqrt{t}}\right) dz d\eta. \tag{1.15}$$

Without loss of generality, assume that the (finite) number sequence $\{a_{jk}\}_{k=1}^{m_j}$, $j = \overline{1,n}$, does not decrease. For any $j \in \overline{1,n}$, denote $\min\limits_{a_{jk}>0} k$ by m_j^0; if j is such that $a_{jk} < 0$ for any $k \in \overline{1, m_j}$, then denote $m_j + 1$ by m_j^0. Denote the differential-difference operator at the right-hand part of Eq. (1.14) by L. Also, consider the operator $\mathcal{L}$ acting as follows:

$$\mathcal{L}u \overset{\text{def}}{=} \Delta u + \sum_{j=1}^{n}\sum_{k<m_j^0} a_{jk} u(x + b_{jk}h_j, t).$$

Denote the operator $\sum\limits_{j=1}^{n}\sum\limits_{k<m_j^0} a_{jk} I - \mathcal{L}$ by R and consider the real part of its symbol (or, which is the same, the symbol of the operator $R + R^*$):

$$\operatorname{Re}R(\xi) = \sum_{j=1}^{n}\sum_{k<m_j^0} a_{jk} + |\xi|^2 - \sum_{j=1}^{n}\sum_{k<m_j^0} a_{jk}\cos b_{jk}\xi_j$$

(see [102, § 8]). We say that $R(\xi)$ is *positive definite* if there exists a positive C such that $\operatorname{Re}R(\xi) \ge C|\xi|^2$ for any ξ from $\mathbb{R}^n$. Similarly to the case of differential operators (see, e.g., [108, p. 66 and p. 78]), any operator R possessing the above property can be called a *second-order operator strong elliptic in the whole space*. Note that, similarly to the case of bounded domains (see [102, § 9]), the strong ellipticity of differential operators differs essentially from the strong ellipticity of differential-difference ones. Therefore, the impact of difference terms is principally important.

The main result of the section is the following theorem.

Theorem 1.6.1. *Let $R(\xi)$ be positive definite. Then, for any x from $\mathbb{R}^n$, we have*

$$\lim_{t\to+\infty}\left[e^{-t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}} u(x,t) - w\left(\frac{x_1 + q_1 t}{p_1}, \dots, \frac{x_n + q_n t}{p_n}, t\right)\right] = 0, \tag{1.16}$$

where $w(x,t)$ is the bounded solution of the Cauchy problem for the heat equation with the initial-value function $u_0(p_1x_1, \dots, p_nx_n)$,

$$p_j = \sqrt{1 + \frac{1}{2}\sum_{k=1}^{m_j} a_{jk}b_{jk}^2}, \qquad \textit{and} \qquad q_j = \sum_{k=1}^{m_j} a_{jk}b_{jk},\ j = \overline{1,n}.$$

Proof. First, we note that $p_1, \dots, p_n$ are well defined and different from zero under the assumptions of the theorem. Take an arbitrary $j \in \overline{1,n}$. From the assumption of the theorem, it follows that

$$\sum_{k<m_j^0} a_{jk} + \xi_j^2 - \sum_{k<m_j^0} a_{jk} \cos b_{jk}\xi_j \geq C\xi_j^2$$

for any positive ξ_j (the positive definiteness condition with $\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_n$ assigned to be equal to zero). This implies that

$$C\xi_j^2 \leq \xi_j^2 + \sum_{k<m_j^0} a_{jk}(1 - \cos b_{jk}\xi_j) = \xi_j^2 + 2\sum_{k<m_j^0} a_{jk} \sin^2 \frac{b_{jk}\xi_j}{2} = \xi_j^2 + \frac{\xi_j^2}{2} \sum_{k<m_j^0} a_{jk} b_{jk}^2 \left(\frac{\sin \frac{b_{jk}\xi_j}{2}}{\frac{b_{jk}\xi_j}{2}}\right)^2.$$

Hence, $1 + \frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 \left(\frac{\sin \frac{b_{jk}\xi_j}{2}}{\frac{b_{jk}\xi_j}{2}}\right)^2 \geq C$ for any $\xi_j > 0$. Therefore, $\frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 > -1$. Indeed, assuming, to the contrary, that $\frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 \leq -1$, we see that the following inequality holds for any positive ξ_j:

$$C \leq 1 + \frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 - \frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 + \frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 \left(\frac{\sin \frac{b_{jk}\xi_j}{2}}{\frac{b_{jk}\xi_j}{2}}\right)^2 \leq \frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 \left[\left(\frac{\sin \frac{b_{jk}\xi_j}{2}}{\frac{b_{jk}\xi_j}{2}}\right)^2 - 1\right].$$

Since we deal with a finite sum, it follows that one can select a small positive ξ_j such that the last expression does not exceed $\frac{C}{2}$. The obtained contradiction proves the positivity of $\frac{1}{2}\sum_{k<m_j^0} a_{jk} b_{jk}^2 + 1$

Therefore, p_j is defined and it is positive. □

Now, we must prove two preliminary assertions.

Lemma 1.6.1. *Let the assumptions of Theorem* 1.6.1 *be satisfied and* $j \in \overline{1,n}$. *Then*

$$\int_0^\infty e^{-z^2 + t\sum_{k=1}^{m_j} a_{jk}\left(\cos \frac{b_{jk}z}{\sqrt{t}} - 1\right)} \cos\left(2z\eta - t\sum_{k=1}^{m_j} a_{jk} \sin \frac{b_{jk}z}{\sqrt{t}}\right) dz - \frac{\sqrt{\pi}}{2p_j} e^{-\frac{(2\eta - q_j\sqrt{t})^2}{4p_j^2}} \overset{t\to\infty}{\longrightarrow} 0$$

uniformly with respect to $\eta \in \mathbb{R}^1$.

Proof. Let $j \in \overline{1,n}$. Redenote m_j by m, m_j^0 by m_0, p_j by p, q_j by q, a_{jk} by a_k, and b_{jk} by b_k; $k = \overline{1,m}$. The relation

$$\frac{\sqrt{\pi}}{2p} e^{-\frac{(2\eta - q\sqrt{t})^2}{4p^2}} = \int_0^\infty e^{-p^2z^2} \cos(2\eta - q\sqrt{t})z dz,$$

holds for any real η and any positive t. Therefore,

$$\int_0^\infty e^{-z^2 + t\sum_{k=1}^{m} a_k\left(\cos \frac{b_k z}{\sqrt{t}} - 1\right)} \cos\left(2z\eta - t\sum_{k=1}^{m} a_k \sin \frac{b_k z}{\sqrt{t}}\right) dz - \frac{\sqrt{\pi}}{2p} e^{-\frac{(2\eta - q\sqrt{t})^2}{4p^2}} = I_1 + I_2,$$

where

$$I_1 = \int_0^\infty e^{-z^2}\left(\cos 2\eta z \left[e^{-2t\sum_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin \frac{b_k z}{\sqrt{t}}\right) - e^{(1-p^2)z^2}\cos(qz\sqrt{t})\right]\right) dz$$

and

$$I_2 = \sin 2\eta z \left[e^{-2t\sum\limits_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}}} \sin\left(t\sum_{k=1}^{m} a_k \sin \frac{b_k z}{\sqrt{t}} \right) - e^{(1-p^2)z^2} \sin(qz\sqrt{t}) \right] dz.$$

Consider the first term:

$$I_1 = \int_0^\infty e^{-z^2} \cos 2\eta z \left[e^{-2t\sum\limits_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}}} \cos\left(t\sum_{k=1}^{m} a_k \sin \frac{b_k z}{\sqrt{t}} \right) - e^{(1-p^2)z^2} \cos(qz\sqrt{t}) \right] dz$$

$$= \int_0^\delta + \int_\delta^\infty \stackrel{\text{def}}{=} I_{3,\delta} + I_{4,\delta}.$$

Take an arbitrary positive ε. First, we estimate $I_{4,\delta}$.

The absolute value of the second term of its integrand does not exceed $e^{-p^2z^2}$. Consider its first term:

$$2t\sum_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}} = 2t\sum_{k=1}^{m} a_k \frac{\sin^2 \frac{b_k z}{2\sqrt{t}}}{\frac{b_k^2 z^2}{4t}} \frac{b_k^2 z^2}{4t} = \frac{z^2}{2}\sum_{k=1}^{m} a_k b_k^2 \left(\frac{\sin \frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}} \right)^2.$$

Therefore, the absolute value of the specified term of the integrand does not exceed

$$e^{-z^2 - \frac{z^2}{2}\sum\limits_{k=1}^{m} a_k b_k^2 \left(\frac{\sin \frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}} \right)^2}.$$

By virtue of the assumption of Theorem 1.6.1, the inequality

$$-\frac{1}{2}\sum_{k<m_0} a_k b_k^2 < 1$$

holds. Then

$$-\frac{1}{2}\sum_{k<m_0} a_k b_k^2 \left(\frac{\sin \frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}} \right)^2 < 1$$

because any a_k of the last sum is negative. Therefore, the power of the last exponential function can be represented as

$$-z^2 - \frac{z^2}{2}\sum_{k<m_0} a_k b_k^2 \left(\frac{\sin \frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}} \right)^2 - \frac{z^2}{2}\sum_{k\ge m_0} a_k b_k^2 \left(\frac{\sin \frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}} \right)^2 < -\frac{z^2}{2}\left[1 + \sum_{k\ge m_0} a_k b_k^2 \left(\frac{\sin \frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}} \right)^2 \right].$$

Thus, the absolute value of the last integrand does not exceed $2e^{-\gamma z^2}$, where

$$\gamma = \min\left(p^2, 1 + \frac{1}{2}\inf_{\substack{z>0\\ t>0}} \sum_{k\ge m_0} a_k b_k^2 \left(\frac{\sin \frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}} \right)^2 \right) = \min(p^2, 1)$$

by virtue of the positivity of any a_k of the last sum. Hence, $\gamma > 0$. Therefore, there exists a positive δ such that $\left| I_{4,\delta} \right| < \dfrac{\varepsilon}{4}$. Fix that δ and estimate $I_{3,\delta}$. The third factor of its integrand is equal to

$$e^{-2t\sum\limits_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}}} \left[\cos\left(t\sum_{k=1}^{m} a_k \sin \frac{b_k z}{\sqrt{t}} \right) - \cos(qz\sqrt{t}) \right] + \cos(qz\sqrt{t}) \left[e^{-2t\sum\limits_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}}} - e^{(1-p^2)z^2} \right]. \tag{1.17}$$

Let us estimate the second term of sum (1.17):

$$e^{-2t\sum\limits_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}}} - e^{(1-p^2)z^2} = e^{-2t\sum\limits_{k=1}^{m} a_k \sin^2 \frac{b_k z}{2\sqrt{t}}} - e^{-\frac{z^2}{2}\sum\limits_{k=1}^{m} a_k b_k^2}$$

$$= e^{-\frac{z^2}{2}\sum\limits_{k=1}^{m} a_k b_k^2}\left[e^{\sum\limits_{k=1}^{m} a_k\left(\frac{b_k^2 z^2}{2}-2t\sin^2\frac{b_k z}{2\sqrt{t}}\right)}-1\right];$$

$$\frac{b_k^2 z^2}{2}-2t\sin^2\frac{b_k z}{2\sqrt{t}}=\frac{b_k^2 z^2}{2}-\frac{b_k^2 z^2}{2}\left(\frac{2\sqrt{t}}{b_k z}\right)^2\sin^2\frac{b_k z}{2\sqrt{t}}$$
$$=\frac{b_k^2 z^2}{2}\left[1-\left(\frac{\sin\frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}}\right)^2\right]=\frac{b_k^2 z^2}{2}\left(1+\frac{\sin\frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}}\right)\left(1-\frac{\sin\frac{b_k z}{2\sqrt{t}}}{\frac{b_k z}{2\sqrt{t}}}\right).$$

Form any positive ε_1 and any $k=\overline{1,m}$ there exists a positive $\delta_{1,k}$ such that for any $x\in(-\delta_{1,k},\delta_{1,k})$ the inequality $\left|\frac{\sin x}{x}-1\right|<\frac{\varepsilon_1}{m|a_k|b_k^2\delta^2}$ holds. On the other hand, $\left|\frac{\sin x}{x}+1\right|<3$ for any x. Hence,

$$|a_k|\left|\frac{b_k^2 z^2}{2}-2t\sin^2\frac{b_k z}{2\sqrt{t}}\right|<\frac{3\varepsilon_1}{2m}$$

for any $t>\left(\frac{b_k\delta}{2\delta_{1,k}}\right)^2$ and any $z\in[0,\delta]$. Select a small ε_1 such that

$$e^{\frac{3\varepsilon_1}{2}}, e^{-\frac{3\varepsilon_1}{2}}\in\left(1-\frac{\varepsilon e^{-\delta^2}}{4\sqrt{\pi}}, 1+\frac{\varepsilon e^{-\delta^2}}{4\sqrt{\pi}}\right).$$

Then

$$e^{-\frac{z^2}{2}\sum\limits_{k=1}^{m} a_k b_k^2}=e^{-\frac{z^2}{2}\sum\limits_{k<m_0} a_k b_k^2-\frac{z^2}{2}\sum\limits_{k\geq m_0} a_k b_k^2}\leq e^{-\frac{z^2}{2}\sum\limits_{k<m_0} a_k b_k^2}\leq e^{z^2}\leq e^{\delta^2}$$

for any $z\in[0,\delta]$. Hence, for the specified z and for any $t>\max\limits_{1\leq k\leq m}\left(\frac{b_k\delta}{2\delta_{1,k}}\right)^2$, we have

$$\left|\cos(qz\sqrt{t})\left[e^{-2t\sum\limits_{k=1}^{m} a_k\sin^2\frac{b_k z}{2\sqrt{t}}}-e^{(1-p^2)z^2}\right]\right|<\frac{\varepsilon}{4\sqrt{\pi}}=\frac{\varepsilon}{8}\left(\int\limits_0^\infty e^{-z^2}dz\right)^{-1}. \tag{1.18}$$

Now, estimate the first term of (1.17).

$$\cos\left(t\sum_{k=1}^{m} a_k\sin\frac{b_k z}{\sqrt{t}}\right)-\cos(qz\sqrt{t})=\cos\left(q\sqrt{t}z+t\sum_{k=1}^{m} a_k\sin\frac{b_k z}{\sqrt{t}}-q\sqrt{t}z\right)-\cos(qz\sqrt{t})$$
$$=\cos(q\sqrt{t}z)\left[\cos\left(t\sum_{k=1}^{m} a_k\sin\frac{b_k z}{\sqrt{t}}-q\sqrt{t}z\right)-1\right]$$
$$-\sin(q\sqrt{t}z)\sin\left(t\sum_{k=1}^{m} a_k\sin\frac{b_k z}{\sqrt{t}}-q\sqrt{t}z\right). \tag{1.19}$$

$$t\sum_{k=1}^{m} a_k\sin\frac{b_k z}{\sqrt{t}}-q\sqrt{t}z=\sum_{k=1}^{m} a_k\left(t\sin\frac{b_k z}{\sqrt{t}}-b_k\sqrt{t}z\right)$$
$$=\sum_{k=1}^{m} a_k b_k\sqrt{t}z\left(\frac{\sqrt{t}}{b_k z}\sin\frac{b_k z}{\sqrt{t}}-1\right)=\sum_{k=1}^{m} a_k(b_k z)^2\frac{\sqrt{t}}{b_k z}\left(\frac{\sin\frac{b_k z}{\sqrt{t}}}{\frac{b_k z}{\sqrt{t}}}-1\right).$$

$$\lim_{x\to 0}\frac{1}{x}\left(\frac{\sin x}{x}-1\right)=\lim_{x\to 0}\frac{\sin x-x}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{2x}=\lim_{x\to 0}\frac{-\sin x}{2}=0;$$

therefore, for any positive ε_1 and any $k = \overline{1,m}$ there exists a positive T_k such that

$$\left|\frac{\sqrt{t}}{b_k z}\left(\frac{\sin\frac{b_k z}{\sqrt{t}}}{\frac{b_k z}{\sqrt{t}}} - 1\right)\right| < \frac{\varepsilon_1}{m|a_k|b_k^2\delta^2}$$

for any $t > T_k$ and any $z \in [0,\delta]$. Therefore, for any $t > \max\limits_{1\le k\le m} T_k$, we have

$$\left|\sin(q\sqrt{t}z)\sin\left(t\sum_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}} - q\sqrt{t}z\right)\right| < |\sin\varepsilon_1|.$$

Further,

$$\left|\cos(q\sqrt{t}z)\left[\cos\left(t\sum_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}} - q\sqrt{t}z\right) - 1\right]\right| = \left| - 2\cos(q\sqrt{t}z)\sin^2\frac{t\sum\limits_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}} - q\sqrt{t}z}{2}\right|$$

$$\le 2\sin^2\frac{t\sum\limits_{k=1}^m a_k \sin\frac{b_k z}{\sqrt{t}} - q\sqrt{t}z}{2}.$$

As above, one can select a large t such that the last expression is less than $2\sin^2\dfrac{\varepsilon_1}{2}$ for any $z \in [0,\delta]$. Thus, selecting a small ε_1 such that the inequality $|\sin\varepsilon_1| + 2\sin^2\dfrac{\varepsilon_1}{2} < \dfrac{\varepsilon e^{-\delta^2}}{4\sqrt{\pi}}$ holds and taking into account the inequality

$$e^{-2t\sum\limits_{k=1}^m a_k \sin^2\frac{b_k z}{2\sqrt{t}}} \le e^{-2t\sum\limits_{k<m_0} a_k \sin^2\frac{b_k z}{2\sqrt{t}}} \le e^{-\frac{z^2}{2}\sum\limits_{k<m_0} a_k b_k^2} \le e^{z^2},$$

we see that the absolute value of expression (1.17) does not exceed $\dfrac{\varepsilon}{4}\left(\int\limits_0^\infty e^{-z^2}dz\right)^{-1}$ provided that t is sufficiently large. Hence, there exists a positive T such that if $t > T$, then $\left|I_{3,\delta}\right| < \dfrac{\varepsilon}{4}$, i.e., $\left|I_1\right| < \dfrac{\varepsilon}{2}$.

The term I_2 is estimated in the same way:

$$e^{-2t\sum\limits_{k=1}^m a_k \sin^2\frac{b_k z}{2\sqrt{t}}}\sin\left(t\sum_{k=1}^m a_k\sin\frac{b_k z}{\sqrt{t}}\right) - e^{(1-p^2)z^2}\sin(qz\sqrt{t})$$

$$= \left[e^{-2t\sum\limits_{k=1}^m a_k \sin^2\frac{b_k z}{2\sqrt{t}}} - e^{(1-p^2)z^2}\right]\sin(qz\sqrt{t})$$

$$+ e^{-2t\sum\limits_{k=1}^m a_k \sin^2\frac{b_k z}{2\sqrt{t}}}\left[\sin\left(t\sum_{k=1}^m a_k\sin\frac{b_k z}{\sqrt{t}}\right) - \sin(qz\sqrt{t})\right].$$

The first of those terms is estimated in the same way as (1.18). It remains to estimate the second one:

$$\sin\left(t\sum_{k=1}^m a_k\sin\frac{b_k z}{\sqrt{t}}\right) - \sin(qz\sqrt{t}) = \sin\left(qz\sqrt{t} + t\sum_{k=1}^m a_k\sin\frac{b_k z}{\sqrt{t}} - qz\sqrt{t}\right) - \sin(qz\sqrt{t})$$

$$= \sin(qz\sqrt{t})\left[\cos\left(t\sum_{k=1}^m a_k\sin\frac{b_k z}{\sqrt{t}} - qz\sqrt{t}\right) - 1\right] + \cos(qz\sqrt{t})\sin\left(t\sum_{k=1}^m a_k\sin\frac{hz}{\sqrt{t}} - qz\sqrt{t}\right);$$

this expression is estimated in the same way as (1.19). Thus, there exists a positive T such that $\left|I_2\right| < \dfrac{\varepsilon}{2}$ for any $t > T$.

This completes the proof of Lemma 1.6.1. □

Lemma 1.6.2. *Under the assumptions of Theorem* 1.6.1, *for any* $j \in \overline{1,n}$ *there exists* M_j *depending only on the coefficients* $a_{j1}, \dots, a_{jm_j}, b_{j1}, \dots, b_{jm_j}$ *and such that*

$$\left| \int_0^\infty e^{-z^2+t\sum\limits_{k=1}^{m_j} a_{jk}\left(\cos\frac{b_{jk}z}{\sqrt{t}}-1\right)} \cos\left(yz - q_j\sqrt{t}z + t\sum_{k=1}^{m_j} a_{jk}\sin\frac{b_{jk}z}{\sqrt{t}}\right) dz \right| \le \frac{M_j}{y^2}$$

for any $y \in (0,+\infty)$ *and* $t \in [1,+\infty)$.

Proof. Take an arbitrary $j \in \overline{1,n}$ and redenote m_j by m, m_j^0 by m_0, p_j by p, q_j by q, a_{jk} by a_k, and b_{jk} by b_k; $k = \overline{1,m}$. It suffices to estimate one term of the last integral (the second one is estimated in the same way). Let us estimate

$$\int_0^\infty e^{-z^2+t\sum\limits_{k=1}^{m} a_k\left(\cos\frac{b_kz}{\sqrt{t}}-1\right)} \cos yz \cos\left(q\sqrt{t}z - t\sum_{k=1}^{m} a_k\sin\frac{b_kz}{\sqrt{t}}\right) dz$$

or, which is equivalent,

$$\int_0^\infty e^{-z^2+\frac{1}{t^2}\sum\limits_{k=1}^{m} a_k(\cos b_kzt-1)} \cos\left(\frac{qz}{t} - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right) \cos yz dz.$$

Integrating by parts two times, we obtain that the last integral is equal to $-\frac{1}{y^2}\int\limits_0^\infty g''(z)\cos yz dz$, where

$$g(z) = e^{-z^2+\frac{1}{t^2}\sum\limits_{k=1}^{m} a_k(\cos b_kzt-1)} \cos\left(\frac{qz}{t} - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right)$$

(it is easy to check that the integrated terms vanish). Therefore, it suffices to show that for arbitrary fixed values of the (vector) parameters a and b satisfying the assumptions of Theorem 1.6.1, the last integral is bounded uniformly with respect to $t > 0$.

We have

$$g'(z) = \frac{1}{t}e^{-z^2+\frac{1}{t^2}\sum\limits_{k=1}^{m} a_k(\cos b_kzt-1)} \left(\sum_{k=1}^{m} a_kb_k\left[\sin\left(\frac{qz}{t} - b_kzt - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right)\right.\right.$$
$$\left.\left. - \sin\left(\frac{qz}{t} - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right)\right] - 2zt\cos\left(\frac{qz}{t} - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right)\right)$$

and

$$g''(z) = \frac{1}{t}e^{-z^2+\frac{1}{t^2}\sum\limits_{k=1}^{m} a_k(\cos b_kzt-1)} \left(-2z - \frac{1}{t}\sum_{k=1}^{m} a_kb_k\sin b_kzt\right)$$
$$\times\left(\sum_{k=1}^{m} a_kb_k\left[\sin\left(\frac{qz}{t} - b_kzt - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right) - \sin\left(\frac{qz}{t} - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right)\right]\right.$$
$$\left. - 2zt\cos\left(\frac{qz}{t} - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right)\right) + \frac{1}{t}e^{-z^2+\frac{1}{t^2}\sum\limits_{k=1}^{m} a_k(\cos b_kzt-1)}$$
$$\times\left(\sum_{k=1}^{m} a_kb_k\left[\left(\frac{q}{t} - b_kt - \frac{1}{t}\sum_{k=1}^{m} a_kb_k\cos b_kzt\right)\cos\left(\frac{qz}{t} - b_kzt - \frac{1}{t^2}\sum_{k=1}^{m} a_k\sin b_kzt\right)\right.\right.$$

$$- \left(\frac{q}{t} - \frac{1}{t} \sum_{k=1}^{m} a_k b_k \times \cos b_k zt \right) \cos \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) \Bigg]$$

$$- 2t \cos \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right)$$

$$+ 2tz \sin \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) \left(\frac{q}{t} - \frac{1}{t} \sum_{k=1}^{m} a_k b_k \times \cos b_k zt \right) \Bigg) .$$

Note that

$$e^{\frac{1}{t^2} \sum\limits_{k=1}^{m} a_k (\cos b_k zt - 1)} = e^{\frac{1}{t^2} \sum\limits_{k<m_0} a_k (\cos b_k zt - 1) + \frac{1}{t^2} \sum\limits_{k \geq m_0} a_k (\cos b_k zt - 1)} \leq e^{\frac{1}{t^2} \sum\limits_{k<m_0} a_k (\cos b_k zt - 1)} \leq e^{-2 \sum\limits_{k<m_0} a_k} .$$

Further, the terms

$$\int_0^\infty e^{-z^2} z^2 \cos \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) dz,$$

$$\int_0^\infty e^{-z^2} \cos \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) dz,$$

and

$$\int_0^\infty e^{-z^2} \sum_{k=1}^{m} a_k b_k^2 \cos \left(\frac{qz}{t} - b_k zt - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) dz$$

are bounded uniformly with respect to $t > 0$. Thus, it suffices to estimate the integral

$$\int_0^\infty e^{-z^2} |\Psi(z;t)| dz,$$

where the function $\Psi(z;t)$ is a sum of terms of the form

$$\left(\frac{2z}{t} + \frac{1}{t^2} \sum_{k=1}^{m} a_k b_k \sin b_k zt \right) \sum_{k=1}^{m} a_k b_k \left[\sin \left(\frac{qz}{t} - b_k tz - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) \right.$$

$$\left. - \sin \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) \right],$$

$$\sum_{k=1}^{m} a_k b_k \left[\left(\frac{1}{t^2} \sum_{k=1}^{m} a_k b_k \cos b_k zt - \frac{q}{t^2} \right) \cos \left(\frac{qz}{t} - b_k tz - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) \right],$$

and

$$\left(\frac{q}{t^2} - \frac{1}{t^2} \sum_{k=1}^{m} a_k b_k \cos b_k zt \right) \cos \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right)$$

$$-2z \sin \left(\frac{qz}{t} - \frac{1}{t^2} \sum_{k=1}^{m} a_k \sin b_k zt \right) \frac{1}{t} \left(q - \sum_{k=1}^{m} a_k b_k \cos b_k zt \right) .$$

We have

$$q - \sum_{k=1}^{m} a_k b_k \cos b_k zt = \sum_{k=1}^{m} a_k b_k (1 - \cos b_k zt) = 2 \sum_{k=1}^{m} a_k b_k \sin^2 \frac{b_k tz}{2}$$

and

$$\frac{1}{t^2}\int\limits_0^\infty e^{-z^2}\sin^2\frac{b_ktz}{2}\,dz=\frac{b_k^2}{4}\int\limits_0^\infty e^{-z^2}\left(\frac{\sin^2\frac{b_ktz}{2}}{\frac{b_ktz}{2}}\right)^2 z^2dz\le\frac{b_k^2}{4}\int\limits_0^\infty z^2e^{-z^2}dz,$$

which bounded uniformly with respect to $t>0$.

Also, we have

$$\int\limits_0^\infty \frac{z}{t}e^{-z^2}\sin^2\frac{b_ktz}{2}\,dz\le\int\limits_0^\infty ze^{-z^2}\frac{\left|\sin\frac{b_ktz}{2}\right|}{t}dz=\frac{|b_k|}{2}\int\limits_0^\infty z^2e^{-z^2}\left|\frac{\sin\frac{b_ktz}{2}}{\frac{b_ktz}{2}}\right|dz\le\frac{|b_k|}{2}\int\limits_0^\infty z^2e^{-z^2}dz,$$

which is bounded uniformly with respect to $t>0$.

Thus, it suffices to estimate the integral

$$\int\limits_0^\infty e^{-z^2}\left|\left(\frac{2z}{t}+\frac{1}{t^2}\sum_{k=1}^m a_kb_k\sin b_kzt\right)\sum_{k=1}^m a_kb_k\left[\sin\left(\frac{qz}{t}-\frac{1}{t^2}\sum_{k=1}^m a_kb_k\sin b_kzt\right)\right.\right.$$
$$\left.\left.-\sin\left(\frac{qz}{t}-\frac{1}{t^2}\sum_{k=1}^m a_kb_k\sin b_kzt-b_ktz\right)\right]\right|dz.$$

The difference of the sines in the last integral is equal to

$$2\sin^2\frac{b_ktz}{2}\sin\left(\frac{qz}{t}-\frac{1}{t^2}\sum_{k=1}^m a_kb_k\sin b_kzt\right)+\sin b_ktz\cos\left(\frac{qz}{t}-\frac{1}{t^2}\sum_{k=1}^m a_kb_k\sin b_kzt\right);$$

therefore, it suffices to estimate the integrals of

$$e^{-z^2}\frac{z}{t}\sin^2\frac{b_ktz}{2},\;\; e^{-z^2}\frac{|\sin b_ktz|}{t^2}\sin^2\frac{b_ktz}{2},\;\; e^{-z^2}\frac{|\sin b_ktz|}{t^2}|\sin b_ktz|,\;\text{ and } e^{-z^2}\frac{z}{t}|\sin b_ktz|.$$

The initial three integrals are reduced to integrals estimated above, while the last one is equal to $|b_k|\int\limits_0^\infty z^2e^{-z^2}\left|\frac{\sin b_ktz}{b_ktz}\right|dz$, i.e., it does not exceed $|b_k|\int\limits_0^\infty z^2e^{-z^2}dz$; hence, it is bounded uniformly with respect to $t>0$.

This completes the proof of Lemma 1.6.2. □

Now, we can pass directly to the proof of Theorem 1.6.1.

Let $x_0=(x_1^0,\dots,x_n^0)$ be an arbitrary point of $\mathbb{R}^n$. Then, using the integral representation for the solution of the Cauchy problem for the heat equation, we have the relation

$$w\left(\frac{x_1^0+q_1t}{p_1},\dots,\frac{x_n^0+q_nt}{p_n},t\right)=\frac{1}{\pi^{\frac{n}{2}}\prod\limits_{j=1}^n p_j}\int\limits_{\mathbb{R}^n}e^{-\sum\limits_{j=1}^n\frac{(2\eta_j+q_j\sqrt{t})^2}{4p_j^2}}u_0(x_1^0-2\eta_1\sqrt{t},\dots,x_n^0-2\eta_n\sqrt{t})d\eta.$$

Hence, the difference

$$e^{-t\sum\limits_{j=1}^n\sum\limits_{k=1}^{m_j}a_{jk}}u(x_0,t)-w\left(\frac{x_1^0+q_1t}{p_1},\dots,\frac{x_n^0+q_nt}{p_n},t\right)$$

can be represented as

$$\frac{1}{\pi^{\frac{n}{2}}}\int\limits_{\mathbb{R}^n} u_0(x_1^0-2\eta_1\sqrt{t},\dots,x_n^0-2\eta_n\sqrt{t})$$

$$\times\left[\prod_{j=1}^{n}\frac{2}{\sqrt{\pi}}\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^{m_j}a_{jk}\left(\cos\frac{b_{jk}z}{\sqrt{t}}-1\right)}\cos\left(2\eta_j z+t\sum_{k=1}^{m_j}a_{jk}\sin\frac{b_{jk}z}{\sqrt{t}}\right)dz-\frac{1}{\prod\limits_{j=1}^{n}p_j}e^{-\sum\limits_{j=1}^{n}\frac{(2\eta_j+q_j\sqrt{t})^2}{4p_j^2}}\right]d\eta.$$

Change the variables: $y_j=2\eta_j+q_j\sqrt{t},\ j=\overline{1,n}$. This reduces the last expression to the form

$$\frac{1}{\pi^n}\int\limits_{\mathbb{R}^n} u_0(x_1^0+q_1t-y_1\sqrt{t},\dots,x_n^0+q_nt-y_n\sqrt{t})\left[\prod_{j=1}^{n}\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^{m_j}a_{jk}\left(\cos\frac{b_{jk}z}{\sqrt{t}}-1\right)}\right.$$

$$\left.\times\cos\left(y_jz-q_j\sqrt{t}z+t\sum_{k=1}^{m_j}a_{jk}\sin\frac{b_{jk}z}{\sqrt{t}}\right)dz-\prod_{j=1}^{n}\frac{\sqrt{\pi}}{2p_j}e^{-\frac{y_j^2}{4p_j^2}}\right]dy, \tag{1.20}$$

which can be represented as a sum

$$\frac{1}{\pi^n}\left(\int\limits_{Q(A)}+\int\limits_{\mathbb{R}^n\setminus Q(A)}\right)\stackrel{\text{def}}{=}J_1+J_2,$$

where A is a positive parameter.

Let $\varepsilon>0$. Each internal (one-dimensional) integral in (1.20) is a bounded function of y_j and t. Indeed, the power of the exponent in the integrand does not exceed

$$-z^2+t\sum_{k<m_0}a_{jk}\left(\cos\frac{b_{jk}z}{\sqrt{t}}-1\right)$$

$$=-z^2+t\sum_{k<m_0}a_{jk}\left(\frac{\sin\frac{b_{jk}z}{2\sqrt{t}}}{\frac{b_{jk}z}{2\sqrt{t}}}\right)^2\left(\frac{b_{jk}z}{2\sqrt{t}}\right)^2\stackrel{\text{def}}{=}-z^2\left[1+\frac{1}{2}\sum_{k<m_0}a_{jk}b_{jk}^2\left(\frac{\sin\alpha_{z,t}}{\alpha_{z,t}}\right)^2\right].$$

All a_{jk} in the last sum are negative; therefore,

$$-z^2+t\sum_{k<m_0}a_{jk}\left(\cos\frac{b_{jk}z}{\sqrt{t}}-1\right)\le -z^2\left(1+\frac{1}{2}\sum_{k<m_0}a_{jk}b_{jk}^2\right)\stackrel{\text{def}}{=}-\gamma z^2,$$

where $\gamma>0$ by virtue of the assumption of the theorem. Then Lemma 1.6.2 implies that the absolute value of each specified (one-dimensional) integral is bounded from above by the function $g_j(\eta_j)\stackrel{\text{def}}{=}\frac{M_j}{1+\eta_j^2}$, where M_j is a positive constant. Now, using the boundedness of the function u_0, select A such that $J_2<\frac{\varepsilon}{2}$ for any t from $[1,+\infty)$. Fix the selected A and consider J_1. By virtue of Lemma 1.6.1 and the boundedness of the internal integrals of expression (1.20), the difference in the square brackets of expression (1.20) tends to zero as $t\to\infty$ uniformly with respect to $y\in\mathbb{R}^n$. Indeed, by virtue of Lemma 1.6.1, there exists a positive T such that for any $t\in(T,+\infty)$, any $j\in\overline{1,n}$, and any $\eta_j\in(-\infty,+\infty)$, we have

$$\left|\int\limits_0^\infty e^{-z^2+t\sum\limits_{k=1}^{m_j}a_{jk}\left(\cos\frac{b_{jk}z}{\sqrt{t}}-1\right)}\cos\left(2\eta_jz+t\sum_{k=1}^{m_j}a_{jk}\sin\frac{b_{jk}z}{\sqrt{t}}\right)dz-\frac{\sqrt{\pi}}{2p_j}e^{-\frac{(2\eta_j+q_j\sqrt{t})^2}{4p_j^2}}\right|<\frac{\varepsilon\pi^n}{2^{n+1}A^n\sup|u_0|}$$

(note that no assumptions for the signs of the coefficients b_{jk} are imposed in Lemma 1.6.1). Hence, the last inequality holds if we take any real y_j and assign $\eta_j = \dfrac{y_j - q_j\sqrt{t}}{2}$. Therefore, for any t from $(T, +\infty)$, we have the inequality

$$\left|\prod_{j=1}^{n}\int_0^{\infty} e^{-z^2+t\sum\limits_{k=1}^{m_j} a_{jk}\left(\cos\frac{b_{jk}z}{\sqrt{t}}-1\right)}\cos\left(y_j z - q_j\sqrt{t}z + t\sum_{k=1}^{m_j} a_{jk}\sin\frac{b_{jk}z}{\sqrt{t}}\right)dz - \prod_{j=1}^{n}\frac{\sqrt{\pi}}{2p_j}e^{-\frac{y_j^2}{2p_j^2}}\right|$$
$$\le \frac{\varepsilon\pi^n}{2^{n+1}A^n \sup|u_0|}.$$

This completes the proof of the theorem because x_0 is chosen arbitrarily.

Remark 1.6.1. The exponential weight arising in the obtained closeness theorem for solutions is caused not by the presence of difference terms in the equation but by the dissipativity of the problem. The specified weight arises in the classical case as well: if all the coefficients b_{jk} vanish, then the limit relation (1.16) becomes the identity (for any t). Once we add low-order (more exactly, zero-order) terms to a parabolic equation, the solution leaves the class of bounded functions (even if the initial-value function is bounded), but, multiplying it by the corresponding exponential (with respect to t) weight, we return the solution to the specified class.

Note that closeness theorems for solutions are, in general, stronger than stabilization theorems. Thus, Theorem 1.6.1 establishes a more general type of behavior of the solution as $t \to \infty$ than the stabilization. However, it is worth showing an important special case where the classical pointwise stabilization of the solution takes place: this is the case where the operator L is symmetric. In the specified case, we can apply the following Repnikov–Eidelman result (see [95]): the stabilization of the Cauchy problem solution for the heat equation (denote this solution by $v(x,t)$) takes place if and only if the following limit relation holds for the bounded initial-value function (denoted by $v_0(x)$ here):

$$\lim_{t\to\infty}\frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}t^n}\int\limits_{|x|<t} v_0(x)dx = l, \tag{1.21}$$

where l is a real constant.

This implies the following assertion.

Corollary 1.6.1. *Let the assumptions of Theorem* 1.6.1 *be satisfied,* L *be symmetric, and* $l \in \mathbb{R}^1$. *Then*

$$\lim_{t\to\infty} e^{-t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}} u(x,t) = l \textit{ for any } x \in \mathbb{R}^n$$

if and only if

$$\lim_{t\to\infty}\frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}t^n\prod\limits_{j=1}^{n}p_j}\int\limits_{\sum\limits_{j=1}^{n}\frac{x_j^2}{p_j^2}<t^2} u_0(x)dx = l.$$

To prove this, it suffices to note that, since the operator L is symmetric, it follows that it can be represented (see [102, Lemma 8.2]) as follows: $Lu = \Delta u + \sum\limits_{h\in\mathcal{M}} a_h u(x-h,t)$, where $\mathcal{M}$ is a finite set of vectors from $\mathbb{R}^n$ such that for any h belonging to $\mathcal{M}$, the vector $-h$ belongs to $\mathcal{M}$ as well and $a_h = a_{-h}$ for any h from $\mathcal{M}$. This implies that $q_1 = \cdots = q_n = 0$. It remains to apply the specified stabilization theorem from [95] to the function $w(x,t)$.

Remark 1.6.2. It follows from Corollary 1.6.1 that surfaces bounding the averaging domains of the initial-value function are not spheres in the differential-difference case: they become ellipsoids. Recall that if we deal with the classical case of differential equations, then such an effect arises if the Laplace operator is replaced by an elliptic operator with different coefficients at different second variables: $\sum\limits_{j=1}^{n} p_j^2 \frac{\partial^2}{\partial x_j^2}$.

Remark 1.6.3. In Corollary 1.6.1, the symmetry assumption for the operator L can be weakened as follows: we replace it by the assumption that $a_j \perp b_j$ for any $j \in \overline{1,n}$, where $a_j = (a_{j1}, \dots, a_{jm_j})$ and $b_j = (b_{j1}, \dots, b_{jm_j})$.

It is known from [26] that if the function $v_0(x)$ satisfies the condition

$$\lim_{t\to\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} t^n} \int\limits_{|x|<t} v_0(x+y)dx = l \text{ uniformly with respect to } y \in \mathbb{R}^n$$

(this condition is stronger than (1.21)), then the uniform stabilization of the function $v(x,t)$ takes place. This implies the following assertion.

Corollary 1.6.2. *Let*

$$\lim_{t\to\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} t^n \prod\limits_{j=1}^{n} p_j} \int\limits_{\sum\limits_{j=1}^{n} \frac{x_j^2}{p_j^2} < t^2} u_0(x+y)dx = l$$

uniformly with respect to $y \in \mathbb{R}^n$ *and the assumption of Theorem* 1.6.1 *be satisfied. Then*

$$\lim_{t\to\infty} e^{-t \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{m_j} a_{jk}} u(x,t) = l$$

for any $x \in \mathbb{R}^n$.

1.7. The Sense of the Positive Definiteness Condition

The positive definiteness condition imposed on the auxiliary operator in Theorem 1.6.1 (as well as the introduction of the specified auxiliary operator itself) looks rather artificial. Let us show that it has a substantial sense.

As a prototype, consider the problem

$$\frac{\partial u}{\partial t} = Lu \overset{\text{def}}{=} \frac{\partial^2 u}{\partial x^2} + au(x+b,t), \ x \in \mathbb{R}^1, t > 0; \tag{1.22}$$

$$u\Big|_{t=0} = u_0(x), \ x \in \mathbb{R}^1 \tag{1.23}$$

(the coefficients a and b are supposed to be real and the initial-value function u_0 is supposed to be continuous and bounded) and consider the positive definiteness condition for the auxiliary operator, providing the validity of the theorem on the (weighted) asymptotic closeness (stabilization) of solutions.

Then the operator $\mathcal{L}$ and the operator R obeying the positive definiteness condition act as follows:

$$\mathcal{L}u = \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + au(x+b,t), \ a < 0, \\ \dfrac{\partial^2 u}{\partial x^2}, \ a \geq 0, \end{cases}$$

and

$$Ru = \begin{cases} au - \dfrac{\partial^2 u}{\partial x^2} - au(x+b,t), \ a < 0, \\ -\dfrac{\partial^2 u}{\partial x^2}, \ a \geq 0. \end{cases}$$

Therefore,

$$\mathrm{Re}R(\xi) = \begin{cases} \xi^2, \ a < 0, \\ a + \xi^2 - a\cos b\xi, \ a \geq 0. \end{cases}$$

Thus, the positive definiteness condition for the operator R is satisfied for any nonnegative a; for any negative a, the specified condition is equivalent to the existence of a positive C such that the inequality $\xi^2 + a(1 - \cos b\,\xi) \geq C\xi^2$ holds for any real ξ. The last inequality is reduced to the form $\xi^2 + 2a\sin^2 \frac{b\xi}{2} \geq C\xi^2$. In the sequel, we assume that $a < 0$. Let $\frac{ab^2}{2} > -1$. Denote the positive constant $1 + \frac{ab^2}{2}$ by C. Taking into account that $\sin^2 \frac{b\xi}{2} \leq \frac{b^2\xi^2}{4}$ and the coefficient a is nonnegative, we obtain the inequality $2a\sin^2 \frac{b\xi}{2} \geq \frac{ab^2\xi^2}{2}$; therefore,

$$\xi^2 + 2a\sin^2 \frac{b\xi}{2} \geq \xi^2 \left(1 + \frac{ab^2}{2}\right) = C\xi^2.$$

Hence, the condition $\frac{ab^2}{2} > -1$ implies the positive definiteness condition for the operator R.

Now, assume that the constant $1 + \frac{ab^2}{2}$ is nonpositive and prove that the positive definiteness condition is not valid for the operator R. To do this, we represent $\xi^2 + 2a\sin^2 \frac{b\xi}{2}$ as

$$\xi^2 \left[1 + \frac{ab^2}{2}\left(\frac{\sin \frac{b\xi}{2}}{\frac{b\xi}{2}}\right)^2\right]$$

(on $\mathbb{R}^1 \backslash \{0\}$) and assume the inverse, i.e., that the operator R is positive definite. Then there exists a positive definite C such that

$$\xi^2 \left[1 + \frac{ab^2}{2}\left(\frac{\sin \frac{b\xi}{2}}{\frac{b\xi}{2}}\right)^2\right] \geq C\xi^2 \text{ for any } \xi \neq 0,$$

i.e.,

$$1 + \frac{ab^2}{2}\left(\frac{\sin \frac{b\xi}{2}}{\frac{b\xi}{2}}\right)^2 \geq C \text{ for any } \xi \neq 0.$$

Since the function $g(\xi) \stackrel{\text{def}}{=} 1 + \frac{ab^2}{2}\left(\frac{\sin \frac{b\xi}{2}}{\frac{b\xi}{2}}\right)^2$ tends to $1 + \frac{ab^2}{2}$ as $\xi \to 0$, it follows that there exists a positive ξ_0 such that $g(\xi) < \frac{C}{2}$ for any $\xi \in (0, \xi_0)$. We arrive at a contradiction, which proves that the assumption about the positive definiteness of the operator R is wrong.

Thus, for Eq. (1.22), the positive definiteness condition for the operator R is equivalent to the condition $1 + \frac{ab^2}{2} > 0$ (regardless the signs of the coefficients a and b).

In the sequel, we assume that the above condition is satisfied.

Then the weighted asymptotic closeness of the solution of problem (1.22)-(1.23) and the function $e^{at}w\left(\frac{x+abt}{\sqrt{C}},t\right)$ takes place, where $w(x,t)$ is the solution of the problem

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2},\ x\in\mathbb{R}^1, t>0, \tag{1.24}$$

$$w\big|_{t=0}=w_0(x)\overset{\text{def}}{=}u_0(\sqrt{C}x),\ x\in\mathbb{R}^1, \tag{1.25}$$

$C=1+\frac{ab^2}{2}$, and the weight is equal to e^{-at}.

Together with Eq. (1.22), which is a *differential-difference* equation, and Eq. (1.24), which is a *differential* one, consider the differential equation

$$\frac{\partial v}{\partial t}=\left(1+\frac{ab^2}{2}\right)\frac{\partial^2 v}{\partial x^2}+ab\frac{\partial v}{\partial x}+av,\ x\in\mathbb{R}^1, t>0. \tag{1.26}$$

Define the function

$$v(x,t)\overset{\text{def}}{=}e^{at}w\left(\frac{x+abt}{\sqrt{C}},t\right)\overset{\text{def}}{=}e^{at}w(\cdot,t) \tag{1.27}$$

and substitute it in Eq. (1.26):

$$\frac{\partial v}{\partial t}=ae^{at}w(\cdot,t)+\frac{ab}{\sqrt{C}}e^{at}\frac{\partial w}{\partial x}(\cdot,t)+e^{at}\frac{\partial w}{\partial t}(\cdot,t)$$

and

$$\frac{\partial v}{\partial x}=\frac{1}{\sqrt{C}}e^{at}\frac{\partial w}{\partial x}(\cdot,t),\qquad \frac{\partial^2 v}{\partial x^2}=\frac{1}{C}e^{at}\frac{\partial^2 w}{\partial x^2}(\cdot,t).$$

Hence,

$$\left(1+\frac{ab^2}{2}\right)\frac{\partial^2 v}{\partial x^2}+ab\frac{\partial v}{\partial x}+av=e^{at}\frac{\partial^2 w}{\partial x^2}(\cdot,t)+\frac{ab}{\sqrt{C}}e^{at}\frac{\partial w}{\partial x}(\cdot,t)+ae^{at}w(\cdot,t).$$

Since

$$\frac{\partial^2 w}{\partial x^2}(\cdot,t)=\frac{\partial w}{\partial t}(\cdot,t)$$

(this holds not only at the point $(\cdot,t)$ but at any point of $\mathbb{R}^1\times(0,+\infty)$), it follows that function (1.27) is a solution of Eq. (1.26).

Further, $v\big|_{t=0}=w\left(\frac{x}{\sqrt{C}},0\right)=u_0(x)$. Therefore, the asymptotic closeness (with the same weight e^{-at}) holds for the solution of problem (1.22)-(1.23) and the solution of problem (1.26), (1.23); note that e^{-at} is the weight function returning the function $v(x,t)$ (which solves a *differential* equation) into the class of bounded functions.

The differential equation (1.26) is the differential-difference equation (1.22), where the nonlocal term is changed for its Taylor expansion up to the order 2 (i.e., the order of the equation) inclusively. The considered that the positive definiteness condition for the auxiliary operator is equivalent to the parabolicity condition for the specified differential equation (i.e., the ellipticity condition for its right-hand part). This holds for all (more general) cases of nonlocal low-order terms considered above as well (the proof is the same).

Thus, the positive definiteness condition for the auxiliary operator, ensuring the validity of the theorem on the (weighted) asymptotic closeness (stabilization) of solutions, is as follows: if all nonlocal terms of the original *differential-difference* equation are changed for their Taylor expansions up to the order 2 inclusively, then the obtained *differential* equation should be *parabolic.*

Note that this clearly illustrates the dual nature of low-order nonlocal terms: they play no role for the solvability investigation because the solvability of the Cauchy problem depends only on the principal terms (only the parabolicity of the equation obtained from the original one by means of the eliminating of all nonlocal terms is important), but, investigating asymptotic properties, we cannot treat them as *low-order* terms anymore (the parabolicity of the equation constructed according to the coefficients of those nonlocal terms is important)

Chapter 2

EQUATIONS WITH NONLOCAL PRINCIPAL TERMS

2.1. The Case of Factorable Fundamental Solutions

Let $a, h \in \mathbb{R}^m$. In $\mathbb{R}^1 \times (0, +\infty)$, consider the following equation:

$$\frac{\partial u}{\partial t} = Lu \stackrel{\text{def}}{=} \frac{\partial^2 u}{\partial x^2} + \sum_{k=1}^{m} a_k \frac{\partial^2 u}{\partial x^2}(x + h_k, t). \tag{2.1}$$

Consider the real part of the symbol of the operator L:

$$\mathrm{Re}L(\xi) = -\xi^2 - \xi^2 \sum_{k=1}^{m} a_k \cos h_k \xi$$

(cf. Sec. 1.6). As in Sec. 1.6, we say that $-L(\xi)$ is *positive definite* if there exists a positive C such that $-\mathrm{Re}L(\xi) \geq C\xi^2$ for any $\xi \in \mathbb{R}^1$; any operator $-L$ possessing the specified property is called a second-order operator *strongly elliptic in the whole space* (see also [108, p. 66 and p. 78]).

In the sequel, we assume that the operator $-L$ is strongly elliptic.

Note that the coefficients of the equation can be arbitrarily large under the strong ellipticity assumption (see, e.g., [102, Ex. 8.1]).

Together with Eq. (2.1), consider condition (1.4), assuming that the initial-value function $u_0(x)$ is continuous and bounded in $\mathbb{R}^1$.

On $\mathbb{R}^1 \times (0, +\infty)$, define the following function:

$$\mathcal{E}(x,t) \stackrel{\text{def}}{=} \mathcal{E}_{a,h}(x,t) \stackrel{\text{def}}{=} \int_0^\infty e^{-t\xi^2(1+\sum\limits_{k=1}^{m} a_k \cos h_k \xi)} \cos(x\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k \xi) d\xi. \tag{2.2}$$

Obviously, if the operator $-L$ is strongly elliptic, then the inequality $1 + \sum\limits_{k=1}^{m} a_k \cos h_k \xi \geq C$ holds for $\xi \neq 0$. Let us show that it is valid for $\xi = 0$ as well (perhaps, with another positive constant), i.e., $1 + \sum\limits_{k=1}^{m} a_k > 0$. Assume the converse: $1 + \sum\limits_{k=1}^{m} a_k \leq 0$. Then, for any $\xi \neq 0$, we have

$$C \leq 1 + \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} a_k \cos h_k \xi = 1 + \sum_{k=1}^{m} a_k + \sum_{k=1}^{m} a_k (\cos h_k \xi - 1)$$

$$= 1 + \sum_{k=1}^{m} a_k - 2 \sum_{k=1}^{m} a_k \sin^2 \frac{h_k \xi}{2} = 1 + \sum_{k=1}^{m} a_k - \frac{1}{2} \sum_{k=1}^{m} a_k h_k^2 \xi^2 \left(\frac{\sin \frac{h_k \xi}{2}}{\frac{h_k \xi}{2}} \right)^2 \leq -\frac{\xi^2}{2} \sum_{k=1}^{m} a_k h_k^2 \left(\frac{\sin \frac{h_k \xi}{2}}{\frac{h_k \xi}{2}} \right)^2.$$

Now, selecting sufficiently small positive ξ, we arrive at a contradiction with the positivity of the constant C.

Therefore,

$$|\mathcal{E}(x,t)| \leq \int_0^\infty e^{-Ct\xi^2} d\xi = \sqrt{\frac{\pi}{4Ct}},$$

i.e., for any t_0, T from $(0, +\infty)$, integral (2.2) converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^1 \times [t_0, T]$; hence, $\mathcal{E}(x,t)$ is well defined on $\mathbb{R}^1 \times (0, +\infty)$.

Formally differentiate $\mathcal{E}$ with respect to the variable t under the integral sign:

$$\frac{\partial \mathcal{E}}{\partial t} = -\int_0^\infty \xi^2 (1 + \sum_{k=1}^m a_k \cos h_k \xi) e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \cos(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) d\xi$$

$$+ \int_0^\infty e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \sin(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) \sum_{k=1}^m a_k \xi^2 \sin h_k \xi d\xi$$

$$= \sum_{k=1}^m a_k \int_0^\infty \xi^2 e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \Bigg[\sin(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) \sin h_k \xi$$

$$- \cos(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) \cos h_k \xi \Bigg] d\xi - \int_0^\infty \xi^2 e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \cos(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) d\xi$$

$$= \int_0^\infty \xi^2 e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \Bigg(\sum_{k=1}^m a_k \cos \Bigg[(x + h_k)\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi \Bigg]$$

$$+ \cos(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) \Bigg) d\xi.$$

Further, formal differentiation of $\mathcal{E}$ with respect to the variable x under the integral sign yields:

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} = -\int_0^\infty \xi^2 e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \cos(x\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) d\xi.$$

The absolute value of both integrals is bounded from above by a linear combination of the form

$$\int_0^\infty \xi^2 e^{-Ct\xi^2} d\xi = \frac{\sqrt{\pi}}{4Ct^{\frac{3}{2}}},$$

i.e., they converge absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^1 \times [t_0, T]$ for all $t_0, T \in (0, +\infty)$. Hence, differentiating under the integral is valid, and the following relation holds in $\mathbb{R}^1 \times (0, +\infty)$:

$$\frac{\partial \mathcal{E}}{\partial t} - \frac{\partial^2 \mathcal{E}}{\partial x^2} = -\int_0^\infty \xi^2 e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \sum_{k=1}^m a_k \cos \Bigg[(x + h_k)\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi \Bigg] d\xi$$

$$= -\sum_{k=1}^m a_k \int_0^\infty \xi^2 e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \cos \Bigg[(x + h_k)\xi - t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi \Bigg] d\xi = \sum_{k=1}^m a_k \frac{\partial^2 \mathcal{E}}{\partial x^2}(x + h_k, t).$$

Therefore, $\mathcal{E}(x,t)$ satisfies (in the classical sense) Eq. (2.1) in $\mathbb{R}^1 \times (0, +\infty)$.

2.2. Cauchy Problem: Unique Solvability

Let us estimate the behavior of $\mathcal{E}(x,t)$ and its derivatives as $x \to \infty$ (assuming that a positive t is fixed). To do this, decompose it into the even and odd (with respect to x) terms $\mathcal{E}_1(x,t)$ and $\mathcal{E}_2(x,t)$:

$$\mathcal{E}_1(x,t) = \int_0^\infty e^{-t\xi^2(1+\sum_{k=1}^m a_k \cos h_k \xi)} \cos x\xi \, \cos(t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) d\xi$$

and

$$\mathcal{E}_2(x,t) = \int\limits_0^\infty e^{-t\xi^2(1+\sum\limits_{k=1}^m a_k \cos h_k\xi)} \sin x\xi \, \sin(t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) d\xi.$$

Let us prove the following assertion.

Lemma 2.2.1. *If* $t > 0$, *then the function* $x^2\mathcal{E}(x,t)$ *is bounded in* $\mathbb{R}^1$.

Proof. Fix an arbitrary positive t and integrate

$$\int\limits_0^\infty e^{-t\xi^2(1+\sum\limits_{k=1}^m a_k \cos h_k\xi)} \cos(t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) \cos x\xi \, d\xi$$

by parts two times. This yields

$$\frac{1}{x^2} \int\limits_0^\infty \left[e^{-t\xi^2(1+\sum\limits_{k=1}^m a_k \cos h_k\xi)} \cos(t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) \right]'' \cos x\xi \, d\xi$$

(it is easy to check that all the integrated terms vanish).

The last integral is a bounded function of the variable x; therefore, $x^2\mathcal{E}_1(x,t)$ is bounded. The boundedness of the function $x^2\mathcal{E}_2(x,t)$ is proved in the same way. Lemma 2.2.1 is proved. □

Thus, the following function is defined in $\mathbb{R}^1 \times (0, +\infty)$:

$$u(x,t) \stackrel{\text{def}}{=} \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t) u_0(\xi) d\xi. \tag{2.3}$$

The following assertion is valid.

Lemma 2.2.2. *If* $t > 0$, *then the function* $x^2 \frac{\partial^2 \mathcal{E}}{\partial x^2}(x,t)$ *is bounded in* $(-\infty, +\infty)$.

To prove this, we decompose $\frac{\partial^2 \mathcal{E}}{\partial x^2}$ into its even and odd (with respect to x) terms and integrate the former term

$$-\int\limits_0^\infty \xi^2 e^{-t\xi^2(1+\sum\limits_{k=1}^m a_k \cos h_k\xi)} \cos(t\xi^2 \sum_{k=1}^m a_k \sin h_k \xi) \cos x\xi \, d\xi$$

by parts two times. The remaining part of the proof is similar to the proof of Lemma 2.2.1.

Obviously, Lemma 2.2.2 remains valid if we take $\frac{\partial^2 \mathcal{E}}{\partial x^2}$ at the point $(x + h_k, t), k = \overline{1,m}$, instead of the point (x,t). Taking into account that $\mathcal{E}(x,t)$ satisfies Eq. (2.1) in $\mathbb{R}^1 \times (0, +\infty)$ (see the previous section), we deduce the following assertion from Lemma 2.2.1 and 2.2.2:

Lemma 2.2.3. *If* $t > 0$, *then* $x^2 \frac{\partial \mathcal{E}}{\partial t}(x,t)$ *is bounded in* $(-\infty, +\infty)$.

Lemmas 2.2.1–2.2.3 and the fact that $\mathcal{E}(x,t)$ satisfies Eq. (2.1) in $\mathbb{R}^1 \times (0, +\infty)$ imply the following assertion:

Theorem 2.2.1. *Let* $-L$ *be a strongly elliptic operator in* $\mathbb{R}^1$. *Then function* (2.3) *is a classical solution of Eq.* (2.1) *in* $\mathbb{R}^1 \times (0, +\infty)$.

Remark 2.2.1. The fact that function (2.3) satisfies problem (2.1), (1.4) *in the sense of generalized functions* is known (see, e.g., [16]). The only new value of Theorem 2.2.1 is the fact that the specified function is a *classical* solution in $\mathbb{R}^1 \times (0, +\infty)$.

Let us prove the uniqueness of the specified solution. Following [16], investigate the real part of the symbol of the elliptic operator L contained in Eq. (2.1). The specified symbol $\mathcal{P}(z) \stackrel{\text{def}}{=} \mathcal{P}(\sigma + i\tau)$ is equal to

$$-z^2\left(1+\sum_{k=1}^{m} a_k e^{-ih_k z}\right) = (\tau^2-\sigma^2-2i\sigma\tau)\left(1+\sum_{k=1}^{m} a_k e^{-ih_k z}\right)$$
$$= (\tau^2-\sigma^2-2i\sigma\tau)\left(1+\sum_{k=1}^{m} a_k e^{h_k\tau - ih_k\sigma}\right)$$
$$= (\tau^2-\sigma^2-2i\sigma\tau)\left(1+\sum_{k=1}^{m} a_k e^{h_k\tau}\cos h_k\sigma - i\sum_{k=1}^{m} a_k e^{h_k\tau}\sin h_k\sigma\right).$$

Thus,

$$\mathrm{Re}\mathcal{P}(z) = (\tau^2-\sigma^2)\left(1+\sum_{k=1}^{m} a_k e^{h_k\tau}\cos h_k\sigma\right) - 2\sigma\tau\sum_{k=1}^{m} a_k e^{h_k\tau}\sin h_k\sigma.$$

The function $\mathcal{Q}(z,t_0,t) \stackrel{\text{def}}{=} e^{(t-t_0)\mathcal{P}(z)}$ satisfies the following inequality

$$|\mathcal{Q}(z,t_0,t)| \le e^{(t-t_0)\left[C_1(1+\sigma^4)+C_2 e^{C_3\tau}\right]}.$$

The last estimate implies (see [16, Ch. 2, Appendix 1]) that problem (2.1), (1.4) has at most one solution in the sense of generalized functions.

Remark 2.2.2. As in the case of low-order nonlocal terms, the uniqueness of the solution of problem (2.1), (1.4) (in corresponding spaces of generalized functions) holds for a much wider classes of initial-value functions than the class of continuous bounded functions; in particular, it holds for Tikhonov classes and their generalizations (cf. Remark 1.5.1 and see [2] and [40]). However, we consider only the case of continuous bounded initial-value functions because we investigate the closeness of solutions of the specified problem and *classical* parabolic problems.

Remark 2.2.3. The uniqueness of the solution allows us to find the integral of the fundamental solution over the whole real axis: the following assertion is valid.

Lemma 2.2.4. $\int\limits_{-\infty}^{\infty} \mathcal{E}(x,t)dx = \pi.$

Proof. Consider the function $u_0(x) \equiv 1$; it is continuous and bounded. Hence, the function

$$y(x,t) \stackrel{\text{def}}{=} \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t)d\xi$$

satisfies Eq. (2.1) in $\mathbb{R}^1\times(0,+\infty)$ and satisfies the initial-value condition

$$y(x,0) \equiv 1.$$

However, $y(x,t)$ does not depend on x :

$$\int\limits_{-\infty}^{+\infty} \mathcal{E}(x-\xi,t)d\xi = \int\limits_{-\infty}^{+\infty} \mathcal{E}(\xi,t)d\xi = \pi y(t),$$

i.e., $y(t)$ satisfies the *ordinary* differential equation $y' = 0$ and the initial-value condition $y(0) = 1$. Hence, $y(t) \equiv 1$. □

2.3. Long-Time Behavior of Solutions

In this section, we study the behavior of $u(x,t)$ as $t \to \infty$. Together with problem (2.1), (1.4), consider the heat equation with the same initial-value condition (1.4). Denote its classical bounded solution by $v(x,t)$; denote the positive constant $1 + \sum\limits_{k=1}^{m} a_k$ by p.

The following assertion is valid:

Theorem 2.3.1. *The limit relation* $\lim\limits_{t\to\infty} [u(x,t) - v(x,pt)] = 0$ *holds for any real* x.

Proof. Take an arbitrary real x_0 and consider $u(x_0,t)$. Change the variable: $\eta = \dfrac{x_0 - \xi}{2\sqrt{t}}$; we obtain that

$$u(x_0,t) = \frac{2\sqrt{t}}{\pi} \int\limits_{-\infty}^{+\infty} \mathcal{E}(2\sqrt{t}\eta, t) u_0(x_0 - 2\sqrt{t}\eta)d\eta.$$

Further,

$$\sqrt{t}\,\mathcal{E}(2\sqrt{t}\eta, t) = \sqrt{t}\int\limits_0^{\infty} e^{-t\xi^2(1+\sum\limits_{k=1}^{m} a_k \cos h_k\xi)} \cos(2\sqrt{t}\eta\xi - t\xi^2 \sum_{k=1}^{m} a_k \sin h_k\xi)d\xi$$

$$= \int\limits_0^{\infty} e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}})} \cos\left(2z\eta - z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}}\right) dz.$$

This implies that the function $u(x_0,t)$ can be represented as

$$\frac{2}{\pi}\int\limits_{-\infty}^{+\infty} u_0(x_0 - 2\sqrt{t}\eta) \int\limits_0^{\infty} e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}})} \cos\left(2z\eta - z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}}\right) dzd\eta.$$

Then

$$u(x_0,t) - v(x_0,pt) = \frac{2}{\pi}\int\limits_{-\infty}^{+\infty} u_0(x_0 - 2\sqrt{t}\eta) \int\limits_0^{\infty} \Bigg[e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}})}$$

$$\times \cos\left(2z\eta - z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}}\right) - e^{-pz^2}\cos 2z\eta\Bigg] dzd\eta. \tag{2.4}$$

To continue the proof, we need the following two lemmas.

Lemma 2.3.1. *We have*

$$\int\limits_0^{\infty} \left[e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}})} \cos\left(2z\eta - z^2 \sum_{k=1}^{m} a_k \sin \frac{h_k z}{\sqrt{t}}\right) - e^{-pz^2}\cos 2z\eta\right] dz \overset{t\to\infty}{\longrightarrow} 0$$

uniformly with respect to $\eta \in (-\infty, +\infty)$.

Proof. Fix an arbitrary positive ε and decompose the estimated integral as follows:

$$\int\limits_0^{\delta} + \int\limits_{\delta}^{\infty} \overset{\text{def}}{=} I_{1,\delta} + I_{2,\delta}.$$

The absolute of this sum is estimated from above by $2\int\limits_0^\infty e^{-Cz^2}dz$; therefore, there exists a positive δ such that $|I_{2,\delta}| \le \frac{\varepsilon}{2}$ for any real η and any positive t. Fix that δ and consider the integral $I_{1,\delta}$. Its integrand is equal to

$$e^{-pz^2}\left[e^{z^2\sum\limits_{k=1}^m a_k\left(1-\cos\frac{h_kz}{\sqrt t}\right)}\cos\left(2z\eta - z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right) - \cos 2z\eta\right]$$

$$= e^{-pz^2}\left(e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt t}}\left[\cos 2z\eta\cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right)\right.\right.$$

$$\left.\left.+\ \sin 2z\eta\sin\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right)\right] - \cos 2z\eta\right)$$

$$= e^{-pz^2}\left(\cos 2z\eta\left[e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt t}}\cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right) - 1\right]\right.$$

$$\left.+\, e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt t}}\sin 2z\eta\sin\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right)\right) \stackrel{\text{def}}{=} A_1(\eta,t;z) + A_2(\eta,t;z).$$

The inequality

$$\left|\int\limits_0^\delta A_2(\eta,t;z)dz\right| \le e^{2\delta^2\sum\limits_{k=1}^m |a_k|}\int\limits_0^\delta\left|\sin\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right)\right|dz$$

holds for any η and t. Denote the fraction

$$\frac{16\delta^8\left(\sum\limits_{k=1}^m |a_k||h_k|\right)^2 e^{4\delta^2\sum\limits_{k=1}^m |a_k|}}{\varepsilon^2}$$

by T_0. Then the inequality

$$\left|\frac{h_kz}{\sqrt t}\right| \le \frac{\varepsilon}{4\delta^3 e^{2\delta^2\sum\limits_{k=1}^m |a_k|}}\,\frac{|h_k|}{\sum\limits_{k=1}^m |a_k||h_k|},\quad k=\overline{1,m},$$

holds for any $t > T_0$; hence,

$$\left|\sin\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right)\right| \le \left|z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right| \le \frac{\varepsilon}{4\delta e^{2\delta^2\sum\limits_{k=1}^m |a_k|}}$$

(because $0 \le z \le \delta$).

Thus, $\left|\int\limits_0^\delta A_2(\eta,t;z)dz\right| \le \frac{\varepsilon}{4}$ if $t > T_0$ and η is real. It remains to estimate the integral $\int\limits_0^\delta A_1(\eta,t;z)dz$. Its absolute value does not exceed

$$\int\limits_0^\delta e^{-pz^2}\left|e^{2z^2\sum\limits_{k=1}^m a_k\sin^2\frac{h_kz}{2\sqrt t}}\cos\left(z^2\sum_{k=1}^m a_k\sin\frac{h_kz}{\sqrt t}\right) - 1\right|dz.$$

The difference in the integrand can be represented as follows:

$$e^{2z^2\sum\limits_{k=1}^{m} a_k \sin^2 \frac{h_k z}{2\sqrt{t}}} \cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) - \cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) + \cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) -1$$

$$= \cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)\left(e^{2z^2\sum\limits_{k=1}^{m} a_k \sin^2 \frac{h_k z}{2\sqrt{t}}} - 1\right) + \cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) - 1.$$

Select a large T_1 such that $\left|e^{2z^2\sum\limits_{k=1}^{m} a_k \sin^2 \frac{h_k z}{2\sqrt{t}}} - 1\right| \le \frac{\varepsilon}{8\delta}$ for any $t > T_1$ and any $z \in [0, \delta]$. This is possible because there exists a positive δ_1 such that $e^x \in \left(1 - \frac{\varepsilon}{8\delta}, 1 + \frac{\varepsilon}{8\delta}\right)$ for any $x \in (-\delta_1, \delta_1)$. Thus, one can assign $T_1 = \dfrac{\delta^4 \sum\limits_{k=1}^{m} |a_k| h_k^2}{2\delta_1}$.

Further, there exists a positive δ_2 such that the inequality $1 - \frac{\varepsilon}{8\delta} < \cos x < 1 + \frac{\varepsilon}{8\delta}$ holds for any $x \in (-\delta_2, \delta_2)$. Assign

$$T_2 \stackrel{\text{def}}{=} \frac{\delta^6 \left(\sum\limits_{k=1}^{m} |a_k||h_k|\right)^2}{\delta_2^2}.$$

Then, for any $t > T_2$ and any $z \in [0, \delta]$, we have

$$\left|z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right| \le z^2 \sum_{k=1}^{m} \frac{|a_k||h_k z|}{\sqrt{T_2}} \le \frac{\delta^3}{\sqrt{T_2}} \sum_{k=1}^{m} |a_k||h_k| = \delta_2.$$

Therefore, the inequality

$$\left|\cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) - 1\right| < \frac{\varepsilon}{8\delta}$$

holds for any $t > T_2$ and any $z \in [0, \delta]$. Hence, for any $t > \max\{T_0, T_1, T_2\}$ and any $\eta \in (-\infty, +\infty)$, we have

$$\left|\int_0^\delta A_1(\eta, t; z)dz\right| < \frac{\varepsilon}{4}, \text{ i.e., } \left|I_{1,\delta}\right| < \frac{\varepsilon}{2}.$$

This completes the proof of Lemma 2.3.1. □

Lemma 2.3.2. *There exists a positive M depending only on a and h such that*

$$\left|\int_0^\infty e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}})} \cos\left(2z\eta - z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) dz\right| \le \frac{M}{\eta^2}$$

for any $t > 1$ and any $\eta \in \mathbb{R}^1 \setminus \{0\}$.

Proof. Represent the estimated integral as

$$\int_0^\infty e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}})} \cos 2z\eta \cos\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) dz$$

$$+ \int_0^\infty e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos \frac{h_k z}{\sqrt{t}})} \sin 2z\eta \sin\left(z^2\sum_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right) dz \tag{2.5}$$

and, for definiteness, consider the former term.

Denote the function $e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}})} \cos\left(z^2 \sum\limits_{k=1}^{m} a_k \sin\frac{h_k z}{\sqrt{t}}\right)$ by $g(z)$ (t is treated as a positive parameter) and integrate $\int\limits_0^\infty g(z)\cos 2\eta z dz$ by parts. We obtain the relation

$$g(z)\frac{\sin 2\eta z}{2\eta}\Big|_{z=0}^{z=+\infty} - \frac{1}{2\eta}\int\limits_0^\infty g'(z)\sin 2\eta z dz = -\frac{1}{2\eta}\int\limits_0^\infty g'(z)\sin 2\eta z dz$$

because $g(+\infty)=0$ since $1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}} \geq C > 0$. Integrating by parts again, we see that

$$g'(z)\frac{\cos 2\eta z}{4\eta^2}\Big|_{z=0}^{z=+\infty} - \frac{1}{4\eta^2}\int\limits_0^\infty g''(z)\cos 2\eta z dz.$$

Let us prove that the integrated term vanishes. To compute $\lim\limits_{z\to+0} g'(z)\frac{\cos 2\eta z}{4\eta^2}$ and $\lim\limits_{z\to+\infty} g'(z)\frac{\cos 2\eta z}{4\eta^2}$, differentiate the function $g(z)$:

$$g'(z) = e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}})}\left[-2z\left(1+\sum_{k=1}^{m} a_k\cos\frac{h_k z}{\sqrt{t}}\right)\right.$$
$$\left.+\frac{z^2}{\sqrt{t}}\sum_{k=1}^{m} a_k h_k \sin\frac{h_k z}{\sqrt{t}}\right]\cos\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)$$
$$- e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}})}\sin\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)\left(2z\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}} + \frac{z^2}{\sqrt{t}}\sum_{k=1}^{m} a_k h_k\cos\frac{h_k z}{\sqrt{t}}\right).$$

The last expression can be reduced to the following form:

$$-e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}})}\left[\frac{z^2}{\sqrt{t}}\sin\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k h_k\cos\frac{h_k z}{\sqrt{t}}\right.$$
$$-\frac{z^2}{\sqrt{t}}\cos\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k h_k\sin\frac{h_k z}{\sqrt{t}} + 2z\cos\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k\cos\frac{h_k z}{\sqrt{t}}$$
$$\left.+2z\sin\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}} + 2z\cos\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)\right];$$

therefore, $g'(z)$ is equal to

$$e^{-z^2(1+\sum\limits_{k=1}^{m} a_k \cos\frac{h_k z}{\sqrt{t}})}\left[\frac{z^2}{\sqrt{t}}\sum_{k=1}^{m} a_k h_k\sin\left(\frac{h_k z}{\sqrt{t}} - z^2\sum_{l=1}^{m} a_l\sin\frac{h_l z}{\sqrt{t}}\right)\right.$$
$$\left.+2z\sum_{k=1}^{m} a_k\cos\left(\frac{h_k z}{\sqrt{t}} - z^2\sum_{l=1}^{m} a_l\sin\frac{h_l z}{\sqrt{t}}\right) - 2z\cos\left(z^2\sum_{k=1}^{m} a_k\sin\frac{h_k z}{\sqrt{t}}\right)\right].$$

Hence, $g'(0) = g'(+\infty) = 0$; therefore,

$$\int\limits_0^\infty g(z)\cos 2\eta z dz = -\frac{1}{4\eta^2}\int\limits_0^\infty g''(z)\cos 2\eta z dz.$$

Obviously, there exists a polynomial $P(z)$ such that its positive coefficients depend only on a and h and the inequality $|g''(z)| \le e^{-z^2(1+\sum\limits_{k=1}^m a_k \cos\frac{h_k z}{\sqrt{t}})} P(z)$ holds on $[0, +\infty)$ provided that $t \ge 1$. Hence,

$$\left|\int\limits_0^\infty g(z)\cos 2\eta z dz\right| \le \frac{1}{4\eta^2}\int\limits_0^\infty e^{-Cz^2}P(z)dz$$

for any $t > 1$ and any $\eta \in \mathbb{R}^1\backslash\{0\}$. Thus, the claimed estimate is valid for the former term of (2.5). The latter one is estimated in the same way. This completes the proof of Lemma 2.3.2. □

To complete the proof of Theorem 2.3.1, we decompose (2.4) into the sum

$$\frac{2}{\pi}\left(\int\limits_{-\infty}^{-R} + \int\limits_{-R}^{R} + \int\limits_{R}^{+\infty}\right) \overset{\text{def}}{=} \frac{2}{\pi}\left[I_{3,R}(t) + I_{4,R}(t) + I_{5,R}(t)\right],$$

where R is a positive parameter. By virtue of Lemma 2.3.2 (without loss of generality, one can assume that $t > 1$) and the boundedness of the function u_0, we have

$$|I_{5,R}(t)| \le \sup_{\mathbb{R}^1}|u_0(x)| \int\limits_R^{+\infty}\left(\frac{M}{\eta^2} + \frac{\sqrt{\pi}}{\sqrt{4p}}\, e^{-\frac{\eta^2}{p}}\right) d\eta$$

The last integral converges. Hence, for any positive ε there exists R_0 from $(1, +\infty)$ such that $|I_{5,R_0}(t)| \le \dfrac{\pi\varepsilon}{6}$ for any t from $(1, +\infty)$. Obviously, I_{3,R_0} satisfies the same estimate.

Fix that R_0 and consider $I_{4,R_0}(t)$. Its absolute value does not exceed

$$\sup_{\mathbb{R}^1}|u_0(x)| \int\limits_{-R_0}^{+R_0}\left|\int\limits_0^\infty\left[e^{-z^2(1+\sum\limits_{k=1}^m a_k\cos\frac{h_k z}{\sqrt{t}})}\cos\left(2z\eta - z^2\sum_{k=1}^m a_k \sin\frac{h_k z}{\sqrt{t}}\right) - e^{-pz^2}\cos 2z\eta\right] dz\right| d\eta.$$

By virtue of Lemma 2.3.1, there exists $T^* > 1$ such that for any real η and any $t > T^*$, the absolute value of the internal integral in the last expression does not exceed $\pi\varepsilon\left(12R_0 \sup\limits_{\mathbb{R}^1}|u_0(x)|\right)^{-1}$. This implies that the absolute value of (2.4) does not exceed ε for any $t > T^*$. Since ε is selected arbitrarily, it follows that $\lim\limits_{t\to\infty}[u(x_0, t) - v(x_0, pt)] = 0$. This completes the proof of Theorem 2.3.1 because x_0 is selected arbitrarily. □

Corollary 2.3.1. *Let $x, l \in (-\infty, +\infty)$. Then*

$$\lim_{t\to\infty} u(x,t) = l \iff \lim_{R\to\infty}\frac{1}{2R}\int\limits_{-R}^{R} u_0(x)dx = l.$$

To prove this, it suffices to note that the assertion of the corollary is the classical pointwise stabilization theorem (see [95]), i.e., it holds for the function $v(x, t)$; further, it remains to apply Theorem 2.3.1 directly.

Remark 2.3.1. Note that although Theorem 2.3.1 and Corollary 2.3.1 are valid under the same conditions, the assertion of the theorem (treating the *closeness* of solutions) is stronger in the following sense: unlike the assertion of the corollary (which is a *stabilization* theorem), it provides information on the solution behavior even for the case where the (necessary and sufficient) stabilization condition is not satisfied.

2.4. The Case of Several Spatial Variables

Let $n, m_1, \dots, m_n \in \mathbb{N}$ and $a_i, b_i \in \mathbb{R}^{m_i}$, where a_i denotes the vector $(a_{i1}, \dots, a_{im_i})$ and b_i denotes the vector $(b_{i1}, \dots, b_{im_i})$, $i = \overline{1,n}$. In the domain $\left\{x \in \mathbb{R}^n \middle| t > 0\right\}$, consider the equation

$$\frac{\partial u}{\partial t} = L_{(n)} u \stackrel{\text{def}}{=} \Delta u + \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_{ij} \frac{\partial^2 u}{\partial x_i^2}(x_1, \dots, x_{i-1}, x_i + b_{ij}, x_{i+1}, \dots, x_n, t) \tag{2.6}$$

and condition (1.4), assuming that u_0 is continuous and bounded in $\mathbb{R}^n$.

Similarly to Sec. 2.1 (see also [102, §8]), impose the positive definiteness condition on the symbol of the operator $-L_{(n)}$: there exists a positive constant C such that

$$-\mathrm{Re} L_{(n)}(\xi) = |\xi|^2 + \sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i \geq C|\xi|^2$$

for any $\xi \in \mathbb{R}^n$.

As in the one-dimensional case, any operator $-L_{(n)}$ possessing the specified property is called a *strongly elliptic operator in the whole space.*

Note that, as in the one-dimensional case (cf. also [102, Ex. 8.1]), the strong ellipticity condition imposes no restrictions on the values of the coefficients of the equation.

Also, note that, as in the case of a bounded domain (see [102, §9]), the strong ellipticity of differential operators substantially differs from the strong ellipticity of differential-difference ones; therefore, the impact of difference terms has a principal meaning.

In $\mathbb{R}^n \times (0, \infty)$, denote the function

$$\mathcal{E}_{(n)}(x,t) \stackrel{\text{def}}{=} \frac{1}{2^n} \int_{\mathbb{R}^n} e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)} \cos\left(x \cdot \xi - t \sum_{i=1}^{n} \xi_i^2 \sum_{j=1}^{m_i} a_{ij} \sin b_{ij}\xi_i\right) d\xi. \tag{2.7}$$

The power of the last exponent can be represented as

$$-t \sum_{i=1}^{n} \xi_i^2 \left(1 + \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right).$$

There exist positive constants $C_1, \dots, C_n$ such that for any ξ from $\mathbb{R}^n$ and any positive t, the last expression does not exceed $-t \sum\limits_{i=1}^{n} C_i \xi_i^2$. Indeed, take an arbitrary $i \in \overline{1,n}$ and apply the strong ellipticity condition, assuming that $\xi_1 = \dots = \xi_{i-1} = \xi_{i+1} = \dots = \xi_n = 0$. We see that $\xi_i^2 + \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i \geq C\xi_i^2$ for any real ξ_i. Hence, $1 + \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i \geq C$ for any $\xi_i \neq 0$. Let us show that the last inequality holds (perhaps, with another positive constant) for $\xi = 0$ as well, i.e., $1 + \sum\limits_{j=1}^{m_i} a_{ij} > 0$. Assume the converse: $1 + \sum\limits_{j=1}^{m_i} a_{ij} \leq 0$. Then, for any $\xi_i \neq 0$, we have

$$C \leq 1 + \sum_{j=1}^{m_i} a_{ij} - \sum_{j=1}^{m_i} a_{ij} + \sum_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i$$

$$= 1+\sum_{j=1}^{m_i} a_{ij} +\sum_{j=1}^{m_i} a_{ij}(\cos b_{ij}\xi_i -1) = 1+\sum_{j=1}^{m_i} a_{ij} -2\sum_{j=1}^{m_i} a_{ij}\sin^2\frac{b_{ij}\xi_i}{2}$$

$$= 1+\sum_{j=1}^{m_i} a_{ij} - \frac{1}{2}\sum_{j=1}^{m_i} a_{ij}b_{ij}^2\xi_i^2\left(\frac{\sin\frac{b_{ij}\xi_i}{2}}{\frac{b_{ij}\xi_i}{2}}\right)^2 \le -\frac{\xi_i^2}{2}\sum_{j=1}^{m_i} a_{ij}b_{ij}^2\left(\frac{\sin\frac{b_{ij}\xi_i}{2}}{\frac{b_{ij}\xi_i}{2}}\right)^2.$$

Now, we can select a small positive ξ_i such that we arrive at a contradiction with the positivity of the constant C.

Therefore, for any $[t_0, T] \subset (0, +\infty)$, integral (2.7) converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^n \times [t_0, T]$, i.e., the function $\mathcal{E}_{(n)}(x,t)$ is well defined.

Formally differentiate $\mathcal{E}_{(n)}$ with respect to the variable t under the integral sign:

$$2^n \frac{\partial \mathcal{E}_{(n)}}{\partial t} = \int_{\mathbb{R}^n} e^{-t\left(|\xi|^2+\sum\limits_{i=1}^{n}\xi_i^2\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)}$$

$$\times \sin\left(x\cdot\xi - t\sum_{i=1}^{n}\xi_i^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\xi_i\right)\sum_{i=1}^{n}\xi_i^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\xi_i d\xi$$

$$-\int_{\mathbb{R}^n}\left(|\xi|^2+\sum_{i=1}^{n}\xi_i^2\sum_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right) e^{-t\left(|\xi|^2+\sum\limits_{i=1}^{n}\xi_i^2\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)}$$

$$\times \cos\left(x\cdot\xi - t\sum_{i=1}^{n}\xi_i^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\xi_i\right) d\xi.$$

This can be represented as

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} a_{ij}\int_{\mathbb{R}^n} e^{-t\left(|\xi|^2+\sum\limits_{i=1}^{n}\xi_i^2\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)}\xi_i^2\left[\sin b_{ij}\xi_i \sin\left(x\cdot\xi - t\sum_{k=1}^{n}\xi_k^2\sum_{l=1}^{m_k} a_{kl}\sin b_{kl}\xi_k\right)\right.$$

$$\left. - \cos b_{ij}\xi_i \cos\left(x\cdot\xi - t\sum_{k=1}^{n}\xi_k^2\sum_{l=1}^{m_k} a_{kl}\sin b_{kl}\xi_k\right)\right] d\xi$$

$$-\int_{\mathbb{R}^n} |\xi|^2 e^{-t\left(|\xi|^2+\sum\limits_{i=1}^{n}\xi_i^2\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)} \cos\left(x\cdot\xi - t\sum_{k=1}^{n}\xi_k^2\sum_{l=1}^{m_k} a_{kl}\sin b_{kl}\xi_k\right) d\xi,$$

which is equal to

$$-\int_{\mathbb{R}^n} |\xi|^2 e^{-t\left(|\xi|^2+\sum\limits_{i=1}^{n}\xi_i^2\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)} \cos\left(x\cdot\xi - t\sum_{k=1}^{n}\xi_k^2\sum_{l=1}^{m_k} a_{kl}\sin b_{kl}\xi_k\right) d\xi$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{m_i} a_{ij}\int_{\mathbb{R}^n}\xi_i^2 e^{-t\left(|\xi|^2+\sum\limits_{i=1}^{n}\xi_i^2\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)}\cos\left(x\cdot\xi + b_{ij}\xi_i - t\sum_{k=1}^{n}\xi_k^2\sum_{l=1}^{m_k} a_{kl}\sin b_{kl}\xi_k\right) d\xi.$$

Further, formal differentiation of $\mathcal{E}_{(n)}$ with respect to the variable x_i under the integral sign yields the relations

$$2^n \frac{\partial^2 \mathcal{E}_{(n)}}{\partial x_i^2} = -\int_{\mathbb{R}^n} \xi_i^2 e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\cdot\xi - t\sum_{k=1}^{n}\xi_k^2 \sum_{l=1}^{m_k} a_{kl}\sin b_{kl}\xi_k\right) d\xi$$

and

$$2^n \frac{\partial^2 \mathcal{E}_{(n)}}{\partial x_i^2}(x_1,\dots,x_{i-1},x_i+b_{ij},x_{i+1},\dots,x_n,t)$$
$$= -\int_{\mathbb{R}^n} \xi_i^2 e^{-t\left(|\xi|^2 + \sum\limits_{i=1}^{n} \xi_i^2 \sum\limits_{j=1}^{m_i} a_{ij} \cos b_{ij}\xi_i\right)} \cos\left(x\cdot\xi + b_{ij}\xi_i - t\sum_{k=1}^{n}\xi_k^2 \sum_{l=1}^{m_k} a_{kl}\sin b_{kl}\xi_k\right) d\xi.$$

Each of those integrals converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^n \times [t_0, T]$ for any $[t_0, T] \subset (0, +\infty)$; therefore, $\mathcal{E}_{(n)}(x,t)$ satisfies (in the classical sense) Eq. (2.6) in $\mathbb{R}^n \times (0, +\infty)$.

Let us prove the following assertion:

Lemma 2.4.1. *If $x \in \mathbb{R}^n$ and $t > 0$, then*

$$\int_{\mathbb{R}^n} u_0(x-\xi)\mathcal{E}_{(n)}(\xi,t)d\xi \tag{2.8}$$

absolutely converges.

Proof. By virtue of the absolute convergence of integral (2.7), the Fubini theorem is applicable to it, i.e., $\mathcal{E}_{(n)}(x,t)$ can be represented as

$$\frac{1}{2^n} \underbrace{\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty}}_{n \text{ times}} \prod_{i=1}^{n} e^{-t\xi_i^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)} \cos\sum_{i=1}^{n}\left(x_i\xi_i - t\xi_i^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\xi_i\right) d\xi_1\dots d\xi_n.$$

The integrand of the last integral can be decomposed into a finite sum of the form

$$\prod_{i=1}^{n} e^{-t\xi_i^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\xi_i\right)} g_i\left(x_i\xi_i - t\xi_i^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\xi_i\right),$$

where either $g_i(\tau) = \cos\tau$ or $g_i(\tau) = \sin\tau$. Hence, the last integral is a finite sum of terms of the form

$$\prod_{i=1}^{n}\int_{-\infty}^{+\infty} e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\tau\right)} g_i\left(x_i\tau - t\tau^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\tau\right) d\tau$$

and only the term

$$\prod_{i=1}^{n}\int_{-\infty}^{+\infty} e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\tau\right)} \cos\left(x_i\tau - t\tau^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\tau\right) d\tau$$
$$= 2^n\prod_{i=1}^{n}\int_{0}^{\infty} e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\tau\right)} \cos\left(x_i\tau - t\tau^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\tau\right) d\tau$$

is different from zero. Any other term vanishes because it contains at least one zero factor: this is an integral (of an odd function) of the form

$$\int_{-\infty}^{+\infty} e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\tau\right)} \sin\left(x_i\tau - t\tau^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\tau\right) d\tau.$$

Thus, the function $\mathcal{E}_{(n)}(x,t)$ is equal to

$$\prod_{i=1}^{n}\int_{0}^{\infty} e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i} a_{ij}\cos b_{ij}\tau\right)} \cos\left(x_i\tau - t\tau^2\sum_{j=1}^{m_i} a_{ij}\sin b_{ij}\tau\right) d\tau.$$

Each factor of the last product is a function $\mathcal{E}_{a_i,b_i}(x_i,t) = \mathcal{E}(x_i,t)$ of the form (2.2). Fix an arbitrary positive t. Then, for any $i = \overline{1,n}$, the function $\mathcal{E}(x_i,t)$ is bounded on $\mathbb{R}^1$. Moreover, by virtue of Lemma 2.2.1, the function $x_i^2\mathcal{E}(x_i,t)$ is bounded on $\mathbb{R}^1$. Therefore, the function $(1+x_i^2)\mathcal{E}(x_i,t)$ is bounded on $\mathbb{R}^1$ as well, i.e., there exists a positive M such that $|\mathcal{E}(x_i,t)| \le \dfrac{M}{1+x_i^2}$ on $\mathbb{R}^1$ for $i = \overline{1,n}$. Therefore, the following inequality is valid in $\mathbb{R}^n$:

$$|\mathcal{E}_{(n)}(x,t)| \le \frac{(2M)^n}{\prod\limits_{i=1}^{n}(1+x_i^2)}.$$

Now, let Ω be an arbitrarily large domain in $\mathbb{R}^n$. There exists a positive A_0 such that $\Omega \subset Q(A_0)$, where $Q(A_0) = \left\{|x_i| < A_0 \middle| i = \overline{1,n}\right\}$. Then

$$\int_{Q(A_0)} \left|u_0(x-\xi)\mathcal{E}_{(n)}(\xi,t)\right| d\xi \le (2M)^n \sup|u_0| \int_{Q(A_0)} \frac{d\xi}{\prod\limits_{i=1}^{n}(1+\xi_i^2)}$$

$$= (2M)^n \sup|u_0| \left(\int_{-A_0}^{A_0} \frac{d\eta}{1+\eta^2}\right)^n = (4M\arctan A_0)^n \sup|u_0| \le (2\pi M)^n \sup|u_0|.$$

Therefore, integral (2.8) absolutely converges and satisfies the same estimate.

This completes the proof of Lemma 2.4.1. □

Thus, the following function is defined in $\mathbb{R}^n \times (0,+\infty)$:

$$u(x,t) \overset{\text{def}}{=} \frac{1}{\pi^n}\int_{\mathbb{R}^n} \mathcal{E}_{(n)}(x-\xi,t)u_0(\xi)d\xi. \tag{2.9}$$

Similarly to the representation of the function $\mathcal{E}_{(n)}$ in Lemma 2.4.1, we represent $\dfrac{\partial^2\mathcal{E}_{(n)}}{\partial x_i^2}$ as

$$\frac{\partial^2\mathcal{E}_{a_i,b_i}}{\partial x_i^2}\prod_{\substack{k=1\\ k\ne i}}^{n}\mathcal{E}_{a_k,b_k}(x_k,t).$$

It follows from Lemma 2.2.2 and the fact that $\mathcal{E}_{(n)}$ satisfies Eq. (2.6) that function (2.9) can be differentiated under the integral sign. This implies the following assertion.

Theorem 2.4.1. *Let the operator $-L_{(n)}$ be strongly elliptic in $\mathbb{R}^n$. Then function* (2.9) *satisfies* (*in the classical sense*) *Eq.* (2.6) *in* $\mathbb{R}^n \times (0,+\infty)$.

Note that function (2.9) is a solution of problem (2.6), (1.4) *in the sense of generalized functions* (see, e.g., [16]).

To establish the uniqueness of the found solution, investigate (according to [16]) the real part of the symbol of the elliptic operator $L_{(n)}$ contained in (2.6). For the specified symbol $\mathcal{P}(z_1, \dots, z_n)$ we have

$$\mathcal{P}(z_1, \dots, z_n) \stackrel{\text{def}}{=} \mathcal{P}(z) \stackrel{\text{def}}{=} \mathcal{P}(\sigma + i\tau) \stackrel{\text{def}}{=} \mathcal{P}(\sigma_1 + i\tau_1, \dots, \sigma_n + i\tau_n)$$

$$= -\sum_{k=1}^{n} z_k^2 \left(1 + \sum_{j=1}^{m_k} a_{kj} e^{-ib_{kj}z_k}\right)$$

$$= \sum_{k=1}^{n} (\tau_k^2 - \sigma_k^2 - 2i\sigma_k\tau_k) \left(1 + \sum_{j=1}^{m_k} a_{kj} e^{b_{kj}\tau_k} \cos b_{kj}\sigma_k - i \sum_{j=1}^{m_k} a_{kj} e^{b_{kj}\tau_k} \sin b_{kj}\sigma_k\right).$$

Thus,

$$\operatorname{Re}\mathcal{P}(z) = \sum_{k=1}^{n} \left[(\tau_k^2 - \sigma_k^2)\left(1 + \sum_{j=1}^{m_k} a_{kj} e^{b_{kj}\tau_k} \cos b_{kj}\sigma_k\right) - 2\sigma_k\tau_k \sum_{j=1}^{m_k} a_{kj} e^{b_{kj}\tau_k} \sin b_{kj}\sigma_k\right]$$

$$= |\tau|^2 - |\sigma|^2 + \sum_{k=1}^{n} \left[(\tau_k^2 - \sigma_k^2) \sum_{j=1}^{m_k} a_{kj} e^{b_{kj}\tau_k} \cos b_{kj}\sigma_k - 2\sigma_k\tau_k \sum_{j=1}^{m_k} a_{kj} e^{b_{kj}\tau_k} \sin b_{kj}\sigma_k\right].$$

Now, estimate the function $\mathcal{Q}(z, t_0, t) \stackrel{\text{def}}{=} e^{(t-t_0)\mathcal{P}(z)}$:

$$|\mathcal{Q}(z, t_0, t)| \le e^{(t-t_0)\left[C_1(1+|\sigma|^4) + C_2 e^{C_3|\tau|}\right]}.$$

From the last estimate, it follows (see [16, Ch. 2, Appendix 1]) that problem (2.6), (1.4) has at most one solution in the sense of generalized functions.

Note that, as in the one-dimensional case, the uniqueness takes place for a wider classes of initial-value functions as well (see Remark 2.2.2); however, due to the same reason, we consider only continuous bounded initial-value functions. Similarly to Lemma 2.2.4, we can compute the integral of $\mathcal{E}_{(n)}$ over the space $\mathbb{R}^n$; it is equal to π^n.

2.5. The Case of Several Spatial Variables: Stabilization of Solutions

In this section, we study the long-time behavior of $u(x, t)$ for the case of several spatial variables. Together with the *differential-difference* equation (2.6), consider the *differential* equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} p_i \frac{\partial^2 u}{\partial x_i^2}, \tag{2.10}$$

where $p_i = 1 + \sum_{j=1}^{m_i} a_{ij}$, $i = \overline{1, n}$ (note that the positivity of all such constants p_i is proved above).

Denote the classical bounded solution of problem (2.10), (1.4) by $v(x, t)$.

The following assertion is valid:

Theorem 2.5.1. *For any* $x \in \mathbb{R}^n$, *the limit relation* $\lim\limits_{t\to\infty} [u(x, t) - v(x, t)] = 0$ *holds.*

Proof. Take an arbitrary $x_0 \stackrel{\text{def}}{=} (x_1^0, \dots, x_n^0)$ from $\mathbb{R}^n$.

In (2.9), change the variables: $\eta_i \stackrel{\text{def}}{=} \dfrac{x_i^0 - \xi_i}{2\sqrt{t}}$ $(i = \overline{1, n})$. This yields the representation

$$u(x_0, t) = \left(\frac{2\sqrt{t}}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 - 2\sqrt{t}\eta)\mathcal{E}_{(n)}(2\sqrt{t}\eta, t)d\eta. \tag{2.11}$$

Taking into account that

$$t^{\frac{n}{2}}\mathcal{E}_{(n)}(2\sqrt{t}\eta,t)=t^{\frac{n}{2}}\prod_{i=1}^{n}\int_{0}^{\infty}e^{-t\tau^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos b_{ij}\tau\right)}\cos\left(2\eta_i\tau\sqrt{t}-t\tau^2\sum_{j=1}^{m_i}a_{ij}\sin b_{ij}\tau\right)d\tau$$

$$=\prod_{i=1}^{n}\int_{0}^{\infty}e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz,$$

we obtain that

$$u(x_0,t)=\left(\frac{2}{\pi}\right)^n\int_{\mathbb{R}^n}u_0(x_0-2\sqrt{t}\eta)\prod_{i=1}^{n}\int_{0}^{\infty}e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dzd\eta.$$

We have

$$v(x_0,t)=\frac{1}{\pi^{\frac{n}{2}}}\int_{\mathbb{R}^n}u_0(x_1^0-2\sqrt{p_1t}\xi_1,\dots,x_n^0-2\sqrt{p_nt}\xi_n)e^{-|\xi|^2}d\xi.$$

The change of variables $\sqrt{p_i}\xi_i=\eta_i$, $i=\overline{1,n}$, reduces the last expression to the form

$$\frac{1}{\pi^{\frac{n}{2}}\prod\limits_{i=1}^{n}\sqrt{p_i}}\int_{\mathbb{R}^n}u_0(x_1^0-2\sqrt{t}\eta_1,\dots,x_n^0-2\sqrt{t}\eta_n)e^{-\sum\limits_{i=1}^{n}\frac{\eta_i^2}{p_i}}d\eta.$$

Thus,

$$u(x_0,t)-v(x_0,t)=\left(\frac{2}{\pi}\right)^n\int_{\mathbb{R}^n}u_0(x_0-2\sqrt{t}\eta)\left[\prod_{i=1}^{n}\int_{0}^{\infty}e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\right.$$

$$\left.\times\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz-\prod_{i=1}^{n}\frac{\sqrt{\pi}}{2\sqrt{p_i}}e^{-\frac{\eta_i^2}{p_i}}\right]d\eta$$

$$=\left(\frac{2}{\pi}\right)^n\left(\int_{Q(A)}+\int_{\mathbb{R}^n\setminus Q(A)}\right)\overset{\text{def}}{=}\left(\frac{2}{\pi}\right)^n(J_1+J_2),\tag{2.12}$$

where A is a positive parameter and $Q(A)$ denotes the cube $\left\{|x_i|<A\middle|i=\overline{1,n}\right\}$.

Let $\varepsilon>0$. By virtue of Lemma 2.3.2, for any $i=\overline{1,n}$ there exists a positive M_i such that for any $\eta_i\geq 1$ and any $t>1$, we have

$$\left|\int_{0}^{\infty}e^{-z^2\left(1+\sum\limits_{j=1}^{m_i}a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)}\cos\left(2z\eta_i-z^2\sum_{j=1}^{m_i}a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right)dz\right|\leq\frac{M_i}{\eta_i^2}\leq\frac{2M_i}{1+\eta_i^2}.$$

Moreover, for $\eta_i\in[0,1]$, the left-hand part of the last inequality does not exceed

$$\int_{0}^{\infty}e^{-Cz^2}dz\overset{\text{def}}{=}\frac{\sqrt{\pi}}{2\sqrt{C}}\leq\frac{\sqrt{\pi}}{\sqrt{C}(1+\eta_i^2)};$$

hence, for any real η_i, we have

$$\left| \int_0^\infty e^{-z^2\left(1+\sum_{j=1}^{m_i} a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)} \cos\left(2z\eta_i - z^2\sum_{j=1}^{m_i} a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right) dz \right| \le \frac{M_i^*}{1+\eta_i^2},$$

where $M_i^* = \max\left(2M_i, \sqrt{\frac{\pi}{C}}\right)$.

Thus, the absolute value of the integrand in (2.12) does not exceed

$$\sup|u_0| \left[\prod_{i=1}^n \frac{M_i^*}{1+\eta_i^2} + \left(\frac{\pi}{4}\right)^{\frac{n}{2}} \prod_{i=1}^n \frac{1}{\sqrt{p_i}} e^{-\frac{\eta_i^2}{p_i}}\right].$$

Hence, integral (2.12) converges absolutely and uniformly with respect to $t \in (1, +\infty)$; therefore, there exists a positive A such that $|J_2| < \frac{\varepsilon\pi^n}{2^{n+1}}$ for any $t > 1$. Fix that A and consider J_1 for $t > 1$.

By virtue of Lemma 2.3.1, for any $i = \overline{1,n}$,

$$\int_0^\infty e^{-z^2\left(1+\sum_{j=1}^{m_i} a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)} \cos\left(2z\eta_i - z^2\sum_{j=1}^{m_i} a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right) dz \xrightarrow{t\to\infty} \sqrt{\frac{\pi}{4p_i}}\, e^{-\frac{\eta_i^2}{p_i}}$$

uniformly with respect to $\eta_i \in (-\infty, +\infty)$.

Since any internal (one-dimensional) integral of (2.12) is bounded (e.g., by the constant M_i^*), it follows that

$$\lim_{t\to\infty} \prod_{i=1}^n \int_0^\infty e^{-z^2\left(1+\sum_{j=1}^{m_i} a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)} \cos\left(2z\eta_i - z^2\sum_{j=1}^{m_i} a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right) dz = \prod_{i=1}^n \sqrt{\frac{\pi}{4p_i}}\, e^{-\frac{\eta_i^2}{p_i}}$$

uniformly with respect to $\eta \in \mathbb{R}^n$.

Hence, there exists a positive T such that for any $t \in (T, +\infty)$, we have

$$\left| \prod_{i=1}^n \int_0^\infty e^{-z^2\left(1+\sum_{j=1}^{m_i} a_{ij}\cos\frac{b_{ij}z}{\sqrt{t}}\right)} \cos\left(2z\eta_i - z^2\sum_{j=1}^{m_i} a_{ij}\sin\frac{b_{ij}z}{\sqrt{t}}\right) dz - \prod_{i=1}^n \sqrt{\frac{\pi}{4p_i}}\, e^{-\frac{\eta_i^2}{p_i}} \right|$$
$$\le \frac{\varepsilon\pi^n}{2^{2n+1}A^n \sup|u_0|},$$

i.e., $|J_1| \le \frac{\varepsilon\pi^n}{2^{n+1}}$; therefore, $|u(x_0,t) - v(x_0,t)| < \varepsilon$.

Since ε is selected arbitrarily, it follows that

$$\lim_{t\to\infty} [u(x_0,t) - v(x_0,t)] = 0.$$

Since x_0 is selected arbitrarily, this completes the proof of Theorem 2.5.1. □

Similarly to Sec. 2.3, this implies the following assertion:

Corollary 2.5.1. *If $x \in \mathbb{R}^n$ and $l \in (-\infty, +\infty)$, then*

$$\lim_{t\to\infty} u(x,t) = l \iff \lim_{R\to\infty} \frac{1}{R^n} \int_{B_R(p_1,\dots,p_n)} u_0(x)\,dx = \frac{2\pi^{\frac{n}{2}} \prod_{i=1}^n \sqrt{p_i}}{n\Gamma(\frac{n}{2})}\, l,$$

where $B_R(p_1,\dots,p_n)=\left\{x\in\mathbb{R}^n\;\middle|\;\frac{x_1^2}{p_1}+\dots+\frac{x_n^2}{p_n}<R\right\}$.

Remark 2.5.1. Theorem 2.5.1 is valid for the case $n=1$ as well, i.e., the asymptotic closeness of solutions for Eq. (2.1) and the equation

$$\frac{\partial u}{\partial t}=p\frac{\partial^2 u}{\partial x^2}$$

takes place apart from Theorem 2.2.1; however, this provides no new information on the stabilization of the solution: the necessary and sufficient condition of the stabilization of the solution of problem (2.1), (1.4), implied by such a closeness theorem, coincides with the assertion of Corollary 2.3.1.

2.6. The General Case of Inhomogeneous Elliptic Operators

In this section, the investigation is extended to the case where the right-hand part of Eq. (2.6) contains low-order (nonlocal) terms as well. In detail, we consider only the aspects substantially different from the prototype case of homogeneous elliptic operators, considered in detail in Secs. 2.4 and 2.5. Thus, instead of (2.6), consider the equation

$$\frac{\partial u}{\partial t}=\Delta u+\sum_{i=1}^{n}\sum_{j=1}^{m_{2,i}}a_{ij}\frac{\partial^2 u}{\partial x_i^2}(x+h_{ij}^{(2)}e_i,t)+\sum_{i=1}^{n}\sum_{j=1}^{m_{1,i}}b_{ij}\frac{\partial u}{\partial x_i}(x+h_{ij}^{(1)}e_i,t)+\sum_{i=1}^{n}\sum_{j=1}^{m_{0,i}}c_{ij}u(x+h_{ij}^{(0)}e_i,t). \quad (2.13)$$

Here e_i denotes the ith coordinate vector in the space $\mathbb{R}^n$, $m_{k,i}\in\mathbb{N}$ for $i=\overline{1,n}$ and $k=\overline{0,2}$, and the coefficients a_{ij},b_{ij},c_{ij}, and $h_{ij}^{(k)}$ are assumed to be real for $i=\overline{1,n}$, $k=\overline{0,2}$, and $j=\overline{1,m_k}$.

Instead of (2.7), define the fundamental solution as follows:

$$\mathcal{E}_{(n)}(x,t)\overset{\text{def}}{=}\frac{1}{2^n}\int\limits_{\mathbb{R}^n}e^{-t\left[|\xi|^2+G_1(\xi)\right]}\cos\left[x\cdot\xi-tG_2(\xi)\right]d\xi, \quad (2.14)$$

where

$$G_1(\xi)=\sum_{i=1}^{n}\xi_i^2\sum_{j=1}^{m_{2,i}}a_{ij}\cos h_{ij}^{(2)}\xi_i+\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m_{1,i}}b_{ij}\sin h_{ij}^{(1)}\xi_i-\sum_{i=1}^{n}\sum_{j=1}^{m_{0,i}}c_{ij}\cos h_{ij}^{(0)}\xi_i$$

and

$$G_2(\xi)=\sum_{i=1}^{n}\xi_i^2\sum_{j=1}^{m_{2,i}}a_{ij}\sin h_{ij}^{(2)}\xi_i-\sum_{i=1}^{n}\xi_i\sum_{j=1}^{m_{1,i}}b_{ij}\cos h_{ij}^{(1)}\xi_i-\sum_{i=1}^{n}\sum_{j=1}^{m_{0,i}}c_{ij}\sin h_{ij}^{(0)}\xi_i.$$

The following assertion is valid:

Theorem 2.6.1. *Let the operator* $-L_{(n)}$ *be strongly elliptic in* $\mathbb{R}^n$. *Then function* (2.9) *with* $\mathcal{E}_{(n)}$ *defined by relation* (2.14) *satisfies* (*in the classical sense*) *Eq.* (2.13) *in* $\mathbb{R}^n\times(0,+\infty)$ *and is a unique solution* (*in the sense of generalized functions*) *of problem* (2.13), (1.4).

To prove this theorem, we substitute function (2.14) in Eq. (2.13):

$$2^n\frac{\partial\mathcal{E}_n}{\partial t}=-\int\limits_{\mathbb{R}^n}\left[|\xi|^2+G_1(\xi)\right]e^{-t\left[|\xi|^2+G_1(\xi)\right]}\cos\left[x\cdot\xi-tG_2(\xi)\right]d\xi$$

$$+\int\limits_{\mathbb{R}^n}G_2(\xi)e^{-t\left[|\xi|^2+G_1(\xi)\right]}\sin\left[x\cdot\xi-tG_2(\xi)\right]d\xi.$$

This can be reduced to

$$\int\limits_{\mathbb{R}^n}e^{-t\left[|\xi|^2+G_1(\xi)\right]}\left(G_2(\xi)\sin\left[x\cdot\xi-tG_2(\xi)\right]\right.$$

$$-G_1(\xi)\cos\left[x\cdot\xi - tG_2(\xi)\right] - |\xi|^2\cos\left[x\cdot\xi - tG_2(\xi)\right]\Big)\,d\xi.$$

Further,

$$\sin h_{ij}^{(2)}\xi_i \sin\left[x\cdot\xi - tG_2(\xi)\right] - \cos h_{ij}^{(2)}\xi_i \cos\left[x\cdot\xi - tG_2(\xi)\right] = -\cos\left[x\cdot\xi + h_{ij}^{(2)}\xi - tG_2(\xi)\right],$$

$$-\cos h_{ij}^{(1)}\xi_i \sin\left[x\cdot\xi - tG_2(\xi)\right] - \sin h_{ij}^{(1)}\xi_i \cos\left[x\cdot\xi - tG_2(\xi)\right] = -\sin\left[x\cdot\xi + h_{ij}^{(1)}\xi - tG_2(\xi)\right],$$

and

$$-\sin h_{ij}^{(0)}\xi_i \sin\left[x\cdot\xi - tG_2(\xi)\right] + \cos h_{ij}^{(0)}\xi_i \cos\left[x\cdot\xi - tG_2(\xi)\right] = \cos\left[x\cdot\xi + h_{ij}^{(0)}\xi - tG_2(\xi)\right].$$

Therefore,

$$2^n\frac{\partial\mathcal{E}_n}{\partial t} = -\sum_{i=1}^{n}\sum_{j=1}^{m_{2,i}} a_{ij}\int\limits_{\mathbb{R}^n} \xi_i^2 e^{-t\left[|\xi|^2+G_1(\xi)\right]}\cos\left[(x+h_{ij}^{(2)}e_i)\cdot\xi - tG_2(\xi)\right]d\xi$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{m_{1,i}} b_{ij}\int\limits_{\mathbb{R}^n} \xi_i e^{-t\left[|\xi|^2+G_1(\xi)\right]}\sin\left[(x+h_{ij}^{(1)}e_i)\cdot\xi - tG_2(\xi)\right]d\xi$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{m_{0,i}} c_{ij}\int\limits_{\mathbb{R}^n} e^{-t\left[|\xi|^2+G_1(\xi)\right]}\cos\left[(x+h_{ij}^{(0)}e_i)\cdot\xi - tG_2(\xi)\right]d\xi$$
$$-\int\limits_{\mathbb{R}^n} |\xi|^2 e^{-t\left[|\xi|^2+G_1(\xi)\right]}\cos\left[x\cdot\xi - tG_2(\xi)\right]d\xi,$$

$$2^n\frac{\partial\mathcal{E}_n}{\partial x_i} = -\int\limits_{\mathbb{R}^n} \xi_i e^{-t\left[|\xi|^2+G_1(\xi)\right]}\sin\left[x\cdot\xi - tG_2(\xi)\right]d\xi,$$

and

$$2^n\frac{\partial^2\mathcal{E}_n}{\partial x_i^2} = -\int\limits_{\mathbb{R}^n} \xi_i^2 e^{-t\left[|\xi|^2+G_1(\xi)\right]}\cos\left[x\cdot\xi - tG_2(\xi)\right]d\xi.$$

Thus, the function $\mathcal{E}_{(n)}(x,t)$ satisfies Eq. (2.13) in $\mathbb{R}^n\times(0,+\infty)$.

Now, represent $G_1(\xi)$ as

$$\sum_{i=1}^{n}\left(\xi_i^2\sum_{j=1}^{m_{2,i}} a_{ij}\cos h_{ij}^{(2)}\xi_i + \xi_i\sum_{j=1}^{m_{1,i}} b_{ij}\sin h_{ij}^{(1)}\xi_i - \sum_{j=1}^{m_{0,i}} c_{ij}\cos h_{ij}^{(0)}\xi_i\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} G_{1,i}(\xi_i),$$

and represent $G_2(\xi)$ as

$$\sum_{i=1}^{n}\left(\xi_i^2\sum_{j=1}^{m_{2,i}} a_{ij}\sin h_{ij}^{(2)}\xi_i - \xi_i\sum_{j=1}^{m_{1,i}} b_{ij}\cos h_{ij}^{(1)}\xi_i - \sum_{j=1}^{m_{0,i}} c_{ij}\sin h_{ij}^{(0)}\xi_i\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} G_{2,i}(\xi_i).$$

Then function (2.14) is equal to

$$\frac{1}{2^n}\underbrace{\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}}_{n\text{ times}}\prod_{i=1}^{n} e^{-t\left[\xi_i^2+G_{1,i}(\xi_i)\right]}\cos\sum_{i=1}^{n}\left[x_i\xi_i - tG_{2,i}(\xi_i)\right]d\xi_1\ldots d\xi_n.$$

Taking into account that the function $G_{1,i}$ is even and the function $G_{2,i}$ is odd for any $i = \overline{1,n}$, we reduce the last expression (similarly to the proof of Lemma 2.4.1) to

$$\prod_{i=1}^{n} \int_{0}^{\infty} e^{-t\left[\tau^2 + G_{1,i}(\tau)\right]} \cos\left[x_i\tau - tG_{2,i}(\tau)\right] d\tau. \tag{2.15}$$

hence, to prove the solvability, it remains (see the proof of Lemma 2.4.1) to prove analogs of Lemmas 2.2.1 and 2.2.2 for the case where the function (of one spatial variable) $\mathcal{E}(x,t)$ has the form

$$\int_{0}^{\infty} e^{-t\left[\tau^2 + G_1(\tau)\right]} \cos\left[x\tau - tG_2(\tau)\right] d\tau,$$

where G_1 and G_2 are $G_{1,i}$ and $G_{2,i}$ with arbitrary $i = \overline{1,n}$, respectively. To do this, as in Lemma 2.2.1, fix a positive t and consider

$$\int_{0}^{\infty} e^{-t\left[\tau^2 + G_1(\tau)\right]} \cos\left[tG_2(\tau)\right] \cos x\tau d\tau.$$

Integrating by parts, reduce the last expression to

$$\frac{\sin x\tau}{x} e^{-t\left[\tau^2 + G_1(\tau)\right]} \cos\left[tG_2(\tau)\right] \Big|_{\tau=0}^{\tau=\infty} - \frac{1}{x} \int_{0}^{\infty} \sin x\tau \left(e^{-t\left[\tau^2 + G_1(\tau)\right]} \cos\left[tG_2(\tau)\right]\right)' d\tau$$

$$= -\frac{1}{x} \int_{0}^{\infty} \sin x\tau \left(e^{-t\left[\tau^2 + G_1(\tau)\right]} \cos\left[tG_2(\tau)\right]\right)' d\tau$$

(the former factor of the integrated term vanishes at zero, while the latter vanishes at infinity);

$$\left(e^{-t\left[\tau^2 + G_1(\tau)\right]} \cos\left[tG_2(\tau)\right]\right)' = -e^{-t\left[\tau^2 + G_1(\tau)\right]} \left(\left[2\tau + G_1'(\tau)\right] \cos\left[tG_2(\tau)\right] + tG_2'(\tau) \sin\left[tG_2(\tau)\right]\right).$$

Obviously, $G_1'(0) = G_2(0) = 0$; hence, integrating by parts again, we obtain

$$-\frac{1}{x^2} \int_{0}^{\infty} \cos x\tau \left(e^{-t\left[\tau^2 + G_1(\tau)\right]} \cos\left[tG_2(\tau)\right]\right)'' d\tau$$

because the integrated term vanishes again. The last integral is a bounded function of x; therefore, Lemma 2.2.1 is valid for the specified case. In the same way, we prove the boundedness of the functions $x^2 \frac{\partial \mathcal{E}}{\partial x}$ and $x^2 \frac{\partial^2 \mathcal{E}}{\partial x^2}$ for any positive t.

Further, arguing exactly as in the proof of Theorem 2.4.1, we prove the solvability.

As above, to prove the uniqueness, consider the symbol of the corresponding elliptic operator:

$$\mathcal{P}(z) = -|z|^2 - \sum_{k=1}^{n} z_k^2 \sum_{j=1}^{m_{2,k}} a_{kj} e^{-ih_{kj}^{(2)} z_k} - i \sum_{k=1}^{n} z_k \sum_{j=1}^{m_{1,k}} b_{kj} e^{-ih_{kj}^{(1)} z_k}$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{m_{0,k}} c_{kj} e^{-ih_{kj}^{(0)} z_k} = \sum_{k=1}^{n} (\tau_k^2 - \sigma_k^2 - 2i\sigma_k\tau_k) \left(1 + \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \cos h_{kj}^{(2)} \sigma_k\right.$$

$$\left. - i \sum_{j=1}^{m_{2,k}} a_{kj} e^{h_{kj}^{(2)}\tau_k} \sin h_{kj}^{(2)} \sigma_k\right) + \sum_{k=1}^{n} (\tau_k - i\sigma_k) \left(\sum_{j=1}^{m_{1,k}} b_{kj} e^{h_{kj}^{(1)}\tau_k} \cos h_{kj}^{(1)} \sigma_k\right.$$

$$- i\sum_{j=1}^{m_{1,k}} b_{kj}e^{h_{kj}^{(1)}\tau_k}\sin h_{kj}^{(1)}\sigma_k\Bigg) + \sum_{k=1}^{n}\sum_{j=1}^{m_{0,k}} c_{kj}e^{h_{kj}^{(0)}\tau_k}\cos h_{kj}^{(0)}\sigma_k - i\sum_{k=1}^{n}\sum_{j=1}^{m_{0,k}} c_{kj}e^{h_{kj}^{(0)}\tau_k}\sin h_{kj}^{(0)}\sigma_k.$$

Hence,

$$\begin{aligned}\mathrm{Re}\mathcal{P}(z) = \sum_{k=1}^{n}\Bigg[(\tau_k^2 - \sigma_k^2)\Bigg(1 + \sum_{j=1}^{m_{2,k}} a_{kj}e^{h_{kj}^{(2)}\tau_k}\cos h_{kj}^{(2)}\sigma_k\Bigg)\\ -2\sigma_k\tau_k\sum_{j=1}^{m_{2,k}} a_{kj}e^{h_{kj}^{(2)}\tau_k}\sin h_{kj}^{(2)}\sigma_k + \tau_k\sum_{j=1}^{m_{1,k}} b_{kj}e^{h_{kj}^{(1)}\tau_k}\cos h_{kj}^{(1)}\sigma_k\\ - \sigma_k\sum_{j=1}^{m_{1,k}} b_{kj}e^{h_{kj}^{(1)}\tau_k}\sin h_{kj}^{(1)}\sigma_k + \sum_{j=1}^{m_{0,k}} c_{kj}e^{h_{kj}^{(0)}\tau_k}\cos h_{kj}^{(0)}\sigma_k\Bigg].\end{aligned}$$

This is equal to

$$\begin{aligned}|\tau|^2 - |\sigma|^2 + \sum_{k=1}^{n}\Bigg[(\tau_k^2 - \sigma_k^2)\sum_{j=1}^{m_{2,k}} a_{kj}e^{h_{kj}^{(2)}\tau_k}\cos h_{kj}^{(2)}\sigma_k\\ -2\sigma_k\tau_k\sum_{j=1}^{m_{2,k}} a_{kj}e^{h_{kj}^{(2)}\tau_k}\sin h_{kj}^{(2)}\sigma_k + \tau_k\sum_{j=1}^{m_{1,k}} b_{kj}e^{h_{kj}^{(1)}\tau_k}\cos h_{kj}^{(1)}\sigma_k\\ - \sigma_k\sum_{j=1}^{m_{1,k}} b_{kj}e^{h_{kj}^{(1)}\tau_k}\sin h_{kj}^{(1)}\sigma_k + \sum_{j=1}^{m_{0,k}} c_{kj}e^{h_{kj}^{(0)}\tau_k}\cos h_{kj}^{(0)}\sigma_k\Bigg].\end{aligned}$$

Thus, the function $\mathcal{Q}(z, t_0, t)$ satisfies the same estimate as in Sec. 2.4 (generally, with different constants), which proves the uniqueness of the constructed solution.

To investigate the long-time behavior of problem (2.13), (1.4), consider (apart from the specified problem) the problem

$$\frac{\partial w}{\partial t} = \Delta w,\ x \in \mathbb{R}^n, t > 0, \tag{2.16}$$

$$w\big|_{t=0} = w_0(x),\ x \in \mathbb{R}^n, \tag{2.17}$$

where $w_0(x) = u_0(\sqrt{p_1}\,x_1, \ldots, \sqrt{p_n}\,x_n)$, i.e., the initial-value function depends on positive parameters $p_1, \ldots, p_n$.

The classical bounded solution of the last problem (it exists and is unique due to the continuity and boundedness of the function $w_0(x)$) is denoted by $w(x, t)$.

In the sequel, without loss of generality, we assume that for any $k = \overline{1, n}$, the number sets $\{b_{kj}h_{kj}^{(1)}\}_{j=1}^{m_{1,k}}$ and $\{c_{kj}\}_{j=1}^{m_{0,k}}$ do not increase. For any $k = \overline{1, n}$, denote $\min\limits_{b_{kj}h_{kj}^{(1)}>0} j$ by $\widetilde{m}_{1,k}$ and denote $\min\limits_{c_{kj}>0} j$ by $\widetilde{m}_{0,k}$; if k is such that $b_{kj}h_{kj}^{(1)} \le 0$ ($c_{kj} \le 0$) for any $j = \overline{1, m_{1,k}}$ ($j = \overline{1, m_{0,k}}$), then we assign $\widetilde{m}_{i,k} = m_{i,k} + 1, i = 0, 1$. Denote the positive constant $1 + \sum\limits_{j=1}^{m_{2,k}} a_{kj} + \sum\limits_{j\ge\widetilde{m}_{1,k}} b_{kj}h_{kj}^{(1)}$ by σ_k, $k = \overline{1, n}$. By $L_{(n)}$ denote the elliptic operator at the right-hand part of (2.13). Together with $L_{(n)}$, consider the operator $\mathcal{L}$ acting as follows:

$$\mathcal{L}u \overset{\text{def}}{=} \Delta u + \sum_{k=1}^{n}\left[\sum_{j<\widetilde{m}_{0,k}} \frac{c_{kj}}{\sigma_k} u(x + h_{kj}^{(0)} e_k, t) - \sum_{j<\widetilde{m}_{1,k}} \frac{2|b_{kj}|}{\sigma_k} u(x + \sqrt{|h_{kj}^{(1)}|}\, e_k, t)\right].$$

Note that nonlocal terms of the differential-difference operator $\mathcal{L}$ have only zero order, but it depends on the coefficients at high-order nonlocal terms of the original operator $L_{(n)}$.

Denote $\sum\limits_{k=1}^{n} \frac{1}{\sigma_k}\left(\sum\limits_{j<\widetilde{m}_{0,k}} c_{kj} - 2\sum\limits_{j<\widetilde{m}_{1,k}} |b_{kj}|\right) I - \mathcal{L}$ by R. The following assertion is valid.

Theorem 2.6.2. *Let $R(\xi)$ be positive definite. Then*

$$\lim_{t\to\infty}\left[e^{-t\sum\limits_{k=1}^{n}\sum\limits_{j=1}^{m_{0,k}} c_{kj}} u(x_0, t) - w\left(\frac{x_1^0 + q_1 t}{\sqrt{p_1}}, \dots, \frac{x_n^0 + q_n t}{\sqrt{p_n}}, t\right)\right] = 0$$

for any $x_0 \overset{\text{def}}{=} (x_1^0, \dots, x_n^0)$ from $\mathbb{R}^n$, where

$$p_i = 1 + \sum_{j=1}^{m_{2,i}} a_{ij} + \sum_{j=1}^{m_{1,i}} b_{ij} h_{ij}^{(1)} + \frac{1}{2}\sum_{j=1}^{m_{0,i}} c_{ij}\left[h_{ij}^{(0)}\right]^2 \quad \textit{and} \quad q_i = \sum_{j=1}^{m_{1,i}} b_{ij} + \sum_{j=1}^{m_{0,i}} c_{ij} h_{ij}^{(0)},\ i = \overline{1, n}.$$

Proof. First, we prove that $p_1, \dots, p_n$ are positive under the conditions of the theorem. To do this, we consider the positive definiteness condition for $R(\xi)$:

$$\sum_{k=1}^{n} \frac{1}{\sigma_k}\left(\sum_{j<\widetilde{m}_{0,k}} c_{kj} - 2\sum_{j<\widetilde{m}_{1,k}} |b_{kj}|\right) + |\xi|^2 - \sum_{k=1}^{n}\frac{1}{\sigma_k}\left(\sum_{j<\widetilde{m}_{0,k}} c_{kj}\cos h_{kj}^{(0)}\xi_k - 2\sum_{j<\widetilde{m}_{1,k}} |b_{kj}|\cos\sqrt{|h_{kj}^{(1)}|}\,\xi_k\right) \geq C|\xi|^2.$$

Then we take an arbitrary $k \in \overline{1, n}$. The last inequality remains valid if we set $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$ to be equal to zero. Therefore, the inequality

$$\sum_{j<\widetilde{m}_{0,k}} c_{kj} - 2\sum_{j<\widetilde{m}_{1,k}} |b_{kj}| + \sigma_k\xi_k^2 + 2\sum_{j<\widetilde{m}_{1,k}} |b_{kj}|\cos\sqrt{|h_{kj}^{(1)}|}\,\xi_k - \sum_{j<\widetilde{m}_{0,k}} c_{kj}\cos h_{kj}^{(0)}\xi_k \geq C\xi_k^2$$

holds for any positive ξ_k. This implies the following inequality:

$$C\xi_k^2 \leq \sigma_k\xi_k^2 - 2\sum_{j<\widetilde{m}_{1,k}} |b_{kj}|\left(1 - \cos\sqrt{|h_{kj}^{(1)}|}\,\xi_k\right) + \sum_{j<\widetilde{m}_{0,k}} c_{kj}\left(1 - \cos h_{kj}^{(0)}\xi_k\right).$$

Its right-hand part is equal to

$$\sigma_k\xi_k^2 - 4\sum_{j<\widetilde{m}_{1,k}} |b_{kj}|\sin^2\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2} + 2\sum_{j<\widetilde{m}_{0,k}} c_{kj}\sin^2\frac{h_{kj}^{(0)}\xi_k}{2}$$

$$= \sigma_k\xi_k^2 - \xi_k^2\sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}|\left(\frac{\sin\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}\right)^2 + \frac{\xi_k^2}{2}\sum_{j<\widetilde{m}_{0,k}} c_{kj}\left[h_{kj}^{(0)}\right]^2\left(\frac{\sin\frac{h_{kj}^{(0)}\xi_k}{2}}{\frac{h_{kj}^{(0)}\xi_k}{2}}\right)^2;$$

hence,

$$\sigma_k - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}|\left(\frac{\sin\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}\right)^2 + \frac{1}{2}\sum_{j<\widetilde{m}_{0,k}} c_{kj}\left[h_{kj}^{(0)}\right]^2\left(\frac{\sin\frac{h_{kj}^{(0)}\xi_k}{2}}{\frac{h_{kj}^{(0)}\xi_k}{2}}\right)^2 \geq C$$

for any positive ξ_k.

This yields the inequality

$$\sigma_k - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 > 0.$$

Indeed, assume the converse:

$$\sigma_k - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \leq 0.$$

Then for any positive ξ_k, the constant C does not exceed

$$\begin{aligned}
&\sigma_k + \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 - \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \\
&- \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| \left(\frac{\sin \frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}} \right)^2 + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \left(\frac{\sin \frac{h_{kj}^{(0)}\,\xi_k}{2}}{\frac{h_{kj}^{(0)}\,\xi_k}{2}} \right)^2 \\
&= \sigma_k - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \\
&- \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| \left[\left(\frac{\sin \frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}} \right)^2 - 1 \right] + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \left[\left(\frac{\sin \frac{h_{kj}^{(0)}\,\xi_k}{2}}{\frac{h_{kj}^{(0)}\,\xi_k}{2}} \right)^2 - 1 \right],
\end{aligned}$$

which does not exceed

$$\frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \left[\left(\frac{\sin \frac{h_{kj}^{(0)}\,\xi_k}{2}}{\frac{h_{kj}^{(0)}\,\xi_k}{2}} \right)^2 - 1 \right] - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| \left[\left(\frac{\sin \frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}}{\frac{\sqrt{|h_{kj}^{(1)}|}\,\xi_k}{2}} \right)^2 - 1 \right].$$

Since all the sums are finite, one can select a small positive ξ_k such that the last expression does not exceed $\frac{C}{2}$. The obtained contradiction proves the positivity of

$$\begin{aligned}
&\sigma_k - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \\
&= 1 + \sum_{j=1}^{m_{2,k}} a_{kj} + \sum_{j\geq\widetilde{m}_{1,k}} b_{kj} h_{kj}^{(1)} - \sum_{j<\widetilde{m}_{1,k}} |b_{kj}||h_{kj}^{(1)}| + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2 \\
&= 1 + \sum_{j=1}^{m_{2,k}} a_{kj} + \sum_{j=1}^{m_{1,k}} b_{kj} h_{kj}^{(1)} + \frac{1}{2} \sum_{j<\widetilde{m}_{0,k}} c_{kj} \left[h_{kj}^{(0)}\right]^2.
\end{aligned}$$

Hence, p_k is positive a fortiori.

Now, fix an arbitrary x_0 from $\mathbb{R}^n$. Then

$$w\left(\frac{x_1^0 + q_1 t}{\sqrt{p_1}}, \ldots, \frac{x_n^0 + q_n t}{\sqrt{p_n}}, t \right) = \frac{1}{(2\sqrt{\pi t})^n} \int\limits_{\mathbb{R}^n} u_0(\sqrt{p_1}\,\xi_1, \ldots, \sqrt{p_n}\,\xi_n) e^{-\frac{1}{4t} \sum\limits_{i=1}^{n} \left(\frac{x_i^0 + q_i t}{\sqrt{p_i}} - \xi_i \right)^2} d\xi.$$

The last expression is equal to

$$\frac{1}{(2\sqrt{\pi t})^n \prod\limits_{i=1}^{n}\sqrt{p_i}} \int\limits_{\mathbb{R}^n} u_0(\eta) e^{-\frac{1}{t}\sum\limits_{i=1}^{n}\frac{(x_i^0+q_it-\eta_i)^2}{4p_i}} d\eta = \frac{1}{\pi^{\frac{n}{2}}\prod\limits_{i=1}^{n}\sqrt{p_i}} \int\limits_{\mathbb{R}^n} u_0(x_0 - 2\sqrt{t}\xi) e^{-\sum\limits_{i=1}^{n}\frac{(2\xi_i+q_i\sqrt{t})^2}{4p_i}} d\xi$$

$$= \frac{1}{2^n\pi^{\frac{n}{2}}\prod\limits_{i=1}^{n}\sqrt{p_i}} \int\limits_{\mathbb{R}^n} u_0(x_0 + tq - \sqrt{t}y) e^{-\sum\limits_{i=1}^{n}\frac{y_i^2}{4p_i}} dy,$$

where q denotes the vector $(q_1, \dots, q_n)$. Further, by virtue of (2.11) and (2.15), we have

$$u(x_0,t) = \left(\frac{2\sqrt{t}}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 - 2\sqrt{t}\eta) \prod_{i=1}^{n} \int\limits_0^\infty e^{-t\left[\tau^2+G_{1,i}(\tau)\right]} \cos\left[2\sqrt{t}\eta_i\tau - tG_{2,i}(\tau)\right] d\tau d\eta$$

$$= \left(\frac{\sqrt{t}}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 + tq - \sqrt{t}y) \prod_{i=1}^{n} \int\limits_0^\infty e^{-t\left[\tau^2+G_{1,i}(\tau)\right]} \cos\left[y_i\tau\sqrt{t} - q_i\tau t - tG_{2,i}(\tau)\right] d\tau dy.$$

Hence,

$$e^{-t\sum\limits_{k=1}^{n}\sum\limits_{j=1}^{m_{0,k}} c_{kj}} u(x_0,t) = \left(\frac{\sqrt{t}}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 + tq - \sqrt{t}y)$$

$$\times \prod_{i=1}^{n} \int\limits_0^\infty e^{-t\left[\tau^2+G_{1,i}(\tau)+\sum\limits_{j=1}^{m_{0,i}} c_{ij}\right]} \cos\left[y_i\tau\sqrt{t} - q_i\tau t - tG_{2,i}(\tau)\right] d\tau dy,$$

which is equal to

$$\left(\frac{1}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 + tq - \sqrt{t}y) \prod_{i=1}^{n} \int\limits_0^\infty e^{-z^2 - tG_{1,i}\left(\frac{z}{\sqrt{t}}\right) - t\sum\limits_{j=1}^{m_{0,i}} c_{ij}} \cos\left[y_iz - q_iz\sqrt{t} - tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right] dzdy.$$

Thus,

$$e^{-t\sum\limits_{k=1}^{n}\sum\limits_{j=1}^{m_{0,k}} c_{kj}} u(x_0,t) - w\left(\frac{x_1^0+q_1t}{\sqrt{p_1}}, \dots, \frac{x_n^0+q_nt}{\sqrt{p_n}}, t\right)$$

$$= \left(\frac{1}{\pi}\right)^n \int\limits_{\mathbb{R}^n} u_0(x_0 + tq - \sqrt{t}y) \left(\prod_{i=1}^{n} \int\limits_0^\infty e^{-z^2 - tG_{1,i}\left(\frac{z}{\sqrt{t}}\right) - t\sum\limits_{j=1}^{m_{0,i}} c_{ij}}\right.$$

$$\left. \times \cos\left[y_iz - q_iz\sqrt{t} - tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right] dz - \prod_{i=1}^{n} \frac{\sqrt{\pi}}{2\sqrt{p_i}} e^{-\frac{y_i^2}{4p_i}}\right) dy. \tag{2.18}$$

The following assertions are valid:

Lemma 2.6.1. *Suppose that the conditions of Theorem 2.6.2 are satisfied and $i \in \overline{1,n}$. Then*

$$\int\limits_0^\infty e^{-z^2 - tG_{1,i}\left(\frac{z}{\sqrt{t}}\right) - t\sum\limits_{j=1}^{m_{0,i}} c_{ij}} \cos\left[yz - q_iz\sqrt{t} - tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right] dz - \frac{\sqrt{\pi}}{2\sqrt{p_i}} e^{-\frac{y^2}{4p_i}} \overset{t\to\infty}{\longrightarrow} 0$$

uniformly with respect to $y \in (-\infty, +\infty)$.

Lemma 2.6.2. *Suppose that the conditions of Theorem* 2.6.2 *are satisfied. Then for any* $i = \overline{1,n}$ *there exists* M_i *depending only on the coefficients of Eq.* (2.13) *such that*

$$\left| \int\limits_0^\infty e^{-z^2 - tG_{1,i}\left(\frac{z}{\sqrt{t}}\right) - t\sum\limits_{j=1}^{m_{0,i}} c_{ij}} \cos\left[yz - q_i z\sqrt{t} - tG_{2,i}\left(\frac{z}{\sqrt{t}}\right)\right] dz \right| < \frac{M}{y^2}$$

for any $y \in \mathbb{R}^1 \backslash \{0\}$ *and any* $t \in [1, \infty)$.

To prove Lemma 2.6.2, we use the same scheme as in the proof of Lemma 2.3.2 (see also [57, Lemma 5]).

To prove Lemma 2.6.1, we represent the power of the exponential function contained in the integral as follows:

$$-z^2 - z^2 \sum_{j=1}^{m_{2,i}} a_{ij} \cos\frac{h_{ij}^{(2)} z}{\sqrt{t}} - z\sqrt{t}\sum_{j=1}^{m_{1,i}} b_{ij} \sin\frac{h_{ij}^{(1)} z}{\sqrt{t}} + t\sum_{j=1}^{m_{0,i}} c_{ij}\left(\cos\frac{h_{ij}^{(0)} z}{\sqrt{t}} - 1\right)$$

$$= -z^2\left(1 + \sum_{j=1}^{m_{2,i}} a_{ij}\cos\frac{h_{ij}^{(2)} z}{\sqrt{t}}\right) - z\sqrt{t}\sum_{j=1}^{m_{1,i}} b_{ij}\frac{\sin\frac{h_{ij}^{(1)} z}{\sqrt{t}}}{\frac{h_{ij}^{(1)} z}{\sqrt{t}}}\frac{h_{ij}^{(1)} z}{\sqrt{t}} - 2t\sum_{j=1}^{m_{0,i}} c_{ij}\sin^2\frac{h_{ij}^{(0)} z}{2\sqrt{t}}$$

$$= -z^2\left[1 + \sum_{j=1}^{m_{2,i}} a_{ij}\cos\frac{h_{ij}^{(2)} z}{\sqrt{t}} + \sum_{j=1}^{m_{1,i}} b_{ij}h_{ij}^{(1)}\frac{\sin\frac{h_{ij}^{(1)} z}{\sqrt{t}}}{\frac{h_{ij}^{(1)} z}{\sqrt{t}}} + \frac{1}{2}\sum_{j=1}^{m_{0,i}} c_{ij}\left[h_{ij}^{(0)}\right]^2\left(\frac{\sin\frac{h_{ij}^{(0)} z}{2\sqrt{t}}}{\frac{h_{ij}^{(0)} z}{2\sqrt{t}}}\right)^2\right].$$

The independent variable of the cosine contained in the integral is represented as

$$z\left(y - q_i\sqrt{t} + \sqrt{t}\sum_{j=1}^{m_{1,i}} b_{ij}\cos\frac{h_{ij}^{(1)} z}{\sqrt{t}}\right) - z^2\sum_{j=1}^{m_{2,i}} a_{ij}\sin\frac{h_{ij}^{(2)} z}{\sqrt{t}} + t\sum_{j=1}^{m_{0,i}} c_{ij}\frac{\frac{\sin h_{ij}^{(0)} z}{\sqrt{t}}}{\frac{h_{ij}^{(0)} z}{\sqrt{t}}}\frac{h_{ij}^{(0)} z}{\sqrt{t}},$$

which is equal to

$$z\left(y - q_i\sqrt{t} + \sqrt{t}\sum_{j=1}^{m_{1,i}} b_{ij}\cos\frac{h_{ij}^{(1)} z}{\sqrt{t}} + \sqrt{t}\sum_{j=1}^{m_{0,i}} c_{ij}h_{ij}^{(0)}\frac{\frac{\sin h_{ij}^{(0)} z}{\sqrt{t}}}{\frac{h_{ij}^{(0)} z}{\sqrt{t}}}\right) - z^2\sum_{j=1}^{m_{2,i}} a_{ij}\sin\frac{h_{ij}^{(2)} z}{\sqrt{t}}.$$

The remaining part of Lemma 2.6.1 is the same as the proof of Lemma 2.3.1 (see also [57, Lemma 4]). Now, we can decompose (2.18) into sum (2.12) and estimate it in the same way as in the proof of Theorem 2.5.1 (using Lemmas 2.6.1 and 2.6.2 instead of Lemmas 2.3.1 and 2.3.2 respectively). This completes the proof of Theorem 2.6.2. □

Note that Remark 1.6.1 remains valid for Theorem 2.6.2 (i.e., for the case where principal terms of the equation are nonlocal) as well.

Remark 2.6.1. It is easy to see that the function

$$\omega(x,t) \stackrel{\text{def}}{=} w\left(\frac{x_1 + q_1 t}{\sqrt{p_1}}, \dots, \frac{x_n + q_n t}{\sqrt{p_n}}, t\right)$$

is a classical bounded solution of the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^n p_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^n q_i \frac{\partial u}{\partial x_i}, \tag{2.19}$$

satisfying condition (1.4); therefore, in the theorem on the (weighted) closeness of solutions, one can use problem (2.19), (1.4) instead of problem (2.16)-(2.17). Note that Theorem 2.6.2 establishes a qualitatively new behavior of the solution compared with the prototype case of the homogeneous elliptic operator at the right-hand part of the equation (see Theorem 2.5.1), and this qualitative novelty is preserved even if there are no nonlocal high-order terms (cf. [57, Th. 2]). Thus, adding low-order terms to a parabolic differential-difference equation, we can encounter qualitatively new effects (as in the classical parabolic theory, see [29]).

2.7. The General Case of Nonfactorable Fundamental Solutions

Let $a_{kj}, h_{kj} \in \mathbb{R}^1$, $k,j = \overline{1,n}$. In $\mathbb{R}^n\times(0,+\infty)$, consider the equation

$$\frac{\partial u}{\partial t} = Lu \overset{\text{def}}{=} \sum_{k,j=1}^{n} a_{kj} \frac{\partial^2 u}{\partial x_k^2}(x + h_{kj}e_j, t), \tag{2.20}$$

where e_j denotes the unit vector of the jth coordinate direction.

As in Sec. 2.1, consider the real part of the symbol of the operator L (cf. Sec. 1.6 and [102, §8]):

$$\mathrm{Re}L(\xi) = -\sum_{k,j=1}^{n} a_{kj}\xi_k^2 \cos h_{kj}\xi_j.$$

We say that $-L(\xi)$ is *positive definite* if there exists a positive C such that $-\mathrm{Re}L(\xi) \geq C|\xi|^2$ for $\xi \in \mathbb{R}^n$. Any operator $-L$ possessing the specified property is called a second-order operator *strongly elliptic* in the whole space.

In the sequel, we assume that the operator $-L$ is strongly elliptic in the whole space.

Consider problem (2.20), (1.4), assuming that $u_0(x)$ is continuous and bounded in $\mathbb{R}^n$.

On $\mathbb{R}^n\times(0,+\infty)$, define the following function:

$$\mathcal{E}(x,t) \overset{\text{def}}{=} \mathcal{E}_{a,h}(x,t) \overset{\text{def}}{=} \int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)} \cos[x\cdot\xi - tG_2(\xi)]d\xi, \tag{2.21}$$

where $G_1(\xi) = \sum\limits_{k,j=1}^{n} a_{kj}\xi_k^2 \cos h_{kj}\xi_j$ and $G_2(\xi) = \sum\limits_{k,j=1}^{n} a_{kj}\xi_k^2 \sin h_{kj}\xi_j$.

The strong ellipticity for the operator $-L$ implies the inequality $|\mathcal{E}(x,t)| \leq \int\limits_{\mathbb{R}^n} e^{-Ct|\xi|^2}d\xi$, i.e., for any t_0, T from $(0,+\infty)$, integral (2.21) converges absolutely and uniformly with respect to $(x,t) \in \mathbb{R}^n\times[t_0,T]$. Therefore, $\mathcal{E}(x,t)$ is well defined on $\mathbb{R}^n\times(0,+\infty)$.

Formally differentiate $\mathcal{E}$ with respect to t under the integral sign:

$$\frac{\partial\mathcal{E}}{\partial t} = -\int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)}G_1(\xi)\cos[x\cdot\xi - tG_2(\xi)]d\xi + \int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)}G_2(\xi)\sin[x\cdot\xi - tG_2(\xi)]d\xi.$$

Taking into account that

$$\sin h_{kj}\xi_j \, \sin[x\cdot\xi - tG_2(\xi)] - \cos h_{kj}\xi_j \, \cos[x\cdot\xi - tG_2(\xi)] = -\cos[(x + h_{kj}e_j)\cdot\xi - tG_2(\xi)],$$

we obtain the relation

$$\frac{\partial\mathcal{E}}{\partial t} = -\int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)} \sum_{k,j=1}^{n} a_{kj}\xi_k^2 \cos[(x + h_{kj}e_j)\cdot\xi - tG_2(\xi)]d\xi$$

$$= -\sum_{k,j=1}^{n} a_{kj} \int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)}\xi_k^2 \cos[(x + h_{kj}e_j)\cdot\xi - tG_2(\xi)]d\xi. \tag{2.22}$$

Further, formally differentiating $\mathcal{E}$ with respect to spatial variables under the integral sign, we obtain the relation

$$\frac{\partial^2 \mathcal{E}}{\partial x_k^2} = -\int\limits_{\mathbb{R}^n} \xi_k^2 e^{-tG_1(\xi)} \cos[x\cdot\xi - tG_2(\xi)]d\xi;$$

hence,

$$\frac{\partial^2 \mathcal{E}}{\partial x_k^2}(x + h_{kj}e_j, t) = -\int\limits_{\mathbb{R}^n} \xi_k^2 e^{-tG_1(\xi)} \cos[(x + h_{kj}e_j)\cdot\xi - tG_2(\xi)]d\xi. \tag{2.23}$$

The absolute value of each of those improper integrals is bounded from above by

$$\text{const}\int\limits_{\mathbb{R}^n} |\xi|^2 e^{-Ct|\xi|^2}d\xi,$$

i.e., it converges absolutely and uniformly with respect to $(x,t)\in\mathbb{R}^n\times[t_0,T]$ for any $t_0,T\in(0,+\infty)$. Therefore, differentiating under the integral sign is valid, and $\mathcal{E}(x,t)$ satisfies (in the classical sense) Eq. (2.20) in $\mathbb{R}^n\times(0,+\infty)$.

Fixing a positive t, estimate the behavior of the function $\mathcal{E}(x,t)$ and its derivatives as $x\to\infty$. To do this, decompose the specified functions into the terms $\mathcal{E}_1(x,t)$ as $\mathcal{E}_2(x,t)$:

$$\mathcal{E}_1(x,t) \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)}\cos tG_2(\xi)\cos x\cdot\xi\,d\xi \quad\text{and}\quad \mathcal{E}_2(x,t) \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)}\sin tG_2(\xi)\sin x\cdot\xi\,d\xi.$$

Let us prove the following assertion:

Lemma 2.7.1. *If $l\in\mathbb{N}$ and $t>0$, then $|x|^l\mathcal{E}(x,t)$ is bounded in $\mathbb{R}^n$.*

Proof. Let $t>0$ and $i\in\overline{1,n}$; then

$$x_i\mathcal{E}_1(x,t) = \lim_{R\to\infty}\int\limits_{|\xi|<R} e^{-tG_1(\xi)}\cos tG_2(\xi)\frac{\partial}{\partial\xi_i}\sin x\cdot\xi\,d\xi$$

$$= \lim_{R\to\infty}\left(\int\limits_{|\xi|=R} e^{-tG_1(\xi)}\cos tG_2(\xi)\sin x\cdot\xi\cos(\xi,e_i)\,dS_\xi - \int\limits_{|\xi|<R}\frac{\partial}{\partial\xi_i}\left[e^{-tG_1(\xi)}\cos tG_2(\xi)\right]\sin x\cdot\xi\,d\xi\right).$$

The absolute value of the surface integral of the last expression is bounded by

$$\int\limits_{|\xi|=R} e^{-tG_1(\xi)}dS_\xi \le \int\limits_{|\xi|=R} e^{-Ct|\xi|^2}dS_\xi = \text{const}\,R^{n-1}e^{-CtR^2} \xrightarrow{R\to\infty} 0;$$

therefore,

$$x_i\mathcal{E}_1(x,t) = -\int\limits_{\mathbb{R}^n}\frac{\partial}{\partial\xi_i}\left[e^{-tG_1(\xi)}\cos tG_2(\xi)\right]\sin x\cdot\xi\,d\xi.$$

It is easy to see that the absolute value of the integrand does not exceed $|P(\xi)|e^{-tG_1(\xi)}$, where P is a polynomial such that its coefficients depend only on t (which is fixed) and the coefficients of Eq. (2.20); therefore, the function $x_i\mathcal{E}_1(x,t)$ is bounded in $\mathbb{R}^n$.

Further,

$$x_i^2\mathcal{E}_1(x,t) = -x_i\int\limits_{\mathbb{R}^n}\frac{\partial}{\partial\xi_i}\left[e^{-tG_1(\xi)}\cos tG_2(\xi)\right]\sin x\cdot\xi\,d\xi = \int\limits_{\mathbb{R}^n}\frac{\partial}{\partial\xi_i}\left[e^{-tG_1(\xi)}\cos tG_2(\xi)\right]\frac{\partial}{\partial\xi_i}\cos x\cdot\xi\,d\xi.$$

Represent the last expression as follows:

$$\lim_{R\to\infty} \int\limits_{|\xi|<R} \frac{\partial}{\partial \xi_i}\left[e^{-tG_1(\xi)} \cos tG_2(\xi)\right] \frac{\partial}{\partial \xi_i} \cos x\cdot\xi \, d\xi$$

$$= \lim_{R\to\infty} \left(\int\limits_{|\xi|=R} \frac{\partial}{\partial \xi_i}\left[e^{-tG_1(\xi)} \cos tG_2(\xi)\right] \cos x\cdot\xi \, \cos(\xi, e_i)\, dS_\xi \right.$$

$$\left. - \int\limits_{\mathbb{R}^n} \frac{\partial^2}{\partial \xi_i^2}\left[e^{-tG_1(\xi)} \cos tG_2(\xi)\right] \cos x\cdot\xi \, d\xi \right).$$

As we see above, the absolute value of the integrand of the last surface integral does not exceed $|P(\xi)|e^{-tG_1(\xi)}$; hence, the absolute value of the specified integral does not exceed

$$e^{-CtR^2} \int\limits_{|\xi|=R} |P(\xi)| dS_\xi \overset{R\to\infty}{\longrightarrow} 0.$$

This implies that

$$x_i^2 \mathcal{E}_1(x,t) = -\int\limits_{|\xi|<R} \frac{\partial^2}{\partial \xi_i^2}\left[e^{-tG_1(\xi)} \cos tG_2(\xi)\right] \cos x\cdot\xi \, d\xi.$$

Differentiating the integrand, we see that its absolute value does not exceed $|P(\xi)|e^{-tG_1(\xi)}$ (in general, the polynomial P might change), i.e., the function $x_i^2\mathcal{E}_1(x,t)$ is bounded in $\mathbb{R}^n$.

Continuing to integrate by parts and taking into account that

$$\left| \frac{\partial^l}{\partial \xi_i^l}\left[e^{-tG_1(\xi)} \cos tG_2(\xi)\right] \right| \le |P(\xi)|e^{-tG_1(\xi)} \tag{2.24}$$

for any $l \in \mathbb{N}$, while the coefficients of the polynomial P depend only on l, t, and the coefficients of Eq. (2.20), we obtain the boundedness of the function $x_i^l\mathcal{E}_1(x,t)$ for any $i \in \overline{1,n}$; hence, the function $|x|^l\mathcal{E}_1(x,t)$ is bounded as well.

The boundedness of the function $|x|^l\mathcal{E}_2(x,t)$ is proved in the same way.

This completes the proof of Lemma 2.7.1. □

Thus, the following function is defined in $\mathbb{R}^n \times (0,+\infty)$:

$$u(x,t) \overset{\text{def}}{=} \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \mathcal{E}(x-\xi,t) u_0(\xi) d\xi. \tag{2.25}$$

Apply to representations (2.22) and (2.23) the procedure applied to integral (2.21) in Lemma 2.7.1. Taking into account that estimate (2.24) remains valid for the functions $\frac{\partial^l}{\partial \xi_i^l}\left[\xi_k^2 e^{-tG_1(\xi)} \cos tG_2(\xi)\right]$, $k = \overline{1,n}$, we see that the assertion of Lemma 2.7.1 holds for the functions $\frac{\partial \mathcal{E}}{\partial t}$ and $\frac{\partial^2 \mathcal{E}}{\partial x_k^2}$ as well. This means that function (2.25) can be differentiated under the integral sign. Since the function $\mathcal{E}(x,t)$ satisfies Eq. (2.20), this implies the following assertion.

Theorem 2.7.1. *Let the operator $-L$ be strongly elliptic in $\mathbb{R}^n$. Then function* (2.25) *satisfies* (*in the classical sense*) *Eq.* (2.20) *in* $\mathbb{R}^n \times (0,+\infty)$.

Remark 2.7.1. The fact that function (2.25) satisfies problem (2.20), (1.4) *in the sense of generalized functions* is known (see, e.g., [16]). The only novelty of Theorem 2.7.1 is the fact that this solution is *classical* in $\mathbb{R}^n \times (0,+\infty)$.

To establish the uniqueness of this solution, investigate (according to [16]) the real part of the symbol of the elliptic operator L contained in Eq. (2.20). The specified symbol $\mathcal{P}(z_1,\dots,z_n) \overset{\text{def}}{=} \mathcal{P}(z) \overset{\text{def}}{=} \mathcal{P}(\sigma + i\tau) \overset{\text{def}}{=} \mathcal{P}(\sigma_1 + i\tau_1,\dots,\sigma_n + i\tau_n)$ is equal to

$$
\begin{aligned}
-\sum_{k=1}^{n} z_k^2 \sum_{j=1}^{n} a_{kj} e^{-ih_{kj}z_j} &= \sum_{k=1}^{n}(\tau_k^2 - \sigma_k^2 - 2i\sigma_k\tau_k)\sum_{j=1}^{n} a_{kj} e^{-ih_{kj}z_j} \\
&= \sum_{k=1}^{n}(\tau_k^2 - \sigma_k^2 - 2i\sigma_k\tau_k)\sum_{j=1}^{n} a_{kj} e^{h_{kj}\tau_j - ih_{kj}\sigma_j} \\
&= \sum_{k=1}^{n}(\tau_k^2 - \sigma_k^2 - 2i\sigma_k\tau_k)\left(\sum_{j=1}^{n} a_{kj} e^{h_{kj}\tau_j}\cos h_{kj}\sigma_j - i\sum_{j=1}^{n} a_{kj} e^{h_{kj}\tau_j}\sin h_{kj}\sigma_j\right).
\end{aligned}
$$

Thus,

$$
\mathrm{Re}\mathcal{P}(z) = \sum_{k=1}^{n}\left[(\tau_k^2 - \sigma_k^2)\sum_{j=1}^{n} a_{kj} e^{h_{kj}\tau_j}\cos h_{kj}\sigma_j - 2\sigma_k\tau_k\sum_{j=1}^{n} a_{kj} e^{h_{kj}\tau_j}\sin h_{kj}\sigma_j\right].
$$

Therefore, the function $\mathcal{Q}(z,t_0,t) \overset{\text{def}}{=} e^{(t-t_0)\mathcal{P}(z)}$ satisfies the estimate

$$
|\mathcal{Q}(z,t_0,t)| \le e^{(t-t_0)\left[C_1(1+|\sigma|^4) + C_2 e^{C_3|\tau|}\right]},
$$

which implies (see [16, Ch. 2, Appendix 1]) that problem (2.20), (1.4) has at most one solution in the sense of generalized functions.

Remark 2.7.2. In general, the uniqueness theorem for problem (2.20), (1.4) (in corresponding spaces of generalized functions) holds for much more wide classes of initial-value functions than the class of continuous bounded functions. In particular, it holds for Tikhonov classes and their generalizations (see [2] and [40]). However, we consider only the case of continuous bounded initial-value functions because we investigate the closeness of solutions of the specified problem and *classical* parabolic problems.

Now, investigate the behavior of $u(x,t)$ as $t \to \infty$. First, we prove the following assertion.

Lemma 2.7.2. *If the conditions of Theorem* 2.7.1 *are satisfied, then the constant* $p_k \overset{\text{def}}{=} \sum\limits_{j=1}^{n} a_{kj}$ *is positive for any* $k \in \overline{1,n}$.

Proof. Let $k \in \overline{1,n}$. Assume the converse: $\sum\limits_{j=1}^{n} a_{kj} \le 0$. Take the strong ellipticity condition for the operator $-L$ and assign $\xi_1 = \dots = \xi_{k-1} = \xi_{k+1} = \dots = \xi_n = 0$ in that inequality. We obtain the inequality

$$
\xi_k^2\left(\sum_{\substack{j=1 \\ j\ne k}}^{n} a_{kj} + a_{kk}\cos h_{kk}\xi_k\right) \ge C\xi_k^2.
$$

Hence,

$$
C \le \sum_{\substack{j=1 \\ j\ne k}}^{n} a_{kj} + a_{kk}\cos h_{kk}\xi_k + a_{kk} - a_{kk} = \sum_{k,j=1}^{n} a_{kj} + a_{kk}(\cos h_{kk}\xi_k - 1) \le a_{kk}(\cos h_{kk}\xi_k - 1)
$$

provided that ξ_k is different from zero. Now, we can select ξ_k such that it is different from zero, but its absolute value is sufficiently small to obtain a contradiction with the positivity of the constant C.

This completes the proof of Lemma 2.7.2. ∎

Together with the *differential-difference* parabolic equation (2.20), consider the *differential* parabolic equation

$$\frac{\partial u}{\partial t}=\sum_{k=1}^{n}p_k\frac{\partial^2 u}{\partial x_k^2}. \tag{2.26}$$

By $v(x,t)$ denote the classical bounded solution of problem (2.26), (1.4) (it exists and is unique due to the continuity and boundedness of the function u_0).

The following assertion is valid:

Theorem 2.7.2. *If the conditions of Theorem* 2.7.1 *are satisfied, then*

$$\lim_{t\to\infty}[u(x,t)-v(x,t)]=0$$

for any $x\in\mathbb{R}^n$.

Proof. Let $x\in\mathbb{R}^n$. In (2.25), change the variables: $\frac{x-\xi_k}{2\sqrt{t}}=\eta_k$, $k=\overline{1,n}$. This yields the representation

$$u(x,t)=\left(\frac{\sqrt{t}}{\pi}\right)^n\int\limits_{\mathbb{R}^n}\mathcal{E}(2\sqrt{t}\eta,t)u_0(x-2\sqrt{t}\eta)d\eta.$$

Taking into account that

$$t^{\frac{n}{2}}\mathcal{E}(2\sqrt{t}\eta,t)=t^{\frac{n}{2}}\int\limits_{\mathbb{R}^n}e^{-tG_1(\xi)}\cos[2\sqrt{t}\xi\cdot\eta-tG_2(\xi)]d\xi$$
$$=\int\limits_{\mathbb{R}^n}e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]dz$$

and

$$v(x,t)=\frac{1}{(2\sqrt{\pi t})^n\prod\limits_{k=1}^{n}\sqrt{p_k}}\int\limits_{\mathbb{R}^n}u_0(\xi)e^{-\sum\limits_{k=1}^{n}\frac{(x_k-\xi_k)^2}{4p_kt}}d\xi$$

(since it is a solution of the Cauchy problem for a differential parabolic equation with constant coefficients), we obtain the following representation of the estimated difference:

$$u(x,t)-v(x,t)$$
$$=\left(\frac{1}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x-2\sqrt{t}\eta)\int\limits_{\mathbb{R}^n}e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]dzd\eta$$
$$-\frac{1}{(2\sqrt{\pi t})^n\prod\limits_{k=1}^{n}\sqrt{p_k}}\int\limits_{\mathbb{R}^n}u_0(\xi)e^{-\sum\limits_{k=1}^{n}\frac{(x_k-\xi_k)^2}{4p_kt}}d\xi$$
$$=\left(\frac{1}{\pi}\right)^n\int\limits_{\mathbb{R}^n}u_0(x-2\sqrt{t}\eta)\left(\int\limits_{\mathbb{R}^n}e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]dz\right.$$
$$\left.-\frac{\pi^{\frac{n}{2}}}{\prod\limits_{k=1}^{n}\sqrt{p_k}}e^{-\sum\limits_{k=1}^{n}\frac{\eta_k^2}{p_k}}\right)d\eta. \tag{2.27}$$

Let us prove the following two lemmas.

Lemma 2.7.3. *The relation*

$$\int_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] dz - \frac{\pi^{\frac{n}{2}}}{\prod\limits_{k=1}^{n}\sqrt{p_k}}\, e^{-\sum\limits_{k=1}^{n}\frac{\eta_k^2}{p_k}} \xrightarrow{t\to\infty} 0$$

holds uniformly with respect to $\eta \in \mathbb{R}^n$.

Proof. Consider the integral

$$\int_{\mathbb{R}^n} e^{-\sum\limits_{k=1}^{n} p_k z_k^2} \cos 2z\cdot\eta dz = \int_{\mathbb{R}^n} \prod_{k=1}^{n} e^{-p_k z_k^2} \cos\left(2\sum_{k=1}^{n} z_k\eta_k\right) dz.$$

The function $\cos\left(2\sum\limits_{k=1}^{n} z_k\eta_k\right)$ is a finite sum of terms of the form $\prod\limits_{k=1}^{n} f_k(2z_k\eta_k)$, where each function f_k is either the sine or the cosine, and only one of those terms contains no sines; this term is $\prod\limits_{k=1}^{n}\cos 2z_k\eta_k$. Therefore, the last integral is a finite sum of terms of the form

$$\underbrace{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{n \text{ times}} \prod_{k=1}^{n} e^{-p_k z_k^2} f_k(2z_k\eta_k) dz_1 \dots dz_n = \prod_{k=1}^{n}\int_{-\infty}^{+\infty} e^{-p_k\tau^2} f_k(2\eta_k\tau) d\tau.$$

Only one of those terms is different from zero; this is

$$\prod_{k=1}^{n}\int_{-\infty}^{+\infty} e^{-p_k\tau^2}\cos 2\eta_k\tau\, d\tau.$$

All other terms vanish because each of them contains at least one factor of the form

$$\prod_{k=1}^{n}\int_{-\infty}^{+\infty} e^{-p_k\tau^2}\sin 2\eta_k\tau\, d\tau = 0.$$

Thus,

$$\int_{\mathbb{R}^n} e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\cos 2z\cdot\eta dz = 2^n \prod_{k=1}^{n}\int_{0}^{+\infty} e^{-p_k\tau^2}\cos 2\eta_k\tau\, d\tau = \pi^{\frac{n}{2}}\prod_{k=1}^{n}\frac{e^{-\frac{\eta_k^2}{p_k}}}{\sqrt{p_k}}.$$

Hence, the following relation is valid for the second factor of the integrand of the external integral in (2.27):

$$\int_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] dz - \frac{\pi^{\frac{n}{2}}}{\prod\limits_{k=1}^{n}\sqrt{p_k}}\, e^{-\sum\limits_{k=1}^{n}\frac{\eta_k^2}{p_k}}$$

$$= \int_{\mathbb{R}^n}\left(e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] - e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\cos 2z\cdot\eta\right) dz. \quad (2.28)$$

The absolute value of integral (2.28) is estimated from above by the sum

$$\int_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} dz + \int_{\mathbb{R}^n} e^{-\sum\limits_{k=1}^{n} p_k z_k^2} dz \le \int_{\mathbb{R}^n} e^{-C|z|^2} dz + \int_{\mathbb{R}^n} e^{-\min\limits_{k=\overline{1,n}} p_k|z|^2} dz < \infty.$$

Therefore, integral (2.28) converges absolutely and uniformly with respect to $(t, \eta) \in \mathbb{R}^1_+ \times \mathbb{R}^n$.

Fix an arbitrary positive ε and represent (2.28) as the sum $\int\limits_{|z|<\delta} + \int\limits_{|z|\geq\delta} \stackrel{\text{def}}{=} I_{1,\delta} + I_{2,\delta}$. By virtue of the proved uniform convergence, there exists a value of the parameter δ such that $|I_{2,\delta}| < \dfrac{\varepsilon}{2}$ for any $t \in \mathbb{R}^1_+$ and any $\eta \in \mathbb{R}^n$. Fix that δ and consider the integral $I_{1,\delta}$.

Its integrand is equal to

$$e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\left(e^{-t\sum\limits_{k,j=1}^{n} a_{kj}\frac{z_k^2}{t}\cos\frac{h_{kj}z_j}{\sqrt{t}}+\sum\limits_{k=1}^{n} p_k z_k^2}\cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right)$$

$$= e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\left(e^{\sum\limits_{k=1}^{n} z_k^2\left(p_k-\sum\limits_{j=1}^{n} a_{kj}\cos\frac{h_{kj}z_j}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right)$$

$$= e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\left(e^{\sum\limits_{k=1}^{n} z_k^2\sum\limits_{j=1}^{n} a_{kj}\left(1-\cos\frac{h_{kj}z_j}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right)$$

$$= e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\left(e^{2\sum\limits_{k=1}^{n} z_k^2\sum\limits_{j=1}^{n} a_{kj}\sin^2\frac{h_{kj}z_j}{2\sqrt{t}}}\cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right)$$

$$= e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\left(e^{2\sum\limits_{k=1}^{n} z_k^2\sum\limits_{j=1}^{n} a_{kj}\sin^2\frac{h_{kj}z_j}{2\sqrt{t}}}\cos 2z\cdot\eta\,\cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right)$$

$$+ e^{-\sum\limits_{k=1}^{n} p_k z_k^2} e^{2\sum\limits_{k=1}^{n} z_k^2\sum\limits_{j=1}^{n} a_{kj}\sin^2\frac{h_{kj}z_j}{2\sqrt{t}}}\sin 2z\cdot\eta\,\sin\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] = A_1(\eta,t;z) + A_2(\eta,t;z),$$

where

$$A_1(\eta,t;z) = e^{-\sum\limits_{k=1}^{n} p_k z_k^2}\cos 2z\cdot\eta\left(e^{2\sum\limits_{k=1}^{n} z_k^2\sum\limits_{j=1}^{n} a_{kj}\sin^2\frac{h_{kj}z_j}{2\sqrt{t}}}\cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-1\right)$$

and

$$A_2(\eta,t;z) = e^{-\sum\limits_{k=1}^{n} p_k z_k^2} e^{2\sum\limits_{k,j=1}^{n} z_k^2 a_{kj}\sin^2\frac{h_{kj}z_j}{2\sqrt{t}}}\sin 2z\cdot\eta\,\sin\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right].$$

First, estimate the latter term:

$$\left|\int\limits_{|z|<\delta} A_2(\eta,t;z)dz\right| \leq \int\limits_{|z|<\delta} e^{2\sum\limits_{k,j=1}^{n}|a_{kj}|z_k^2}\left|\sin\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]\right|dz$$

$$\leq e^{2\delta^2\sum\limits_{k,j=1}^{n}|a_{kj}|}\int\limits_{|z|<\delta}\left|\sin\left(\sum\limits_{k,j=1}^{n}|a_{kj}|z_k^2\sin\frac{h_{kj}z_j}{\sqrt{t}}\right)\right|dz$$

$$\leq e^{2\delta^2\sum\limits_{k,j=1}^{n}|a_{kj}|}\int\limits_{|z|<\delta}\sum\limits_{k,j=1}^{n}|a_{kj}|z_k^2\left|\sin\frac{h_{kj}z_j}{\sqrt{t}}\right|dz$$

$$\le \delta^2 \sum_{k,j=1}^{n} |a_{kj}| e^{2\delta^2 \sum\limits_{k,j=1}^{n} |a_{kj}|} \int\limits_{|z|<\delta} \left|\sin \frac{h_{kj} z_j}{\sqrt{t}}\right| dz$$

$$\le \frac{\delta^3}{\sqrt{t}} \sum_{k,j=1}^{n} |a_{kj}||h_{kj}| e^{2\delta^2 \sum\limits_{k,j=1}^{n} |a_{kj}|} \int\limits_{|z|<\delta} dz \stackrel{\text{def}}{=} \frac{C_0 \delta^{n+3} e^{C_1 \delta^2}}{\sqrt{t}},$$

where the constants C_0 and C_1 depend only on the coefficients of Eq. (2.20).

Denoting $\frac{16 C_0^2 \delta^{2n+6} e^{2C_1 \delta^2}}{\varepsilon^2}$ by T_0, we obtain that the following inequality holds for any $t > T_0$ and any $\eta \in \mathbb{R}^n$:

$$\left| \int\limits_{|z|<\delta} A_2(\eta, t; z) dz \right| \le \frac{\varepsilon}{4}.$$

It remains to estimate the former term:

$$\left| \int\limits_{|z|<\delta} A_1(\eta, t; z) dz \right| \le \int\limits_{|z|<\delta} \left| e^{2 \sum\limits_{k,j=1}^{n} a_{kj} z_k^2 \sin^2 \frac{h_{kj} z_j}{2\sqrt{t}}} \cos\left(\sum_{k,j=1}^{n} a_{kj} z_k^2 \sin \frac{h_{kj} z_j}{\sqrt{t}} \right) - 1 \right| dz. \tag{2.29}$$

Without loss of generality, we assume that δ is sufficiently large to satisfy the inequality

$$\gamma \stackrel{\text{def}}{=} \frac{n \Gamma\left(\frac{n}{2}\right)}{24 \pi^{\frac{n}{2}} \delta^n} \varepsilon < 1.$$

The inequality

$$\left| 2 \sum_{k,j=1}^{n} a_{kj} z_k^2 \sin^2 \frac{h_{kj} z_j}{2\sqrt{t}} \right| \le 2\delta^2 \sum_{k,j=1}^{n} |a_{kj}| \frac{h_{kj}^2 z_j^2}{4t} \le \frac{\delta^4 \sum\limits_{k,j=1}^{n} |a_{kj}| h_{kj}^2}{2t}$$

is valid in the domain of integration of integral (2.29). Therefore, there exists a positive T_1 such that for $t > T_1$, the value of the exponential function in (2.29) belongs to $(1-\gamma, 1+\gamma)$.

The inequality

$$\left| \sum_{k,j=1}^{n} a_{kj} z_k^2 \sin \frac{h_{kj} z_j}{\sqrt{t}} \right| \le \delta^2 \sum_{k,j=1}^{n} |a_{kj}| \frac{|h_{kj} z_j|}{\sqrt{t}} \le \frac{\delta^3 \sum\limits_{k,j=1}^{n} |a_{kj} h_{kj}|}{\sqrt{t}}$$

is valid in the domain of integration of integral (2.29). Therefore, there exists a positive T_2 such that for $t > T_2$, the value of the cosine in (2.29) belongs to $(1-\gamma, 1+\gamma)$. Thus, for $t > \max\{T_1, T_2\}$, we have

$$1 - 3\gamma \le (1-\gamma)^2 < e^{2 \sum\limits_{k,j=1}^{n} a_{kj} z_k^2 \sin^2 \frac{h_{kj} z_j}{2\sqrt{t}}} \cos\left(\sum_{k,j=1}^{n} a_{kj} z_k^2 \sin \frac{h_{kj} z_j}{\sqrt{t}} \right) < (1+\gamma)^2 \le 1 + 3\gamma.$$

Hence, the integrand in (2.29) does not exceed 3γ and integral (2.29) does not exceed $3\gamma \frac{2\pi^{\frac{n}{2}} \delta^n}{n\Gamma\left(\frac{n}{2}\right)} = \frac{\varepsilon}{4}$.

Therefore, $|I_{1,\delta}| \le \frac{\varepsilon}{2}$ for any $t > \max\{T_0, T_1, T_2\}$ and any $\eta \in \mathbb{R}^n$.

This completes the proof of Lemma 2.7.3 because the positive ε is selected arbitrarily. □

Lemma 2.7.4. *There exists a positive M such that*

$$\left|\int\limits_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] dz\right| < \frac{M}{|\eta|^{n+1}}$$

for any $t > 1$ and any $\eta \in \mathbb{R}^n$.

Proof. Let $t > 1, i \in \overline{1,n}$. Represent the estimated integral as

$$\int\limits_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] \cos 2z\cdot\eta\, dz$$

$$+\int\limits_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \sin\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] \sin 2z\cdot\eta\, dz \overset{\text{def}}{=} f_1(t,\eta) + f_2(t,\eta)$$

and estimate

$$\eta_i f_1(t,\eta) = \eta_i \int\limits_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] \cos 2z\cdot\eta\, dz.$$

The last expression is equal to

$$\frac{1}{2}\lim_{R\to\infty} \int\limits_{|z|<R} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] \frac{\partial}{\partial z_i} \sin 2z\cdot\eta\, dz$$

$$= \frac{1}{2}\lim_{R\to\infty} \left[\int\limits_{|z|=R} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] \sin 2z\cdot\eta \cos(z,e_i)\, dS_z\right.$$

$$\left. - \int\limits_{|z|<R} \frac{\partial}{\partial z_i}\left(e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]\right) \sin 2z\cdot\eta\, dz\right]. \tag{2.30}$$

The absolute value of the surface integral in (2.30) does not exceed

$$\int\limits_{|z|=R} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} dS_z \le \int\limits_{|z|=R} e^{-C|z|^2} dS_z = \text{const } R^{n-1}e^{-CR^2} \overset{R\to\infty}{\longrightarrow} 0;$$

hence, (2.30) is equal to

$$-\frac{1}{2}\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial z_i}\left(e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)} \cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]\right) \sin 2z\cdot\eta\, dz$$

$$\overset{\text{def}}{=} -\frac{1}{2}\int\limits_{\mathbb{R}^n} g(z;t) \sin 2z\cdot\eta\, dz. \tag{2.31}$$

Let us compute

$$g(z;t) = \frac{\partial}{\partial z_i}\left(e^{-\sum\limits_{k,j=1}^{n} a_{kj} z_k^2 \cos\frac{h_{kj}z_j}{\sqrt{t}}} \cos \sum_{k,j=1}^{n} a_{kj} z_k^2 \sin\frac{h_{kj}z_j}{\sqrt{t}}\right).$$

We obtain

$$\left(-2z_i \sum_{j=1}^{n} a_{ij}\cos\frac{h_{ij}z_j}{\sqrt{t}} + \sum_{k=1}^{n} z_k^2 \frac{a_{ki}h_{ki}}{\sqrt{t}} \sin\frac{h_{ki}z_i}{\sqrt{t}}\right) e^{-\sum\limits_{k,j=1}^{n} a_{kj} z_k^2 \cos\frac{h_{kj}z_j}{\sqrt{t}}} \cos\sum_{k,j=1}^{n} a_{kj} z_k^2 \sin\frac{h_{kj}z_j}{\sqrt{t}}$$

$$- e^{-\sum\limits_{k,j=1}^{n} a_{kj} z_k^2 \cos \frac{h_{kj} z_j}{\sqrt{t}}} \left(2z_i \sum_{j=1}^{n} a_{ij} \sin \frac{h_{ij} z_j}{\sqrt{t}} + \sum_{k=1}^{n} z_k^2 \frac{a_{ki} h_{ki}}{\sqrt{t}} \cos \frac{h_{ki} z_i}{\sqrt{t}} \right) \sin \sum_{k,j=1}^{n} a_{kj} z_k^2 \sin \frac{h_{kj} z_j}{\sqrt{t}}.$$

Its absolute value does not exceed

$$\left(4|z_i| \sum_{j=1}^{n} |a_{ij}| + 2 \sum_{k=1}^{n} |a_{ki} h_{ki}| z_k^2 \right) e^{-C|z|^2}$$

(because $t > 1$). Hence, integral (2.31) converges absolutely and uniformly with respect to $(t, \eta) \in (1, +\infty) \times \mathbb{R}^n$; therefore, $\eta_i f_1(t, \eta)$ is a function bounded on the set $\{\eta \in \mathbb{R}^n, t > 1\}$.

Further,

$$\eta_i^2 f_1(t, \eta) = -\frac{\eta_i}{2} \int\limits_{\mathbb{R}^n} g(z; t) \sin 2z \cdot \eta \, dz$$

$$= \frac{1}{4} \lim_{R \to \infty} \left(\int\limits_{|z|=R} g(z; t) \cos 2z \cdot \eta \, \cos(z, e_i) dS_z - \int\limits_{|z|<R} \frac{\partial}{\partial z_i} g(z; t) \cos 2z \cdot \eta \, dz \right). \tag{2.32}$$

The absolute value of the integrand of the last surface integral does not exceed $|g(z; t)|$; it follows from the estimate obtained above that the absolute value of the specified integral does not exceed const $(1 + R) R^n e^{-CR^2}$. Therefore, (2.32) is equal to $-\frac{1}{4} \int\limits_{\mathbb{R}^n} \frac{\partial}{\partial z_i} g(z; t) \cos 2z \cdot \eta \, dz$.

Differentiating $g(z; t)$ and taking into account that $t > 1$, we see that the absolute value of the last integrand does not exceed $P(|z|) e^{-C|z|^2}$, where P is a polynomial with positive coefficients. Therefore, the function $\eta_i^2 f_1(t, \eta)$ is bounded on the set $\{\eta \in \mathbb{R}^n, t > 1\}$.

Continuing to integrate by parts, we obtain that the function $\eta_i^m f_1(t, \eta)$ is bounded on the set $\{\eta \in \mathbb{R}^n, t > 1\}$ for any $i = \overline{1, n}$ and any $m \in \mathbb{N}$; we take into account that $t > 1$ and the function g is such that the absolute value of the integrand is estimated from above by the function $P(|z|) e^{-C|z|^2}$, where P is a polynomial (in general, it depends on m and i) with positive coefficients.

In the same way, we prove the boundedness of the function $\eta_i^m f_2(t, \eta)$ on the set $\{\eta \in \mathbb{R}^n, t > 1\}$ for any $i = \overline{1, n}$ and any $m \in \mathbb{N}$.

Since $|\eta|^{n+1} \leq \text{const} \sum\limits_{i=1}^{n} |\eta_i|^{n+1}$, Lemma 2.7.4 is proved. □

Now, we can get back to the proof of Theorem 2.7.2. To do this, we take an arbitrary positive ε and represent (2.27) as

$$\left(\frac{1}{\pi} \right)^n \left(\int\limits_{|\eta|<R} + \int\limits_{|\eta| \geq R} \right) \stackrel{\text{def}}{=} I_{3,R}(t) + I_{4,R}(t),$$

where R is a positive parameter. The integrand of (2.27) does not exceed

$$\sup_{\mathbb{R}^n} |u_0| \left(\left| \int\limits_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}}, \dots, \frac{z_n}{\sqrt{t}}\right)} \cos \left[2z \cdot \eta - tG_2 \left(\frac{z_1}{\sqrt{t}}, \dots, \frac{z_n}{\sqrt{t}} \right) \right] dz \right| + \frac{\pi^{\frac{n}{2}}}{\prod\limits_{k=1}^{n} \sqrt{p_k}} e^{-\sum\limits_{k=1}^{n} \frac{\eta_k^2}{p_k}} \right).$$

Hence, by virtue of Lemma 2.7.4 (without loss of generality, we assume that $t > 1$), the absolute value of the integrand of $I_{4,R}(t)$ is bounded from above by the function const $\left(\frac{1}{|\eta|^{n+1}} + e^{-\sum\limits_{k=1}^{n} \frac{\eta_k^2}{p_k}} \right)$. Since all of the functions $\frac{1}{|\eta|^{n+1}}$ and $e^{-\sum\limits_{k=1}^{n}\frac{\eta_k^2}{p_k}}$ are integrable over the set $\{|\eta|>1\}$, it follows that there exists $R>1$ such that $|I_{4,R}(t)|<\pi^n\frac{\varepsilon}{2}$ for any $t>1$. Fix that R and consider $I_{3,R}(t)$. By virtue of Lemma 2.7.3, there exists $T^*>1$ such that the inequality

$$\left|\int\limits_{\mathbb{R}^n} e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta - tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]dz - \frac{\pi^{\frac{n}{2}}}{\prod\limits_{k=1}^{n}\sqrt{p_k}}\,e^{-\sum\limits_{k=1}^{n}\frac{\eta_k^2}{p_k}}\right| < \frac{n\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\varepsilon}{4R^n\sup\limits_{\mathbb{R}^n}|u_0|}.$$

holds for any $t>T^*$ and any $\eta\in\mathbb{R}^n$. Then

$$|I_{3,R}(t)|\le\frac{n\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\varepsilon}{4R^n\sup\limits_{\mathbb{R}^n}|u_0|}\int\limits_{|\eta|<R}u_0(x-2\sqrt{t}\eta)\,d\eta\le\pi^n\frac{\varepsilon}{2}$$

for any $t>T^*$.

Thus, we found a positive T^* such that the absolute value of (2.27) does not exceed ε once $t>T^*$. Since the positive ε is selected arbitrarily, it follows that $\lim\limits_{t\to\infty}[u(x,t)-v(x,t)]=0$. This completes the proof of Theorem 2.7.2 because x is arbitrarily selected from $\mathbb{R}^n$. □

This implies the following assertion:

Corollary 2.7.1. *Let $x\in\mathbb{R}^n$ and $l\in(-\infty,+\infty)$. Then*

$$\lim_{t\to\infty}u(x,t)=l \iff \lim_{R\to\infty}\frac{1}{R^n}\int\limits_{B_R(p_1,\dots,p_n)}u_0(x)\,dx=\frac{2\pi^{\frac{n}{2}}\prod\limits_{i=1}^{n}\sqrt{p_i}}{n\Gamma(\frac{n}{2})}\,l,$$

where $B_R(p_1,\dots,p_n)=\left\{x\in\mathbb{R}^n\,\middle|\,\frac{x_1^2}{p_1}+\dots+\frac{x_n^2}{p_n}<R\right\}$.

The proof consists of the direct application of Theorem 2.7.2 and the classical stabilization theorem for the classical bounded solution of problem (2.26), (1.4) (see, e.g., [9]).

For the case where the spatial variable is unique (the unique positive constant p_1 is redenoted by p then), the following assertion is valid as well:

Corollary 2.7.2. *If $w(x,t)$ is the classical bounded solution of the Cauchy problem for the equation $\frac{\partial u}{\partial t}=\Delta u$ with the initial-value condition* (1.4)*, then $\lim\limits_{t\to\infty}[u(x,t)-w(x,pt)]=0$ for any real x.*

To prove this, it suffices to consider the integral representation of the function $w(x,pt)$; we see that it coincides with the integral representation of the function $v(x,t)$.

Remark 2.7.3. Note that Theorem 2.7.2 and Corollary 2.7.1 are valid under same conditions, but the theorem treating the *closeness* of solutions is a stronger assertion: unlike the corollary treating the *stabilization* of solutions, it provides information on the behavior of the solution even in the case where the (necessary and sufficient) stabilization condition is not satisfied. In the same way, for $n=1$, Corollary 2.7.2 is a stronger assertion (in the same sense) than Corollary 2.7.1.

Now, we extend the investigation to a more general case of homogeneous elliptic differential-difference operators containing *mixed* second-order derivatives as well. In detail, we consider only the aspects substantially different from the prototype case considered above. Thus, instead of Eq. (2.20), we consider the equation

$$\frac{\partial u}{\partial t} = L_h u \stackrel{\text{def}}{=} \sum_{k,j,m=1}^{n} a_{kjm} \frac{\partial^2 u}{\partial x_k \partial x_j} (x + h_{kjm} e_m, t). \tag{2.33}$$

As above, the coefficients a_{kjm} and h_{kjm} are assumed to be real and the operator $-L_h$ is assumed to be strongly elliptic, i.e., there exists a positive C_h such that $G_1(\xi) \geq C_h |\xi|^2$ for any ξ from $\mathbb{R}^n$, where

$$G_{\left\{ {1 \atop 2} \right\}}(\xi) = \sum_{k,j,m=1}^{n} a_{kjm} \xi_k \xi_j \left\{ {\cos \atop \sin} \right\} h_{kjm} \xi_m. \tag{2.34}$$

Then the fundamental solution (2.21) is well defined on $\mathbb{R}^n \times (0, +\infty)$.

The following assertion is valid:

Theorem 2.7.3. *Let the operator $-L_h$ be strongly elliptic in the space $\mathbb{R}^n$ and the functions G_1 and G_2 be defined by relations* (2.34). *Then function* (2.25) *satisfies* (*in the classical sense*) *Eq.* (2.33) *in the subspace $\mathbb{R}^n \times (0, +\infty)$ and is the unique solution* (*in the sense of generalized functions*) *of problem* (2.33), (1.4).

To prove this, we take the fundamental solution (2.21) with the functions $G_1(\xi)$ and $G_2(\xi)$ defined by relations (2.34) and substitute (2.21) in Eq. (2.33). Taking into account that

$$\begin{aligned} &\sin h_{kjm}\xi_m \, \sin[x \cdot \xi - tG_2(\xi)] - \cos h_{kjm}\xi_m \, \cos[x \cdot \xi - tG_2(\xi)] \\ &= -\cos[(x + h_{kjm} e_m) \cdot \xi - tG_2(\xi)], \end{aligned}$$

we obtain the relation

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} &= -\int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)} \sum_{k,j,m=1}^{n} a_{kjm} \xi_k \xi_j \cos[(x + h_{kjm} e_m) \cdot \xi tG_2(\xi)] d\xi \\ &= -\sum_{k,j,m=1}^{n} a_{kjm} \int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)} \xi_k \xi_j \cos[(x + h_{kjm} e_m) \cdot \xi - tG_2(\xi)] d\xi. \end{aligned}$$

Further, we have

$$\frac{\partial^2 \mathcal{E}}{\partial x_k x_j} = -\int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)} \xi_k \xi_j \cos[x \cdot \xi - tG_2(\xi)] d\xi.$$

Therefore, the function $\mathcal{E}(x, t)$ satisfies Eq. (2.33) in $\mathbb{R}^n \times (0, +\infty)$ (formal differentiation under the integral sign is valid because, by virtue of the strong ellipticity of the operator $-L_h$, all the integrals obtained by means of the specified formal differentiating converge absolutely and uniformly with respect to $(x, t) \in \mathbb{R}^n \times [t_0, T]$ provided that $0 < t_0 < T < \infty$).

For the considered function $\mathcal{E}(x, t)$, Lemma 2.7.1 is proved in the same way as in the prototype case of *pure* second-order derivatives. Only the coefficients of the polynomials $P(\xi)$ are changed in the general case, but those coefficients still depend only on l, t, and the coefficients of Eq. (2.33).

The proof of the uniqueness is entirely the same as in the prototype case.

To investigate the long-time behavior of the solution, we introduce the *differential* operator

$$L_0 \stackrel{\text{def}}{=} \sum_{k,j,m=1}^{n} a_{kjm} \frac{\partial^2}{\partial x_k \partial x_j} \stackrel{\text{def}}{=} \sum_{k,j=1}^{n} b_{kj} \frac{\partial^2}{\partial x_k \partial x_j}$$

and prove the following analog of Lemma 2.7.2:

Lemma 2.7.5. *If the operator $-L_h$ is strongly elliptic in $\mathbb{R}^n$, then the operator $-L_0$ is elliptic.*

Proof. Assume the converse, i.e., for any (sufficiently small) positive C there exists ξ from $\mathbb{R}^n$ such that $L_0(\xi) \overset{\text{def}}{=} \sum\limits_{k,j=1}^{n} b_{kj}\xi_k\xi_j < C|\xi|^2$. Then, since the polynomial $L_0(\xi)$ is homogeneous, it follows that a stronger assertion is valid as well: *for any* (*sufficiently small*) *positive* C *and* r *there exists* ξ *from* $\mathbb{R}^n$ *such that* $|\xi| = r$ *and* $L_0(\xi) < C|\xi|^2$.

Indeed, fix arbitrary positive C and r and take η from $\mathbb{R}^n$ such that $L_0(\eta) < C|\eta|^2$. If $\eta = 0$, then the continuous function $L_0(\xi)$ is strictly negative at the origin. Hence, there exists a ball centered at the origin such that it is strictly negative in that ball, i.e., there exists $\eta \neq 0$ such that the last inequality is still valid. Therefore, taking η from $\mathbb{R}^n$ such that $L_0(\eta) < C|\eta|^2$, we can assume (without loss of generality) that $\eta \neq 0$. Then we can define $\xi_j \overset{\text{def}}{=} r\dfrac{\eta_j}{|\eta|}$, $j = \overline{1,n}$. This yields the relation

$$L_0(\xi) = \frac{r^2}{|\eta|^2}\sum_{k,j=1}^{n} b_{kj}\eta_k\eta_j < C\frac{r^2}{|\eta|^2}|\eta|^2 = C\left[\left(r\frac{\eta_1}{|\eta|}\right)^2 + \dots + \left(r\frac{\eta_n}{|\eta|}\right)^2\right] = C|\xi|^2,$$

where $|\xi| = r$. On the other hand, $G_1(\xi)$ can be represented as

$$\begin{aligned}
&\sum_{k,j,m=1}^{n} a_{kjm}\xi_k\xi_j\left(1 - 2\sin^2\frac{h_{kjm}\xi_m}{2}\right) = \sum_{k,j,m=1}^{n} a_{kjm}\xi_k\xi_j - 2\sum_{k,j,m=1}^{n} a_{kjm}\xi_k\xi_j\sin^2\frac{h_{kjm}\xi_m}{2}\\
&= \sum_{k,j=1}^{n}\xi_k\xi_j\sum_{m=1}^{n} a_{kjm} - 2\sum_{k,j,m=1}^{n} a_{kjm}\xi_k\xi_j\sin^2\frac{h_{kjm}\xi_m}{2}\\
&= \sum_{k,j=1}^{n} b_{kj}\xi_k\xi_j - 2\sum_{k,j,m=1}^{n} a_{kjm}\xi_k\xi_j\sin^2\frac{h_{kjm}\xi_m}{2} \overset{\text{def}}{=} L_0(\xi) + R_{0,h}(\xi),
\end{aligned}$$

where $|R_{0,h}(\xi)| \leq \dfrac{1}{2}\sum\limits_{k,j,m=1}^{n} |a_{kjm}||\xi_k||\xi_j|\xi_m^2 h_{kjm}^2$. Thus, for all (sufficiently small) positive C and r there exists ξ from $\mathbb{R}^n$ such that $|\xi| = r$ and

$$G_1(\xi) < C|\xi|^2 + \frac{1}{2}\sum_{k,j,m=1}^{n} |a_{kjm}|h_{kjm}^2|\xi_k||\xi_j|\xi_m^2;$$

hence, for all (sufficiently small) positive C and r there exists ξ from $\mathbb{R}^n$ such that $|\xi| = r$ and

$$G_1(\xi) < Cr^2 + C_{0,h}r^4,$$

where $C_{0,h}$ depends only on the coefficients a_{kjm} and h_{kjm} of Eq. (2.33).

However, the inequality $G_1(\xi) \geq C_h|\xi|^2 = C_h r^2$ holds for the found ξ (as well as for any ξ from $\mathbb{R}^n$) by virtue of the strong ellipticity of the operator L_h.

Thus, for all (sufficiently small) positive C and r there exists ξ from $\mathbb{R}^n$ such that

$$C_h r^2 \leq G_1(\xi) < Cr^2 + C_{0,h}r^4, \quad \text{i.e.,} \quad C_h < C + C_{0,h}r^2.$$

Then, selecting sufficiently small positive C and r, we obtain a contradiction.

This completes the proof of Lemma 2.7.5. □

Thus,

$$\frac{\partial u}{\partial t} = L_0 u \tag{2.35}$$

is a parabolic differential equation with constant coefficients. Hence, problem (2.35), (1.4) has a unique classical bounded solution; denote it by $v(x,t)$.

The following assertion is valid:

Theorem 2.7.4. *If the conditions of Theorem* 2.7.3 *are satisfied, then*

$$\lim_{t\to\infty}[u(x,t)-v(x,t)]=0$$

for any $x\in\mathbb{R}^n$.

The scheme of the proof is the same as for the proof of Theorem 2.7.2. For the case where the functions $G_1(\xi)$ and $G_2(\xi)$ are defined by relations (2.34), it suffices to prove the following analogs of Lemmas 2.7.3 and 2.7.4 respectively:

Lemma 2.7.6. *The limit relation*

$$\int_{\mathbb{R}^n}\left(e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-e^{-\sum\limits_{k,j=1}^{n}b_{kj}z_kz_j}\cos 2z\cdot\eta\right)dz\overset{t\to\infty}{\longrightarrow}0$$

holds uniformly with respect to $\eta\in\mathbb{R}^n$.

Lemma 2.7.7. *There exists a positive* M *such that*

$$\left|\int_{\mathbb{R}^n}e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]dz\right|\leq\frac{M}{|\eta|^{n+1}}$$

for any $t>1$ *and any* $\eta\in\mathbb{R}^n$.

Proof of Lemma 2.7.6. The absolute value of the considered integral is estimated from above by the sum

$$\int_{\mathbb{R}^n}e^{-tG_1\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)}dz+\int_{\mathbb{R}^n}e^{-\sum\limits_{k,j=1}^{n}b_{kj}z_kz_j}dz=\int_{\mathbb{R}^n}e^{-C_h|z|^2}dz+\int_{\mathbb{R}^n}e^{-C_0|z|^2}dz<\infty,$$

where C_0 denotes the ellipticity constant of the operator L_0; hence, the specified integral converges absolutely and uniformly with respect to $(t,\eta)\in\mathbb{R}^1_+\in\mathbb{R}^n$. Fix an arbitrary positive ε and represent the specified integral as $\int\limits_{|z|<\delta}+\int\limits_{|z|\geq\delta}\overset{\text{def}}{=}I_{1,\delta}+I_{2,\delta}$, where δ is a positive parameter. By virtue of the uniform convergence of the integral, there exists δ such that $|I_{2,\delta}|<\frac{\varepsilon}{2}$ for any positive t and any η from $\mathbb{R}^n$. Fix that δ and consider the integral $I_{1,\delta}$. Its integrand is equal to

$$e^{-\sum\limits_{k,j,m=1}^{n}a_{kjm}z_kz_j\cos\frac{h_{kjm}z_m}{\sqrt{t}}}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-e^{-\sum\limits_{k,j=1}^{n}b_{kj}z_kz_j}\cos 2z\cdot\eta$$

$$=e^{-\sum\limits_{k,j=1}^{n}z_kz_j\sum\limits_{m=1}^{n}a_{kjm}\cos\frac{h_{kjm}z_m}{\sqrt{t}}}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-e^{-\sum\limits_{k,j=1}^{n}b_{kj}z_kz_j}\cos 2z\cdot\eta$$

$$=e^{-L_0(z)}\left(e^{-\sum\limits_{k,j=1}^{n}z_kz_j\left(\sum\limits_{m=1}^{n}a_{kjm}\cos\frac{h_{kjm}z_m}{\sqrt{t}}-b_{kj}\right)}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right),$$

which can be reduced to the form

$$e^{-L_0(z)}\left(e^{-\sum\limits_{k,j=1}^{n}z_kz_j\sum\limits_{m=1}^{n}a_{kjm}\left(\cos\frac{h_{kjm}z_m}{\sqrt{t}}-1\right)}\cos\left[2z\cdot\eta-tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right)$$

$$=e^{-L_0(z)}\left(e^{2\sum\limits_{k,j=1}^{n}z_kz_j\sum\limits_{m=1}^{n}a_{kjm}\sin^2\frac{h_{kjm}z_m}{2\sqrt{t}}}\cos 2z\cdot\eta\cos\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right]-\cos 2z\cdot\eta\right)$$

$$+\, e^{-L_0(z)} e^{2\sum\limits_{k,j=1}^{n} z_k z_j \sum\limits_{m=1}^{n} a_{kjm} \sin^2 \frac{h_{kjm} z_m}{2\sqrt{t}}} \sin 2z\cdot\eta \sin\left[tG_2\left(\frac{z_1}{\sqrt{t}},\dots,\frac{z_n}{\sqrt{t}}\right)\right] \stackrel{\text{def}}{=} A_1(\eta,t;z)+A_2(\eta,t;z).$$

The further proof is entirely similar to the remaining part of the proof of Lemma 2.7.3. □

Proof of Lemma 2.7.7. Let $t > 1, i \in \overline{1,n}$. Repeat the arguments in the proof of Lemma 2.7.4 until relation (2.31). In the specified relation, assign

$$g(z;t) = \frac{\partial}{\partial z_i}\left(e^{-\sum\limits_{k,j,m=1}^{n} a_{kjm} z_k z_j \cos \frac{h_{kjm} z_m}{\sqrt{t}}} \cos \sum_{k,j,m=1}^{n} a_{kjm} z_k z_j \sin \frac{h_{kjm} z_m}{\sqrt{t}}\right).$$

Differentiating and taking into account that $t > 1$, we obtain (as in the proof of Lemma 2.7.4) the inequality

$$|g(z;t)| \le |P(z)| e^{-C_h|z|^2},$$

where P is a polynomial with coefficients depending only on the coefficients a_{kjm} and h_{kjm} of Eq. (2.33). The further proof is entirely similar to the remaining part of the proof of Lemma 2.7.4. □

Remark 2.7.4. For the general case of Eq. (2.33), a stabilization theorem is valid as well. It is an analog of Corollary 2.7.1 valid for Eq. (2.20). The domains of integration of the initial-value function, included in the (necessary and sufficient) condition of the stabilization of the solution, are determined by the coefficients b_{kj} of Eq. (2.35), which is a *parabolic differential equation with constant coefficients* (see [9]).

Chapter 3

SINGULAR INTEGRODIFFERENTIAL EQUATIONS

In this chapter, nonlocal terms of studied equations are special *generalized translation operators* introduced in [41]; they play a role of translation operators in the theory of equations containing the Bessel operator. The specified generalized translation operators are integral ones. Therefore, the studied equations are not *differential-difference* anymore: they are *integrodifferential*. Thus, the development of this research direction is motivated both by the interest to extend models of [91, 103, 120–123] to the singular case and by the interest in purely theoretical aspects of passage from differential-difference equations to integrodifferential ones.

3.1. Basic Definitions and Notation

In this chapter, we use the following notation:

$$B_y \stackrel{\text{def}}{=} \frac{1}{y^{2\nu+1}}\frac{\partial}{\partial y}\left(y^{2\nu+1}\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial y^2} + \frac{2\nu+1}{y}\frac{\partial}{\partial y}$$

is the Bessel operator with respect to y;

$$T_y^h f(y) \stackrel{\text{def}}{=} \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^\pi f\left(\sqrt{y^2+h^2-2yh\cos\theta}\right)\sin^{2\nu}\theta d\theta$$

is the corresponding generalized translation operator;

$$j_\nu(y) \stackrel{\text{def}}{=} \frac{2^\nu \Gamma(\nu+1)}{y^\nu} J_\nu(y)$$

is the corresponding (uniformly) normalized Bessel function of the first type.

We investigate the case of Bessel operators with positive parameters at the singularity; therefore, we assume that $\nu > -\dfrac{1}{2}$.

The following problem is considered:

$$\frac{\partial u}{\partial t} = B_x u + \sum_{k=1}^{s} a_k T_x^{h_k} u, \ x > 0, t > 0, \tag{3.1}$$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0, \ t > 0, \tag{3.2}$$

$$u\Big|_{t=0} = u_0(x), \ x > 0. \tag{3.3}$$

Here u_0 is a continuous and bounded function and $a, h \in \mathbb{R}^s$.

Note that, in general, any solution of problem (3.1)–(3.3) is defined only in the quarter $(0, +\infty) \times (0, +\infty)$ of the plane, while, to apply the generalized translation operator, we need it to be defined for negative values of the variable x as well. To provide this, we use the even (with respect to x) extension of the solution; the specified extension exists by virtue of the *evenness condition* (3.2). In other words, we can consider problem (3.1)–(3.3) in the whole half-plane $(-\infty, +\infty) \times (0, +\infty)$, replacing condition (3.2) by the evenness (with respect to the variable x) requirement imposed on the solution. For *differential* parabolic equations containing Bessel operators, such problems are well defined (see, e.g., [36–38, 42–45, 47] and references therein).

3.2. Fundamental Solutions of Singular Integrodifferential Equations

On $(0, +\infty)\times(0, +\infty)$, define the following function:

$$\mathcal{E}(x, t) \overset{\text{def}}{=} \mathcal{E}_{a,h}(x, t) \overset{\text{def}}{=} \int_0^\infty \xi^{2\nu+1} e^{-t\left[\xi^2 - \sum\limits_{k=1}^{s} a_k j_\nu(h_k\xi)\right]} j_\nu(x\xi) d\xi. \tag{3.4}$$

Since $|j_\nu(z)| \le 1$, it follows that

$$|\mathcal{E}(x, t)| \le e^{\sum\limits_{k=1}^{s} |a_k| t} \int_0^\infty \xi^{2\nu+1} e^{-t\xi^2} d\xi = \frac{e^{\sum\limits_{k=1}^{s} |a_k| t}}{2t^{\nu+1}} \int_0^\infty z^\nu e^{-z} dz = \frac{\Gamma(\nu+1) e^{\sum\limits_{k=1}^{s} |a_k| t}}{2t^{\nu+1}}.$$

Thus, for all $t_0, T \in (0, +\infty)$, integral (3.4) converges absolutely and uniformly with respect to $(x, t) \in [0, +\infty)\times[t_0, T]$; hence, $\mathcal{E}(x, t)$ is well defined on $[0, +\infty)\times(0, +\infty)$. Formally differentiate $\mathcal{E}$ under the integral sign:

$$\frac{\partial \mathcal{E}}{\partial t} = \int_0^\infty \xi^{2\nu+1} \left[\sum_{k=1}^{s} a_k j_\nu(h_k\xi) - \xi^2\right] j_\nu(x\xi) e^{-t\left[\xi^2 - \sum\limits_{k=1}^{s} a_k j_\nu(h_k\xi)\right]} d\xi.$$

Since $T_x^y j_\nu(ax) = j_\nu(ax) j_\nu(ay)$ (see, e.g., [34, p. 19]), it follows that

$$\frac{\partial \mathcal{E}}{\partial t} = \int_0^\infty \xi^{2\nu+1} \sum_{k=1}^{s} a_k T_x^{h_k} j_\nu(x\xi) e^{-t\left[\xi^2 - \sum\limits_{k=1}^{s} a_k j_\nu(h_k\xi)\right]} d\xi - \int_0^\infty \xi^{2\nu+3} j_\nu(x\xi) e^{-t\left[\xi^2 - \sum\limits_{k=1}^{s} a_k j_\nu(h_k\xi)\right]} d\xi$$

$$= \sum_{k=1}^{s} a_k T_x^{h_k} \mathcal{E} - \int_0^\infty \xi^{2\nu+3} j_\nu(x\xi) e^{-t\left[\xi^2 - \sum\limits_{k=1}^{s} a_k j_\nu(h_k\xi)\right]} d\xi.$$

Further, $B_x j_\nu(x\xi) = -\xi^2 j_\nu(x\xi)$ (see, e.g., [34, p. 18]); hence,

$$B_x \mathcal{E} = -\int_0^\infty \xi^{2\nu+3} j_\nu(x\xi) e^{-t\left[\xi^2 - \sum\limits_{k=1}^{s} a_k j_\nu(h_k\xi)\right]} d\xi.$$

Thus, $\mathcal{E}(x,t)$ formally satisfies Eq. (3.1).

Moreover, we have

$$\left|B_x\mathcal{E}\right| \le \frac{\Gamma(\nu+2)e^{\sum\limits_{k=1}^{s}|a_k|t}}{2t^{\nu+2}}$$

and

$$\left|T_x^{h_k}\mathcal{E}\right| \le \int\limits_0^\infty \xi^{2\nu+1}\left|T_x^{h_k}j_\nu(x\xi)\right|e^{-t\left[\xi^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_k\xi)\right]}d\xi$$

$$\le \int\limits_0^\infty \xi^{2\nu+1}e^{-t\left[\xi^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_k\xi)\right]}d\xi \le \frac{\Gamma(\nu+1)e^{\sum\limits_{k=1}^{s}|a_k|t}}{2t^{\nu+1}}.$$

Therefore,

$$\left|\frac{\partial\mathcal{E}}{\partial t}\right| \le \frac{e^{\sum\limits_{k=1}^{s}|a_k|t}}{2}\left[\frac{\Gamma(\nu+1)}{t^{\nu+1}}+\frac{\Gamma(\nu+2)}{t^{\nu+2}}\right],$$

i.e., the formal differentiation and formal generalized translation under the integral sign are valid for all terms of Eq. (3.1). Hence, function (3.4) satisfies (in the classical sense) Eq. (3.1) on $(0,+\infty)\times(0,+\infty)$.

We call $\mathcal{E}(x,t)$ the *fundamental solution* of Eq. (3.1). To show the reasonability of this term, we prove below that the generalized convolution (see [34, §1.8]) of $\mathcal{E}_{a,h}$ with any bounded initial-value function coincides with that initial-value function on the initial semiaxis.

3.3. Generalized Convolutions of Fundamental Solutions and Bounded Functions

Let us estimate the behavior of the function $\mathcal{E}(x,t)$ as $x\to\infty$ (assuming that a positive t is fixed). To do this, introduce the function $g_\nu(z) \overset{\text{def}}{=} z^\nu J_\nu(z)$. Then $\frac{1}{z}g'_\nu(z) = g_{\nu-1}(z)$ (see, e.g., [80, p. 333]), i.e., $g'_{\nu+1}(z) = zg_\nu(z)$; therefore, $g'_{\nu+1}(az) = a^2zg_\nu(az)$. Hence,

$$\frac{1}{2^\nu\Gamma(\nu+1)}\mathcal{E}(x,t) = \int\limits_0^\infty z^{2\nu+1}e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\frac{J_\nu(xz)}{(xz)^\nu}dz$$

$$= \frac{1}{x^{2\nu}}\int\limits_0^\infty (xz)^\nu ze^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}J_\nu(xz)dz = \frac{1}{x^{2\nu+2}}\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}x^2zg_\nu(xz)dz$$

$$= \frac{1}{x^{2\nu+2}}\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}g'_{\nu+1}(xz)dz = \frac{1}{x^{2\nu+2}}\left[g_{\nu+1}(xz)e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\Bigg|_{z=0}^{z=+\infty}\right.$$

$$\left.+\,t\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\left[2z+\sum_{k=1}^{s}a_kh_k^2zj_{\nu+1}(h_kz)\right]g_{\nu+1}(xz)dz\right]$$

$$= \frac{t}{x^{2\nu+4}}\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\left[2+\sum_{k=1}^{s}a_kh_k^2j_{\nu+1}(h_kz)\right]x^2zg_{\nu+1}(xz)dz$$

$$= \frac{t}{x^{2\nu+4}}\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\left[2+\sum_{k=1}^{s}a_kh_k^2j_{\nu+1}(h_kz)\right]g'_{\nu+2}(xz)dz.$$

The last expression is reduced to the form

$$\frac{t}{x^{2\nu+4}}\left[g_{\nu+2}(xz)\left[2+\sum_{k=1}^{s}a_kh_k^2j_{\nu+1}(h_kz)\right]e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\Bigg|_{z=0}^{z=+\infty}\right.$$

$$\left.-\int_0^\infty\left(e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\left[2+\sum_{k=1}^{s}a_kh_k^2j_{\nu+1}(h_kz)\right]\right)'g_{\nu+2}(xz)dz\right]$$

$$=\frac{t}{x^{2\nu+4}}\int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\left(t\left[2+\sum_{k=1}^{s}a_kh_k^2j_{\nu+1}(h_kz)\right]^2z\right.$$

$$\left.+\sum_{k=1}^{s}a_kh_k^4zj_{\nu+2}(h_kz)\right)g_{\nu+2}(xz)dz$$

$$=\frac{t}{x^{2\nu+6}}\int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\left(t\left[2+\sum_{k=1}^{s}a_kh_k^2j_{\nu+1}(h_kz)\right]^2\right.$$

$$\left.+\sum_{k=1}^{s}a_kh_k^4j_{\nu+2}(h_kz)\right)x^2zg_{\nu+2}(xz)dz$$

$$=\frac{t}{x^{2\nu+6}}\int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\left(t\left[2+\sum_{k=1}^{s}a_kh_k^2j_{\nu+1}(h_kz)\right]^2\right.$$

$$\left.+\sum_{k=1}^{s}a_kh_k^4j_{\nu+2}(h_kz)\right)g'_{\nu+3}(xz)dz.$$

Continuing to integrate by parts, we obtain (assuming that the positive t is fixed) that for any positive integer m there exists a bounded function f_m such that

$$x^{2\nu+2m}\mathcal{E}(x,t)=\int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_m(z)zg_{\nu+m}(xz)dz.$$

This implies that

$$x^{2\nu+2m}\mathcal{E}(x,t)=\int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_m(z)zx^{\nu+m}z^{\nu+m}J_{\nu+m}(xz)dz,$$

i.e.,

$$x^{\nu+m}\mathcal{E}(x,t)=\int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_m(z)z^{\nu+m+1}J_{\nu+m}(xz)dz$$

$$=\int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_m(z)\sqrt{xz}J_{\nu+m}(xz)z^{m+\nu+\frac{1}{2}}dz\frac{1}{\sqrt{x}};$$

therefore,

$$x^{\nu+m+\frac{1}{2}}\mathcal{E}(x,t)=\int\limits_0^\infty z^{m+\nu+\frac{1}{2}}e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_m(z)f(xz)dz,$$

where $f(\tau)=\sqrt{\tau}J_{\nu+m}(\tau)\in L_\infty(0,+\infty)$ for any $m\geq 1$.

Thus, by virtue of the boundedness of the functions f, f_m, and j_ν and the fact that m is selected arbitrarily, the following assertion is proved:

Lemma 3.3.1. *Let* $\alpha>0$, $t>0$, *and* $a,h\in\mathbb{R}^m$. *Then*

$$\lim_{x\to\infty}x^\alpha\mathcal{E}(x,t)=0.$$

Due to the evenness of the normalized Bessel function and the continuity of the generalized translation operator (see, e.g., [34, p. 18-19]), this implies that the function

$$\int\limits_0^\infty \xi^{2\nu+1}\mathcal{E}(\xi,t)T_\xi^x u_0(\xi)d\xi \tag{3.5}$$

is well defined on $(0,+\infty)\times(0,+\infty)$.

Now, let us estimate the behavior of the functions $B_x\mathcal{E}, T_x^{h_k}\mathcal{E}$, and $\dfrac{\partial\mathcal{E}}{\partial t}$ at infinity:

$$\frac{1}{2^\nu\Gamma(\nu+1)}B_x\mathcal{E}(x,t)=-\int\limits_0^\infty z^{2\nu+3}e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\frac{J_\nu(xz)}{(xz)^\nu}dz$$

$$=-\frac{1}{x^\nu}\int\limits_0^\infty z^{\nu+3}e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}J_\nu(xz)dz=-\frac{1}{x^{2\nu+2}}\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}z^2x^2zg_\nu(xz)dz$$

$$=-\frac{1}{x^{2\nu+2}}\int\limits_0^\infty z^2e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}g'_{\nu+1}(xz)dz=\frac{1}{x^{2\nu+2}}\int\limits_0^\infty\left(z^2e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}\right)'g_{\nu+1}(xz)dz.$$

As above, continuing to integrate by parts, we obtain (assuming that a positive t is fixed) that for any positive integer m, the function $x^{2\nu+2m}B_x\mathcal{E}(x,t)$ is a finite sum of terms of the form

$$\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_\beta(z)z^\beta g_{\nu+m}(xz)dz,$$

where $\beta\geq 1$ and f_β is a bounded function. Then $x^{\nu+m}B_x\mathcal{E}(x,t)$ is a finite sum of terms of the form

$$\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_\beta(z)z^{m+\nu+\beta}J_{\nu+m}(xz)dz$$

$$=\frac{1}{\sqrt{x}}\int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_kz)\right]}f_\beta(z)z^{m+\nu+\beta-\frac{1}{2}}\sqrt{xz}J_{\nu+m}(xz)dz;$$

therefore, $x^{\nu+m+\frac{1}{2}}B_x\mathcal{E}(x,t)$ is a finite sum of terms of the form $\int\limits_0^\infty\psi(z)f(xz)dz$, where

$$\psi(\tau)=\tau^{m+\nu+\beta-\frac{1}{2}}f_\beta(\tau)e^{-t\left[\tau^2-\sum\limits_{k=1}^{s}a_kj_\nu(h_k\tau)\right]},$$

i.e., $\psi \in L_1(0,+\infty)$ and $f(\tau)=\sqrt{\tau}J_{m+\nu}(\tau)$ (as above); hence, $f \in L_\infty(0,+\infty)$.

This implies the following assertion.

Lemma 3.3.2. *Let* $\alpha > 0$, $t > 0$, *and* $a, h \in \mathbb{R}^m$. *Then* $\lim\limits_{x\to\infty} x^\alpha B_x \mathcal{E}(x,t) = 0$.

Further,

$$T_x^{h_k}\mathcal{E}(x,t) = \int\limits_0^\infty z^{2\nu+1} e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} j_\nu(h_k z) j_\nu(xz)dz$$

$$= \frac{1}{x^{2\nu}} \int\limits_0^\infty j_\nu(h_k z) e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} z g_\nu(xz)dz$$

$$= \frac{1}{x^{2\nu+2}} \int\limits_0^\infty j_\nu(h_k z) e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} g'_{\nu+1}(xz)dz$$

$$= \frac{1}{x^{2\nu+2}} \left(g_{\nu+1}(xz) j_\nu(h_k z) e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} \Bigg|_{z=0}^{z=+\infty} \right.$$

$$+ \int\limits_0^\infty g_{\nu+1}(xz) e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} \left[h_k^2 z j_{\nu+1}(h_k z) \right.$$

$$\left.\left. + \ t z j_\nu(h_k z) \sum_{k=1}^{s} h_k^2 a_k j_\nu(h_k z) + 2tz j_\nu(h_k z) \right] dz \right).$$

The last expression is equal to

$$\frac{1}{x^{2\nu+2}} \int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} \left[h_k^2 j_{\nu+1}(h_k z) \right.$$

$$\left. + \ t j_\nu(h_k z) \sum_{k=1}^{s} h_k^2 a_k j_\nu(h_k z) + 2t j_\nu(h_k z) \right] z g_{\nu+1}(xz)dz$$

$$= \frac{1}{x^{2\nu+4}} \int\limits_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} \left[h_k^2 j_{\nu+1}(h_k z) \right.$$

$$\left. + \ t j_\nu(h_k z) \sum_{k=1}^{s} h_k^2 a_k j_\nu(h_k z) + 2t j_\nu(h_k z) \right] g'_{\nu+2}(xz)dz$$

$$= \frac{1}{x^{2\nu+4}} \left(g_{\nu+2}(xz) e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} \left[h_k^2 j_{\nu+1}(h_k z) \right. \right.$$

$$\left. + \ t j_\nu(h_k z) \sum_{k=1}^{s} h_k^2 a_k j_\nu(h_k z) + 2t j_\nu(h_k z) \right] \Bigg|_{z=0}^{z=+\infty}$$

$$- \int\limits_0^\infty g_{\nu+2}(xz) \left(e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} \left[h_k^2 j_{\nu+1}(h_k z) \right. \right.$$

$$+ \quad tj_\nu(h_k z)\sum_{k=1}^{s} h_k^2 a_k j_\nu(h_k z) + 2tj_\nu(h_k z)\Big]\Big)' dz\Big)$$

$$= -\frac{1}{x^{2\nu+4}}\int_0^\infty g_{\nu+2}(xz)\left(e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]}\Big[h_k^2 j_{\nu+1}(h_k z)\right.$$

$$\left. + \; tj_\nu(h_k z)\sum_{k=1}^{s} h_k^2 a_k j_\nu(h_k z) + 2tj_\nu(h_k z)\Big]\right)' dz.$$

Continuing to integrate by parts and taking into account the boundedness of the function $j_\nu(x)$, we obtain (assuming that a positive t is fixed) that for any positive integers m and k there exist nonnegative M and β such that

$$x^{2\nu+2m} T_x^{h_k}\mathcal{E}(x,t) = \int_0^\infty e^{-t\left[z^2-\sum\limits_{k=1}^{s} a_k j_\nu(h_k z)\right]} f_0(z) g_{\nu+m}(xz)dz,$$

where $|f_0(z)| \leq M(1+z^\beta)$.

Then

$$x^{2\nu+2m+\frac{1}{2}}\sum_{k=1}^{s} a_k T_x^{h_k}\mathcal{E}(x,t) = \int_0^\infty \psi(z) f(xz)dz,$$

where $\psi \in L_1(0,+\infty)$ and $f(\tau) \in L_\infty(0,+\infty)$. Taking into account that $\mathcal{E}(x,t)$ satisfies Eq. (3.1) in $(0,+\infty)\times(0,+\infty)$, we obtain the following assertion.

Lemma 3.3.3. *Let* $\alpha > 0$, $t > 0$, *and* $a, h \in \mathbb{R}^m$. *Then*

$$\lim_{x\to\infty} x^\alpha \frac{\partial\mathcal{E}}{\partial t} = 0.$$

Further, we note that the Bessel operator and the generalized translation operator commute each other (see, e.g., [34, p. 35]), which implies the following assertion:

Theorem 3.3.1. *Function* (3.5) *satisfies* (*in the classical sense*) *Eq.* (3.1).

3.4. Solutions of Nonclassical Cauchy Problems

Introduce the following notation:

$$u(x,t) \overset{\text{def}}{=} \frac{1}{4^\nu\Gamma^2(\nu+1)}\int_0^\infty \xi^{2\nu+1}\mathcal{E}(\xi,t)T_\xi^x u_0(\xi)d\xi. \tag{3.6}$$

Since the generalized translation operator is self-adjoint (see, e.g., [34, p. 19]), we have the relation

$$u(x,t) = \frac{1}{4^\nu\Gamma^2(\nu+1)}\int_0^\infty \xi^{2\nu+1}u_0(\xi)T_\xi^x\mathcal{E}(\xi,t)d\xi.$$

Since the function $T_\xi^x\mathcal{E}(\xi,t)$ is even with respect to the variable x (see, e.g., [34, p. 35]), it follows that the function $u(x,t)$ satisfies condition (3.2).

Let us show that it satisfies condition (3.3) as well.

The function $u(x,t)$ is defined on $(0,+\infty)\times(0,+\infty)$. Take an arbitrary nonnegative x_0 and investigate the behavior of $u(x_0,t)$ as $t \longrightarrow +0$.

The change of variables $\eta = \dfrac{\xi}{2\sqrt{t}}$ yields the relation

$$u(x_0,t) = \frac{2^{2\nu+2}t^{\nu+1}}{4^\nu\Gamma^2(\nu+1)} \int\limits_0^\infty \eta^{2\nu+1}\mathcal{E}(2\eta\sqrt{t},t) T^{x_0}_{2\eta\sqrt{t}} u_0(2\eta\sqrt{t}) d\eta.$$

Further,

$$\begin{aligned}\mathcal{E}(2\sqrt{t}\eta,t) &= \int\limits_0^\infty \xi^{2\nu+1} e^{-t\left[\xi^2 - \sum\limits_{k=1}^{s} a_k j_\nu(h_k\xi)\right]} j_\nu(2\xi\eta\sqrt{t}) d\xi \\ &= t^{-\nu-1}\int\limits_0^\infty z^{2\nu+1} e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z) dz. \end{aligned} \tag{3.7}$$

Thus,

$$u(x_0,t) = \frac{4}{\Gamma^2(\nu+1)} \int\limits_0^\infty \eta^{2\nu+1} T^{2\eta\sqrt{t}}_{x_0} u_0(x_0) \int\limits_0^\infty z^{2\nu+1} e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z) dz d\eta. \tag{3.8}$$

Now, let us prove the following auxiliary assertions.

Lemma 3.4.1. *There exist $C > 0$ and $\alpha > 1$ such that*

$$\left| \eta^{2\nu+1} \int\limits_0^\infty z^{2\nu+1} e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z) dz \right| \le \frac{C}{\eta^\alpha}$$

for any t from $(0,1)$ and any positive η.

Proof. We have

$$\begin{aligned} &\frac{1}{2^\nu\Gamma(\nu+1)} \int\limits_0^\infty z^{2\nu+1} e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z) dz \\ &= \frac{1}{(2\eta)^\nu} \int\limits_0^\infty z^{\nu+1} e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} J_\nu(2\eta z) dz \\ &= \frac{1}{(2\eta)^{2\nu}} \int\limits_0^\infty (2\eta z)^{\nu+1} J_\nu(2\eta z) z e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz \\ &= \frac{1}{(2\eta)^{2\nu+2}} \int\limits_0^\infty e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} (2\eta)^2 z g_\nu(2\eta z) dz \\ &= \frac{1}{(2\eta)^{2\nu+2}} \int\limits_0^\infty e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} g'_{\nu+1}(2\eta z) dz \\ &= \frac{1}{(2\eta)^{2\nu+2}} \left[g_{\nu+1}(2\eta z) e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} \Bigg|_{z=0}^{z=\infty} - \int\limits_0^\infty \left[e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} \right]' g_{\nu+1}(2\eta z) dz \right] \\ &= \frac{1}{(2\eta)^{2\nu+2}} \int\limits_0^\infty g_{\nu+1}(2\eta z) \left[2z + z\sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right) \right] e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz \end{aligned}$$

$$= \frac{1}{(2\eta)^{2\nu+4}} \int_0^\infty (2\eta)^2 z g_{\nu+1}(2\eta z) \left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right] e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz.$$

The last expression is equal to

$$\frac{1}{(2\eta)^{2\nu+4}} \int_0^\infty g'_{\nu+2}(2\eta z) \left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right] e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz.$$

Integrating it by parts, we obtain the expression

$$-\frac{1}{(2\eta)^{2\nu+4}} \int_0^\infty g_{\nu+2}(2\eta z) \left(\left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right] e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)}\right)' dz$$

$$= \frac{1}{(2\eta)^{2\nu+4}} \int_0^\infty g_{\nu+2}(2\eta z) e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)}$$

$$\times \left(\left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right]^2 z + \frac{z}{t}\sum_{k=1}^{s} h_k^4 a_k j_{\nu+2}\left(\frac{h_k z}{\sqrt{t}}\right)\right) dz$$

$$= \frac{1}{(2\eta)^{2\nu+6}} \int_0^\infty (2\eta)^2 z g_{\nu+2}(2\eta z) e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)}$$

$$\times \left(\left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right]^2 + \frac{1}{t}\sum_{k=1}^{s} h_k^4 a_k j_{\nu+2}\left(\frac{h_k z}{\sqrt{t}}\right)\right) dz$$

$$= \frac{1}{(2\eta)^{2\nu+6}} \int_0^\infty g'_{\nu+3}(2\eta z) e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)}$$

$$\times \left(\left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right]^2 + \frac{1}{t}\sum_{k=1}^{s} h_k^4 a_k j_{\nu+2}\left(\frac{h_k z}{\sqrt{t}}\right)\right) dz.$$

Hence, the estimated integral can be represented as follows:

$$-\frac{1}{(2\eta)^{2\nu+6}} \int_0^\infty g_{\nu+3}(2\eta z) \left[e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)}\right.$$

$$\left.\times \left(\left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right]^2 + \frac{1}{t}\sum_{k=1}^{s} h_k^4 a_k j_{\nu+2}\left(\frac{h_k z}{\sqrt{t}}\right)\right)\right]' dz$$

$$= \frac{1}{(2\eta)^{2\nu+6}} \int_0^\infty g_{\nu+3}(2\eta z) e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} \left(\left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right]^3 z\right.$$

$$+ \left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right]^2 \frac{z}{t}\sum_{k=1}^{s} h_k^4 a_k j_{\nu+2}\left(\frac{h_k z}{\sqrt{t}}\right)$$

$$\left.+ \frac{2z}{t}\left[2 + \sum_{k=1}^{s} h_k^2 a_k j_{\nu+1}\left(\frac{h_k z}{\sqrt{t}}\right)\right] \sum_{k=1}^{s} h_k^4 a_k j_{\nu+2}\left(\frac{h_k z}{\sqrt{t}}\right) + \frac{z}{t^2}\sum_{k=1}^{s} h_k^6 a_k j_{\nu+3}\left(\frac{h_k z}{\sqrt{t}}\right)\right) dz.$$

Continuing to integrate by parts, we obtain that for any positive integer m, the integral

$$\int\limits_0^\infty z^{2\nu+1} e^{-z^2+t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z)dz$$

is a finite sum of terms of the form

$$\frac{1}{\eta^{2\nu+2m}t^l}\int\limits_0^\infty z g_{\nu+m}(2\eta z) e^{-z^2+t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_{\nu+l+1}\left(\frac{h_k z}{\sqrt{t}}\right) f_l(z,t)dz, \tag{3.9}$$

where l is a positive integer not exceeding $m-1$, while f_l is a bounded function.

Estimate (3.9), assuming (without loss of generality) that $t \leq 1$:

$$\frac{j_{\nu+l+1}\left(\frac{h_k z}{\sqrt{t}}\right)}{t^l} = \frac{2^\nu\Gamma(\nu+1)J_{\nu+l+1}\left(\frac{h_k z}{\sqrt{t}}\right)}{\frac{h_k^{\nu+l+1}z^{\nu+l+1}}{t^{\frac{\nu+l+1}{2}}}t^l} = \frac{2^\nu\Gamma(\nu+1)\sqrt{\frac{h_k z}{\sqrt{t}}}J_{\nu+l+1}\left(\frac{h_k z}{\sqrt{t}}\right)}{h_k^{\nu+l+1}z^{\nu+l+1}t^{\frac{l-\nu-1}{2}}\sqrt{\frac{h_k z}{\sqrt{t}}}}.$$

The absolute value of the last expression does not exceed $z^{-\nu-l-\frac{3}{2}}t^{\frac{1}{4}+\frac{\nu-l+1}{2}}$ because the function $\sqrt{\tau}J_{\nu+l+1}(\tau)$ is bounded.

Further, $\dfrac{1}{4}+\dfrac{\nu-l+1}{2} \geq 0$ provided that $l \leq \nu+\dfrac{3}{2}$, i.e., to satisfy the last inequality, it suffices to assume that $m \leq \nu+\dfrac{5}{2}$; then the absolute value of (3.9) does not exceed

$$\frac{\text{const}}{\eta^{2\nu+2m}}\int\limits_0^\infty \frac{|g_{\nu+m}(2\eta z)|}{z^{\nu+l+\frac{1}{2}}} e^{-z^2+t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz$$

$$= \frac{\text{const}}{\eta^{2\nu+2m}}\int\limits_0^\infty (2\eta z)^{m+\nu}\frac{|J_{\nu+m}(2\eta z)|}{z^{\nu+l+\frac{1}{2}}} e^{-z^2+t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz$$

$$= \frac{\text{const}}{\eta^{\nu+m}}\int\limits_0^\infty z^{m-l-\frac{1}{2}}|J_{\nu+m}(2\eta z)| e^{-z^2+t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz$$

$$= \frac{\text{const}}{\eta^{\nu+m+\frac{1}{2}}}\int\limits_0^\infty z^{m-l-1}\sqrt{2\eta z}|J_{\nu+m}(2\eta z)| e^{-z^2+t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} dz \leq \frac{\text{const}}{\eta^{\nu+m+\frac{1}{2}}}.$$

Note that $2\nu+1-m-\nu-\dfrac{1}{2} = \nu+\dfrac{1}{2}-m < -1$ provided that $m > \nu+\dfrac{3}{2}$.

Thus, to satisfy the assertion of the lemma, it suffices to select a positive integer $m \in \left(\nu+\frac{3}{2},\, \nu+\frac{5}{2}\right]$. Such m exists for any $\nu > -\dfrac{1}{2}$, which completes the proof of Lemma 3.4.1. □

Lemma 3.4.2. *The limit relation*

$$\int\limits_0^\infty z^{2\nu+1} e^{-z^2+t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z)dz \overset{t\to+0}{\longrightarrow} \frac{\Gamma(\nu+1)}{2}e^{-\eta^2}$$

holds uniformly with respect to $\eta \geq 0$.

Proof. We have

$$\int\limits_0^\infty z^{2\nu+1}e^{-z^2}j_\nu(2\eta z)dz = \frac{\Gamma(\nu+1)}{\eta^\nu}\int\limits_0^\infty z^{\nu+1}e^{-z^2}J_\nu(2\eta z)dz = \frac{\Gamma(\nu+1)}{2}e^{-\eta^2}$$

(see, e.g., [88, p. 186]).

Therefore,

$$\left| \int_0^\infty z^{2\nu+1} e^{-z^2+t\sum\limits_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z)dz - \frac{\Gamma(\nu+1)}{2}\, e^{-\eta^2} \right|$$

$$= \left| \int_0^\infty z^{2\nu+1} e^{-z^2} j_\nu(2\eta z) \left[e^{t\sum\limits_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} - 1 \right] dz \right| \le \int_0^\infty z^{2\nu+1} e^{-z^2} \left| e^{t\sum\limits_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} - 1 \right| dz.$$

Let $\varepsilon > 0$. Select a small t_0 such that

$$e^{-t_0 \sum\limits_{k=1}^s |a_k|}, \quad e^{t_0 \sum\limits_{k=1}^s |a_k|} \in \left(1 - \frac{2\varepsilon}{\Gamma(\nu+1)}, 1 + \frac{2\varepsilon}{\Gamma(\nu+1)} \right).$$

Then

$$t\sum_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right) \in \left(-t\sum_{k=1}^s |a_k|, t\sum_{k=1}^s |a_k| \right)$$

for any t from $(0, t_0)$; hence, due to the monotonicity of the exponential function, we have

$$\int_0^\infty z^{2\nu+1} e^{-z^2} \left| e^{t\sum\limits_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} - 1 \right| dz \le \frac{2\varepsilon}{\Gamma(\nu+1)} \int_0^\infty z^{2\nu+1} e^{-z^2} dz = \varepsilon.$$

Since ε is selected arbitrarily, the proof of Lemma 3.4.2 is completed. □

Take an arbitrary nonnegative x_0 and consider the difference

$$u(x_0,t) - u_0(x_0) = \frac{4}{\Gamma^2(\nu+1)} \int_0^\infty \eta^{2\nu+1} T_{x_0}^{2\eta\sqrt{t}} u_0(x_0) \int_0^\infty z^{2\nu+1} e^{-z^2+t\sum\limits_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z)dzd\eta$$

$$- \frac{4}{\Gamma^2(\nu+1)} \int_0^\infty \eta^{2\nu+1} u_0(x_0) \frac{\Gamma(\nu+1)}{2}\, e^{-\eta^2} d\eta$$

$$= \frac{4}{\Gamma^2(\nu+1)} \int_0^\infty \eta^{2\nu+1} \left[T_{x_0}^{2\eta\sqrt{t}} u_0(x_0) \int_0^\infty z^{2\nu+1} e^{-z^2+t\sum\limits_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} \right.$$

$$\left. \times\, j_\nu(2\eta z)dz - u_0(x_0) \frac{\Gamma(\nu+1)}{2}\, e^{-\eta^2} \right] d\eta = \frac{4}{\Gamma^2(\nu+1)} \left(\int_0^A + \int_A^\infty \right) \stackrel{\text{def}}{=} \frac{4}{\Gamma^2(\nu+1)} (I_1 + I_2).$$

Take an arbitrary positive ε. The following inequality is valid:

$$|I_2| \le \sup|u_0| \int_A^\infty \left| \eta^{2\nu+1} \int_0^\infty z^{2\nu+1} e^{-z^2+t\sum\limits_{k=1}^s a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z)dz \right| d\eta + \sup|u_0| \frac{\Gamma(\nu+1)}{2} \int_A^\infty \eta^{2\nu+1} e^{-\eta^2} d\eta.$$

By virtue of Lemma 3.4.1 (without loss of generality, we assume that $t \le 1$), we obtain that the former integral at the right-hand part does not exceed $C \int\limits_A^\infty \frac{d\eta}{\eta^\alpha} = \frac{C}{A^{1-\alpha}}$, where $\alpha > 1$. This and the convergence of the integral $\int\limits_0^\infty \eta^{2\nu+1} e^{-\eta^2} d\eta$ imply that there exists a positive A such that $|I_2| \le \frac{\Gamma^2(\nu+1)}{8}\,\varepsilon$

for any t from $(0,1)$. Select such A and fix it. It remains to estimate I_1. To do this, we note that

$$\begin{aligned}
&T_{x_0}^{2\eta\sqrt{t}} u_0(x_0) - u_0(x_0) \\
&= \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\pi u_0\Big(\sqrt{x_0^2 + 4\eta^2 t - 4x_0\eta\sqrt{t}\cos\theta}\Big) \sin^{2\nu}\theta d\theta - \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\pi u_0(x_0)\sin^{2\nu}\theta d\theta \\
&= \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^\pi \left[u_0\left(\sqrt{x_0^2 + 4\eta^2 t - 4x_0\eta\sqrt{t}\cos\theta}\right) - u_0(x_0)\right] \sin^{2\nu}\theta d\theta.
\end{aligned}$$

Let $\delta > 0$. By virtue of the continuity of the function u_0 at the point x_0, one can select a small t_0 such that for any t from $(0, t_0)$, any η from $[0, A]$, and any θ from $[0, \pi]$, the following inequality holds:

$$\left| u_0\left(\sqrt{x_0^2 + 4\eta^2 t - 4x_0\eta\sqrt{t}\cos\theta}\right) - u_0(x_0)\right| < \delta.$$

Since δ is selected arbitrarily, it follows that $T_{x_0}^{2\eta\sqrt{t}} u_0(x_0) \overset{t\to+0}{\longrightarrow} u_0(x_0)$ uniformly with respect to $\eta \in [0, A]$. This and Lemma 3.4.2 imply that there exists a positive t_0 such that for any t from $(0, t_0)$ and any η from $[0, A]$, we have

$$\left| T_{x_0}^{2\eta\sqrt{t}} u_0(x_0) \int_0^\infty z^{2\nu+1} e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z)dz - u_0(x_0)\frac{\Gamma(\nu+1)}{2}\, e^{-\eta^2}\right| < \frac{(\nu+1)\Gamma^2(\nu+1)}{4A^{2\nu+2}}\,\varepsilon,$$

i.e.,

$$|I_1| \leq \frac{4}{\Gamma^2(\nu+1)} \frac{(\nu+1)\Gamma^2(\nu+1)}{4A^{2\nu+2}}\,\varepsilon \int_0^A \eta^{2\nu+1} d\eta = \varepsilon \frac{\nu+1}{A^{2\nu+2}} \frac{A^{2\nu+2}}{2\nu+2} = \frac{\varepsilon}{2}.$$

Since ε is selected arbitrarily, it follows that

$$u(x_0, t) - u_0(x_0) \overset{t\to+0}{\longrightarrow} 0.$$

Thus, the function $u(x,t)$ satisfies condition (3.3) because x_0 is selected arbitrarily.

Thus, the following assertion is proved.

Theorem 3.4.1. *Let a function $u_0(x)$ be continuous and bounded for nonnegative x. Then the function $u(x,t)$ defined by relation* (3.6) *is a classical solution of problem* (3.1)–(3.3).

In particular, using the proved theorem, one can compute the weight integral of the fundamental solution over the whole positive semiaxis:

Lemma 3.4.3. *The following relation is valid*:

$$\int_0^\infty x^{2\nu+1}\mathcal{E}(x,t)dx = 4^\nu \Gamma^2(\nu+1) e^{t\sum\limits_{k=1}^{s} a_k}.$$

Proof. Consider the function $u_0(x) \equiv 1$. It is continuous and bounded. Therefore, by virtue of Theorem 3.4.1, the function

$$y(x,t) \overset{\text{def}}{=} \frac{1}{4^\nu \Gamma^2(\nu+1)} \int_0^{+\infty} \xi^{2\nu+1}\mathcal{E}(\xi,t)d\xi$$

satisfies problem (3.1)–(3.3) with the initial-value condition $y\big|_{t=0} \equiv 1$. However, $y(x,t)$ does not depend on x; hence, $y(t)$ satisfies the *ordinary differential equation* $y' - y\sum\limits_{k=1}^{s} a_k = 0$ and the initial-value condition $y(0) = 1$. Therefore, $y(t) = e^{t\sum\limits_{k=1}^{s} a_k}$, which completes the proof of Lemma 3.4.3. □

3.5. Inhomogeneous Equations

Consider the equation

$$\frac{\partial u}{\partial t} - B_x u + \sum_{k=1}^{s} a_k T_x^{h_k} u = f(x,t), \ \ x > 0, t > 0, \tag{3.10}$$

assuming that f continuous and bounded.

Let us show that, using the fundamental solution defined by relation (3.4), one can obtain an integral representation for the (classical) solution of problem (3.10), (3.2), (3.3) as well.

To do this, we fix an arbitrary positive x_0 and introduce the function

$$G(t,\tau) \overset{\text{def}}{=} \frac{1}{4^{\nu+1}} \int\limits_0^\infty \xi^{2\nu+1} f(\xi, t-\tau) T_{x_0}^{\xi} \mathcal{E}(x_0,\tau)d\xi$$

defined for $t > \tau > 0$.

The following assertion is valid:

Lemma 3.5.1. *There exists a positive* $t_0 > 0$ *such that* $G(t,\tau)$ *is bounded in the domain* $(0,t_0)\times(0,t)$.

Proof. Take into account the self-adjointness of the generalized translation operator and change the variable: $\eta = \dfrac{\xi}{2\sqrt{t}}$. This yields the relation

$$\begin{aligned} G(t,\tau) &= \tau^{\nu+1} \int\limits_0^\infty \eta^{2\nu+1} \mathcal{E}(2\eta\sqrt{\tau},\tau) T_{x_0}^{2\eta\sqrt{\tau}} f(x_0, t-\tau) d\eta \\ &= \int\limits_0^\infty \eta^{2\nu+1} \int\limits_0^\infty z^{2\nu+1} e^{-z^2 + \tau\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{\tau}}\right)} j_\nu(2\eta z) dz T_{x_0}^{2\eta\sqrt{\tau}} f(x_0, t-\tau) d\eta = \int\limits_0^1 + \int\limits_1^\infty \overset{\text{def}}{=} I_3 + I_4. \end{aligned}$$

To estimate $|I_4|$, we apply Lemma 3.4.1 (without loss of generality, we assume that $t \le 1$) and obtain that there exists $\alpha > 1$ such that

$$|I_4| \le C \sup|f| \int\limits_1^\infty \frac{d\eta}{\eta^\alpha} = \frac{C\sup|f|}{\alpha - 1}.$$

Under the same assumptions, we have

$$\begin{aligned} |I_3| &\le \sup|f| \int\limits_0^\infty \eta^{2\nu+1} e^{\tau\sum\limits_{k=1}^{s}|a_k|} \int\limits_0^\infty z^{2\nu+1} e^{-z^2} dz d\eta \\ &\le \sup|f| e^{\sum\limits_{k=1}^{s}|a_k|} \int\limits_0^\infty \eta^{2\nu+1} d\eta \int\limits_0^\infty z^{2\nu+1} e^{-z^2} dz = \frac{\sup|f|\Gamma(\nu+1)}{4(\nu+1)} e^{\sum\limits_{k=1}^{s}|a_k|}, \end{aligned}$$

which completes the proof of Lemma 3.5.1. □

Therefore, the following function is defined on $[0,+\infty)\times(0,+\infty)$:

$$v(x,t) \overset{\text{def}}{=} \frac{1}{4^{\nu}\Gamma^2(\nu+1)} \int_0^t \int_0^\infty \xi^{2\nu+1}\mathcal{E}(\xi,t)T_x^{\xi} f(x,t-\tau)d\xi d\tau. \tag{3.11}$$

Let us show that the specified function satisfies Eq. (3.10) and the homogeneous initial-value condition.

To prove the former assertion, we note that it is proved in Secs. 3.2-3.3 that the function $\mathcal{E}(x,t)$ satisfies Eq. (3.1) in $[0,+\infty)\times(0,+\infty)$ and tends to zero (as well as the functions $\dfrac{\partial\mathcal{E}}{\partial t}$ and $B_x\mathcal{E}$) as $x\to+\infty$ faster than any negative power of $|x|$.

Therefore, it remains to prove the following lemma:

Lemma 3.5.2. *If* $x_0 \geq 0$ *and* $t_0 > 0$, *then*

$$\lim_{\tau\to+0} \frac{1}{4^{\nu}\Gamma^2(\nu+1)} \int_0^\infty \xi^{2\nu+1} f(\xi,t_0-\tau)T_{x_0}^{\xi}\mathcal{E}(x_0,\tau)d\xi = f(x_0,t_0).$$

Proof. We have

$$\int_0^\infty \xi^{2\nu+1} f(\xi,t_0-\tau)T_{x_0}^{\xi}\mathcal{E}(x_0,\tau)d\xi = 4^{\nu+1}G(t_0,\tau)$$

$$= 4^{\nu+1}\int_0^\infty \eta^{2\nu+1}T_{x_0}^{2\eta\sqrt{\tau}} f(x_0,t-\tau)\int_0^\infty z^{2\nu+1}e^{-z^2+\tau\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{\tau}}\right)} j_\nu(2\eta z)dzd\eta$$

(see the proof of Lemma 3.5.1).

Therefore,

$$\frac{1}{4^{\nu}\Gamma^2(\nu+1)} \int_0^\infty \xi^{2\nu+1} f(\xi,t_0-\tau)T_{x_0}^{\xi}\mathcal{E}(x_0,\tau)d\xi - f(x_0,t_0)$$

$$= \frac{4}{\Gamma^2(\nu+1)}\int_0^\infty \eta^{2\nu+1}\Bigg[T_{x_0}^{2\eta\sqrt{\tau}} f(x_0,t_0-\tau)\int_0^\infty z^{2\nu+1}$$

$$\times e^{-z^2+\tau\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{\tau}}\right)} j_\nu(2\eta z)dz - \frac{\Gamma(\nu+1)}{2}e^{-\eta^2}f(x_0,t_0)\Bigg]d\eta$$

$$= \frac{4}{\Gamma^2(\nu+1)}\left(\int_0^A + \int_0^\infty\right) \overset{\text{def}}{=} \frac{4}{\Gamma^2(\nu+1)}(I_5+I_6),$$

where A is a positive parameter.

Let $\varepsilon > 0$. We have

$$|I_6| \leq \sup|f|\int_A^\infty \eta^{2\nu+1}\left|\int_0^\infty z^{2\nu+1}e^{-z^2+\tau\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{\tau}}\right)} j_\nu(2\eta z)dz\right|d\eta + \sup|f|\frac{\Gamma(\nu+1)}{2}\int_A^\infty e^{-\eta^2}d\eta.$$

By virtue of Lemma 3.4.1 (without loss of generality, we assume that $\tau < 1$), there exists $\alpha > 1$ such that the first term of the right-hand part of the last inequality is less than or equal to

$$C\sup|f|\int_1^\infty \frac{d\eta}{\eta^\alpha} = \frac{C\sup|f|}{\alpha-1}.$$

This and the convergence of the integral $\int\limits_0^\infty e^{-\eta^2}d\eta$ imply that there exists a positive A such that $|I_6| < \frac{\varepsilon}{2}$ for any $\tau \in (0,1)$. Fix such A and estimate I_5.

Consider

$$T_{x_0}^{2\eta\sqrt{\tau}} f(x_0, t_0 - \tau) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^\pi f\left(\sqrt{x_0^2 + 4\eta^2(t_0-\tau) - 4x_0\eta\sqrt{t_0-\tau}\cos\theta}\right)\sin^{2\nu}\theta d\theta.$$

By virtue of the continuity and boundedness of the function f, the last expression tends to $f(x_0, t_0)$ as $\tau \to +0$ uniformly with respect to $\eta \in [0, A]$. This and Lemma 3.4.2 imply that there exists a positive τ_0 such that for any $\tau < \tau_0$ and any $\eta \in [0, A]$, we have

$$\left| T_{x_0}^{2\eta\sqrt{t}} f(x_0, t_0-\tau) \int\limits_0^\infty z^{2\nu+1} e^{-z^2 + t\sum\limits_{k=1}^{s} a_k j_\nu\left(\frac{h_k z}{\sqrt{t}}\right)} j_\nu(2\eta z)dz - \frac{\Gamma(\nu+1)}{2} f(x_0,t_0)e^{-\eta^2}\right| < \frac{\varepsilon}{4}\frac{\nu+1}{A^{2\nu+2}\Gamma^2(\nu+1)},$$

i.e., $|I_5| \le \frac{\varepsilon}{2}$. This completes the proof of Lemma 3.5.2. □

It remains to prove that $v(x_0,t) \overset{t\to+0}{\longrightarrow} 0$ for any nonnegative x_0.

To do this, we represent $v(x_0,t)$ as $\frac{4}{\Gamma^2(\nu+1)}\int\limits_0^t G(t,\tau)d\tau$ and use Lemma 3.5.1. This yields that there exists a positive t_0 such that

$$|v(x_0,t)| \le \frac{4}{\Gamma^2(\nu+1)} \sup_{t\in[0,t_0]} |G|t$$

for any t from $(0, t_0)$.

Taking into account that the nonnegative x_0 is selected arbitrarily and the function $T_\xi^x \mathcal{E}(\xi,t)$ is even with respect to the variable x, we prove the following assertion.

Theorem 3.5.1. *Let u_0 be continuous and bounded in $[0,+\infty)$ and f be continuous and bounded in $[0,+\infty)\times(0,+\infty)$. Then the function*

$$\frac{1}{4^\nu\Gamma^2(\nu+1)}\left[\int\limits_0^\infty \xi^{2\nu+1}\mathcal{E}(\xi,t)T_x^\xi u_0(x)d\xi + \int\limits_0^t\int\limits_0^\infty \xi^{2\nu+1}\mathcal{E}(\xi,t)T_x^\xi f(x,t-\tau)d\xi d\tau\right]$$

is a classical solution of problem (3.10), (3.2), (3.3).

Chapter 4

SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

In this chapter, we study the nonclassical Cauchy problem for singular parabolic equations of the most general type: they are not only integrodifferential, but differential-difference as well. Apart from the specified theoretical aspect, this problem is interesting from the point of view of applications: the motivation is to extend models of [91, 103, 120–123] for the case of media with characteristics degenerated along selected directions.

We find fundamental solutions of the specified equations, investigate their properties, and obtain integral representations of solutions of the investigated problem (the initial-value function and the right-hand part are assumed to be continuous and bounded). Thus, we prove the solvability theorem.

To prove the uniqueness of the solution, the method of Fourier transforms is applied. The function-theory technique necessary to apply the specified method (the Fourier–Bessel transformation and the scale of generalized functions, corresponding to the degenerated measure $\prod\limits_{l} y_l^{k_l} dxdy$) is deeply and comprehensively developed in [34] (see also references therein); therefore, following the general scheme of [16], one could apply the specified method to investigate the solvability as well. However, the specified method yields only solutions in the sense of generalized function. Moreover, it is not guaranteed that such a solution belongs to any Sobolev class or Schwartz class of generalized function. Unlike this case, we obtain a *classical* solution, i.e., a function differentiable (up to the order of the equation) and satisfying the equation and the boundary-value conditions at each point.

4.1. Statement of the Problem

We use the following notation.

$k_l = 2\nu_l + 1$ is a positive parameter ($l \in 1, n$);

$$B_{k_l,y_l} \stackrel{\text{def}}{=} \frac{1}{y_l^{k_l}} \frac{\partial}{\partial y_l} \left(y_l^{k_l} \frac{\partial}{\partial y_l} \right) = \frac{\partial^2}{\partial y_l^2} + \frac{k_l}{y_l} \frac{\partial}{\partial y_l}$$

is the Bessel operator with respect to the variable y_l;

$$T_y^h f(y) \stackrel{\text{def}}{=} \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_0^\pi f\left(\sqrt{y^2 + h^2 - 2yh\cos\theta}\right) \sin^{2\nu}\theta d\theta$$

is the corresponding generalized translation operator (with scalar variable y).

In the case where y and h are vectors, the generalized translation operator is defined as the superposition of the one-dimensional operators: $T_y^h = T_{y_1}^{h_1} \cdots T_{y_n}^{h_n}$.

Let $\mathbb{R}_+^{m+n}$ denote the set

$$\left\{ (x,y) \middle| x \in \mathbb{R}^m, y_1 > 0, \ldots, y_n > 0 \right\}.$$

In $\mathbb{R}_+^{m+n} \times (0,\infty)$, consider the equation

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{m} \left[\frac{\partial^2 u}{\partial x_i^2} + \sum_{s=1}^{m_i} a_{is} u(x + h_{is}, y, t) \right] - \sum_{l=1}^{n} \left(B_{k_l,y_l} u + \sum_{r=1}^{n_l} b_{lr} T_{y_l}^{g_{lr}} u \right) = f(x,y,t) \tag{4.1}$$

with the boundary-value conditions

$$\left.\frac{\partial u}{\partial y_l}\right|_{y_l=0} = 0\,(l = \overline{1,n}),\ t > 0, \tag{4.2}$$

and

$$u\Big|_{t=0} = u_0(x,y),\ (x,y) \in \overline{\mathbb{R}_+^{m+n}}. \tag{4.3}$$

Here $u_0, f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}$, and $\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n}$ are continuous and bounded functions, f satisfies condition (4.2), h_{is} are vectors parallel to the ith coordinate axis of the space $\mathbb{R}^m$, $i \in \overline{1,m}$, for any s, and the coefficients a_{is}, b_{lr}, and g_{lr} are assumed to be real for all values of their indices.

Similarly to Chap 3, problem (4.1)–(4.3) can be considered in the whole subspace $\mathbb{R}^{m+n} \times (0,+\infty)$ if condition (4.2) is replaced by the requirement that the function u is even with respect to each variable y_l; for *differential* parabolic equations with Bessel operator, such problems are well defined (see, e.g., [37, 38, 42–45, 47]).

Let $f(x,y,t)\equiv 0$.

Assigning

$$\mathcal{E}_1(x,t)\overset{\text{def}}{=}\int\limits_{\mathbb{R}^m} e^{-t\left(|\xi|^2-\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m_i}a_{is}\cos h_{is}\cdot\xi\right)}\cos\left(x\cdot\xi+t\sum_{i=1}^{m}\sum_{s=1}^{m_i}a_{is}\sin h_{is}\cdot\xi\right)d\xi \tag{4.4}$$

and

$$\mathcal{E}_2(y,t)\overset{\text{def}}{=}\prod_{l=1}^{n}\int\limits_0^\infty \eta_l^{k_l}e^{-t\left[\eta_l^2-\sum\limits_{r=1}^{n_l}b_{lr}j_{\nu_l}(g_{lr}\eta_l)\right]}j_{\nu_l}(y_l\eta_l)\,d\eta_l, \tag{4.5}$$

define the function

$$\mathcal{E}(x,y,t)\overset{\text{def}}{=}\mathcal{E}_1(x,y,t)\mathcal{E}_2(x,y,t)$$

on $\overline{\mathbb{R}^{m+n}_+}\times(0,\infty)$.

For all $t_0,T\in(0,+\infty)$, integrals (4.4) and (4.5) converge absolutely and uniformly with respect to $(x,y,t)\in\overline{\mathbb{R}^{m+n}_+}\times[t_0,T]$ (note that $|j_\nu(z)|\le 1$); therefore, the function $\mathcal{E}(x,y,t)$ is well defined. Substitute (formally) $\mathcal{E}$ in Eq. (4.1):

$$\frac{\partial\mathcal{E}}{\partial t}=\frac{\partial\mathcal{E}_1}{\partial t}\,\mathcal{E}_2+\mathcal{E}_1\frac{\partial\mathcal{E}_2}{\partial t},\ \frac{\partial^2\mathcal{E}}{\partial x_i^2}=\frac{\partial^2\mathcal{E}_1}{\partial x_i^2}\,\mathcal{E}_2\,(i=\overline{1,m}),$$

$$B_{k_l,y_l}\mathcal{E}=\mathcal{E}_1\frac{\partial^2\mathcal{E}_2}{\partial y_l^2}+\frac{k_l}{y_l}\mathcal{E}_1\frac{\partial\mathcal{E}_2}{\partial y_l}=\mathcal{E}_1B_{k_l,y_l}\mathcal{E}_2\,(l=\overline{1,n}),$$

$$\mathcal{E}(x+h,y,t)=\mathcal{E}_1(x+h,y,t)\mathcal{E}_2 \text{ for any } h\in\mathbb{R}^m,$$

and

$$T_y^g\mathcal{E}=\mathcal{E}_1T_y^g\mathcal{E}_2(x,y,t)\text{ for any } g\in\mathbb{R}^n.$$

Thus,

$$\begin{aligned}&\frac{\partial\mathcal{E}}{\partial t}-\sum_{i=1}^{m}\left[\frac{\partial^2\mathcal{E}}{\partial x_i^2}+\sum_{s=1}^{m_i}a_{is}\mathcal{E}(x+h_{is},y,t)\right]-\sum_{l=1}^{n}\left(B_{k_l,y_l}\mathcal{E}+\sum_{r=1}^{n_l}b_{lr}T_{y_l}^{g_{lr}}\mathcal{E}\right)\\&=\mathcal{E}_2\left[\frac{\partial\mathcal{E}_1}{\partial t}-\Delta_x\mathcal{E}_1-\sum_{i=1}^{m}\sum_{s=1}^{m_i}a_{is}\mathcal{E}_1(x+h_{is},y,t)\right]\\&+\mathcal{E}_1\left[\frac{\partial\mathcal{E}_2}{\partial t}-\sum_{l=1}^{n}\left(B_{k_l,y_l}\mathcal{E}_2+\sum_{r=1}^{n_l}b_{lr}T_{y_l}^{g_{lr}}\mathcal{E}_2\right)\right].\end{aligned} \tag{4.6}$$

It is known from [67] that the former term of sum (4.6) vanishes; consider the latter one:

$$\frac{\partial\mathcal{E}_2}{\partial t}=\underbrace{\int\limits_0^\infty\dots\int\limits_0^\infty}_{n\text{ times}}\left[\sum_{l=1}^{n}\sum_{r=1}^{n_l}b_{lr}j_{\nu_l}(g_{lr}\eta_l)-|\eta|^2\right]e^{\left[\sum\limits_{l=1}^{n}\sum\limits_{r=1}^{n_l}b_{lr}j_{\nu_l}(g_{lr}\eta_l)-|\eta|^2\right]t}\prod_{l=1}^{n}\eta_l^{k_l}j_{\nu_l}(y_l\eta_l)\,d\eta_l$$

$$=\underbrace{\int\limits_0^\infty\dots\int\limits_0^\infty}_{n\text{ times}}\sum_{l=1}^{n}\sum_{r=1}^{n_l}b_{lr}T_{y_l}^{g_{lr}}j_{\nu_l}(y_l\eta_l)\prod_{\substack{\kappa=1\\\kappa\ne l}}^{n}j_{\nu_\kappa}(y_\kappa\eta_\kappa)\,e^{\left[\sum\limits_{l=1}^{n}\sum\limits_{r=1}^{n_l}b_{lr}j_{\nu_l}(g_{lr}\eta_l)-|\eta|^2\right]t}\prod_{l=1}^{n}\eta_l^{k_l}d\eta_l$$

$$-\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n\text{ times}} \sum_{l=1}^n \eta_l^{k_l+2} \prod_{\substack{\kappa=1\\ \kappa\neq l}}^n \eta_\kappa^{k_\kappa} e^{\left[\sum\limits_{l=1}^n \sum\limits_{r=1}^{n_l} b_{lr} j_{\nu_l}(g_{lr}\eta_l) - |\eta|^2\right]t} \prod_{l=1}^n j_{\nu_l}(y_l\eta_l)\, d\eta_l$$

because $T_x^y j_\nu(ax) = j_\nu(ax) j_\nu(ay)$ (see, e.g., [34, p. 19]).

Further, $B_{k_l,y_l} j_{\nu_l}(y_l\eta_l) = -\eta_l^2 j_{\nu_l}(y_l\eta_l)$ for any l (see, e.g., [34, p. 18]); hence,

$$B_{k_l,y_l}\mathcal{E}_2 = -\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n\text{ times}} e^{\left[\sum\limits_{l=1}^n \sum\limits_{r=1}^{n_l} b_{lr} j_{\nu_l}(g_{lr}\eta_l) - |\eta|^2\right]t} \eta_l^2 \prod_{\kappa=1}^n \eta_\kappa^{k_\kappa} j_{\nu_\kappa}(y_\kappa\eta_\kappa)\, d\eta_\kappa.$$

Thus, the latter term of sum (4.6) vanishes in $\overline{\mathbb{R}_+^{m+n}} \times (0,+\infty)$ as well. This means that the function $\mathcal{E}(x,t)$ formally satisfies Eq. (4.1).

Note that the inequalities

$$\left|B_{k_l,y_l}\mathcal{E}_2\right| \leq \text{const} \prod_{\substack{\kappa=1\\ \kappa\neq l}}^n \int_0^\infty \eta_\kappa^{k_\kappa} e^{-\eta_\kappa^2 t} d\eta_\kappa \int_0^\infty \eta_l^{k_l+2} e^{-\eta_l^2 t} d\eta_l$$

and

$$\left|T_{y_l}^{g_{lr}}\mathcal{E}_2\right| \leq \text{const} \prod_{l=1}^n \int_0^\infty \eta_l^{k_l} e^{-\eta_l^2 t} d\eta_l$$

are valid for all l and r. Therefore,

$$\left|\frac{\partial\mathcal{E}_2}{\partial t}\right| \leq \text{const}\ t^{-1-\frac{n}{2}-\frac{1}{2}\sum\limits_{l=1}^n k_l}.$$

In the same way,

$$\left|\frac{\partial\mathcal{E}_1}{\partial t}\right| \leq \text{const}\ t^{-1-\frac{m}{2}};$$

hence, the formal differentiation and the formal generalized translation under the integral sign are valid for all terms of Eq. (4.1). Therefore, the function $\mathcal{E}$ satisfies Eq. (4.1) in $\overline{\mathbb{R}_+^{m+n}} \times (0,+\infty)$.

We call $\mathcal{E}(x,t)$ the *fundamental solution* of Eq. (4.1). To show the reasonability of this term, we prove below that the generalized convolution (see [34, §1.8]) of $\mathcal{E}$ with any bounded initial-value function coincides with that initial-value function at the initial half-plane.

4.3. Generalized Convolutions of Fundamental Solutions and Bounded Functions

On $\overline{\mathbb{R}_+^{m+n}} \times (0,+\infty)$, consider the function

$$\int_{\mathbb{R}_+^{m+n}} \prod_{l=1}^n \eta_l^{k_l} \mathcal{E}(\xi,\eta,t) T_y^\eta u_0(x-\xi,y)\, d\xi d\eta. \tag{4.7}$$

The following assertion is valid:

Theorem 4.3.1. *Function* (4.7) *satisfies* (*in the classical sense*) *Eq.* (4.1).

Proof. First, we prove that function (4.7) is well defined. To do this, we apply the following estimates established for functions (4.4) and (4.5) in Secs. 1.4 and 3.3 respectively:

$$|x|^{m+2}|\mathcal{E}_1(x,t)| \leq C \tag{4.8}$$

and

$$y_l^\alpha \left| \int_0^\infty \eta_l^{k_l} e^{-t\left[\eta_l^2 - \sum\limits_{r=1}^{n_l} b_{lr} j_{\nu_l}(g_{lr}\eta_l)\right]} j_{\nu_l}(y_l\eta_l)\, d\eta_l \right| \leq C \tag{4.9}$$

(those estimates are valid for any positive t and α and any $l \in \overline{1,n}$).

For all positive t_0 and T, the constants of inequalities (4.8) and (4.9) depend only on t_0 and T but do not depend on $t \in [t_0, T]$. This and the boundedness of the function u_0 imply that integral (4.7) converges absolutely and uniformly with respect to $t \in [t_0, T]$ for any fixed T. Indeed,

$$\int\limits_{\mathbb{R}_+^{m+n}} \prod_{l=1}^n \eta_l^{k_l} |\mathcal{E}(\xi,\eta,t) T_y^\eta u_0(x-\xi,y)| d\xi d\eta \leq \frac{1}{2} \sup |u_0| \int\limits_{\mathbb{R}^{m+n}} \prod_{l=1}^n |\eta_l|^{k_l} |\mathcal{E}(\xi,\eta,t)| d\xi d\eta. \tag{4.10}$$

The integrand function at the right-hand part of the last inequality is extended to the whole space $\mathbb{R}^{m+n}$ as a function even with respect to each variable y_l. The inequality itself is understood in the following sense: if its right-hand side converges, then its left-hand side converges as well, and the inequality is valid; note that the normalized Bessel function is even and the generalized translation operator is continuous (see, e.g., [34, p. 18-19]).

By virtue of the smoothness of the factors of the function $\mathcal{E}(\xi,\eta,t)$ and estimates (4.8)-(4.9), the integrand function of the last integral can be represented as $\left| f_{0,t}(\xi) \prod\limits_{l=1}^n f_{l,t}(\eta_l) \right|$ such that its factors satisfy the following inequalities for $t \in [t_0, T]$:

$$|f_{0,t}(\xi)| \leq \frac{M_0}{1 + |\xi|^{m+1}}$$

and

$$|f_{l,t}(\xi)| \leq \frac{M_l}{1 + \eta_l^2},$$

where $M_0, \ldots, M_n$ are positive constants.

Let Ω be an arbitrary large bounded domain in $\mathbb{R}^{m+n}$. Without loss of generality, we assume that it contains the domain $Q(1) \stackrel{\text{def}}{=} \{|\xi| < 1, |\eta_l| \leq 1, \, l = \overline{1,n}\}$. There exists A_0 from the interval $(1, +\infty)$ such that

$$\Omega \subset Q(A_0) \stackrel{\text{def}}{=} \{|\xi| < A_0, |\eta_l| \leq A_0, \, l = \overline{1,n}\}.$$

The function $\left| f_{0,t}(\xi) \prod\limits_{l=1}^n f_{l,t}(\eta_l) \right|$ is integrable over $Q(A_0)$ by virtue of the boundedness of that domain; hence, the Fubini theorem is applicable:

$$\begin{aligned}
\int\limits_{Q(A_0)} \left| f_{0,t}(\xi) \prod_{l=1}^n f_{l,t}(\eta_l) \right| d\xi d\eta &= \int\limits_{|\xi|<A_0} |f_0(\xi)| d\xi \prod_{l=1}^n \int\limits_{-A_0}^{A_0} |f_{l,t}(\eta_l)| d\eta_l \\
&\leq M_0 \left(\max_{l=\overline{1,n}} M_l \right)^n \left[\frac{2\pi^m}{m\Gamma\left(\frac{m}{2}\right)} + \int\limits_{|\xi|>1} \frac{d\xi}{|\xi|^{m+1}} \right] \left(2 + 2 \int\limits_1^{A_0} \frac{dz}{z^2} \right)^n \\
&\leq M_0 \left(4 \max_{l=\overline{1,n}} M_l \right)^n \left[\frac{2\pi^m}{m\Gamma\left(\frac{m}{2}\right)} + \frac{2\pi^m}{\Gamma\left(\frac{m}{2}\right)} \int\limits_1^\infty \frac{dr}{r^2} \right] = \frac{2\pi^m M_0}{m\Gamma\left(\frac{m}{2}\right)} \left(4 \max_{l=\overline{1,n}} M_l \right)^n (1+m).
\end{aligned}$$

Therefore, the integral at the right-hand part of inequality (4.10) converges and satisfies the same estimate. This implies that function (4.7) is well defined on $\overline{\mathbb{R}_+^{m+n}} \times (0, +\infty)$. Further, by virtue of the self-adjointness of the generalized translation operator in the corresponding weight space (see, e.g., [34, p. 19]), function (4.7) is equal to

$$\int\limits_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} u_0(\xi,\eta) T_y^\eta \mathcal{E}(x-\xi, y, t) d\xi d\eta.$$

To complete the proof, it remains to justify the validity of differentiating and applying the generalized translation operator under the integral sign in (4.7). To do this, we must estimate the behavior of the functions $\Delta_x \mathcal{E}, B_{k_l,y_l}\mathcal{E}$, and $T_{y_l}^{g_{lr}}\mathcal{E}$ at infinity. The inequality $\left|T_{y_l}^{g_{lr}}\mathcal{E}\right| = \left|j_{\nu_l}(g_{lr}y_l)\mathcal{E}\right| \le \left|\mathcal{E}\right|$ is valid. Further, it is proved in Sec. 1.4 that $|x|^{m+1}\Delta_x \mathcal{E}_1(x,t) \overset{|x|\to\infty}{\longrightarrow} 0$ for any positive t and it is proved in Sec. 3.3 that

$$y_l^\alpha B_{k_l,y_l} \int\limits_0^\infty \eta_l^{k_l} e^{-t\left[\eta_l^2 - \sum\limits_{r=1}^{n_l} b_{lr} j_{\nu_l}(g_{lr}\eta_l)\right]} j_{\nu_l}(y_l\eta_l)\, d\eta_l \overset{y\to\infty}{\longrightarrow} 0$$

for any positive t, any positive α, and any $l \in \overline{1,n}$. This and inequalities (4.8)-(4.9) imply (as above) that the differentiation and generalized translation under the integral sign are valid in (4.7), which completes the proof. □

4.4. Solutions of the Nonclassical Cauchy Problem for Singular Functional Differential Equations

Introduce the following notation:

$$u(x,y,t) \overset{\text{def}}{=} \frac{2^{n-m}}{\pi^m \prod\limits_{l=1}^{n} 2^{k_l}\Gamma^2\left(\frac{k_l+1}{2}\right)} \int\limits_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} u_0(x-\xi,\eta) T_y^\eta \mathcal{E}(\xi, y, t) d\xi d\eta. \tag{4.11}$$

The following assertion is valid:

Theorem 4.4.1. *The function defined by relation* (4.11) *is a solution of problem* (4.1)–(4.3).

Proof. It follows from Theorem 4.3.1 that the function $u(x,y,t)$ satisfies Eq. (4.1). By virtue of the evenness of the function $T_y^\eta \mathcal{E}(\xi,y,t)$ with respect to the variables $y_1,\dots,y_n$ (see, e.g., [34, p. 35]), it follows that $u(x,y,t)$ satisfies condition (4.2). It remains to show that it satisfies condition (4.3) as well.

Take an arbitrary $(x_0,y_0) \overset{\text{def}}{=} (x_1^0,\dots,x_m^0,y_1^0,\dots,y_n^0)$ from $\overline{\mathbb{R}_+^{m+n}}$ and investigate the behavior of the function $u(x_0,y_0,t)$ as $t\to+0$.

Noting that $T_y^\eta f(y) = T_\eta^y f(\eta)$ (see, e.g., [34, p. 19]) and denoting $\dfrac{2^{n-m}}{\pi^m \prod\limits_{l=1}^{n} 2^{k_l}\Gamma^2\left(\frac{k_l+1}{2}\right)}$ by C, we obtain that

$$u(x_0,y_0,t) = C \int\limits_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} u_0(x_0-\xi,\eta) T_\eta^{y_0} \mathcal{E}(\xi,\eta,t) d\xi d\eta.$$

Change the variables as follows: $\zeta_i = \dfrac{\xi_i}{2\sqrt{t}}$ $(i=\overline{1,m})$ and $\rho_l = \dfrac{\eta_l}{2\sqrt{t}}$ $(l=\overline{1,n})$; this reduces the last relation to the form

$$u(x_0,y_0,t) = 2^{m+n+|k|} C t^{\frac{m+n+|k|}{2}} \int\limits_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \rho_l^{k_l} u_0(x_0-2\zeta\sqrt{t}, 2\rho\sqrt{t}) T_{2\rho\sqrt{t}}^{y_0} \mathcal{E}(2\zeta\sqrt{t}, 2\rho\sqrt{t}, t) d\zeta d\rho,$$

where $|k| \overset{\text{def}}{=} k_1 + \dots + k_n$ is the length of the multi-index.

Without loss of generality, we assume that

$$m_1 = \dots = m_m = n_1 = \dots = n_n = 1.$$

Redenote b_{l1} by b_l. Redenote g_{l1} by g_l $(l = \overline{1,n})$. Redenote a_{i1} by a_i. Let h_i denote $|h_{i1}|$ if the vector h_{i1}, $i = \overline{1,m}$, coincides with the positive direction of the ith coordinate axis of the space $\mathbb{R}^m$ and denote $-|h_{i1}|$ otherwise. Then $\mathcal{E}_1(x,t)$ can be represented as

$$2^m \prod_{i=1}^{m} \int_0^{+\infty} e^{-t(\tau^2 - a_i \cos h_i \tau)} \cos(x_i\tau + a_i t \sin h_i \tau) d\tau$$

(see Sec. 1.4 and [67]); therefore,

$$\mathcal{E}(2\zeta\sqrt{t}, 2\rho\sqrt{t}, t) = 2^m \prod_{i=1}^{m} \int_0^{+\infty} e^{-t(\tau^2 - a_i \cos h_i \tau)} \cos(2\sqrt{t}\zeta_i\tau + a_i t \sin h_i \tau) d\tau$$
$$\times \prod_{l=1}^{n} \int_0^{\infty} \eta_l^{k_l} e^{-t\left[\eta_l^2 - b_l j_{\nu_l}(g_l \eta_l)\right]} j_{\nu_l}\left(2\sqrt{t}\rho_l\eta_l\right) d\eta_l.$$

Change the variables as follows: $\tau\sqrt{t} = z$ and $\eta_l\sqrt{t} = \xi, l = \overline{1,n}$. This reduces the last expression to the following form:

$$\frac{2^m}{t^{\frac{m+n+|k|}{2}}} \prod_{i=1}^{m} \int_0^{+\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos\left(2z\zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz$$
$$\times \prod_{l=1}^{n} \int_0^{\infty} \xi^{k_l} e^{-\xi^2 + b_l t j_{\nu_l}\left(\frac{g_l \xi}{\sqrt{t}}\right)} j_{\nu_l}(2\xi\rho_l) d\xi. \tag{4.12}$$

Thus, taking into account the self-adjointness of the generalized translation operator, we see that

$$u(x_0, y_0, t) = 2^{2m+n+|k|} C \int_{\mathbb{R}_+^{m+n}} T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0)$$
$$\times \prod_{i=1}^{m} \int_0^{+\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos\left(2z\zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz$$
$$\times \prod_{l=1}^{n} \rho_l^{k_l} \int_0^{\infty} \xi^{k_l} e^{-\xi^2 + b_l t j_{\nu_l}\left(\frac{g_l \xi}{\sqrt{t}}\right)} j_{\nu_l}(2\xi\rho_l) d\xi d\zeta d\rho. \tag{4.13}$$

Further, we use the following assertions:

Lemma 4.4.1. *The limit relation*

$$\prod_{i=1}^{m} \int_0^{\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos\left(2z\zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz \overset{t \to +0}{\longrightarrow} \left(\frac{\sqrt{\pi}}{2}\right)^m e^{-|\zeta|^2}$$

holds uniformly with respect to $\zeta \in \mathbb{R}^m$.

Lemma 4.4.2. *For any $i \in \overline{1,m}$ and any positive A there exists M_i depending only on a_i and h_i such that for any t from $(0,1)$ and any ζ_i from $(A,+\infty)$ the inequality*

$$\left| \int_0^\infty e^{-z^2+a_i t\cos\frac{h_i z}{\sqrt{t}}} \cos\left(2z\zeta_i + a_i t \sin\frac{h_i z}{\sqrt{t}}\right) dz \right| \le \frac{M_i}{\zeta_i^2}$$

is valid.

Lemma 4.4.3. *For any $l \in \overline{1,n}$, the limit relation*

$$\int_0^\infty \xi^{k_l} e^{-\xi^2+b_l t j_{\nu_l}\left(\frac{g_l\xi}{\sqrt{t}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi \overset{t\to+0}{\longrightarrow} \frac{\Gamma(\nu_l+1)}{2}\, e^{-\rho_l^2}$$

holds uniformly with respect to ρ_l from $[0,+\infty)$.

Lemma 4.4.4. *For any $l \in \overline{1,n}$ there exist a positive C_l and α from $(1,+\infty)$ such that the inequality*

$$\left| \rho_l^{k_l} \int_0^\infty \xi^{k_l} e^{-\xi^2+b_l t j_{\nu_l}\left(\frac{g_l\xi}{\sqrt{t}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi \right| \le \frac{C_l}{\rho_l^\alpha}$$

is valid for any t from $(0,1)$ and any positive ρ_l.

Lemmas 4.4.1 and 4.4.2 are proved in Sec. 1.3 and [67] respectively. Lemmas 4.4.3 and 4.4.4 are proved in Sec. 3.4.

We have

$$\int_0^\infty \xi^{k_l} e^{-\xi^2} j_{\nu_l}(2\xi\rho_l) d\xi = \frac{\Gamma(\nu_l+1)}{2}\, e^{-\rho_l^2}$$

(see, e.g., [88, p. 186]); hence,

$$u_0(x_0,y_0) = \frac{2^{m+2n}}{\pi^m \prod\limits_{l=1}^n \Gamma^2(\nu_l+1)} \int\limits_{\mathbb{R}_+^{m+n}} u_0(x_0,y_0) \prod_{i=1}^m \int_0^{+\infty} e^{-z^2} \cos 2z\zeta_i\, dz \prod_{l=1}^n \rho_l^{k_l} \int_0^\infty \xi^{k_l} e^{-\xi^2} j_{\nu_l}(2\xi\rho_l)\, d\xi d\zeta d\rho.$$

Now, consider the difference $u(x_0,y_0,t) - u_0(x_0,y_0)$; it is equal to

$$2^{2m+n+|k|} C \int\limits_{\mathbb{R}_+^{m+n}} \prod_{l=1}^n \rho_l^{k_l} \Bigg[T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0)$$

$$\times \prod_{i=1}^m \int_0^{+\infty} e^{-z^2+a_i t\cos\frac{h_i z}{\sqrt{t}}} \cos\left(2z\zeta_i + a_i t\sin\frac{h_i z}{\sqrt{t}}\right) dz$$

$$\times \prod_{l=1}^n \int_0^\infty \xi^{k_l} e^{-\xi^2+b_l t j_{\nu_l}\left(\frac{g_l\xi}{\sqrt{t}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi d\zeta d\rho$$

$$- u_0(x_0,y_0) \prod_{i=1}^m \int_0^{+\infty} e^{-z^2} \cos 2z\zeta_i\, dz \prod_{l=1}^n \int_0^\infty \xi^{k_l} e^{-\xi^2} j_{\nu_l}(2\xi\rho_l)\, d\xi \Bigg] d\zeta d\rho$$

$$= 2^{2m+n+|k|}C\left(\int\limits_{Q(A)} + \int\limits_{\mathbb{R}_+^{m+n}\setminus Q(A)}\right) \stackrel{\text{def}}{=} 2^{2m+n+|k|}C(I_1+I_2), \tag{4.14}$$

where A is a positive parameter and $Q(A)$ denotes the domain

$$\left\{(\zeta,\rho)\in\mathbb{R}_+^{m+n}\middle| |\zeta|<1, \rho_l<A;\ l=\overline{1,n}\right\}.$$

Take an arbitrary positive ε.

Integral (4.14) converges absolutely and uniformly with respect to $t\in(0,1)$. Indeed, by virtue of Lemmas 4.4.2 and 4.4.4 and the boundedness of the function u_0, for any A from $(0,+\infty)$ and any t from $(0,1)$, the absolute value of its integrand function is estimated from above by

$$\sup|u_0|\left[\prod_{i=1}^m f_{1,i}(\zeta_i)\prod_{l=1}^n f_{2,l}(\rho_l) + \frac{\pi^{\frac{m}{2}}}{2^{m+n}}\prod_{l=1}^n\Gamma(\nu_l+1)e^{-|\zeta|^2-|\rho|^2}\right],$$

where $0\le f_{1,i}(\zeta_i)\le\dfrac{M_i}{1+\zeta_i^2}$, $i=\overline{1,m}$, and $0\le f_{2,l}(\rho_l)\le\dfrac{C_l}{1+\rho_l^\alpha}$, $l=\overline{1,n}$. Therefore, one can select a positive A such that the inequality $|I_2|<\dfrac{\varepsilon}{2^{m+n+|k|+1}C}$ is satisfied for any t from $(0,1)$. Fix the selected A and consider I_1:

$$T_{y_0}^{2\rho\sqrt{t}}u_0(x_0-2\zeta\sqrt{t},y_0)-u_0(x_0,y_0) = \pi^{-\frac{n}{2}}\prod_{l=1}^n\frac{\Gamma(\nu_l+1)}{\Gamma(\nu_l+\frac12)}$$

$$\times\underbrace{\int\limits_0^\pi\cdots\int\limits_0^\pi}_{n\text{ times}} u_0\Big[x_1^0-2\zeta_1\sqrt{t},\dots,x_m^0-2\zeta_m\sqrt{t},$$

$$\sqrt{(y_1^0)^2+4\rho_1^2t-4y_1^0\rho_1\sqrt{t}\cos\theta_1},\dots,$$

$$\sqrt{(y_n^0)^2+4\rho_n^2t-4y_n^0\rho_n\sqrt{t}\cos\theta_n}\Big]\prod_{l=1}^n\sin^{2\nu_l}\theta_l d\theta_l$$

$$-\pi^{-\frac{n}{2}}\prod_{l=1}^n\frac{\Gamma(\nu_l+1)}{\Gamma(\nu_l+\frac12)}\underbrace{\int\limits_0^\pi\cdots\int\limits_0^\pi}_{n\text{ times}} u_0(x_0,y_0)\prod_{l=1}^n\sin^{2\nu_l}\theta_l d\theta_l$$

$$=\pi^{-\frac{n}{2}}\prod_{l=1}^n\frac{\Gamma(\nu_l+1)}{\Gamma(\nu_l+\frac12)}\underbrace{\int\limits_0^\pi\cdots\int\limits_0^\pi}_{n\text{ times}}\Big(u_0\Big[x_1^0-2\zeta_1\sqrt{t},\dots,x_m^0-2\zeta_m\sqrt{t},$$

$$\sqrt{(y_1^0)^2+4\rho_1^2t-4y_1^0\rho_1\sqrt{t}\cos\theta_1},\dots,$$

$$\sqrt{(y_n^0)^2+4\rho_n^2t-4y_n^0\rho_n\sqrt{t}\cos\theta_n}\Big]u_0(x_0,y_0)\Big)\prod_{l=1}^n\sin^{2\nu_l}\theta_l d\theta_l.$$

Let $\delta>0$. By virtue of the continuity of the function u_0 at the point (x_0,y_0), one can select sufficiently small t_0 such that for any t from $(0,t_0)$, any (ζ,ρ) from $Q(A)$, and any θ_l from $[0,\pi]$, $l=\overline{1,n}$, the following inequality is valid:

$$\left|u_0\left[x_1^0-2\zeta_1\sqrt{t},\dots,x_m^0-2\zeta_m\sqrt{t},\sqrt{(y_1^0)^2+4\rho_1^2t-4y_1^0\rho_1\sqrt{t}\cos\theta_1},\dots,\right.\right.$$

$$\left.\left.\sqrt{(y_n^0)^2+4\rho_n^2t-4y_n^0\rho_n\sqrt{t}\cos\theta_n}\right]-u_0(x_0,y_0)\right|<\delta.$$

This means (take into account that the positive δ is selected arbitrarily) that

$$T_{y_0}^{2\rho\sqrt{t}}u_0(x_0-2\zeta\sqrt{t},y_0)\overset{t\to+0}{\longrightarrow}u_0(x_0,y_0)$$

uniformly with respect to (ζ,ρ) from $Q(A)$. This and Lemmas 4.4.1 and 4.4.3 imply that there exists a positive t_0 such that for any t from $(0,t_0)$ and any (ζ,ρ) from $Q(A)$, we have the inequality

$$\left|\int\left[T_{y_0}^{2\rho\sqrt{t}}u_0(x_0-2\zeta\sqrt{t},y_0)\prod_{i=1}^{m}\int_0^{+\infty}e^{-z^2+a_it\cos\frac{h_iz}{\sqrt{t}}}\cos\left(2z\zeta_i+a_it\sin\frac{h_iz}{\sqrt{t}}\right)dz\right.\right.$$

$$\times\prod_{l=1}^{n}\int_0^{\infty}\xi^{k_l}e^{-\xi^2+b_ltj_{\nu_l}\left(\frac{g_l\xi}{\sqrt{t}}\right)}j_{\nu_l}(2\xi\rho_l)\,d\xi d\zeta d\rho-u_0(x_0,y_0)\prod_{i=1}^{m}\int_0^{+\infty}e^{-z^2}\cos 2z\zeta_i\,dz$$

$$\left.\left.\times\prod_{l=1}^{n}\int_0^{\infty}\xi^{k_l}e^{-\xi^2}j_{\nu_l}(2\xi\rho_l)\,d\xi\right]d\zeta d\rho\right|<\frac{m\Gamma\left(\frac{m}{2}\right)\prod\limits_{l=1}^{n}(k_l+1)}{\pi^{\frac{m}{2}}A^{m+n+|k|}2^{2m+n+|k|+1}C}\varepsilon,$$

i.e.,

$$|I_1|\le\frac{\pi^{\frac{m}{2}}A^{m+n+|k|}\varepsilon}{2^{2m+n+|k|+1}Cm\Gamma\left(\frac{m}{2}\right)\prod\limits_{l=1}^{n}(k_l+1)}\int\limits_{Q(A)}\prod_{l=1}^{n}\rho_l^{k_l}d\zeta d\rho=\frac{\varepsilon}{2^{2m+n+|k|+1}C}.$$

Since the positive ε is selected arbitrarily, it follows that

$$u(x_0,y_0,t)-u_0(x_0,y_0)\overset{t\to+0}{\longrightarrow}0.$$

Since the point (x_0,y_0) is selected arbitrarily, it follows that the function $u(x,y,t)$ satisfies condition (4.3), which completes the proof of Theorem 4.4.1. □

In particular, using the proved theorem, one can compute the weight integral of the fundamental solution over $\mathbb{R}_+^{m+n}$: the following assertion holds.

Lemma 4.4.5. *The relation*

$$\int\limits_{\mathbb{R}_+^{m+n}}\prod_{l=1}^{n}y^{k_l}\mathcal{E}(x,y,t)dxdy=\frac{\pi^m\prod\limits_{l=1}^{n}\Gamma^2\left(\frac{k_l+1}{2}\right)}{2^{n-m-|k|}}e^{t\left(\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m_i}a_{is}+\sum\limits_{l=1}^{n}\sum\limits_{r=1}^{n_l}b_{lr}\right)}$$

is valid.

Proof. Consider $u_0(x,y)\equiv 1$. It is continuous and bounded; hence, by virtue of Theorem 4.4.1, the function

$$y(x,y,t)\overset{\text{def}}{=}\frac{2^{n-m}}{\pi^m\prod\limits_{l=1}^{n}2^{k_l}\Gamma^2\left(\frac{k_l+1}{2}\right)}\int\limits_{\mathbb{R}_+^{m+n}}\prod_{l=1}^{n}\eta_l^{k_l}\mathcal{E}(\xi,\eta,t)d\xi d\eta$$

satisfies problem (4.1)–(4.3) with the initial-value condition

$$y(x, y, 0) \equiv 1.$$

However, $y(x, y, t)$ does not depend on x and y. Hence, the function $y(t)$ satisfies the *ordinary differential* equation

$$y' - y\left(\sum_{i=1}^{m}\sum_{s=1}^{m_i} a_{is} + \sum_{l=1}^{n}\sum_{r=1}^{n_l} b_{lr}\right) = 0$$

and the initial-value condition

$$y(0) = 1.$$

Hence, the relation $y(t) = e^{t\left(\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m_i} a_{is} + \sum\limits_{l=1}^{n}\sum\limits_{r=1}^{n_l} b_{lr}\right)}$ holds. This completes the proof of Lemma 4.4.5. □

4.5. Inhomogeneous Singular Equations

In this section, we assume that the right-hand part of Eq. (4.1) is different from the identical zero. Let us show that, under the specified assumption, the fundamental solution $\mathcal{E}(x, y, t)$ can still be used to obtain integral representations of solutions of problem (4.1)–(4.3).

Take an arbitrary (x_0, y_0) from $\overline{\mathbb{R}^{m+n}_+}$ and define the following function for $t > \tau > 0$:

$$G(t, \tau) \overset{\text{def}}{=} 2^{-2m-n-|k|} \int\limits_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta_l^{k_l} f(\xi, \eta, t-\tau) T_{y_0}^{\eta} \mathcal{E}(x_0 - \xi, y_0, \tau) d\xi d\eta.$$

The following assertion is valid.

Lemma 4.5.1. *There exists a positive t_0 such that the function $G(t, \tau)$ is bounded in the domain $(0, t_0) \times (0, t)$.*

Proof. Change the variables as follows: $\zeta_i = \dfrac{\xi_i}{2\sqrt{\tau}}$, $i = \overline{1, m}$, and $\rho_l = \dfrac{\eta_l}{2\sqrt{\tau}}$, $l = \overline{1, n}$. Then, taking into account the self-adjointness of the generalized translation operator, we obtain the relation

$$G(t, \tau) = 2^{-m} \tau^{\frac{m+n+|k|}{2}} \int\limits_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \rho_l^{k_l} T_{y_0}^{2\rho\sqrt{\tau}} f(x_0 - 2\zeta\sqrt{\tau}, y_0, t-\tau) \mathcal{E}(2\zeta\sqrt{\tau}, 2\rho\sqrt{\tau}, \tau) d\zeta d\rho.$$

Now, similarly to the proof of Theorem 4.4.1, we assume (without loss of generality) that

$$m_1 = \cdots = m_m = n_1 = \cdots = n_n = 1;$$

redenote b_{l1} by b_l, redenote g_{l1} by g_l, $l = \overline{1, n}$, and redenote a_{i1} by a_i. Also, by h_i we denote $|h_{i1}|$ if the vector h_{i1} coincides with the positive direction of the ith coordinate axis of the space $\mathbb{R}^m$, $i = \overline{1, m}$; if it coincides with its negative direction, then h_i denotes $-|h_{i1}|$. Then $\mathcal{E}(2\zeta\sqrt{t}, 2\rho\sqrt{t}, t)$ is equal to (4.12). Therefore,

$$G(t, \tau) = \int\limits_{\mathbb{R}^{m+n}_+} \prod_{i=1}^{m} \int\limits_{0}^{+\infty} e^{-z^2 + a_i\tau\cos\frac{h_i z}{\sqrt{\tau}}} \cos\left(2z\zeta_i + a_i\tau\sin\frac{h_i z}{\sqrt{\tau}}\right) dz$$

$$\times \prod_{l=1}^{n} \rho_l^{k_l} \int\limits_{0}^{\infty} \xi^{k_l} e^{-\xi^2 + b_l\tau j_{\nu_l}\left(\frac{g_l\xi}{\sqrt{\tau}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi\, T_{y_0}^{2\rho\sqrt{\tau}} f(x_0 - 2\zeta\sqrt{\tau}, y_0, t-\tau) d\zeta d\rho.$$

Since the function f is bounded, it follows that the last integral converges absolutely and uniformly in the triangle $\{0 < \tau < t < 1\}$ (the proof is totally identical to the proof of the absolute and uniform convergence of the first term of integral (4.14)).

Thus, we have the following estimate of the function G:

$$|G(t,\tau)| \le 2^{-n}\sup|f| \prod_{i=1}^{m} \int_{-\infty}^{+\infty} \int_{0}^{\infty} e^{-z^2+a_i\tau\cos\frac{h_i z}{\sqrt{\tau}}} \cos\left(2z\zeta_i + a_i\tau \sin\frac{h_i z}{\sqrt{\tau}}\right) dz d\zeta_i$$

$$\times \prod_{l=1}^{n} \int_{-\infty}^{+\infty} |\rho_l|^{k_l} \int_0^{\infty} \xi^{k_l} e^{-\xi^2+b_l\tau j_{\nu_l}\left(\frac{g_l\xi}{\sqrt{\tau}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi\, d\rho_l.$$

In the last expression, each external (one-dimensional) integral (i.e., each integral over the real axis) can be represented as $\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{+\infty}$. Taking into account the boundedness of the internal integrals, we assign $A = 1$, assume (without loss of generality) that $t < 1$, and apply Lemmas 4.4.2 and 4.4.4. We obtain that there exists α from $(1, +\infty)$ such that

$$|G(t,\tau)| \le \text{const}\left(2 + 2\int_1^{+\infty} \frac{dr}{r^2}\right)^m \left(2 + 2\int_1^{+\infty}\frac{dr}{r^\alpha}\right)^n.$$

This completes the proof of Lemma 4.5.1. □

Therefore, the following function is defined on $\overline{\mathbb{R}^{m+n}_+} \times (0, +\infty)$:

$$v(x,y,t) \overset{\text{def}}{=} \frac{2^{n-m-|k|}}{\pi^m \prod\limits_{l=1}^{n}\Gamma^2\left(\frac{k_l+1}{2}\right)} \int_0^t \int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta_l^{k_l} \mathcal{E}(\xi,\eta,\tau) T_y^\eta f(x-\xi, y, t-\tau) d\xi d\eta d\tau. \tag{4.15}$$

Since the function $T_y^\eta \mathcal{E}(\xi, y, t)$ is even with respect to the variables $y_1, \dots, y_n$, it follows that function (4.15) satisfies condition (4.2). Let us show that the specified function satisfies Eq. (4.1) and the homogeneous initial-value function.

To prove the former assertion, we note that it is proved in Sec. 4.2 that the function $\mathcal{E}(x, y, t)$ satisfies Eq. (4.1) in $\overline{\mathbb{R}^{m+n}_+} \times (0, +\infty)$. Taking into account the decay estimates (established in Sec. 4.3) for its factors and their derivatives of the corresponding order as $|x| \to \infty$ and $|y| \to \infty$, we see that it remains to prove the following lemma:

Lemma 4.5.2. *Let* $(x_0, y_0) \in \overline{\mathbb{R}^{m+n}_+}$ *and* $t_0 > 0$. *Then*

$$\lim_{\tau\to+0} \frac{2^{n-m-|k|}}{\pi^m \prod\limits_{l=1}^{n}\Gamma^2\left(\frac{k_l+1}{2}\right)} \int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta_l^{k_l} f(\xi,\eta,t_0-\tau) T_{y_0}^\eta \mathcal{E}(x_0-\xi, y_0, \tau) d\xi d\eta = f(x_0, y_0, t_0).$$

Proof. We have

$$\int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta_l^{k_l} f(\xi,\eta,t_0-\tau) T_{y_0}^\eta \mathcal{E}(x_0-\xi, y_0, \tau) d\xi d\eta = 2^{2m+n+|k|} G(t_0,\tau)$$

$$= 2^{2m+n+|k|} \int_{\mathbb{R}^{m+n}_+} \prod_{i=1}^{m} \int_0^{+\infty} e^{-z^2+a_i\tau\cos\frac{h_i z}{\sqrt{\tau}}} \cos\left(2z\zeta_i + a_i\tau\sin\frac{h_i z}{\sqrt{\tau}}\right) dz$$

$$\times \prod_{l=1}^{n} \rho_l^{k_l} \int_0^{\infty} \xi^{k_l} e^{-\xi^2 + b_l \tau j_{\nu_l}\left(\frac{g_l \xi}{\sqrt{\tau}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi T_{y_0}^{2\rho\sqrt{\tau}} f(x_0 - 2\zeta\sqrt{\tau}, y_0, t_0 - \tau) d\zeta d\rho$$

(see the proof of Lemma 4.5.1).

Therefore, the difference

$$\frac{2^{n-m-|k|}}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l+1}{2}\right)} \int_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} f(\xi, \eta, t_0 - \tau) T_{y_0}^{\eta} \mathcal{E}(x_0 - \xi, y_0, \tau) d\xi d\eta - f(x_0, y_0, t_0)$$

can be represented as follows:

$$\frac{2^{m+2n}}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l+1}{2}\right)} \Bigg[\int_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \rho_l^{k_l} \Big[T_{y_0}^{2\rho\sqrt{\tau}} f(x_0 - 2\zeta\sqrt{\tau}, y_0, t_0 - \tau)$$

$$\times \prod_{i=1}^{m} \int_0^{+\infty} e^{-z^2 + a_i \tau \cos \frac{h_i z}{\sqrt{\tau}}} \cos\left(2z\zeta_i + a_i \tau \sin \frac{h_i z}{\sqrt{\tau}}\right) dz$$

$$\times \prod_{l=1}^{n} \int_0^{\infty} \xi^{k_l} e^{-\xi^2 + b_l \tau j_{\nu_l}\left(\frac{g_l \xi}{\sqrt{\tau}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi - \frac{\pi^{\frac{m}{2}} \prod_{l=1}^{n} \Gamma\left(\frac{k_l+1}{2}\right)}{2^{m+n}} e^{-|\zeta|^2 - |\rho|^2} f(x_0, y_0, t_0) \Big] d\zeta d\rho \Bigg]$$

$$= \frac{2^{m+2n}}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l+1}{2}\right)} \left(\int_{Q(A)} + \int_{\mathbb{R}_+^{m+n} \setminus Q(A)} \right) \stackrel{\text{def}}{=} \widetilde{C}(I_3 + I_4), \tag{4.16}$$

where A is a positive parameter.

Let $\varepsilon > 0$.

By virtue of the boundedness of the function f, integral (4.16) converges absolutely and uniformly with respect to $\tau \in (0,1)$; the proof is totally identical to the proof of the absolute and uniform convergence of integral (4.14). Therefore, one can select a positive A such that $|I_4| < \frac{\varepsilon}{2\widetilde{C}}$ for any τ from $(0,1)$. Fix the selected A and consider I_3.

The generalized translation

$$T_{y_0}^{2\rho\sqrt{\tau}} f(x_0 - 2\zeta\sqrt{\tau}, y_0, t_0 - \tau)$$

can be represented as follows:

$$\pi^{-\frac{n}{2}} \prod_{l=1}^{n} \frac{\Gamma(\nu_l+1)}{\Gamma(\nu_l+\frac{1}{2})} \underbrace{\int_0^{\pi} \cdots \int_0^{\pi}}_{n \text{ times}} f\Big[x_1^0 - 2\zeta_1\sqrt{\tau}, \dots, x_m^0 - 2\zeta_m\sqrt{\tau},$$

$$\sqrt{(y_1^0)^2 + 4\rho_1^2\tau - 4y_1^0\rho_1\sqrt{\tau}\cos\theta_1},$$

$$\dots, \sqrt{(y_n^0)^2 + 4\rho_n^2\tau - 4y_n^0\rho_n\sqrt{\tau}\cos\theta_n}, t_0 - \tau\Big] \prod_{l=1}^{n} \sin^{2\nu_l}\theta_l d\theta_l.$$

By virtue of the continuity and boundedness of the function f, the last expression tends to $f(x_0, y_0, t_0)$ as $\tau \to +0$ uniformly with respect to $(\zeta, \rho) \in Q(A)$. This and Lemmas 4.4.1 and 4.4.3 imply that there exists a positive τ_0 such that for any $\tau < \tau_0$ and any η from $Q(A)$, we have

$$\left| T_{y_0}^{2\rho\sqrt{\tau}} f(x_0 - 2\zeta\sqrt{\tau}, y_0, t_0 - \tau) \prod_{i=1}^{m} \int_0^{+\infty} e^{-z^2 + a_i\tau\cos\frac{h_i z}{\sqrt{\tau}}} \cos\left(2z\zeta_i + a_i\tau\sin\frac{h_i z}{\sqrt{\tau}}\right) dz \right.$$

$$\left. \times \prod_{l=1}^{n} \int_0^{\infty} \xi^{k_l} e^{-\xi^2 + b_l\tau j_{\nu_l}\left(\frac{g_l\xi}{\sqrt{\tau}}\right)} j_{\nu_l}(2\xi\rho_l)\, d\xi - \frac{\pi^{\frac{m}{2}} \prod_{l=1}^{n} \Gamma\left(\frac{k_l+1}{2}\right)}{2^{m+n}} e^{-|\zeta|^2 - |\rho|^2} f(x_0, y_0, t_0) \right|$$

$$< \frac{m\Gamma\left(\frac{m}{2}\right) \prod_{l=1}^{n} (k_l + 1)}{4\widetilde{C}\pi^{\frac{m}{2}} A^{m+n+|k|}} \varepsilon, \text{ i.e., } |I_3| \le \frac{\varepsilon}{2}.$$

This completes the proof of Lemma 4.5.2. □

It remains to proof that $v(x_0, y_0, t) \overset{t\to+0}{\longrightarrow} 0$ for any (x_0, y_0) from $\overline{\mathbb{R}_+^{m+n}}$.

To do that, we represent $v(x_0, y_0, t)$ as

$$\frac{2^{m+2n}}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l+1}{2}\right)} \int_0^t G(t, \tau)\, d\tau$$

and use Lemma 4.5.1: there exists a positive t_0 such that

$$|v(x_0, t)| \le \frac{2^{m+2n} \sup\limits_{t\in[0,t_0]} |G|}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l+1}{2}\right)} t$$

for any t from $(0, t_0)$. Since the point (x_0, y_0) is selected arbitrarily, this implies the following assertion:

Theorem 4.5.1. *Let u_0 be a continuous and bounded in $\overline{\mathbb{R}_+^{m+n}}$ function. Let f be a function continuous and bounded in $\overline{\mathbb{R}_+^{m+n}} \times (0, +\infty)$ such that it satisfies condition* (4.2) *and the functions $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}$ and $\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}$ are continuous and bounded in $\overline{\mathbb{R}_+^{m+n}} \times (0, +\infty)$. Then the function*

$$u(x, y, t) = \frac{2^{n-m-|k|}}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l+1}{2}\right)} \left[\int_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} \mathcal{E}(\xi, \eta, t) T_y^{\eta} u_0(x - \xi, y)\, d\xi d\eta \right.$$

$$\left. + \int_0^t \int_{\mathbb{R}_+^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} \mathcal{E}(\xi, \eta, \tau) T_y^{\eta} f(x - \xi, y, t - \tau)\, d\xi d\eta d\tau \right] \qquad (4.17)$$

is a solution of problem (4.1)–(4.3).

4.6. The Uniqueness of the Solution of the Singular Problem

First, we prove the following assertion:

Lemma 4.6.1. *For any positive T, function (4.17) is bounded in $\overline{\mathbb{R}^{m+n}_+}\times[0,T]$.*

Proof. Using (4.13), we represent the solution of problem (4.1)–(4.3) as follows:

$$u(x,y,t)=C_1\Bigg[\int\limits_{\mathbb{R}^{m+n}_+} T_y^{2\rho\sqrt{t}}u_0(x-2\zeta\sqrt{t},y)\prod_{i=1}^m\int\limits_0^{+\infty}e^{-z^2+a_it\cos\frac{h_iz}{\sqrt{t}}}\cos\left(2z\zeta_i+a_it\sin\frac{h_iz}{\sqrt{t}}\right)dz$$

$$\times\prod_{l=1}^n\rho_l^{k_l}\int\limits_0^\infty\xi^{k_l}e^{-\xi^2+b_ltj_{\nu_l}\left(\frac{g_l\xi}{\sqrt{t}}\right)}j_{\nu_l}(2\xi\rho_l)\,d\xi d\zeta d\rho+\int\limits_0^t\int\limits_{\mathbb{R}^{m+n}_+}T_y^{2\rho\sqrt{\tau}}f(x-2\zeta\sqrt{\tau},y,t-\tau)$$

$$\times\prod_{i=1}^m\int\limits_0^{+\infty}e^{-z^2+a_i\tau\cos\frac{h_iz}{\sqrt{\tau}}}\cos\left(2z\zeta_i+a_i\tau\sin\frac{h_iz}{\sqrt{\tau}}\right)dz$$

$$\times\prod_{l=1}^n\rho_l^{k_l}\int\limits_0^\infty\xi^{k_l}e^{-\xi^2+b_l\tau j_{\nu_l}\left(\frac{g_l\xi}{\sqrt{\tau}}\right)}j_{\nu_l}(2\xi\rho_l)\,d\xi d\zeta d\rho d\tau\Bigg]\stackrel{\text{def}}{=}C_1\left[I_5(x,y,t)+I_6(x,y,t)\right].$$

Here we impose (without loss of generality) the same assumptions regarding the coefficients of the equations as in the proof of Theorem 4.4.1, but we take into account that, in general, the right-hand part of the equation is different from the identical zero.

Integrate

$$\int\limits_0^{+\infty}e^{-z^2+a_it\cos\frac{h_iz}{\sqrt{t}}}\cos\left(2z\zeta_i+a_it\sin\frac{h_iz}{\sqrt{t}}\right)dz,\ i=\overline{1,m},$$

by parts twice. We obtain (see [67]) that if $0\le t\le T$ and $\zeta_i\ne0$, then the absolute value of the last integral does not exceed $\dfrac{M_i(1+T)e^{|a_i|T}}{\zeta_i^2}$, where the constant M_i depends only on the coefficients of Eq. (4.1).

Denote by n_0 the only positive integer lying in $\left(\nu_l+\dfrac{3}{2},\nu_l+\dfrac{5}{2}\right]$ and integrate

$$\int\limits_0^\infty\xi^{k_l}e^{-\xi^2+b_ltj_{\nu_l}\left(\frac{g_l\xi}{\sqrt{t}}\right)}j_{\nu_l}(2\xi\rho_l)\,d\xi,\ l=\overline{1,n},$$

by parts n_0 times. We obtain (see [59]) that if $0\le t\le T$ and $\rho_l>0$, then the absolute value of the last integral does not exceed $\dfrac{\widetilde{M}_l(T)e^{|b_l|T}}{\rho_l^{k+\alpha}}$, where the function $\widetilde{M}_l$ is a linear combination of power functions with nonnegative powers such that its coefficients depend only on the coefficients of Eq. (4.1).

Use the obtained estimates for $|\zeta_i|>1$, $i=\overline{1,m}$, and $\rho_l>1$, $l=\overline{1,n}$. If $|\zeta_i|\le1$ and $\rho_l\le1$, then the absolute values of the specified integrals are obviously estimated from above by $\dfrac{\sqrt{\pi}e^{|a_i|T}}{2}$ and $\dfrac{\Gamma(\nu_l+1)e^{|b_l|T}}{2}$ respectively.

Using the obtained estimates and the boundedness of the functions u_0 and f, we complete the proof of the lemma. □

Now, we can investigate the uniqueness of the found solution of problem (4.1)–(4.3) by means of the Fourier transforms (see [16, Ch. 2, §4, and Appendix]), using the Fourier–Bessel transformation (see, e.g., [34, Ch. 1]). To do this, following [16, Ch. 1], we introduce special spaces of test functions (cf. [34, §1.1]) below, assuming that condition (4.2) is replaced by the *equivalent* condition of evenness of the function u with respect to each variable y_l, $l = \overline{1,n}$, and, correspondingly, considering the problem in the subspace $\mathbb{R}^{m+n} \times (0, +\infty)$ (see Sec. 4.1).

Let μ_i and ω_i be continuous increasing on $[0, +\infty)$ functions such that

$$\mu_i(0) = \omega_i(0) = 0 \text{ and } \lim_{r\to\infty} \mu_i(r) = \lim_{r\to\infty} \omega_i(r) = \infty, i = \overline{1, m+n}.$$

Define the following concave functions on $[0, +\infty)$:

$$M_i(r) \overset{\text{def}}{=} \int\limits_0^r \mu_i(\rho)d\rho \text{ and } \Omega_i(r) \overset{\text{def}}{=} \int\limits_0^r \omega_i(\rho)d\rho.$$

Define the space of test functions $W_M^\Omega \overset{\text{def}}{=} W_{M_1,\dots,M_{m+n}}^{\Omega_1,\dots,\Omega_{m+n}}$ as the set of entire functions of complex variables $x_1, \dots, x_m$ and $y_1, \dots, y_n$ even with respect to each variable y_l, $l = \overline{1,n}$, and satisfying the estimate

$$|\varphi(x_1, \dots, x_m, y_1, \dots, y_n)| \le Ce^{-\sum\limits_{i=1}^m M_i(\alpha_i \mathrm{Re} x_i) - \sum\limits_{l=1}^n M_{m+l}(\alpha_l \mathrm{Re} y_l) + \sum\limits_{i=1}^m \Omega_i(\beta_i \mathrm{Im} x_i) + \sum\limits_{l=1}^n \Omega_{m+l}(\beta_l \mathrm{Im} y_l)}, \tag{4.18}$$

where the constants C, $\alpha_1, \dots, \alpha_{m+n}$, and $\beta_1, \dots, \beta_{m+n}$ may depend on the test function φ.

Introduce the classical topology of test functions: we say that a sequence $\{\varphi_\nu\}_{\nu=1}^\infty$ converges to zero in W_M^Ω if it uniformly converges to zero in any bounded domain of $\mathbb{C}^{m+n}$ and the constants C, $\alpha_1, \dots, \alpha_{m+n}$, and $\beta_1, \dots, \beta_{m+n}$ (from the definition of test functions) do not depend on the index ν.

Correspondingly, a set $Q \subset W_M^\Omega$ is called bounded if there exist absolute constants C, $\alpha_1, \dots, \alpha_{m+n}$, and $\beta_1, \dots, \beta_{m+n}$ such that all elements of Q satisfy estimate (4.18).

The Fourier–Bessel transformation is defined on W_M^Ω as follows:

$$\hat{f}(\xi, \eta) \overset{\text{def}}{=} \mathcal{F}_b f \overset{\text{def}}{=} \int\limits_{\mathbb{R}_+^{m+n}} \prod_{l=1}^n y_l^{k_l} j_{\nu_l}(\eta_l y_l) e^{-ix\cdot\xi} f(x,y)dxdy.$$

Denote by $W_{M,\alpha}^{\Omega,\beta}$ the subset of W_M^Ω such that each its element satisfies inequality (4.18) with α and β replaced by $\tilde{\alpha}$ and $\tilde{\beta}$ respectively for all

$$\tilde{\alpha}_1 < \alpha_1, \dots, \tilde{\alpha}_{m+n} < \alpha_{m+n} \text{ and } \tilde{\beta}_1 > \beta_1, \dots, \tilde{\beta}_{m+n} > \beta_{m+n}.$$

The following assertion is valid:

Lemma 4.6.2. *Suppose that functions $\widetilde{M}_i$ and $\widetilde{\Omega}_i$ are dual in the Young sense to the functions Ω_i and M_i respectively, $i = \overline{1, m+n}$. Then the Fourier–Bessel transformation is a bounded operator mapping $W_{M,\alpha}^{\Omega,\beta}$ into $W_{\widetilde{M},\frac{1}{\beta}}^{\widetilde{\Omega},\frac{1}{\alpha}}$, where*

$$\frac{1}{\alpha} = \left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_{m+n}}\right) \text{ and } \frac{1}{\beta} = \left(\frac{1}{\beta_1}, \dots, \frac{1}{\beta_{m+n}}\right).$$

Proof. For any $\nu > -\frac{1}{2}$, we have

$$j_\nu(x+iy) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int\limits_{-1}^1 e^{(ix-y)t}(1-t^2)^{\nu-\frac{1}{2}}dt$$

(see, e.g., [34, (1.5.8)]). Therefore, the following estimate holds:

$$|j_\nu(x+iy)| \le \text{const } e^{|y|}.$$

The remaining part of the proof is totally identical to the proof of Theorem 4 from [16, Ch. 1, §3]. This completes the proof of Lemma 4.6.2. □

Consider the elliptic operator $\mathcal{A}$ contained in Eq. 4.4.1:

$$\mathcal{A}u \stackrel{\text{def}}{=} \sum_{i=1}^{m}\left[\frac{\partial^2 u}{\partial x_i^2} + \sum_{s=1}^{m_i} a_{is}u(x+h_{is},y,t)\right] + \sum_{l=1}^{n}\left(B_{k_l,y_l}u + \sum_{r=1}^{n_l} b_{lr}T_{y_l}^{g_{lr}}u\right). \tag{4.19}$$

Let us find its symbol

$$\mathcal{P}(z) \stackrel{\text{def}}{=} \mathcal{P}(z_1,\dots,z_{m+n}) \stackrel{\text{def}}{=} \mathcal{P}(\sigma_1+i\tau_1,\dots,\sigma_{m+n}+i\tau_{m+n}).$$

It suffices to consider the case where $m_1 = n_1 = 1$ (i.e., the case where there are one special and one nonspecial spatial variables). Then

$$\mathcal{P}(z_1,z_2) = -z_1^2 + \sum_{s=1}^{m_1} a_{1s}e^{-ih_{1s}z_1} - z_2^2 + \sum_{r=1}^{n_1} b_{1r}j_{\nu_1}(g_{1r}z_2)$$

(see [34, (1.3.5) and (1.3.8)]) and

$$\mathrm{Re}\mathcal{P}(z) = |\sigma|^2 - |\tau|^2 + \sum_{s=1}^{m_1} a_{1s}e^{h_{1s}\tau_1}\cos h_{1s}\sigma_1 + \sum_{r=1}^{n_1} b_{1r}\mathrm{Re}j_{\nu_1}(g_{1r}z_2).$$

Using [34, (1.5.8)] again, we see that

$$\mathrm{Re}j_{\nu_1}(z_2) = \text{const}\int_{-1}^{1}(1-t^2)^{\nu_1-\frac{1}{2}}e^{-\tau_2 t}\cos\sigma_2 t\,dt.$$

Therefore,

$$\mathrm{Re}j_{\nu_1}(g_{1r}z_2) = \text{const}\int_{-1}^{1}(1-t^2)^{\nu_1-\frac{1}{2}}e^{-g_{1r}\tau_2 t}\cos(g_{1r}\sigma_2 t)\,dt.$$

Now, estimate the function $\mathcal{Q}(z,t_0,t) \stackrel{\text{def}}{=} e^{(t-t_0)\mathcal{P}(z)}$:

$$|\mathcal{Q}(z,t_0,t)| \le e^{(t-t_0)\left(|\sigma|^2+\sum\limits_{s=1}^{m_1}|a_{1s}|e^{h_{1s}\tau_1}+\text{const}\sum\limits_{r=1}^{n_1}|b_{1r}|e^{|g_{1r}|\tau_2}\right)} e^{(t-t_0)\left[(1+|\sigma|)^2+C_2e^{C_3\tau}\right]}.$$

The last estimate and Lemma 4.6.2 imply (see [16, Ch. 2, Appendix 1]) that problem (4.1)–(4.3) from Sec. 4.1 has at most one solution bounded in any layer $\overline{\mathbb{R}^{m+n}_+}\times[0,T]$. Then, taking into account that Eq. (4.1) is linear, we deduce the following assertion from Lemma 4.6.1.

Theorem 4.6.1. *Function* (4.17) *is the unique solution of problem* (4.1)–(4.3) *such that it is bounded in* $\overline{\mathbb{R}^{m+n}_+}\times[0,T]$ *for any positive* T.

Remark 4.6.1. The requirement of the boundedness of the function f and its derivatives can be weakened: it can be replaced by the requirement of their boundedness in any layer $\overline{\mathbb{R}^{m+n}_+}\times[0,T]$.

4.7. Long-Time Behavior of Solutions of Singular Problems

In this section, we investigate the long-time behavior of the solution of problem (4.1)–(4.3). To do this, we consider the operator $\mathcal{A}$ introduced by relation (4.19) and introduce the operator $\mathcal{L}$ acting as follows:

$$\mathcal{L}u \overset{\text{def}}{=} \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{l=1}^{n} B_{k_l,y_l} u + \sum_{a_{is}<0} a_{is} u(x + h_{is}, y, t) + \sum_{b_{lr}<0} b_{lr} T_{y_l}^{g_{lr}} u.$$

Further, without loss of generality, we redenote the vector h_{is} by $(\underbrace{0,\dots,0}_{i-1 \text{ times}}, h_{is}, 0, \dots, 0)$ (note that here h_{is} is a scalar parameter).

Now, denote the operator $\left(\sum\limits_{a_{is}<0} a_{is} + \sum\limits_{b_{lr}<0} b_{lr}\right) I - \mathcal{L}$ by R and consider the real part of its symbol:

$$\mathrm{Re}R(\xi,\eta) = \sum_{a_{is}<0} a_{is} + \sum_{b_{lr}<0} b_{lr} + |\xi|^2 + |\eta|^2 - \sum_{a_{is}<0} a_{is} \cos h_{is}\xi_i - \sum_{b_{lr}<0} b_{lr} j_{\nu_l}(g_{lr}\eta_l)$$

(cf. [102, §8]). We say that $R(\xi,\eta)$ is *positive definite* if there exists a positive C such that

$$\mathrm{Re}R(\xi,\eta) \ge C\left(|\xi|^2 + |\eta|^2\right)$$

for any (ξ,η) from $\overline{\mathbb{R}^{m+n}_+}$.

Together with Eq. (4.1), consider the equation

$$\frac{\partial w}{\partial t} = \sum_{i=1}^{m} \frac{\partial^2 w}{\partial x_i^2} + \sum_{l=1}^{n} B_{k_l,y_l} w, \ (x,y) \in \overline{\mathbb{R}^{m+n}_+}, \ t > 0, \tag{4.20}$$

and the initial-value condition

$$w\big|_{t=0} = w_0(x,y), \ (x,y) \in \overline{\mathbb{R}^{m+n}_+}, \tag{4.21}$$

where w_0 is continuous and bounded.

It is known from [36] (see also [37] and [47]) that problem (4.20), (4.2), (4.21) has a unique classical bounded solution $w(x,y,t)$.

The following assertion is valid:

Theorem 4.7.1. *Let $f(x,y) \equiv 0$ and $R(\xi,\eta)$ be positive definite. Then the following limit relation is valid for any (x,y) from $\overline{\mathbb{R}^{m+n}_+}$:*

$$e^{-t\left(\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m_i} a_{is} + \sum\limits_{l=1}^{n}\sum\limits_{r=1}^{n_l} b_{lr}\right)} u(x,y,t) - w\left(\frac{x_1+q_1t}{p_1}, \dots, \frac{x_m+q_mt}{p_m}, \frac{y_1}{p_{m+1}}, \dots, \frac{y_n}{p_{m+n}}, t\right) \overset{t\to\infty}{\longrightarrow} 0, \tag{4.22}$$

where

$$p_i = \sqrt{1 + \frac{1}{2}\sum_{s=1}^{m_i} a_{is}h_{is}^2}, \ q_i = \sum_{s=1}^{m_i} a_{is}h_{is}, \ i = \overline{1,m},$$

$$p_{m+l} = \sqrt{1 + \frac{1}{2(k_l+1)}\sum_{r=1}^{n_l} b_{lr}g_{lr}^2}, \ l = \overline{1,n},$$

and

$$w_0(x,y) = u_0(p_1x_1, \dots, p_mx_m, p_{m+1}y_1, \dots p_{m+n}y_n).$$

Proof. First, we prove that $p_1, \dots, p_{m+n}$ are well defined and different from zero under the assumptions of the theorem. Assuming, without loss of generality, that the (finite) sequences $\{a_{is}\}_{s=1}^{m_i}$, $i = \overline{1,m}$, and $\{b_{lr}\}_{r=1}^{n_l}$, $l = \overline{1,n}$, do not decrease, we denote $\min\limits_{a_{is}>0} s$ by m_i^0 and denote $\min\limits_{b_{lr}>0} r$ by n_l^0; if i or l is such that all the coefficients are negative, then we assign $m_i^0 = m_i + 1$ ($n_l^0 = n_l + 1$ respectively).

Let $i \in \overline{1,m}$. From the condition of the theorem, we deduce that

$$\sum_{s<m_i^0} a_{is} + \xi_i^2 - \sum_{s<m_i^0} a_{is} \cos h_{is}\xi_i \geq C\xi_i^2$$

for any positive ξ_i (indeed, we take the inequality of the positive definiteness condition and assign $\xi_1 = \dots = \xi_{i-1} = \xi_{i+1} = \xi_m = \eta_1 = \dots = \eta_n = 0$ in that inequality). This implies the inequality $\frac{1}{2}\sum\limits_{s<m_i^0} a_{is}h_{is}^2 > -1$ (see proof of Theorem 1 in [72]), i.e., $p_1, \dots, p_m$ are well defined and positive.

Now, let $l \in \overline{1,n}$. Then

$$\sum_{r<n_l^0} b_{lr} + \eta_l^2 - \sum_{r<n_l^0} b_{lr} j_{\nu_l}(g_{lr}\eta_l) \geq C\eta_l^2$$

for any positive η_l; hence,

$$C\eta_l^2 \leq \eta_l^2 + \sum_{r<n_l^0} b_{lr}\left[1 - j_{\nu_l}(g_{lr}\eta_l)\right].$$

We have the following representation of the normalized Bessel function:

$$j_\nu(z) = 1 - \frac{z^2}{4(\nu+1)} + \mathcal{O}(z^4) \Longrightarrow 1 - j_\nu(z) = \frac{z^2}{4(\nu+1)} + \mathcal{O}(z^4).$$

Therefore, there exists a neighborhood of the origin such that the following inequality holds in that neighborhood:

$$C\eta_l^2 \leq \eta_l^2 + \frac{\eta_l^2}{4(\nu_l+1)} \sum_{r<n_l^0} b_{lr}g_{lr}^2 + \mathcal{O}(\eta_l^4).$$

This implies the inequality

$$C \leq 1 + \frac{1}{4(\nu_l+1)} \sum_{r<n_l^0} b_{lr}g_{lr}^2 + \mathcal{O}(\eta_l^2).$$

Therefore,

$$\frac{C}{2} \leq 1 + \frac{1}{4(\nu_l+1)} \sum_{r<n_l^0} b_{lr}g_{lr}^2 + \mathcal{O}(\eta_l^2) - \frac{C}{2},$$

i.e., there exists a (small) neighborhood of the origin such that

$$0 < \frac{C}{2} \leq 1 + \frac{1}{4(\nu_l+1)} \sum_{r<n_l^0} b_{lr}g_{lr}^2 \Longrightarrow \frac{1}{4(\nu_l+1)} \sum_{r<n_l^0} b_{lr}g_{lr}^2 > -1.$$

Thus, $p_{m+1}, \dots, p_{m+n}$ are well defined and positive.

Now, we prove the following auxiliary lemma.

Lemma 4.7.1. *Let the conditions of Theorem* 4.7.1 *be satisfied. Then for any* $l \in \overline{m+1, m+n}$, *the limit relation*

$$\int_0^\infty \eta_l^{2\nu_l+1} e^{-\eta_l^2 + t\sum\limits_{r=1}^{n_l}\left[j_{\nu_l}\left(\frac{g_{lr}\eta_l}{\sqrt{t}}\right)-1\right]} j_{\nu_l}(2\rho\eta_l)\,d\eta_l \overset{t\to\infty}{\longrightarrow} \frac{\Gamma(\nu_l+1)}{2p_l^{k_l+1}} e^{-\frac{\rho^2}{p_l^2}}$$

holds uniformly with respect to $\rho \geq 0$.

Proof. First, we note that the sum symbol and the index r can be omitted because it is obvious that it suffices to prove the lemma for the case where the summand is single. We omit the index l as well because it is selected arbitrarily and redenote p^2 by p. Further,

$$\int_0^\infty \eta^{2\nu+1} e^{-p\eta^2} j_\nu(2\rho\eta)\,d\eta = \frac{1}{p^{\nu+1}} \frac{\Gamma(\nu+1)}{2} e^{-\frac{\rho^2}{p}}.$$

Therefore, it suffices to prove that

$$\int_0^\infty \eta^{2\nu+1} j_\nu(2\rho\eta) \left(e^{-\eta^2 + bt\left[j_\nu\left(\frac{g\eta}{\sqrt{t}}\right)-1\right]} - e^{-p\eta^2} \right) d\eta \overset{t\to\infty}{\longrightarrow} 0$$

uniformly with respect to $\rho \geq 0$.

First, we prove that the last integral converges absolutely and uniformly with respect to (t, ρ). To do this, we take into account that the parameter p is positive and the function j_ν is bounded and see that it suffices to estimate the power of the first integrated exponential function. We assume that $b < 0$ because the claimed convergence is obvious otherwise.

Let the parameter a be negative. Estimate the function $f(z) \overset{\text{def}}{=} z^2 - a[j_\nu(z) - 1]$:

$$f'(z) = 2z + a\frac{z}{2\nu+2} j_{\nu+1}(z) = 2z \left[1 + a\frac{1}{4\nu+4} j_{\nu+1}(z)\right] \geq 0$$

for $\left|\dfrac{a}{4\nu+4}\right| \leq 1$, which is equivalent to the inequality $a \geq -4\nu - 4$.

Thus, for $a \geq -4\nu - 4$, the function f does not decrease on $[0, +\infty)$. Since $f(0) = 0$, it follows that if $a \geq -4\nu - 4$, then the function f is nonnegative on the real axis (by virtue of its evenness).

Now, let $a > -4\nu - 4$. Then there exists α from $(0, 1)$ such that $\dfrac{a}{1-\alpha} \geq -4\nu - 4$. Therefore,

$$f(z) - \alpha z^2 = (1-\alpha)z^2 - a[j_\nu(z) - 1] = (1-\alpha)\left(z^2 - \frac{a}{1-\alpha}[j_\nu(z) - 1]\right) \geq 0.$$

Thus, for $a > -4\nu - 4$, there exists a positive α such that $f(z) \geq \alpha z^2$ on $\mathbb{R}^1$.

Now, redenote $\dfrac{g\eta}{\sqrt{t}}$ by z. The power to be estimated takes the form

$$-\frac{z^2 t}{g^2} + bt[j_\nu(z) - 1] = -\frac{t}{g^2}\left(z^2 - bg^2[j_\nu(z) - 1]\right).$$

Since $bg^2 > -4\nu - 4$, it follows that there exists a positive α such that the last expression does not exceed $-\dfrac{t\alpha}{g^2} z^2 = -\alpha\eta^2$. Therefore, the last integral converges absolutely and uniformly.

Then decompose it into the sum $\displaystyle\int_0^\delta + \int_\delta^\infty \overset{\text{def}}{=} I_1 + I_2$, where δ is a positive parameter. Take an arbitrary positive ε and, using the proved absolute and uniform convergence of the last integral, select a positive δ such that $|I_2| \leq \dfrac{\varepsilon}{2}$ for any t from $(1, +\infty)$. Fix the selected δ and estimate I_1. Its absolute value does not exceed

$$\int_0^\delta \eta^{2\nu+1} e^{-p\eta^2} \left| e^{(p-1)\eta^2 + bt\left[j_\nu\left(\frac{g\eta}{\sqrt{t}}\right)-1\right]} - 1 \right| d\eta.$$

To estimate the last expression, represent the function $j_\nu(z)$ as

$$j_\nu(0) + j'_\nu(0)z + \frac{j''_\nu(0)}{2} z^2 + \frac{j'''_\nu(\theta)}{6} z^3,$$

where $\theta \in [0, z]$.

Take into account that

$$j_\nu(0) = 1,$$
$$j'_\nu(0) = 0,$$
$$j''_\nu(0) = -\frac{1}{2\nu + 2},$$

and

$$j'''_\nu(\theta) = \frac{3\theta j_{\nu+2}(\theta)}{4(\nu+1)(\nu+2)} - \frac{\theta^3 j_{\nu+3}(\theta)}{8(\nu+1)(\nu+2)(\nu+3)}.$$

Thus,

$$j_\nu\left(\frac{g\eta}{\sqrt{t}}\right) - 1 = -\frac{1}{4(\nu+1)}\frac{g^2\eta^2}{t} + \frac{\psi(\eta,t)g^3\eta^3}{t^{\frac{3}{2}}},$$

where

$$|\psi(\eta,t)| \le \frac{3g\delta}{8(\nu+1)(\nu+2)} + \frac{(g\delta)^3}{48(\nu+1)(\nu+2)(\nu+3)}$$

for any $t \ge 1$ and $\eta \le \delta$.

Therefore, the power of the second exponential function in the last integral is equal to

$$\left[p - 1 - \frac{bg^2}{4(\nu+1)}\right]\eta^2 + \frac{b\psi(\eta,t)g^3\eta^3}{\sqrt{t}} = \frac{b\psi(\eta,t)g^3\eta^3}{\sqrt{t}} \stackrel{\text{def}}{=} \frac{\widetilde{\psi}(\eta,t)}{\sqrt{t}}$$

and there exists a positive M such that $|\widetilde{\psi}(\eta,t)| \le M$ for any $\eta \in [0,\delta]$ and $t \ge 1$.

Thus,

$$|I_1| \le \int\limits_0^\delta \eta^{2\nu+1} e^{-p\eta^2}\left|e^{\frac{\widetilde{\psi}(\eta,t)}{\sqrt{t}}} - 1\right| d\eta.$$

Select t_0 from $[1, +\infty)$ such that $e^{\frac{M}{\sqrt{t_0}}}, e^{-\frac{M}{\sqrt{t_0}}} \in [1-\delta_0, 1+\delta_0]$, where

$$\delta_0 = \frac{\varepsilon}{2}\left(\int\limits_0^\delta \eta^{2\nu+1} e^{-p\eta^2} d\eta\right)^{-1}.$$

Then $|I_1| \le \frac{\varepsilon}{2}$ for any $t \ge t_0$ by virtue of the monotonicity of the exponential function. This completes the proof of Lemma 4.7.1. □

Now, we pass directly to the proof of the limit relation (4.22). Obviously, it suffices to do this for the case where

$$m_1 = \cdots = m_m = n_1 = \cdots = n_n = 1;$$

therefore, we omit the second indices of the coefficients a, b, h, and g.

Let $(x_0, y_0) = (x_1^0, \dots, x_m^0, y_1^0, \dots, y_n^0)$ be an arbitrary element of $\overline{\mathbb{R}_+^{m+n}}$. Applying relation (4.13), we represent

$$e^{-t\left(\sum\limits_{i=1}^m a_i + \sum\limits_{l=1}^n b_l\right)} u(x_0, y_0, t)$$

as follows:

$$\frac{2^{m+2n}}{\pi^m \prod\limits_{i=1}^m \Gamma^2(\nu_l + 1)} \int\limits_{\mathbb{R}_+^{m+n}} T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0)$$

$$\times \prod_{i=1}^{m} \int_{0}^{+\infty} e^{-z^2+a_i t\left(\cos \frac{h_i z}{\sqrt{t}}-1\right)} \cos\left(2z\zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz$$

$$\times \prod_{l=1}^{n} \rho_l^{k_l} \int_{0}^{\infty} \xi^{k_l} e^{-\xi^2+b_l t\left[j_{\nu_l}\left(\frac{g_l \xi}{\sqrt{t}}\right)-1\right]} j_{\nu_l}(2\xi\rho_l)\, d\xi d\zeta d\rho.$$

In the sequel, we assume, without loss of generality, that $m = n = 1$. Then the last expression is equal to

$$\frac{8}{\pi\Gamma^2\left(\frac{k+1}{2}\right)} \int_{0}^{\infty}\int_{-\infty}^{\infty} \rho^k T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0)$$

$$\times \int_{0}^{+\infty} e^{-z^2+at\left(\cos \frac{hz}{\sqrt{t}}-1\right)} \cos\left(2z\zeta + at \sin \frac{hz}{\sqrt{t}}\right) dz$$

$$\times \int_{0}^{\infty} \xi^k e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} j_\nu(2\xi\rho)\, d\xi d\zeta d\rho.$$

Together with this expression, consider

$$\frac{8}{\pi\Gamma^2\left(\frac{k+1}{2}\right)} \int_{0}^{\infty}\int_{-\infty}^{\infty} \rho^k T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0) \int_{0}^{+\infty} e^{-(1+\frac{ah^2}{2})z^2} \cos(2\zeta + ah\sqrt{t})z\, dz$$

$$\times \int_{0}^{\infty} \xi^k e^{-\left[1+\frac{bg^2}{2(k+1)}\right]\xi^2} j_\nu(2\xi\rho)\, d\xi d\zeta d\rho = \frac{8}{\pi\Gamma^2\left(\frac{k+1}{2}\right)} \int_{0}^{\infty}\int_{-\infty}^{\infty} \rho^k T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0)$$

$$\times \int_{0}^{+\infty} e^{-p_1^2 z^2} \cos(2\zeta + ah\sqrt{t})z\, dz \int_{0}^{\infty} \xi^k e^{-p_2^2\xi^2} j_\nu(2\xi\rho)\, d\xi d\zeta d\rho$$

$$= \frac{2}{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right) p_1 p_2^{k+1}} \int_{0}^{\infty}\int_{-\infty}^{\infty} \rho^k T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0) e^{-\frac{(2\zeta+q_1\sqrt{t})^2}{4p_1^2} - \frac{\rho^2}{p_2^2}} d\zeta d\rho. \tag{4.23}$$

On the other hand, it is known from [37, 38, 42–45, 47] that

$$w(x_0, y_0, t) = \frac{2}{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right)} \int_{0}^{\infty}\int_{-\infty}^{\infty} \rho^k e^{-\zeta^2-\rho^2} T_{y_0}^{2\rho\sqrt{t}} w_0(x_0 - 2\zeta\sqrt{t}, y_0) d\zeta d\rho.$$

Therefore,

$$w\left(\frac{x_0 + q_1 t}{p_1}, \frac{y_0}{p_2}, t\right) = \frac{2}{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right) p_1} \int_{0}^{\infty}\int_{-\infty}^{\infty} \rho^k e^{-\frac{(2\zeta+q_1\sqrt{t})^2}{4p_1^2} - \rho^2} T_{y_0}^{2p_2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0) d\zeta d\rho$$

$$= \frac{2}{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right) p_1 p_2^{k+1}} \int_{0}^{\infty}\int_{-\infty}^{\infty} \rho^k T_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta\sqrt{t}, y_0) e^{-\frac{(2\zeta+q_1\sqrt{t})^2}{4p_1^2} - \frac{\rho^2}{p_2^2}} d\zeta d\rho.$$

Thus, expression (4.23) is equal to $w\left(\dfrac{x_0+q_1t}{p_1}, \dfrac{y_0}{p_2}, t\right)$, i.e., to deduce relation (4.22), we must investigate the long-time behavior of

$$\int\limits_0^\infty \int\limits_{-\infty}^\infty \rho^k T_{y_0}^{2\rho\sqrt{t}} u_0(x_0-2\zeta\sqrt{t}, y_0) \left[\int\limits_0^{+\infty} e^{-z^2+at\left(\cos\frac{hz}{\sqrt{t}}-1\right)} \cos\left(2z\zeta + at\sin\frac{hz}{\sqrt{t}}\right) dz \right.$$
$$\left. \times \int\limits_0^\infty \xi^k e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} j_\nu(2\xi\rho)\, d\xi - \frac{\sqrt{\pi}}{2p_1} e^{-\frac{(2\zeta+q_1\sqrt{t})^2}{4p_1^2}} \frac{\Gamma(\nu+1)}{2p_2^{k+1}} e^{-\frac{\rho^2}{p_2^2}} \right] d\zeta d\rho. \tag{4.24}$$

First, we prove that the last integral converges absolutely and uniformly with respect to $t\in[1,+\infty)$. By virtue of the boundedness of the function u_0, the absolute value of the second term of the specified integral is estimated from above by const $\int\limits_{-\infty}^\infty e^{-\frac{\zeta^2}{p_1^2}} d\zeta \int\limits_0^\infty \rho^k e^{-\frac{\rho^2}{p_2^2}} d\rho$; hence, it suffices to prove the absolute and uniform convergence of its first term. Change the variable: $y = 2\zeta + q\sqrt{t}$; this reduces the specified term to the form

$$\frac{1}{2}\int\limits_0^\infty \int\limits_{-\infty}^\infty \rho^k T_{y_0}^{2\rho\sqrt{t}} u_0(x_0-y\sqrt{t}-qt, y_0) \int\limits_0^{+\infty} e^{-z^2+at\left(\cos\frac{hz}{\sqrt{t}}-1\right)} \cos\left(yz - q\sqrt{t}z + at\sin\frac{hz}{\sqrt{t}}\right) dz$$
$$\times \int\limits_0^\infty \xi^k e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} j_\nu(2\xi\rho)\, d\xi dy d\rho.$$

It is proved in Sec. 2.3 that if the conditions of Theorem 4.7.1 are satisfied, then there exists a positive M such that

$$\left| \int\limits_0^{+\infty} e^{-z^2+at\left(\cos\frac{hz}{\sqrt{t}}-1\right)} \cos\left(yz - q\sqrt{t}z + at\sin\frac{hz}{\sqrt{t}}\right) dz \right| \le \frac{M}{1+y^2}$$

provided that $t\ge 1$ and $y>0$.

Further, it is known from 3.3 that

$$\int\limits_0^\infty \xi^k e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} j_\nu(2\xi\rho)\, d\xi$$

is a finite sum of terms of the form

$$\frac{1}{\rho^{2\nu+2m}t^l} \int\limits_0^\infty \xi \mathrm{j}_{\nu+m}(2\rho\xi) e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} j_{\nu+l+1}\left(\frac{g\xi}{\sqrt{t}}\right) f_l(\xi,t) d\xi, \tag{4.25}$$

where $\mathrm{j}_\nu(z) = z^\nu J_\nu(z)$, f_l is a bounded function, and l is a positive integer not exceeding $m-1$.

For $t\ge 1$, the absolute value of (4.25) does not exceed

$$\frac{\text{const}}{\rho^{2\nu+2m}} \int\limits_0^\infty \xi \mathrm{j}_{\nu+m}(2\rho\xi) e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} d\xi. \tag{4.26}$$

Further, we have

$$\xi \mathrm{j}_{\nu+m}(2\rho\xi) = \frac{1}{2\rho}(2\rho\xi)^{\nu+m+1} J_{\nu+m}(2\rho\xi) = \frac{(2\rho\xi)^{\nu+m+\frac{1}{2}}}{2\rho}\sqrt{2\rho\xi}\, J_{\nu+m}(2\rho\xi).$$

However, under the assumptions of Theorem 4.7.1, i.e., for $1 + \dfrac{bg^2}{4\nu+4} > 0$, there exists a positive α such that the power of the exponential function in (4.26) does not exceed $-\alpha\xi^2$ (see the proof of Lemma 4.7.1). Taking into account the boundedness of the function $\sqrt{\tau}J_\nu(\tau)$, we see that this implies that the absolute value of expression (4.26) does not exceed

$$\frac{\text{const}}{\rho^{2\nu+2m+1-\nu-m-\frac{1}{2}}}\int\limits_0^\infty \xi^{m+\nu+\frac{3}{2}}e^{-\alpha\xi^2}d\xi = \frac{\text{const}}{\rho^{m+\nu+\frac{1}{2}}}.$$

Thus, selecting an integer m from the interval $\left(\nu+\dfrac{3}{2},\infty\right)$, we obtain that the there exists β from $(1,+\infty)$ such that

$$\int\limits_0^\infty \xi^k e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} j_\nu(2\xi\rho)\,d\xi \le \frac{\text{const}}{\rho^\beta}.$$

Use this estimate for $\rho \ge 1$; if $\rho \in (0,1)$, then we use the boundedness of the last integral (as a function of variables $t \in [1,\infty)$ and $\rho \in (0,1)$) implied from the boundedness of the function $j_\nu(\cdot)$ and the above-mentioned estimate of the power of the integrated exponential function, obtained in Lemma 4.4.1. By virtue of the boundedness of the function u_0, this completes the proof of the absolute and uniform convergence of the first term of the integral (4.24).

Now, decompose integral (4.24) into the sum

$$\int\limits_{\{|\zeta|<\delta,\,0<\rho<\delta\}} + \int\limits_{\mathbb{R}^2_+\setminus\{|\zeta|<\delta,\,0<\rho<\delta\}} \stackrel{\text{def}}{=} I_3 + I_4.$$

Take an arbitrary positive ε. By virtue of the proved absolute and uniform convergence of the specified integral, there exists a positive δ such that $|I_4| \le \dfrac{\varepsilon}{2}$ for any t from $[1,\infty)$. Fix the selected δ and consider I_3.

By virtue of the boundedness of the function u_0, we have the estimate

$$\begin{aligned}|I_3| \le \text{const}\int\limits_0^\delta\int\limits_{-\delta}^\delta \rho^k\Bigg|&\int\limits_0^{+\infty} e^{-z^2+at\left(\cos\frac{hz}{\sqrt{t}}-1\right)}\cos\left(2z\zeta+at\sin\frac{hz}{\sqrt{t}}\right)dz\\ &\times\int\limits_0^\infty \xi^k e^{-\xi^2+bt\left[j_\nu\left(\frac{g\xi}{\sqrt{t}}\right)-1\right]} j_\nu(2\xi\rho)\,d\xi - \frac{\sqrt{\pi}}{2p_1}e^{-\frac{(2\zeta+q_1\sqrt{t})^2}{4p_1^2}}\frac{\Gamma(\nu+1)}{2p_2^{k+1}}e^{-\frac{\rho^2}{p_2^2}}\Bigg|d\zeta d\rho. \qquad (4.27)\end{aligned}$$

By virtue of [72, Lemma 1], the limit relation

$$\int\limits_0^{+\infty} e^{-z^2+at\left(\cos\frac{hz}{\sqrt{t}}-1\right)}\cos\left(2z\zeta+at\sin\frac{hz}{\sqrt{t}}\right)dz - \frac{\sqrt{\pi}}{2p_1}e^{-\frac{(2\zeta+q_1\sqrt{t})^2}{4p_1^2}} \xrightarrow{t\to\infty} 0$$

holds uniformly with respect to $\zeta \in \mathbb{R}^1$.

This and Lemma 4.7.1 imply that there exists a positive t_0 such that for any t from $(t_0,+\infty)$, the expression under the modulus sign in (4.27) does not exceed $\dfrac{\varepsilon}{2}\left(\int\limits_0^\delta\int\limits_{-\delta}^\delta \rho^k d\zeta d\rho\right)^{-1}$.

This completes the proof of Theorem 4.7.1. □

Similarly to the regular case of [72], imposing the additional condition of the symmetry of the elliptic operator contained in the considered equation, we obtain a weight *stabilization* of the solution

$u(x,y,t)$ (apart from the weight *closeness* of solutions, proved in Theorem 4.7.1). More exactly, the following assertion is valid.

Corollary 4.7.1. *Let the conditions of Theorem* 4.7.1 *be satisfied and the operator* $\mathcal{A}$ *be symmetric. Then for any real* l, *the assertion*

$$\lim_{t\to\infty} e^{-t\left(\sum\limits_{i=1}^{m}\sum\limits_{s=1}^{m_i} a_{is}+\sum\limits_{l=1}^{n}\sum\limits_{r=1}^{n_l} b_{lr}\right)} u(x,y,t)=l \text{ for any } (x,y)\in\overline{\mathbb{R}^{m+n}_+}$$

is valid if and only if

$$\lim_{t\to\infty}\frac{C_{m,n,k}}{t^{m+n+|k|}}\int\limits_{B^+(p,t)}\prod_{l=1}^{n} y_l^{k_l}u_0(x,y)dxdy=l,$$

where

$$C_{m,n,k}=\frac{\Gamma\left(\frac{k_n+1}{2}\right)\Gamma\left(\frac{k_n+k_{n-1}+1}{2}\right)\cdots\Gamma\left(\frac{|k|+1}{2}\right)}{\Gamma\left(\frac{k_n+k_{n-1}}{2}+1\right)\Gamma\left(\frac{k_n+k_{n-1}+k_{n-2}}{2}+1\right)\cdots\Gamma\left(\frac{|k|}{2}+1\right)}\frac{\pi^{\frac{m}{2}}\prod\limits_{l=1}^{n}\Gamma\left(\frac{k_l+1}{2}\right)}{2^{n-1}(m+n+|k|)\prod\limits_{i=1}^{m}p_i\prod\limits_{l=1}^{n}p_{m+l}^{k_l+1}}$$

and

$$B^+(p,t)=\left\{(x,y)\in\mathbb{R}^{m+n}_+\middle|\sum_{i=1}^{m}\frac{x_i^2}{p_i^2}+\sum_{l=1}^{n}\frac{y_l^2}{p_{m+l}^2}<t^2\right\}.$$

To prove this, it suffices to note that $q_1=\cdots=q_m=0$ under the conditions of Corollary 4.7.1 and apply theorems on the stabilization of solutions of singular *differential* parabolic equations. For details, see the Appendix.

Remark 4.7.1. Since $T_y^h f(y)=T_y^{-h}f(y)$, it follows that the singular part of the operator $\mathcal{A}$ is always symmetric. Therefore, the symmetry assumption for the operator $\mathcal{A}$ can be replaced by the symmetry assumption for the following *differential-difference* operator:

$$\mathcal{A}_{reg}u\overset{\text{def}}{=}\sum_{i=1}^{m}\left[\frac{\partial^2 u}{\partial x_i^2}+\sum_{s=1}^{m_i}a_{is}u(x+h_{is},y,t)\right].$$

Remark 4.7.2. If the conditions of Corollary 4.7.1 are satisfied, then the symmetry requirement for the operator $\mathcal{A}_{reg}$ can be weakened: it can be replaced by the requirement that $a_i\perp h_i$, where $a_i=(a_{i1},\dots,a_{im_i})$ and $h_i=(h_{i1},\dots,h_{im_i})$, $i=\overline{1,m}$.

Remark 4.7.3. It follows from Corollary 4.7.1 that in the differential-difference case, surfaces bounding averaging domains of the initial-value function in the stabilization condition are not guaranteed to be segments of spheres anymore: in general, they are segments of ellipsoids. Note that in the classical case of *differential* equations, such an effect arises if the operator

$$\Delta_{\mathcal{B}}\overset{\text{def}}{=}\Delta_x+\sum_{l=1}^{n}B_{k_l,y_l}$$

is replaced by an operator with different coefficients at different second derivatives:

$$\sum_{i=1}^{n}p_i^2\frac{\partial^2}{\partial x_i^2}+\sum_{l=1}^{n}p_{m+l}^2B_{k_l,y_l}.$$

Remark 4.7.4. Remark 1.6.1 is completely correct in the singular case as well.

5.1. Stabilization of Solutions of the Cauchy Problem: Prototype Case

In this section, we study the long-time behavior of Cauchy problem solutions for equations of the form

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k \frac{\partial u}{\partial y}\right) \quad \text{and} \quad \frac{\partial u}{\partial t} = a(t)\left[\Delta u + \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k \frac{\partial u}{\partial y}\right)\right].$$

The solvability and uniqueness of solutions of such questions are investigated in [36, 46, 47] and a number of other papers. In the regular case (i.e., for $k = 0$), the long-time behavior of solutions is investigated in [95, 96].

5.1.1. The Cauchy problem for the singular heat equation. The following notation are used:

- $\mathbb{R}^n$ is the real Euclid n-dimensional space;
- $\mathbb{R}^{n+1}_+$ is the half-space $\{(x, y)|x \in \mathbb{R}^n, y > 0\}$;
- $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \cdots + \dfrac{\partial^2}{\partial x_n^2}$ and $B_{k,y} = \dfrac{\partial^2 u}{\partial y^2} + \dfrac{k}{y}\dfrac{\partial u}{\partial y}$, where k is a positive parameter.

Consider the following problem:

$$\frac{\partial u}{\partial t} = (\Delta + B_{k,y})\, u, \;\; x \in \mathbb{R}^n, \;\; y > 0, \;\; t > 0, \tag{5.1}$$

$$u\big|_{t=0} = \varphi(x, y), \;\; \frac{\partial u}{\partial y}\Big|_{y=0} = 0. \tag{5.2}$$

Here the function $\varphi(x, y)$ is assumed to be continuous and bounded in $\overline{\mathbb{R}^{n+1}_+}$.

It is proved in [36] that problem (5.1)-(5.2) has a unique bounded solution and it is a classical solution, i.e., all its derivatives included in the equation exist in the classical sense and are continuous (for $y = 0$, the right-hand derivative with respect to y is treated as the derivative with respect to y) and relations (5.1) and (5.2) are satisfied pointwise (on the hyperplanes $\{t = 0\}$ and $\{y = 0\}$, they are satisfied in the sense of limit values as $t \to 0+$ and $y \to 0+$). Note that, investigating the Cauchy problem in this chapter, we always mean classical solutions.

Theorem 5.1.1. *Let $u(x, y, t)$ be a bounded solution of problem* (5.1)-(5.2)*, l be real, x belong to $\mathbb{R}^n$, and y be nonnegative. Then the relation*

$$\lim_{t\to\infty} u(x, y, t) = l \tag{5.3}$$

is valid if and only if the following relation is valid:

$$\lim_{t\to\infty} \frac{n + k + 1}{\pi^{\frac{n}{2}} t^{n+k+1}} \int\limits_{B_+(t)} y^k \varphi(x, y)dxdy = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+k+1}{2}\right)}\, l, \tag{5.4}$$

where $B_+(A)$ denotes the semiball $\{|x|^2 + y^2 < A^2|\, y > 0\}$.

Proof. Without loss of generality, we assume that $n = 1$.

We decompose the proof into three stages. In the first stage, we prove the theorem for the case where $x = y = l = 0$. In the second stage, we prove that the assertions "$\lim\limits_{t\to\infty} u(0, 0, t) = 0$" and "$\lim\limits_{t\to\infty} u(x, y, t) = 0$" are equivalent to each other provided that $(x, y) \in \overline{\mathbb{R}^{n+1}_+}$. In the third, we extend the proof to the case of arbitrary real values of l.

Stage 1. *Sufficiency.*

Let

$$\lim_{t\to\infty}\frac{1}{t^{k+2}}\int\limits_{B_+(t)} y^k\varphi(x,y)dxdy=0.$$

It is known (see, e.g., [46]) that

$$u(0,0,t)=\frac{2^{-k-1}t^{-\frac{k}{2}-1}}{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right)}\int\limits_0^\infty\int\limits_{-\infty}^{+\infty} y^k\varphi(x,y)e^{-\frac{x^2+y^2}{4t}}dxdy.$$

Introduce a function $v_0(x,y)$ as follows: $v_0(x,y)=\dfrac{1}{2}[\varphi(x,y)+\varphi(-x,y)]$. Then

$$\frac{1}{t^{k+2}}\int\limits_{B_+(t)} y^k\varphi(x,y)dxdy=\frac{1}{t^{k+2}}\int\limits_0^t\int\limits_0^{\frac{\pi}{2}} r^{k+1}v_0(r\cos\alpha,r\sin\alpha)\sin^k\alpha\;d\alpha dr$$

and

$$\frac{1}{t^{\frac{k}{2}+1}}\int\limits_0^\infty\int\limits_{-\infty}^{+\infty} y^k\varphi(x,y)e^{-\frac{x^2+y^2}{4t}}dxdy=\frac{2^{k+3}}{\tau^{k+2}}\int\limits_0^\infty\int\limits_0^\infty y^kv_0(x,y)e^{-\frac{x^2+y^2}{\tau}}dxdy,$$

where $\tau=2\sqrt{t}$.

Denote $\int\limits_0^\infty\int\limits_0^\infty y^kv_0(tx,ty)e^{-x^2-y^2}dxdy$ by $v(t)$. Let us show that $v(t)\xrightarrow{t\to\infty}0$.

Apply the polar change of variables; this yields the following relations:

$$v(t)=\int\limits_0^\infty\int\limits_0^{\frac{\pi}{2}} r^{k+1}v_0(tr\cos\alpha,tr\sin\alpha)\sin^k\alpha\;e^{-r^2}d\alpha dr$$

$$=2\int\limits_0^\infty r^{k+3}\frac{e^{-r^2}}{(rt)^{k+2}}\int\limits_0^{rt}\int\limits_0^{\frac{\pi}{2}}\eta^{k+1}v_0(\eta\cos\alpha,\eta\sin\alpha)\sin^k\alpha\;d\alpha d\eta dr$$

$$=2\int\limits_0^\delta r^{k+3}\frac{e^{-r^2}}{(rt)^{k+2}}\int\limits_0^{rt}\int\limits_0^{\frac{\pi}{2}}\eta^{k+1}v_0(\eta\cos\alpha,\eta\sin\alpha)\sin^k\alpha\;d\alpha d\eta dr$$

$$+2\int\limits_\delta^\infty r^{k+3}\frac{e^{-r^2}}{(rt)^{k+2}}\int\limits_0^{rt}\int\limits_0^{\frac{\pi}{2}}\eta^{k+1}v_0(\eta\cos\alpha,\eta\sin\alpha)\sin^k\alpha\;d\alpha d\eta dr\stackrel{\text{def}}{=}J_1(t;\delta)+J_2(t;\delta),$$

where δ is a positive parameter.

Since $v_0(x,y)$ is bounded, it follows that $\dfrac{1}{t^{k+2}}\int\limits_0^t\int\limits_0^{\frac{\pi}{2}}\eta^{k+1}v_0(\eta\cos\alpha,\eta\sin\alpha)\sin^k\alpha\;d\alpha d\eta$ is bounded as well. Therefore, there exists M such that the following inequality is valid for all positive r and t:

$$\left|\frac{1}{t^{k+2}}\int\limits_0^t\int\limits_0^{\frac{\pi}{2}}\eta^{k+1}v_0(\eta\cos\alpha,\eta\sin\alpha)\sin^k\alpha\;d\alpha d\eta\right|\le M.$$

Hence, $|J_1(t;\delta)| \le 2M \int\limits_0^\delta r^{k+3}e^{-r^2}dr$ for all positive δ and t. Select δ such that $|J_1(t;\delta)| < \frac{\varepsilon}{2}$ and fix the selected δ.

By the condition, for any positive ε there exists a positive R such that the following inequality is valid for any t from $[R,+\infty)$:

$$\left|\frac{1}{t^{k+2}}\int\limits_0^t\int\limits_0^{\frac{\pi}{2}} y^{k+1}v_0(y\cos\alpha, y\sin\alpha)\sin^k\alpha\, d\alpha dy\right| < \frac{\varepsilon}{2}\left(2\int\limits_\delta^\infty r^{k+3}e^{-r^2}dr\right)^{-1}.$$

Therefore,

$$|J_1(t;\delta)| < \frac{\varepsilon}{2}\left(2\int\limits_\delta^\infty r^{k+3}e^{-r^2}dr\right)^{-1} 2\int\limits_\delta^\infty r^{k+3}e^{-r^2}dr = \frac{\varepsilon}{2}$$

for any t from $[R,+\infty)$.

Thus, for any positive ε there exists a positive R such that $|v(t)| < \varepsilon$ for any t from $[R,+\infty)$. Since τ and t either both tend to infinity or do not, it follows that the sufficiency is proved.

Necessity.

Introduce the function $f_0(r) = \int\limits_{S_+(r)} y^k\varphi(x,y)dS$, where $S_+(A)$ denotes the semicircle $\{x^2+y^2 = A^2 |\, y > 0\}$ and dS denotes the circle measure. Obviously, $f_0(r)$ is continuous and it satisfies the estimate $|f_0(r)| \le Cr^{k+1}$, where $C = \pi \sup\limits_{\mathbb{R}^2_+} |\varphi(x,y)|$. Then

$$\frac{1}{t^{\frac{k}{2}+1}}\int\limits_0^\infty\int\limits_{-\infty}^{+\infty} y^k\varphi(x,y)e^{-\frac{x^2+y^2}{4t}}dxdy = \frac{2^{k+2}}{\tau^{k+1}}\int\limits_0^\infty e^{-r^2}f_0(r\tau)dr.$$

Since τ and t either both tend to infinity or do not, it follows that it suffices to prove the following assertion:

Lemma 5.1.1. *Let a continuous function $f(t)$ satisfy the estimate $|f(t)| \le Ct^{k+1}$ for $t \ge 0$. Let the following relation be valid:*

$$\lim_{t\to\infty}\frac{1}{t^{k+1}}\int\limits_0^\infty e^{-r^2}f(rt)dr = 0.$$

Then

$$\lim_{t\to\infty}\frac{1}{t^{k+2}}\int\limits_0^t f(r)dr = 0.$$

Proof. We use the following corollary from the Wiener Tauberian theorem (see [13, p. 163]):

"*if $\varphi \in L_1(0,\infty)$, $g \in L_\infty(0,\infty)$, and $\int\limits_0^\infty \varphi(t)t^{ix}dt \ne 0$ for any real x and*

$$\lim_{r\to\infty}\frac{1}{r}\int\limits_0^\infty \varphi\left(\frac{t}{r}\right)g(t)dt = 0,$$

then the relation

$$\lim_{r\to\infty}\frac{1}{r}\int\limits_0^\infty \psi\left(\frac{t}{r}\right)g(t)dt=0$$

holds for any function ψ from $L_1(0,\infty)$."

Denote $\dfrac{f(r)}{r^{k+1}}$ by $g(r)$. This function belongs to $L_\infty(0,\infty)$. Change the variables: $rt=\rho$; this yields

$$\frac{1}{t^{k+1}}\int\limits_0^\infty e^{-r^2}f(rt)dr=\frac{1}{t}\int\limits_0^\infty \varphi\left(\frac{r}{t}\right)g(r)dr,\ \text{where } \varphi(x)=x^{k+1}e^{-x^2}\in L_1(0,\infty).$$

On the other hand,

$$\frac{1}{t^{k+2}}\int\limits_0^t f(r)dr=\frac{1}{t}\int\limits_0^\infty \psi\left(\frac{r}{t}\right)f(r)dr,\ \text{where } \psi(x)=\begin{cases}x^{1+k} \text{ if } x\le 1\\ 0,\ \text{if } x>1,\end{cases}\qquad \text{i.e., } \psi\in L_1(0,\infty).$$

Finally,

$$\int\limits_0^\infty \varphi(t)t^{ix}dt=\int\limits_0^\infty e^{-t^2}t^{k+1+ix}dt=\frac{1}{2}\int\limits_0^\infty e^{-\tau}\tau^{\frac{k+ix}{2}}d\tau=\frac{1}{2}\Gamma\left(\frac{k+2+ix}{2}\right)e^{-2\pi mx},$$

where $m=0,\pm1,\pm2,\dots$

The last expression is well defined on the real axis. It does not vanish provided that x is real. Thus, the specified corollary from the Wiener Tauberian theorem is applicable. Hence,

$$\frac{1}{t^{k+2}}\int\limits_0^t f(r)dr\xrightarrow{t\to\infty}0.$$

This completes the proof of Lemma 5.1.1. □

Since the function $f_0(t)$ satisfies the assumption of Lemma 5.1.1, it follows that the necessity is proved.

Let us pass to the second stage of the proof of Theorem 5.1.1.

Following [41], introduce the generalized translation operator with respect to the variable y:

$$T_y^\eta f(y)\stackrel{\text{def}}{=}\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}\right)}\int\limits_0^\pi f\left(\sqrt{y^2+\eta^2-2y\eta\cos\theta}\right)\sin^{k-1}\theta d\theta.$$

This operator commutes with the Bessel operator $B_{k,y}$ (see, e.g., [34, p. 35]).

Introduce the function $\widetilde{u}$ as follows: $\widetilde{u}(x,\xi,y,\eta,t)=T_y^\eta u(x+\xi,y,t)$. Then

$$\frac{\partial\widetilde{u}}{\partial t}-\frac{\partial^2\widetilde{u}}{\partial\xi^2}-B_{k,\eta}\widetilde{u}=T_y^\eta\left[\frac{\partial}{\partial t}u(x+\xi,y,t)-\frac{\partial^2}{\partial x^2}u(x+\xi,y,t)-B_{k,y}u(x+\xi,y,t)\right]=0$$

because $u(x,y,t)$ satisfies problem (5.1)-(5.2).

Denote $T_y^\eta\varphi(x+\xi,y)$ by $\widetilde{\varphi}(x,\xi,y,\eta)$. Obviously, $\widetilde{\varphi}$ is continuous and bounded and

$$\widetilde{u}\big|_{t=0}=\widetilde{\varphi}(x,\xi,y,\eta).$$

Moreover, it is known from [41] that $\dfrac{\partial\widetilde{u}}{\partial\eta}\Big|_{\eta=0}=0$.

Thus, the function $\widetilde{u}(x,\xi,y,\eta,t)$ is the classical bounded solution of the problem

$$\frac{\partial\widetilde{u}}{\partial t}=\frac{\partial^2\widetilde{u}}{\partial\xi^2}+B_{k,\eta}\widetilde{u},\ \xi\in\mathbb{R}^1,\ \eta>0,\ t>0, \tag{5.5}$$

$$\widetilde{u}\big|_{t=0} = \widetilde{\varphi}(x,\xi,y,\eta), \ \frac{\partial \widetilde{u}}{\partial \eta}\big|_{\eta=0} = 0, \tag{5.6}$$

where x is a real parameter and y is a positive parameter.

It is proved in the first stage that $\lim\limits_{t\to\infty} \widetilde{u}(x,0,y,0,t) = 0$ if and only if

$$\lim_{t\to\infty} \frac{1}{t^{k+2}} \int\limits_0^t \int\limits_0^\pi \rho^{k+1} \sin^k\!\alpha\, \widetilde{\varphi}(x,\rho\cos\alpha, y, \rho\sin\alpha)\, d\alpha d\rho = 0.$$

Now, let us prove the equivalence of the condition of stabilization of the solution at an arbitrary point to the condition of its stabilization at the origin. To do this, assume that a is real and b is nonnegative and consider the integral

$$\int\limits_{D_{a,0,0}} \xi^{k-1}\varphi\left[x, \sqrt{\xi^2+(\eta-b)^2}\right] dx d\eta d\xi,$$

where $D_{a,b,c}$ denotes the semiball $\{(x-a)^2+(\eta-b)^2+(\xi-c)^2 < t^2 |\, y>0\}$.

Change the variables as follows: $\xi = y\sin\alpha$ and $\eta = y\cos\alpha$. We obtain that the last integral is equal to

$$\frac{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} \int\limits_{B_+(t)} y^k T_y^b \varphi(x+a,y) dx dy.$$

Thus,

$$\int\limits_{B_+(t)} y^k T_y^b \varphi(x+a,y) dx dy = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{k}{2}\right)} \int\limits_{D_{a,-b,0}} \xi^{k-1}\varphi\left(x, \sqrt{\xi^2+\eta^2}\right) dx d\eta d\xi.$$

Let us prove that for any real a, any nonnegative b, and any function $\varphi(x,y)$ continuous and bounded in $\overline{\mathbb{R}^2_+}$, the relation

$$\lim_{t\to\infty}\frac{1}{t^{k+2}} \int\limits_{D_{a,-b,0}} \xi^{k-1}\varphi\left(x, \sqrt{\xi^2+\eta^2}\right) dx d\eta d\xi = \lim_{t\to\infty}\frac{1}{t^{k+2}} \int\limits_{D_{0,0,0}} \xi^{k-1}\varphi\left(x, \sqrt{\xi^2+\eta^2}\right) dx d\eta d\xi \tag{5.7}$$

is valid in the following sense: if there exists one of the limits, then the other exists as well, and they are equal to each other.

Define the sets

$$\Omega'_{t;a,b} \overset{\text{def}}{=} \left\{(x,\eta,\xi)\in\mathbb{R}^3 \middle| (x-a)^2+\xi^2+(\eta+b)^2 \le t^2; x^2+\xi^2+\eta^2 \ge t^2\right\}$$

and

$$\Omega''_{t;a,b} \overset{\text{def}}{=} \left\{(x,\eta,\xi)\in\mathbb{R}^3 \middle| x^2+\xi^2+\eta^2 \le t^2; (x-a)^2+\xi^2+(\eta+b)^2 \ge t^2\right\}.$$

We have the inequality

$$\left| \frac{1}{t^{k+2}} \int\limits_{D_{a,-b,0}} \xi^{k-1}\varphi\left(x, \sqrt{\xi^2+\eta^2}\right) dx d\eta d\xi - \frac{1}{t^{k+2}} \int\limits_{D_{0,0,0}} \xi^{k-1}\varphi\left(x, \sqrt{\xi^2+\eta^2}\right) dx d\eta d\xi \right|$$

$$= \frac{1}{t^{k+2}} \left| \int\limits_{\Omega'_{t;a,b}} \xi^{k-1}\varphi\left(x, \sqrt{\xi^2+\eta^2}\right) dx d\eta d\xi - \int\limits_{\Omega''_{t;a,b}} \xi^{k-1}\varphi\left(x, \sqrt{\xi^2+\eta^2}\right) dx d\eta d\xi \right| \le \frac{2M}{t^3} S,$$

where $M = \sup\limits_{\mathbb{R}^2_+} |\varphi(x,y)|$ and S is the volume of $\Omega'_{t;a,b}$.

Estimating S as a function of variable t, we obtain the inequality

$$S \leq \frac{4\pi}{3}\left[\left(t+\frac{a}{2}\right)^3-\left(t-\frac{a}{2}\right)^3\right]=\frac{12\pi a t^2+\pi a^3}{3}.$$

This implies (5.7), which completes the proof of Theorem 5.1.1 for $l=0$.

Let us pass to the third stage of the proof of Theorem 5.1.1. To do this, we consider (apart from problem (5.1)-(5.2)) the problem

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}+B_{k,y}w,\ x\in\mathbb{R}^1,\ y>0,\ t>0, \tag{5.8}$$

$$w\big|_{t=0}=w_0(x,y),\ \frac{\partial w}{\partial y}\big|_{y=0}=0, \tag{5.9}$$

where $w_0(x,y)=\varphi(x,y)+l$.

Obviously, if $u(x,y,t)$ and $w(x,y,t)$ are solutions of problems (5.1)-(5.2) and (5.8)-(5.9) respectively, then $w(x,y,t)=u(x,y,t)+l$.

Note that

$$\int\limits_{B_+(t)} y^k l\,dxdy=\frac{lt^{k+2}}{k+2}\cdot\frac{\sqrt{\pi}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}.$$

Hence, the validity of the relation

$$\lim_{t\to\infty}\frac{1}{t^{k+2}}\int\limits_{B_+(t)} y^k w_0(x,y)\,dxdy=\frac{l\sqrt{\pi}\,\Gamma\left(\frac{k+1}{2}\right)}{(k+2)\Gamma\left(\frac{k}{2}+1\right)}$$

implies the validity of the relation

$$\lim_{t\to\infty}\frac{1}{t^{k+2}}\int\limits_{B_+(t)} y^k\varphi(x,y)\,dxdy=0;$$

this implies (as is proved in the second stage of the proof) the validity of the limit relation

$$\lim_{t\to\infty}u(x,y,t)=0,$$

which means that $\lim\limits_{t\to\infty}w(x,y,t)=l$.

Thus, condition (5.4) imposed on the function $w_0(x,y)$ is sufficient for the function $w(x,y,t)$ to satisfy (5.3). The necessity of the specified condition is proved in the same way.

This completes the proof of Theorem 5.1.1. □

Theorem 5.1.1 implies the following fact: if the stabilization of the classical bounded solution of problem (5.1)-(5.2) takes place at at least at one point (x^0,y^0) of the half-space $\overline{\mathbb{R}^{n+1}_+}$, then it takes place at any other point (x,y) of $\overline{\mathbb{R}^{n+1}_+}$ and the limit of the solution is the same as at the point (x^0,y^0). This means that no stabilization of the classical bounded solution of problem (5.1)-(5.2) to a function $V(x,y)$ different from a constant is possible. Thus, the following alternative takes place for the classical bounded solution of problem (5.1)-(5.2): either this solution stabilizes to a constant in the half-space $\overline{\mathbb{R}^{n+1}_+}$ or it stabilizes at no point of the specified half-space, i.e., there is no point $(x,y)\in\overline{\mathbb{R}^{n+1}_+}$ such that a limit of the function $u(x,y,t)$ as $t\to\infty$ exists. For the regular case, the long-time behavior of the solution is investigated in [95, 96].

5.1.2. The Cauchy problem for singular parabolic equations with time-dependent coefficients. Consider the equation

$$\frac{\partial u}{\partial t} = a(t)\left(\Delta + B_{k,y}\right) u, \; x \in \mathbb{R}^n, \; y > 0, \; t > 0, \tag{5.10}$$

where $a(t)$ is continuous and positive for $t \geq 0$.

Let $\varphi(x, y)$ be continuous and bounded for $x \in \mathbb{R}^n$ and $y \geq 0$. Consider problem (5.10), (5.2). The existence and uniqueness of the classical bounded solution of that problem are established in [36]. Investigate the long-time behavior of the solution.

Theorem 5.1.2. *Suppose that* $\int\limits_0^\infty a(t)dt$ *diverges and* $u(x, y, t)$ *is the classical bounded solution of problem* (5.10), (5.2). *Then for any* x *from* $\mathbb{R}^n$, *any nonnegative* y, *and any real* l, *relation* (5.3) *is equivalent to relation* (5.4).

Proof. It is known, e.g., from [110], that

$$u(x, y, t) = \frac{C_{n,k}}{[A(t)]^{\frac{n+k+1}{2}}} \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \eta^k \varphi(\xi, \eta) T_\eta^y e^{-\frac{|\xi - x|^2 + \eta^2}{4A(t)}} d\xi d\eta,$$

where $A(t) = \int\limits_0^t a(\tau)d\tau$ and $C_{n,k}$ depends only on n and k.

The function $A(t)$ tends to infinity if and only if t tends to infinity. Therefore, for any real l, the limit relation

$$\lim_{t\to\infty} \frac{1}{[A(t)]^{\frac{n+k+1}{2}}} \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \eta^k \varphi(\xi, \eta) T_\eta^y e^{-\frac{|\xi - x|^2 + \eta^2}{4A(t)}} d\xi d\eta = l$$

holds if and only if the limit relation

$$\lim_{r\to\infty} \frac{1}{r^{\frac{n+k+1}{2}}} \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \eta^k \varphi(\xi, \eta) T_\eta^y e^{-\frac{|\xi - x|^2 + \eta^2}{4r}} d\xi d\eta = l$$

holds.

Taking into account Theorem 5.1.1, we see that this implies the assertion of Theorem 5.1.2. □

Theorem 5.1.3. *Suppose that* $\int\limits_0^\infty a(t)dt = a_0 < \infty$. *Then* $\lim\limits_{t\to\infty} u(x, y, t) = v(x, y, a_0)$, *where the functions* $u(x, y, t)$ *and* $v(x, y, t)$ *are the classical bounded solutions of problem* (5.10), (5.2) *and problem* (5.1)-(5.2) *respectively.*

Proof. For any positive t_0, the integral

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^n} \frac{\eta^k \varphi(\xi, \eta)}{[A(t)]^{\frac{n+k+1}{2}}} T_\eta^y e^{-\frac{|\xi - x|^2 + \eta^2}{4r}} d\xi d\eta$$

converges uniformly with respect to (x, y, t) from $\mathbb{R}^n \times [0, +\infty) \times [t_0, +\infty)$. Hence, one can pass to the limit as $t \to \infty$. This yields the relation

$$\lim_{t\to\infty} u(x, y, t) = \frac{C_{n,k}}{a_0^{\frac{n+k+1}{2}}} \int\limits_0^\infty \int\limits_{\mathbb{R}^n} \eta^k \varphi(\xi, \eta) T_\eta^y e^{-\frac{|\xi - x|^2 + \eta^2}{4a_0}} d\xi d\eta.$$

Obviously, its right-hand part is equal to $v(x, y, a_0)$.

This completes the proof of Theorem 5.1.3. $\square$

Note that if the integral $\int\limits_0^\infty a(t)dt$ converges, then the stabilization of the solution takes place regardless of the behavior of the initial-value function; however, the limit of the solution is, in general, not a constant anymore: it is a bounded function of x and y.

5.1.3. Properties of weight integral means. Let $\varphi(x, y)$ be a continuous and bounded in $\overline{\mathbb{R}^{n+1}_+}$ function and α be a positive constant. Define the function $S^\alpha_n\varphi(r)$ as follows:

$$S^\alpha_n\varphi(r) = \frac{1}{r^{n+\alpha+1}} \int\limits_{B_+(r)} y^\alpha \varphi(x, y)dxdy.$$

Theorem 5.1.4. *Let $n \geq 1$, $\alpha \geq 0$, $\beta \geq 0$, and $\alpha \neq \beta$. Then there exists a bounded function φ from $C^\infty(\overline{\mathbb{R}^{n+1}_+})$ such that $S^\alpha_n\varphi(r)$ has a limit as $r \to \infty$, while $S^\beta_n\varphi(r)$ has no limit as $r \to \infty$.*

Proof. First, we note that the considered limits can be only finite because the function φ is bounded. The following two lemmas precede the proof.

Lemma 5.1.2. *Let $G = \{\theta = (\theta_1, \dots, \theta_n) | 0 \leq \theta_1 \leq \pi, \dots, 0 \leq \theta_n \leq \pi\}$. Then there exists a function $g(\theta)$ from $C^\infty(G)$ such that $J_\alpha = 0$ and $J_\beta \neq 0$, where J_γ denotes $\int\limits_G g(\theta) \prod\limits_{j=1}^n \sin^{n+\gamma-j}\theta_j \, d\theta$ (γ is a nonnegative parameter).*

Proof. The functions $\prod\limits_{j=1}^n \sin^{n+\alpha-j}\theta_j$ and $\prod\limits_{j=1}^n \sin^{n+\beta-j}\theta_j$ are linearly independent elements of the Hilbert space $L_2(G)$. Therefore, there exists an element $g(\theta)$ of the space $L_2(G)$ such that $g(\theta)$ is orthogonal to $\prod\limits_{j=1}^n \sin^{n+\alpha-j}\theta_j$ but is not orthogonal to $\prod\limits_{j=1}^n \sin^{n+\beta-j}\theta_j$, i.e., $J_\alpha = 0$ and $J_\beta = A > 0$ (for definiteness).

Since $C^\infty(G)$ is dense in $L_2(G)$, it follows that one can select a sequence $\{g_m(\theta)\}_{m=1}^\infty \subset C^\infty(G)$ such that $g_m \xrightarrow{m\to\infty} g(\theta)$ in $L_2(G)$. Assuming that γ is nonnegative, denote $\int\limits_G g_m(\theta) \prod\limits_{j=1}^n \sin^{n+\gamma-j}\theta_j \, d\theta$ by $J_{\gamma,m}$. Then $\lim\limits_{m\to\infty} J_{\alpha,m} = 0$ and $\lim\limits_{m\to\infty} J_{\beta,m} = A$ by virtue of the continuity of the scalar product.

By C_γ denote

$$\int\limits_G \prod_{j=1}^n \sin^{n+\gamma-j}\theta_j \, d\theta = \pi^{\frac{n}{2}} \prod_{j=1}^n \Gamma\left(\frac{n+\gamma+1-j}{2}\right)$$

(see [87, p. 386]). Obviously, C_γ is positive provided that γ is nonnegative. Introduce the following new function: $\widetilde{g}_m(\theta) \stackrel{\text{def}}{=} g_m(\theta) - \frac{J_{\alpha,m}}{C_\alpha}$; this function is infinitely differentiable in G. Now, compute the scalar products

$$\left(\widetilde{g}_m(\theta), \prod_{j=1}^n \sin^{n+\alpha-j}\theta_j\right) \text{ and } \left(\widetilde{g}_m(\theta), \prod_{j=1}^n \sin^{n+\beta-j}\theta_j\right):$$

$$\int\limits_G \widetilde{g}_m(\theta) \prod_{j=1}^n \sin^{n+\alpha-j}\theta_j \, d\theta = J_{\alpha,m} - J_{\alpha,m} = 0$$

and

$$\int_G \widetilde{g}_m(\theta) \prod_{j=1}^{n} \sin^{n+\beta-j}\theta_j \, d\theta = J_{\beta,m} - J_{\alpha,m}\frac{C_\beta}{C_\alpha}.$$

Now, take ε from the interval $\left(0, \frac{A}{2}\right)$. There exists a positive integer m such that $|J_{\alpha,m'}| < \frac{\varepsilon}{2} \cdot \frac{C_\alpha}{C_\beta}$ and $|J_{\beta,m'}| > \frac{A}{2} + \frac{\varepsilon}{2}$. Then

$$\int_G \widetilde{g}_{m'}(\theta) \prod_{j=1}^{n} \sin^{n+\beta-j}\theta_j \, d\theta = J_{\beta,m'} - J_{\alpha,m'}\frac{C_\beta}{C_\alpha} > \frac{A}{2} + \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \frac{A}{2} > 0.$$

Thus, the function $g_{m'}$ belongs to $C^\infty(G)$, is orthogonal to $\prod_{j=1}^{n} \sin^{n+\alpha-j}\theta_j$, and is not orthogonal to $\prod_{j=1}^{n} \sin^{n+\beta-j}\theta_j$.

This completes the proof of Lemma 5.1.2. □

The following assertion is provided (without a proof) in [96] for the case where α and β are positive integers.

Lemma 5.1.3. *Let $f(r)$ be continuous and bounded for nonnegative r. Let $\alpha \geq 0$ and $\beta \geq 0$. Then $S_0^\alpha f(r)$ has a limit as $r \to \infty$ if and only if $S_0^\beta f(r)$ has a limit as $r \to \infty$.*

Proof. We have

$$S_0^\alpha f(r) = \frac{1}{r^{1+\alpha}} \int_0^r \tau^\alpha f(\tau) d\tau = \frac{1}{r} \int_0^\infty \psi_\alpha\left(\frac{\tau}{r}\right) f(\tau) d\tau,$$

where

$$\psi_k(\tau) = \begin{cases} \tau^k \text{ if } \tau \leq 1 \\ 0 \text{ if } \tau > 1 \end{cases}$$

provided that $k \geq 0$.

Obviously, ψ_α belongs to $L_1(0, +\infty)$ and f belongs to $L_\infty(0, +\infty)$. Further,

$$\int_0^\infty \psi_k(\tau)\tau^{ix} d\tau = \int_0^1 \tau^k e^{ix(\log\tau + 2\pi i m)} d\tau = e^{-2\pi m x}\left[\int_0^1 \tau^k \cos(x \log\tau) d\tau + i \int_0^1 \tau^k \sin(x\log\tau) d\tau\right]$$

$$= e^{-2\pi m x}\left[\int_{-\infty}^0 e^{(k+1)t} \cos xt dt + i \int_{-\infty}^0 e^{(k+1)t} \sin xt dt\right] = \frac{e^{-2\pi m x}}{x^2 + (k+1)^2}(k + 1 - ix).$$

The real part of the obtained expression is positive for any integer m, any real x, and any nonnegative k. Therefore, the function $\int_0^\infty \psi_k(\tau)\tau^{ix} d\tau$ has no real zeros provided that k is nonnegative.

Suppose that $\lim_{r\to\infty} S_0^\alpha f(r)$ exists; denote it by C_α. The function ψ_β belongs to the space $L_1(0, +\infty)$; therefore, by virtue of the corollary from the Wiener Tauberian theorem (see [13, p. 163]), there exists

$\lim\limits_{r\to\infty} \frac{1}{r}\int\limits_0^\infty \psi_\beta\left(\frac{\rho}{r}\right) f(\rho)d\tau$, i.e., $\lim\limits_{r\to\infty} S_0^\beta f(r)$, and this limit is equal to

$$\frac{C_\alpha}{\beta+1}\left(\int\limits_0^1 \tau^\alpha d\tau\right)^{-1} = \frac{\alpha+1}{\beta+1}\lim_{r\to\infty} S_0^\alpha f(r).$$

This completes the proof of Lemma 5.1.3 because α and β are selected arbitrarily. □

Let us pass directly to the proof of Theorem 5.1.4.

In $\int\limits_{B_+(r)} y^k \varphi(x,y)dxdy$, change the variables as follows:

$$\begin{aligned}
x_1 &= \rho\cos\theta_1,\\
x_2 &= \rho\sin\theta_1\cos\theta_2,\\
&\dots\\
x_n &= \rho\sin\theta_1\sin\theta_2\dots\sin\theta_{n-1}\cos\theta_n,\\
y &= \rho\sin\theta_1\sin\theta_2\dots\sin\theta_{n-1}\sin\theta_n.
\end{aligned}$$

We obtain the relation

$$S_n^k\varphi(r) = \frac{1}{r^{n+k+1}}\int\limits_0^r\int\limits_G v(\rho,\theta)\rho^{n+k}\prod_{j=1}^n \sin^{n+k-j}\theta_j\, d\theta d\rho,$$

where

$$v(\rho,\theta) = \varphi(\rho\cos\theta_1, \rho\sin\theta_1\cos\theta_2, \dots, \rho\sin\theta_1\sin\theta_2\dots\sin\theta_{n-1}\cos\theta_n, \rho\sin\theta_1\sin\theta_2\dots\sin\theta_{n-1}\sin\theta_n).$$

Take a function $f(r)$ such that it is bounded and infinitely differentiable on the positive semiaxis, $f(r)=0$ for $r\le\frac{1}{2}$, and $\lim\limits_{r\to\infty}\frac{1}{r}\int\limits_0^r f(\rho)d\rho$ does not exist; for example, one can take the Kzhizhanskii function (see [14, p. 337]) and smooth it out. Then, by virtue of Lemma 5.1.3, for any nonnegative k, no limit of $\frac{1}{r^{k+1}}\int\limits_0^r r^k f(\rho)d\rho$ as $r\to\infty$ exists. Assign $v(\rho,\theta) = f(\rho)g(\theta)$, where $g(\theta)$ is the function, the existence of which is proved in Lemma 5.1.2.

The constructed function $v(\rho,\theta)$ uniquely defines a function $\varphi(x,y)$ bounded and infinitely differentiable for x from $\mathbb{R}^n$ and y from $[0,+\infty)$. On the other hand, the following relation is valid:

$$S_n^k\varphi(r) = \frac{1}{r^{n+k+1}}\int\limits_0^r \rho^{n+k}f(\rho)d\rho\int\limits_G g(\theta)\prod_{j=1}^n \sin^{n+k-j}\theta_j\, d\theta.$$

This implies that $S_n^\alpha\varphi(r)$ has a limit as $r\to\infty$ (moreover, it is equal to the identical zero), while $S_n^\beta\varphi(r)$ has no limit as $r\to\infty$.

This completes the proof of Theorem 5.1.4. □

Remark. For $n=0$, the assertion opposite to the assertion of Theorem 5.1.4 is valid: *if α and β are nonnegative and the function $\varphi(y)$ is continuous and bounded on $[0,+\infty)$, then the limit $\lim\limits_{r\to\infty} S_0^\alpha\varphi(r)$ exists if and only if $\lim\limits_{r\to\infty} S_0^\beta\varphi(r)$ exists. If the limit exists, then $\lim\limits_{r\to\infty} S_0^k\varphi(r) = \frac{1}{k+1}\lim\limits_{r\to\infty} S_0^0\varphi(r)$.*

This follows directly from Lemma 5.1.3.

To get back to the stabilization of solutions of singular parabolic equations, we assume that the integral $\int\limits_0^\infty a(t)dt$ diverges and denote the classical solution of problem (5.10), (5.2) by $u_k(x,y,t)$. Theorems 5.1.2–5.1.4 and the remark to the last theorem imply the following two assertions.

Theorem 5.1.5. *Let $n \geq 1$, $\alpha \geq 0$, $\beta \geq 0$, and $\alpha \neq \beta$. Then there exists a bounded function φ from $C^\infty(\overline{\mathbb{R}^{n+1}_+})$ such that for any (x,y) from $\overline{\mathbb{R}^{n+1}_+}$, $\lim\limits_{t\to\infty} u_\alpha(x,y,t)$ exists, but $\lim\limits_{t\to\infty} u_\beta(x,y,t)$ does not exist.*

Theorem 5.1.6. *Let $n = 0$, $\varphi(y)$ be a bounded function, and α and β be nonnegative. Then the existence of $\lim\limits_{t\to\infty} u_\alpha(y,t)$ is equivalent to the existence of $\lim\limits_{t\to\infty} u_\beta(y,t)$. If those limits exist, then*

$$\lim_{t\to\infty} u_\alpha(y,t) = \frac{\beta+1}{\alpha+1}\lim_{t\to\infty} u_\beta(y,t).$$

5.1.4. Cauchy problems with unbounded initial-value functions. Let us prove that if the initial-value function of the Cauchy problem is not bounded, then (similarly to the regular case) the stabilization condition is not necessary anymore. It suffices to consider problem (5.10), (5.2) for the case where $n = 0$.

Define the initial-value function $\varphi(y)$ as follows:

$$\varphi(y) = 2(k+1)\cos y^2 - 4y^2 \sin y^2.$$

Obviously, $\varphi(y) = B_{k,y}\Phi(y)$, where $\Phi(y) = \sin y^2$.

Now, consider

$$\frac{1}{r^{k+1}}\int\limits_0^r y^k\varphi(y)dy = \frac{2(k+1)}{r^{k+1}}\int\limits_0^r y^k \cos y^2 dy - \frac{2}{r^{k+1}}\int\limits_0^r y^{k+1}\cdot 2y\sin y^2 dy$$
$$= \frac{2(k+1)}{r^{k+1}}\int\limits_0^r y^k \cos y^2 dy - \frac{2}{r^{k+1}}\left[-y^{k+1}\cos y^2\Big|_0^r + (k+1)\int\limits_0^r y^k\cos y^2 dy\right] = 2\cos r^2.$$

The last expression has no limit as $r\to\infty$.

Thus, even a pointwise stabilization of $\frac{1}{r^{k+1}}\int\limits_0^r y^k T_x^y\varphi(x)dy$ as $r\to\infty$ does not take place.

On the other hand,

$$\frac{1}{t^{\frac{k+1}{2}}}\int\limits_0^\infty \eta^k T_\eta^y e^{-\frac{\eta^2}{4t}}\varphi(\eta)d\eta = \int\limits_0^\infty \alpha^k T_\alpha^{\frac{y}{\sqrt{t}}} e^{-\frac{\alpha^2}{4}}\varphi(\alpha\sqrt{t})d\alpha.$$

Further, $\varphi(y) = B_{k,y}\Phi(y)$; hence, $\varphi(\alpha\sqrt{t}) = B_{k,\alpha\sqrt{t}}\Phi(\alpha\sqrt{t})$. We have

$$B_{k,\alpha\sqrt{t}}\Phi(\alpha\sqrt{t}) = \frac{1}{\alpha^k}\frac{\partial}{\partial\alpha}\left[\alpha^k\frac{\partial\Phi}{\partial\alpha}(\alpha\sqrt{t})\right] = \frac{\sqrt{t}}{\alpha^k}\frac{\partial}{\partial\alpha}\left[\alpha^k\Phi'(\alpha\sqrt{t})\right]$$
$$= t\Phi''(\alpha\sqrt{t}) + \sqrt{t}\frac{k}{\alpha}\Phi'(\alpha\sqrt{t}) = tB_{k,\alpha\sqrt{t}}\Phi(\alpha\sqrt{t}) = t\varphi(\alpha\sqrt{t}).$$

Therefore, $\varphi(\alpha\sqrt{t}) = \frac{1}{t}B_{k,\alpha}\Phi(\alpha\sqrt{t})$.

This implies that

$$\frac{1}{t^{\frac{k+1}{2}}}\int_0^\infty \eta^k T_\eta^y e^{-\frac{\eta^2}{4t}}\varphi(\eta)d\eta = \frac{1}{t}\int_0^\infty \alpha^k T_\alpha^{\frac{y}{\sqrt{t}}} e^{-\frac{\alpha^2}{4}} B_{k,\alpha}\Phi(\alpha\sqrt{t})d\alpha = \frac{1}{t}\int_0^\infty \alpha^k e^{-\frac{\alpha^2}{4}} T_\alpha^{\frac{y}{\sqrt{t}}} B_{k,\alpha}\Phi(\alpha\sqrt{t})d\alpha$$

by virtue of the self-adjointness of the generalized translation operator in the space $L_{2,k}(0,+\infty)$ (see [33, 34, 41]). Since the generalized translation operator commutes with the Bessel operator, it follows that the last expression is equal to

$$\frac{1}{t}\int_0^\infty \alpha^k e^{-\frac{\alpha^2}{4}} B_{k,\alpha} T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})d\alpha = \frac{1}{t}\int_0^\infty \alpha^k B_{k,\alpha} e^{-\frac{\alpha^2}{4}} T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})d\alpha$$
$$+\frac{\alpha^k}{t}\left[e^{-\frac{\alpha^2}{4}}\frac{\partial}{\partial\alpha}T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t}) + \frac{\alpha}{2}e^{-\frac{\alpha^2}{4}}T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})\right]\Bigg|_0^\infty$$

(we use integration by parts twice).

Let us show that the integrated term is equal to zero for any nonnegative y and any positive t. Obviously, its second term is equal to zero (because the function $\Phi(y)$ is bounded). Further, we have

$$\left|\frac{\partial}{\partial\alpha}T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})\right| = \left|\frac{\partial}{\partial\alpha}\int_0^\infty \Phi\left(\sqrt{y^2+\alpha^2 t - 2\alpha y\sqrt{t}\cos\theta}\right)\sin^{k-1}\theta d\theta\right|$$
$$= \left|\int_0^\infty (2t\alpha - 2y\sqrt{t}\cos\theta)\cos(y^2+\alpha^2 t - 2\alpha y\sqrt{t}\cos\theta)\sin^{k-1}\theta d\theta\right| \le 2(t\alpha + y\sqrt{t})\pi;$$

hence, the first term is equal to zero as well.

Hence, for any nonnegative y and any positive t, the following relation is valid:

$$\frac{1}{t}\int_0^\infty \alpha^k B_{k,\alpha} e^{-\frac{\alpha^2}{4}} T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})d\alpha = \int_0^\infty \alpha^k e^{-\frac{\alpha^2}{4}}\left(\frac{\alpha^2}{4} - \frac{k+1}{2}\right)\frac{T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})}{t}d\alpha.$$

For any positive t_0, the last integral converges uniformly with respect to $(y,t) \in [0,+\infty)\times[t_0,+\infty)$. Since the inequality $\left|T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})\right| \le 1$ holds for any nonnegative y, any nonnegative α, and any positive t, it follows that

$$\int_0^\infty \alpha^k e^{-\frac{\alpha^2}{4}}\left(\frac{\alpha^2}{4} - \frac{k+1}{2}\right)\frac{T_\alpha^{\frac{y}{\sqrt{t}}}\Phi(\alpha\sqrt{t})}{t}d\alpha \xrightarrow{t\to\infty} 0$$

uniformly with respect to $y \in [0,+\infty)$.

Thus, the solution stabilizes to zero *uniformly* on the semiaxis, while even a *pointwise* stabilization of the weight mean of the initial-value function does not take place.

5.1.5. Stabilization of solutions for equations with dissipation. Consider the equation

$$\frac{\partial u}{\partial t} = (\Delta + B_{k,y})u - a(t)u,\ x\in\mathbb{R}^n,\ y>0,\ t>0, \tag{5.11}$$

where $a(t)$ is continuous and positive for $t \ge 0$.

The classical bounded solution of problem (5.11), (5.2) is equal to

$$u(x,y,t) = e^{-A(t)}v(x,y,t),$$

where $v(x, y, t)$ is the classical bounded solution of problem (5.1)-(5.2) and $A(t) = \int\limits_0^t a(\tau)d\tau$.

Then, by virtue of the boundedness of the function $v(x, y, t)$, the following assertion is valid:

Theorem 5.1.7. *If* $\int\limits_0^\infty a(\tau)d\tau$ *diverges, then the classical bounded solution of problem* (5.11), (5.2) *uniformly stabilizes to zero as* $t \to \infty$ *provided that the initial-value function* $\varphi(x, y)$ *is continuous and bounded. If the specified integral converges, then for any* x *from* $\mathbb{R}^n$*, any nonnegative* y*, and any real* l*, the limit relation* $u(x, y, t) \xrightarrow{t\to\infty} l$ *is valid if and only if the limit relation*

$$\lim_{t\to\infty} \frac{n+k+1}{\pi^{\frac{n}{2}} t^{n+k+1}} \int\limits_{B_+(t)} y^k \varphi(x, y)dxdy = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+k+1}{2}\right)} e^{a_0} l$$

is valid, where $u(x, y, t)$ *is the classical bounded solution of problem* (5.11), (5.2) *and* a_0 *denotes* $\int\limits_0^\infty a(\tau)d\tau$*.*

5.2. The Case of Coefficients Depending on Spatial Variables

In this section, we investigate the long-time behavior of solutions of the Cauchy problem for equations of the form

$$p(x, y)\frac{\partial u}{\partial t} = \Delta u + \frac{1}{y^k}\frac{\partial}{\partial y}\left(y^k \frac{\partial u}{\partial y}\right).$$

The solvability of such problems and uniqueness of their solutions are investigated in [36, 46, 47] and a number of other papers. For the regular case (i.e., for $k = 0$), the long-time behavior of solutions is investigated in [124] (see also [9] and references therein).

The main result of this section (Theorem 5.2.1) is proved by a method proposed in [26]. The principal idea of the specified method is to reduce the question on the stabilization of the solution of the original problem to the question on the stabilization of the solution of the Cauchy problem for Eq. (5.1) investigated in the previous section.

Note that in this section (similarly to the previous one), we deal only with the pointwise stabilization of the Cauchy problem with a bounded initial-value function; therefore, the above-mentioned method is applicable.

5.2.1. Main theorem: claim.

Definition. Let Ω be a closed domain of a Euclidean space, m be a positive integer, and α belong to $(0, 1)$. The space $H_m^\alpha(\Omega)$ is the set of functions defined on Ω such that each such function and all its derivatives until order m (inclusively) are continuous and bounded and satisfy the Hölder condition of order α on Ω.

In the sequel, we omit indices of the operator $B_{k,y}$ (if no misunderstanding can arise).

The following notation are used:

$$\Delta_B = \Delta + B,\ D_{x_j}^m = \frac{\partial^m}{\partial x_j^m},\ j = \overline{1, n},\ \widetilde{D}_y^m = \begin{cases} B^{\frac{m}{2}} \text{ if } m \text{ is even} \\ \dfrac{\partial}{\partial y} B^{\frac{m-1}{2}}, \text{ if } m \text{ is odd,} \end{cases}$$

and $\widetilde{D}^\beta = D_{x_1}^{\beta_1} \dots D_{x_n}^{\beta_n} D_y^{\beta_{n+1}}$, where $\beta = (\beta_1, \dots, \beta_n, \beta_{n+1})$ is a multi-index and $|\beta|$ is its length: $|\beta| = \beta_1 + \beta_2 + \dots + \beta_n + \beta_{n+1}$.

Together with the space $H^\alpha_m(\Omega)$, introduce the space $\widetilde{H}^\alpha_m(\Omega)$. It is the set of functions f defined on Ω and such that for any β such that $|\beta| \le m$, the function $\widetilde{D}^\beta f$ is continuous and bounded on Ω and satisfies the Hölder condition of order α on Ω. If $\alpha = 0$, then the requirement to satisfy the Hölder condition is taken off the definition of the spaces H^α_m and $\widetilde{H}^\alpha_m$. By $H^\alpha_0(\Omega) = H^\alpha(\Omega) = \widetilde{H}^\alpha_0(\Omega) = \widetilde{H}^\alpha(\Omega)$ we denote the set of functions that are continuous and bounded on Ω and satisfy the Hölder condition of order α on Ω. By $H^0(\Omega) = H(\Omega) = \widetilde{H}^0(\Omega) = \widetilde{H}(\Omega)$ we denote the set of functions continuous and bounded on Ω.

Remark. Defining spaces $\widetilde{H}^\alpha_m(\Omega)$, we assume that Ω is contained in the subspace $\{y \ge 0\}$.

Consider the equation

$$p(x,y)\frac{\partial u}{\partial t} = \Delta_B u, \ x \in \mathbb{R}^n, \ y > 0, \ t > 0, \tag{5.12}$$

where $p(x,y) \ge p_0 > 0$ and $p(x,y) \in H(\overline{\mathbb{R}^{n+1}_+})$.

The existence and uniqueness of a classical bounded solution of problem (5.12), (5.2) (under the assumption that k is positive and φ is continuous and bounded) is established in [36]. Investigate the long-time behavior of that solution.

Theorem 5.2.1. *Let $u(x,y,t)$ be the classical bounded solution of problem* (5.12), (5.2), $k > 0$, $p(x,y) \ge p_0 > 0$, *and* $\varphi \in H(\overline{\mathbb{R}^{n+1}_+})$. *Let $p(x,y)$ satisfy the following conditions*:

$$p(x,y) \in H^\alpha_{\left[\frac{n+k+1}{2}\right]}(\overline{\mathbb{R}^{n+1}_+}), \ \textit{where } \alpha \in (0,1), \tag{5.13}$$

$$\frac{\partial^m p}{\partial y^m}\Big|_{y=0} = 0 \ \textit{for } m = 1, \dots, \left[\frac{n+k+1}{2}\right] \ \textit{if } n+k \ge 1, \tag{5.14}$$

and there exists a constant b such that

$$\lim_{r\to\infty} \frac{1}{t^{n+k+1}} \int\limits_{B_+(t)} y^k T^\eta_y |p(x+\xi, y) - b| dx dy = 0 \tag{5.15}$$

uniformly with respect to $(\xi,\eta) \in \overline{\mathbb{R}^{n+1}_+}$.

Then for any x from $\mathbb{R}^n$, any nonnegative $y \ge 0$, and any real l, the limit relation $u(x,y,t) \xrightarrow{t\to\infty} l$ is valid if and only if the limit relation

$$\lim_{t\to\infty} \frac{n+k+1}{t^{n+k+1}} \int\limits_{B_+(t)} y^k \varphi(x,y) dx dy = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+k+1}{2}\right)} l$$

is valid.

Note that in the regular case (i.e., in the case where $k = 0$), the Gushchin–Mikhailov condition (see [26, 124]) corresponds to condition (5.15). Therefore, it is reasonable to call condition (5.15) the *weight Gushchin–Mikhailov condition*.

In the sequel, without loss of generality, we assume that $b = 1$.

Introduce the following function: $q(x,y) \overset{\text{def}}{=} p(x,y) - 1$. Then the weight Gushchin–Mikhailov condition takes the following form:

$$\lim_{r\to\infty} \frac{1}{t^{n+k+1}} \int\limits_{B_+(t)} y^k T^\eta_y |q(x+\xi, y)| dx dy = 0 \tag{5.16}$$

uniformly with respect to $(\xi,\eta) \in \overline{\mathbb{R}^{n+1}_+}$.

Let $v(x,y,t)$ be the classical bounded solution of Eq. (5.1), satisfying the following boundary-value conditions:

$$v\Big|_{t=0} = p(x,y)\varphi(x,y), \ \frac{\partial v}{\partial y}\Big|_{y=0} = 0. \tag{5.17}$$

We must prove that $\lim\limits_{t\to\infty}[u(x,y,t) - v(x,y,t)] = 0$ for any (x,y) from $\overline{\mathbb{R}^{n+1}_+}$.

Introduce the following function $f(t)$ depending on parameters x and y:

$$f(t) \stackrel{\text{def}}{=} \int\limits_0^t [u(x,y,\tau) - v(x,y,\tau)]d\tau.$$

Let us prove the following two auxiliary assertions, assuming that the conditions of Theorem 5.2.1 are satisfied.

Theorem 5.2.2. *If* $(x,y) \in \overline{\mathbb{R}^{n+1}_+}$, *then*

$$f(t) = o(t) \ \text{as} \ t \to \infty.$$

Theorem 5.2.3. *If* $(x,y) \in \overline{\mathbb{R}^{n+1}_+}$, *then*

$$f''(t) = O\left(\frac{1}{t}\right) \ \text{as} \ t \to \infty.$$

5.2.2. Proof of Theorem 5.2.2. In this section (as well as in the next one), we assume that all conditions of Theorem 5.2.1 are satisfied.

Apply the Laplace transformation with respect to t to the functions $u(x,y,t)$ and $v(x,y,t)$. The obtained functions (denoted by $\tilde{u}(x,y,\lambda)$ and $\tilde{u}(x,y,\lambda)$ respectively) are solutions of the following problems:

$$-\Delta_B \tilde{u} + \lambda p(x,y)\tilde{u} = p(x,y)\varphi(x,y), \ x \in \mathbb{R}^n, \ y > 0, \tag{5.18}$$

$$\frac{\partial \tilde{u}}{\partial y}\Big|_{y=0} = 0 \tag{5.19}$$

and

$$-\Delta_B \tilde{v} + \lambda \tilde{v} = p(x,y)\varphi(x,y), \ x \in \mathbb{R}^n, \ y > 0, \tag{5.20}$$

$$\frac{\partial \tilde{v}}{\partial y}\Big|_{y=0} = 0. \tag{5.21}$$

Indeed, for any function $g(x,y)$ from $C_0^\infty(\mathbb{R}^{n+1})$, the function $u(x,y,t)$ satisfies the following integral identity:

$$\int\limits_{\mathbb{R}^{n+1}_+} y^k \frac{\partial u}{\partial t} p(x,y)g(x,y)dxdy - \int\limits_{\mathbb{R}^{n+1}_+} y^k u(x,y,t)\Delta_B g(x,y)dxdy = 0.$$

Take a positive ε, N from $(\varepsilon, +\infty)$, and a complex λ such that $\mathrm{Re}\lambda > 0$. Multiply the last identity by $e^{-\lambda t}$ and integrate it with respect to t from ε to N. Then, changing the order of the integration and integrating by parts, we obtain the relation

$$\int\limits_{\mathbb{R}^{n+1}_+} y^k p(x,y)g(x,y)\left[u(x,y,N)e^{-\lambda N} - u(\varepsilon,y,N)e^{-\varepsilon N}\right]dxdy$$

$$+ \int\limits_{\mathbb{R}^{n+1}_+} y^k\left[\lambda p(x,y)g(x,y) - \Delta_B g(x,y)\right]\int\limits_\varepsilon^N e^{-\lambda t}u(x,y,t)dtdxdy = 0.$$

The limit relations

$$e^{-\lambda N}u(x,y,N)\xrightarrow{N\to\infty}0,\quad u(x,y,\varepsilon)\xrightarrow{\varepsilon\to 0}\varphi(x,y),\quad and\quad \int\limits_{\varepsilon}^{N}e^{-\lambda t}u(x,y,t)dt\xrightarrow[N\to\infty]{\varepsilon\to 0}\tilde{u}(x,y,\lambda)$$

hold uniformly with respect to $(x,y)\in K$, where K is any compact set of $\mathbb{R}^{n+1}$. Hence, $\tilde{u}(x,y,\lambda)$ satisfies the integral identity

$$\int\limits_{\overline{\mathbb{R}^{n+1}_+}} y^k\left[\lambda p(x,y)g(x,y)-\Delta_B g(x,y)\right]\tilde{u}(x,y,\lambda)dxdy=\int\limits_{\overline{\mathbb{R}^{n+1}_+}} y^k p(x,y)g(x,y)\varphi(x,y)dxdy$$

for any function $g(x,y)$ from $C_0^\infty(\mathbb{R}^{n+1})$.

Therefore, for any compact set $K\subset\overline{\mathbb{R}^{n+1}}$ and any (fixed) complex λ, the function $\tilde{u}(x,y,\lambda)$ belongs to the Kipriyanov space $W^2_{2,k}(K)$ (see [33]) and satisfies Eq. (5.18) almost everywhere.

This proof is valid for $\tilde{v}(x,y,\lambda)$ as well: it suffices to assign $p(x,y)\equiv 1$.

Since the functions $u(x,y,t)$ and $v(x,y,t)$ are bounded, it follows that the functions $\tilde{u}(x,y,\lambda)$ and $\tilde{v}(x,y,\lambda)$ are bounded with respect to $(x,y)\in\overline{\mathbb{R}^{n+1}_+}$ and are analytic with respect to λ provided that $\mathrm{Re}\lambda>0$. The solution of problem (5.20)-(5.21) (as well as the solution of problem (5.18)-(5.19)) bounded with respect to $(x,y)\in\overline{\mathbb{R}^{n+1}_+}$ and analytic with respect to λ for $\mathrm{Re}\lambda>0$ is unique. On the other hand, one can obtain this solution by directly applying the Laplace transformation to the function

$$v(x,y,t)=\frac{C_{n,k}}{t^{\frac{n+k+1}{2}}}\int\limits_{\overline{\mathbb{R}^{n+1}_+}}\eta^k e^{-\frac{|\xi-x|^2}{4t}}T^y_\eta\left(e^{-\frac{\eta^2}{4t}}\right)p(\xi,\eta)\varphi(\xi,\eta)d\xi d\eta$$

(the constant $C_{n,k}$ depends only on n and k).

Further, we have

$$\tilde{v}(x,y,\lambda)=C_{n,k}\int\limits_0^\infty\int\limits_{\overline{\mathbb{R}^{n+1}_+}}\eta^k\frac{p(\xi,\eta)\varphi(\xi,\eta)}{t^{\frac{n+k+1}{2}}}e^{-\lambda t-\frac{|\xi-x|^2}{4t}}T^y_\eta e^{-\frac{\eta^2}{4t}}d\xi d\eta dt$$

$$=C_{n,k}\int\limits_{\overline{\mathbb{R}^{n+1}_+}}\int\limits_0^\infty\eta^k\frac{p(\xi,\eta)\varphi(\xi,\eta)}{t^{\frac{n+k+1}{2}}}T^y_\eta\left(e^{-\lambda t-\frac{|\xi-x|^2+\eta^2}{4t}}\right)dtd\xi d\eta.$$

The change of the order of integration is valid because for $\mathrm{Re}\lambda>0$, the integral converges absolutely (moreover, this convergence is uniform with respect to (x,y) from $\overline{\mathbb{R}^{n+1}_+}$). Arguing in the same way, we can change the order of the following two operations: integrating with respect to t and the generalized translation. Finally, this yields the relation

$$\tilde{v}(x,y,\lambda)=2^{\frac{n+k+1}{2}}C_{n,k}\lambda^{\frac{n+k-1}{4}}\int\limits_{\overline{\mathbb{R}^{n+1}_+}}\eta^k p(\xi,\eta)\varphi(\xi,\eta)T^y_\eta\frac{K_{\frac{n+k-1}{2}}\left(\sqrt{\lambda(|\xi-x|^2+\eta^2)}\right)}{(|\xi-x|^2+\eta^2)^{\frac{n+k-1}{4}}}d\xi d\eta.\tag{5.22}$$

For $\mathrm{Re}\lambda>0$, define the operator M_λ on $H(\overline{\mathbb{R}^{n+1}_+})$ as follows:

$$M_\lambda f(x,y)=2^{\frac{n+k+1}{2}}C_{n,k}\lambda^{\frac{n+k-1}{4}}\int\limits_{\overline{\mathbb{R}^{n+1}_+}}\eta^k f(\xi,\eta)T^y_\eta\frac{K_{\frac{n+k-1}{2}}\left(\sqrt{\lambda(|\xi-x|^2+\eta^2)}\right)}{(|\xi-x|^2+\eta^2)^{\frac{n+k-1}{4}}}d\xi d\eta.$$

This is the resolving operator of problem (5.20)-(5.21) with the right-hand part $f(x,y)$. Therefore, if $\mathrm{Re}\lambda > 0$, $g \in W^2_{2,k,loc}(\overline{\mathbb{R}^{n+1}_+})$, and $(-\Delta_B + \lambda)g \in H(\overline{\mathbb{R}^{n+1}_+})$, then $M_\lambda(-\Delta_B + \lambda)g = g$. For $\mathrm{Re}\lambda > 0$, define the operator L_λ on $H(\overline{\mathbb{R}^{n+1}_+})$ as follows:

$$L_\lambda f(x,y) = \lambda M_\lambda[q(x,y)f(x,y)]$$

$$= 2^{\frac{n+k+1}{2}} C_{n,k} \lambda^{\frac{n+k+3}{2}} \int\limits_{\mathbb{R}^{n+1}_+} \eta^k q(\xi,\eta) f(\xi,\eta) T^y_\eta \frac{K_{\frac{n+k-1}{2}}\left(\sqrt{\lambda(|\xi-x|^2+\eta^2)}\right)}{(|\xi-x|^2+\eta^2)^{\frac{n+k-1}{4}}} d\xi d\eta.$$

The function $\widetilde{u}(x,y,\lambda)$ is a solution bounded with respect to $(x,y) \in \overline{\mathbb{R}^{n+1}_+}$ and analytic with respect to λ, $\mathrm{Re}\lambda > 0$, of the equation

$$(-\Delta_B + \lambda)\widetilde{u} = p(x,y)\varphi(x,y) - \lambda q(x,y)\widetilde{u}.$$

Here the right-hand part is analytic with respect to λ, $\mathrm{Re}\lambda > 0$, and is bounded with respect to $(x,y) \in \overline{\mathbb{R}^{n+1}_+}$ for any fixed λ, $\mathrm{Re}\lambda > 0$. Therefore, the operator M_λ can be applied both to the right-hand part and left-hand part of the last relation.

Since $\widetilde{u}$ belongs to $W^2_{2,k,loc}(\overline{\mathbb{R}^{n+1}_+})$, it follows that $M_\lambda(-\Delta_B + \lambda)\widetilde{u} = \widetilde{u}$.

Since M_λ is the resolving operator of problem (5.20)-(5.21) and $p(x,y)\varphi(x,y)$ belongs to $H(\overline{\mathbb{R}^{n+1}_+})$, it follows that $M_\lambda[p(x,y)\varphi(x,y)] = \widetilde{v}$.

Since $\widetilde{u}$ belongs to $H(\overline{\mathbb{R}^{n+1}_+})$ for any λ, $\mathrm{Re}\lambda > 0$, it follows that $M_\lambda[\lambda q(x,y)\widetilde{u}] = \lambda M_\lambda[q(x,y)\widetilde{u}] = L_\lambda \widetilde{u}$.

Thus, $\widetilde{u}(x,y,\lambda)$ satisfies the integral equation

$$\widetilde{u}(x,y,\lambda) + L_\lambda \widetilde{u}(x,y,\lambda) = v(x,y,\lambda). \tag{5.23}$$

In the sequel, we assume (in this section) that all the constants depend only on n and k unless otherwise stated.

Lemma 5.2.1. *Let $D_\sigma = \{\lambda \in \mathbb{C} \big| |\arg\lambda| < \pi - \sigma\}$, where $0 < \sigma < \pi$. Then, if $\lambda \in D_\sigma$, then L_λ is a bounded operator acting in $H(\overline{\mathbb{R}^{n+1}_+})$ and $\lim\limits_{\substack{|\lambda|\to 0\\ \lambda\in D_\sigma}} \|L_\lambda\| = 0$.*

Proof. It is known (see, e.g., [26]) that for $\nu > 0$, the function $K_\nu(z)$ satisfies the estimate $K_\nu(z) \leq C\alpha_\nu(|z|)$ in the domain $\left\{|\arg z| \leq \frac{\pi-\sigma}{2}\right\}$, where

$$\alpha_\nu(r) = \begin{cases} r^{-\nu} \text{ for } 0 < r \leq 1 \\ \dfrac{e^{-\gamma_0(r-1)}}{\sqrt{r}} \text{ for } r > 1 \end{cases}$$

and C and γ_0 are positive constants depending only on σ.

Assuming that $\lambda \in D_\sigma$, consider the expression

$$J_{n,k}(x,y;\lambda) = \int\limits_{\mathbb{R}^{n+1}_+} \eta^k T^y_\eta |q(\xi+x,\eta)| \frac{\left|K_{\frac{n+k-1}{2}}\left(\sqrt{\lambda(|\xi|^2+\eta^2)}\right)\right|}{(|\xi|^2+\eta^2)^{\frac{n+k-1}{4}}} d\xi d\eta$$

$$\leq C\int\limits_0^\infty \frac{\alpha_{\frac{n+k-1}{2}}(\sqrt{|\lambda|}\rho)}{\rho^{\frac{n+k-1}{2}}} \frac{\partial}{\partial\rho}\left[\int\limits_{B_+(\rho)} \eta^k T^y_\eta |q(\xi+x,\eta)| d\xi d\eta\right] d\rho.$$

Since $\alpha_\nu(r)$ is a piecewise-smooth function, it follows that it is possible to integrate by parts; this yields the inequality

$$J_{n,k}(x,y;\lambda) \leq -C\int\limits_0^\infty \frac{\partial}{\partial\rho}\left[\frac{\alpha_{\frac{n+k-1}{2}}\left(\sqrt{|\lambda|}\rho\right)}{\rho^{\frac{n+k-1}{2}}}\right]\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta d\rho$$

$$+ C\lim_{\rho\to\infty}\frac{\alpha_{\frac{n+k-1}{2}}\left(\sqrt{|\lambda|}\rho\right)}{\rho^{\frac{n+k-1}{2}}}\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta$$

$$- C\lim_{\rho\to 0}\frac{\alpha_{\frac{n+k-1}{2}}\left(\sqrt{|\lambda|}\rho\right)}{\rho^{\frac{n+k-1}{2}}}\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta.$$

Obviously, the former limit is equal to zero. Taking into account that

$$\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta \leq \operatorname{const}\rho^{n+k+1}$$

and $\alpha_{\frac{n+k-1}{2}}\left(\sqrt{|\lambda|}\rho\right) = \dfrac{1}{|\lambda|^{\frac{n+k-1}{4}}\rho^{\frac{n+k-1}{2}}}$ provided that ρ is sufficiently small, we see that the latter limit is equal to zero as well.

Thus, $J_{n,k}(x,y;\lambda) \leq J_{n,k,1}(x,y;\lambda) + J_{n,k,1}(x,y;\lambda)$, where

$$J_{n,k,1}(x,y;\lambda) = \frac{C(n+k-1)}{\lambda^{\frac{n+k-1}{4}}}\int\limits_0^{\frac{1}{\sqrt{|\lambda|}}} \frac{1}{\rho^{\frac{n+k-1}{2}}}\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta\rho d\rho$$

and

$$J_{n,k,2}(x,y;\lambda) \leq \frac{\operatorname{const}}{|\lambda|^{\frac{1}{4}}}\int\limits_{\frac{1}{\sqrt{|\lambda|}}}^\infty \frac{e^{-\gamma_0\sqrt{|\lambda|}\rho}}{\rho^{\frac{n+k}{2}}}\left(\sqrt{|\lambda|}+\frac{1}{\rho}\right)\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta\rho d\rho;$$

note that the constants in this lemma depend on σ as well.

Thus,

$$J_{n,k}(x,y;\lambda) \leq \operatorname{const}\left[\frac{1}{\lambda^{\frac{n+k-1}{4}}}\int\limits_0^{\frac{1}{\sqrt{|\lambda|}}} \frac{1}{\rho^{\frac{n+k-1}{2}}}\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta\rho d\rho\right.$$

$$\left. + |\lambda|^{\frac{1}{4}}\int\limits_{\frac{1}{\sqrt{|\lambda|}}}^\infty \frac{1}{\rho^{\frac{n+k-1}{2}}}\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta\rho^{\frac{n+k}{2}+1}\left(1+\frac{1}{\sqrt{|\lambda|}\rho}\right)e^{-\gamma_0\sqrt{|\lambda|}\rho}d\rho\right] \leq \frac{\operatorname{const}}{|\lambda|^{\frac{n+k+3}{4}}}$$

by virtue of the boundedness of the function

$$\frac{1}{\rho^{\frac{n+k-1}{2}}}\int\limits_{B_+(\rho)} \eta^k T_\eta^y|q(\xi+x,\eta)|d\xi d\eta.$$

This implies the boundedness of the operator L_λ.

Indeed,

$$|L_\lambda f(x,y)| = \text{const}\,|\lambda|^{\frac{n+k+3}{4}} \left| \int\limits_{\mathbb{R}^{n+1}_+} \eta^k T^y_\eta [q(\xi+x,\eta) f(\xi+x,\eta)] \frac{K_{\frac{n+k-1}{2}}\left(\sqrt{\lambda(|\xi|^2+\eta^2)}\right)}{(|\xi|^2+\eta^2)^{\frac{n+k-1}{4}}} d\xi d\eta \right|$$

$$\leq \text{const}\,|\lambda|^{\frac{n+k+3}{4}} \|f\|_{H(\overline{\mathbb{R}^{n+1}_+})} J_{n,k}(x,y;\lambda) \leq \text{const}\,\|f\|_{H(\overline{\mathbb{R}^{n+1}_+})},$$

i.e., for any fixed σ, the operator L_λ is bounded uniformly with respect to $\lambda \in D_\sigma$.

Let us pass to the proof of the second assertion of Lemma 5.2.1. From the estimate obtained for $|L_\lambda f(x,y)|$, it follows that

$$\|L_\lambda\| \leq \text{const}\,|\lambda|^{\frac{n+k+3}{4}} \sup_{(x,y)\in\overline{\mathbb{R}^{n+1}_+}} J_{n,k}(x,y;\lambda).$$

It follows from (5.16) that for any positive ε there exists $N(\varepsilon)$ such that

$$\frac{1}{\rho^{n+k+1}} \int\limits_{B_+(\rho)} \eta^k T^y_\eta |q(\xi+x,\eta)| d\xi d\eta < \varepsilon$$

for any ρ from $[N(\varepsilon),+\infty)$ and any (x,y) from $\overline{\mathbb{R}^{n+1}_+}$. Therefore, the inequality

$$|\lambda|^{\frac{n+k+3}{4}} J_{n,k}(x,y;\lambda) \leq \text{const}\left[|\lambda|\frac{N^2(\varepsilon)}{2} + \frac{\varepsilon}{2} + \varepsilon \int\limits_1^\infty r^{\frac{n+k}{2}+1}\left(1+\frac{1}{r}\right) e^{-\gamma_0 r} dr\right]$$

is valid for $|\lambda| < \dfrac{1}{N^2(\varepsilon)}$.

Take an arbitrary positive δ and select a positive ε to satisfy the inequality

$$\text{const}\left[\frac{\varepsilon}{2} + \varepsilon \int\limits_1^\infty r^{\frac{n+k}{2}+1}\left(1+\frac{1}{r}\right) e^{-\gamma_0 r} dr\right] < \frac{\delta}{2}$$

(where the constant is the same as in the previous inequality). Obviously, one can select a positive β such that for any λ from D_σ, the inequality $|\lambda| < \beta$ implies the inequality $\text{const}\,|\lambda|\dfrac{N^2(\varepsilon)}{2} < \dfrac{\delta}{2}$, i.e., for any positive δ there exist N and β such that $|\lambda|^{\frac{n+k+3}{2}} J_{n,k}(x,y;\lambda) < \delta$. This proves Lemma 5.2.1 for the case where n is positive.

If $n = 0$, then the proof is similar, but one has to consider three cases separately: $k = 1$, $k < 1$, and $k > 1$. If $k = 1$, then we use the estimate of $|K_0(z)|$ via $\alpha_0(|z|)$ (see, e.g., [26]). If $k < 1$, then we use the evenness of $|K_\nu(z)|$ with respect to ν.

This completes the proof of Lemma 5.2.1. □

The functions $\widetilde{u}(x,y;\lambda)$ and $\widetilde{v}(x,y;\lambda)$, being the Laplace transforms of analytic functions, are defined only for positive $\mathrm{Re}\lambda$. However, the right-hand part of relation (5.22) is analytic in the domain $\{|\arg\lambda| < \pi\}$. Therefore, using relation (5.22), we analytically extend the function $\widetilde{v}(x,y;\lambda)$ to the domain $\{|\arg\lambda| < \pi\}$ such that the extended function belongs to the space $H(\overline{\mathbb{R}^{n+1}_+})$ with respect to the variables x and y. By virtue of Lemma 5.2.1, for any positive σ, there exists a positive δ such that $\|L_\lambda\| < 1$ for any λ from $\{|\lambda| < \delta\} \cap D_\sigma$. Then the function $\widetilde{u}(x,y;\lambda)$ is analytically extended to the domain $\lambda \in \{|\lambda| < \delta\} \cap D_\sigma$ such that the extended function belongs to the space $H(\overline{\mathbb{R}^{n+1}_+})$ with respect to the variables x and y (because it is a solution of the integral equation (5.23)).

Thus, for any positive σ there exists a positive δ such that the functions $\widetilde{u}(x,y;\lambda)$ and $\widetilde{v}(x,y;\lambda)$ are analytic with respect to $\lambda \in \{|\lambda| < \delta\} \cap \{|\arg\lambda| < \pi - \sigma\}$ and continuous and bounded with respect to $(x,y) \in \overline{\mathbb{R}^{n+1}_+}$. From the boundedness of the functions $p(x,y)$ and $\varphi(x,y)$ and from the estimate of $K_\nu(z)$ via $\alpha_\nu(|z|)$ (see Lemma 5.2.1), it follows that the inequality

$$\left| \int\limits_{\mathbb{R}^{n+1}_+} \eta^k p(\xi,\eta)\varphi(\xi,\eta) \frac{K_{\frac{n+k-1}{2}}\left(\sqrt{\lambda(|\xi-x|^2+\eta^2)}\right)}{(|\xi-x|^2+\eta^2)^{\frac{n+k-1}{4}}}\, d\xi d\eta \right|$$

$$\leq \text{const} \int\limits_0^\infty \rho^{\frac{n+k+1}{2}} \alpha_{\frac{n+k+1}{2}}(\sqrt{|\lambda|}\rho) d\rho = \frac{\text{const}}{|\lambda|^{\frac{n+k+3}{4}}}$$

is valid for $n+k>1$. This and relation (5.22) imply that $\|\widetilde{v}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})} \leq \dfrac{\text{const}}{|\lambda|}$ for any $\lambda \in D_\sigma$ provided that $\sigma > 0$.

The last estimate is valid for $n+k \leq 1$ as well. Indeed, if $n=0$ and $k=1$, then

$$|\widetilde{v}(x,y;\lambda)| \leq \frac{\text{const}}{|\lambda|} \int\limits_0^\infty r\alpha_0(r)dr = \frac{\text{const}}{|\lambda|};$$

if $n=0$ and $k<1$, then

$$|\widetilde{v}(x,y;\lambda)| \leq \frac{\text{const}}{|\lambda|} \int\limits_0^\infty \alpha_{\frac{1-k}{2}}(r) r^{\frac{k+1}{2}} dr = \frac{\text{const}}{|\lambda|}$$

(the constants depend only on n, k, and σ).

Then relation (5.23) implies the inequality

$$\|\widetilde{u}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})} \leq \|L_\lambda\| \|\widetilde{u}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})} + \frac{\text{const}}{|\lambda|}.$$

By virtue of Lemma 5.2.1, this implies that the inequality $\|\widetilde{u}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})} \leq \dfrac{\text{const}}{|\lambda|}$ is valid for any λ from $D_\sigma \cap \{|\lambda| < \delta\}$. Further, from relation (5.23), we obtain that

$$\|\widetilde{u}(x,y;\lambda) - \widetilde{v}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})} = \|L_\lambda \widetilde{u}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})}$$

$$\leq \|L_\lambda\| \|\widetilde{u}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})} \leq \frac{\|L_\lambda\| \,\text{const}}{|\lambda|} = o\left(\frac{1}{|\lambda|}\right) \text{ as } |\lambda| \to 0$$

for $\lambda \in D_\sigma \cap \{|\lambda| < \delta\}$ because $\lim\limits_{\substack{|\lambda|\to 0\\ \lambda \in D_\sigma}} \|L_\lambda\| = 0$ due to Lemma 5.2.1.

Thus, the following assertions are valid:

(1) The functions $u(x,y,t)$ and $v(x,y,t)$ are continuous and bounded.
(2) The functions $\widetilde{u}(x,y;\lambda)$ and $\widetilde{v}(x,y;\lambda)$ are their Laplace transforms analytically extended (for positive values of the parameter σ) to the domain $\{|\lambda| < \delta(\sigma)\} \cap \{|\arg\lambda| < \pi - \sigma\}$ such that the extended functions are continuous and bounded with respect to $(x,y) \in \overline{\mathbb{R}^{n+1}_+}$.
(3) For any positive σ there exists a positive $\delta = \delta(\sigma)$ such that the relation

$$\|\widetilde{u}(x,y;\lambda) - \widetilde{v}(x,y;\lambda)\|_{H(\overline{\mathbb{R}^{n+1}_+})} = o\left(\frac{1}{|\lambda|}\right) \text{ as } |\lambda| \to 0$$

is valid in the domain $\{|\lambda| < \delta(\sigma)\} \cap \{|\arg\lambda| < \pi - \sigma\}$.

Then, as is known from [26], the limit relation

$$\frac{1}{t}\int\limits_0^t [u(x,y,\tau)-v(x,y,\tau)]d\tau \xrightarrow{t\to\infty} 0$$

holds for any (x,y) from $\overline{\mathbb{R}^{n+1}_+}$.

This completes the proof of Theorem 5.2.2.

5.2.3. Proof of Theorem 5.2.3. First, we note that, since condition (5.14) is satisfied, it follows that the space $H^{\alpha}_{[\frac{n+k+1}{2}]}(\overline{\mathbb{R}^{n+1}_+})$ in condition (5.13) can be replaced by the space $\widetilde{H}^{\alpha}_{[\frac{n+k+1}{2}]}(\overline{\mathbb{R}^{n+1}_+})$; this is proved in [31]. Further, following [34] (see also [33]), we introduce the following functional spaces.

Let $C^{\infty}_E(\Omega)$ denote the space of functions even with respect to y and infinitely differentiable on Ω (as above, we assume that $\Omega\subset\overline{\mathbb{R}^{n+1}_+}$). Define the set $L_{p,k}(\Omega)$, $p\ge 1$, as the set of functions such that the following their norm is finite:

$$\|f\|_{L_{p,k}(\Omega)}=\left(\int\limits_{\Omega} y^k|f(x,y)|dxdy\right)^{\frac{1}{p}}.$$

The space $W^m_{2,k}(\Omega)$ is the completion of the set $C^{\infty}_E(\Omega)$ with respect to the following norm:

$$\|f\|_{W^m_{2,k}(\Omega)}=\left(\sum_{|\beta|\le m}\|\widetilde{D}^{\beta}f\|^2_{L_{p,k}(\Omega)}\right)^{\frac{1}{2}}.$$

In the sequel, if no misunderstanding regarding the domain can arise, the norm in $W^m_{2,k}$ is denoted by $\|\cdot\|_m$, the norm in $L_{2,k}$ is denoted by $\|\cdot\|_0$, and the norm in H is denoted by $\|\cdot\|$. The variable y is denoted by x_{n+1} whenever it is convenient.

Let us pass to the proof of Theorem 5.2.3.

Take an arbitrary δ_0 and fix it. Denote $\dfrac{\partial u}{\partial t}$ by $w(x,y,t)$ and denote $\dfrac{\partial u}{\partial t}\Big|_{t=\delta_0}$ by $\psi(x,y)$. It follows from [47] that $\psi\in\widetilde{H}_{[\frac{n+k+1}{2}]+2}(\overline{\mathbb{R}^{n+1}_+})$ and $w(x,y,t)$ is the classical bounded solution of the following problem:

$$p(x,y)\frac{\partial w}{\partial t}=\Delta_B w,\ x\in\mathbb{R}^n,\ y>0,\ t>\delta_0, \tag{5.24}$$

$$w\Big|_{t=\delta_0}=\psi(x,y),\ \frac{\partial w}{\partial y}\Big|_{y=0}=0. \tag{5.25}$$

Together with problem (5.24)-(5.25), consider the problem

$$p(x,y)\frac{\partial^2 Z}{\partial t^2}=\Delta_B Z,\ x\in\mathbb{R}^n,\ y>0,\ t>\delta_0, \tag{5.26}$$

$$Z\Big|_{t=\delta_0}=0,\ \frac{\partial Z}{\partial t}\Big|_{t=\delta_0}=\psi(x,y),\ \frac{\partial Z}{\partial y}\Big|_{y=0}=0. \tag{5.27}$$

Lemma 5.2.2. *There exists a positive C such that for any t from $[\delta_0,+\infty)$, for any x from $\mathbb{R}^n$, and for any nonnegative y, the inequality*

$$|Z(x,y,t)|\le Ct^{\frac{n+1}{2}}(t+y)^{\frac{k}{2}+1-\{\frac{n+k+1}{2}\}},$$

where $\{x\}$ denotes the fractional part of x, is valid.

Proof. By the condition, the function ψ belongs to the space $\widetilde{H}_{[\frac{n+k+1}{2}]+2}(\overline{\mathbb{R}^{n+1}_+})$. Therefore, it belongs to the space $W^{[\frac{n+k+1}{2}]+2}_{2,k,loc}(\overline{\mathbb{R}^{n+1}_+})$ as well. Using the Duhamel integral, we deduce from [3] that $Z \in W^{[\frac{n+k+1}{2}]+2}_{2,k,loc}(\overline{\mathbb{R}^{n+2}_{++}})$, where $\mathbb{R}^{n+2}_{++} = \{(x,y,t)\big| t > \delta_0, x \in \mathbb{R}^n, y > 0\}$.

Take an arbitrary point (x_0, y_0) from $\mathbb{R}^{n+1}_+$ and an arbitrary number t_0 from $(\delta_0, +\infty)$. Without loss of generality, one can assume that $p_0 = 1$, i.e., $p(x,y) \geq 1$. Then (see, e.g., [108, p. 93]) the value of the function $Z(x,y,t)$ at the point (x_0, y_0, t_0) depends only on the values of the function $\psi(x,y)$ for $|x-x_0|^2 + (y-y_0)^2 \leq (t_0 - \delta_0)^2$ and $y \geq 0$. Take a function $\psi_0(x,y)$ such that

(a) The support of the function ψ_0 is a subset of $\{|x-x_0|^2 + (y-y_0)^2 \leq t_0^2,\ y \geq 0\}$.
(b) If $|x-x_0|^2 + (y-y_0)^2 \leq (t_0-\delta_0)^2$ and $y \geq 0$, then $\psi_0(x,y) = \psi(x,y)$.
(c) We have $\dfrac{\partial \psi_0}{\partial y}\Big|_{y=0} = 0$.
(d) The function ψ_0 has the same smoothness as the function ψ.

To find a function $\psi_0(x,y)$ possessing the properties (a) and (b), it suffices to multiply $\psi(x,y)$ by an appropriate cut-off function.

Let $u_0(x,y,t)$ be a solution of the following problem:

$$p(x,y)\frac{\partial^2 u_0}{\partial t^2} = \Delta_B u_0,\ x \in \mathbb{R}^n,\ y > 0,\ t > \delta_0, \tag{5.28}$$

$$u_0\Big|_{t=\delta_0} = 0,\ \frac{\partial u_0}{\partial t}\Big|_{t=\delta_0} = \psi_0(x,y),\ \frac{\partial u_0}{\partial y}\Big|_{y=0} = 0. \tag{5.29}$$

Taking into account that the function u_0 belongs to $W^2_{2,k,loc}(\overline{\mathbb{R}^{n+2}_{++}})$ and the support of the function $u_0(x,y,t)$ is compact for any fixed t, we obtain the following relation between energy integrals:

$$\int\limits_{\overline{\mathbb{R}^{n+1}_+}} y^k \left[p(x,y)\left(\frac{\partial u_0}{\partial t}\right)^2 + \sum_{j=1}^n \left(\frac{\partial u_0}{\partial x_j}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2\right]\Bigg|_{t=t_0} dxdy$$
$$= \int\limits_{\overline{\mathbb{R}^{n+1}_+}} y^k \left[p(x,y)\left(\frac{\partial u_0}{\partial t}\right)^2 + \sum_{j=1}^n \left(\frac{\partial u_0}{\partial x_j}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2\right]\Bigg|_{t=\delta_0} dxdy.$$

Taking into account that

$$\operatorname{supp}\psi \subset \{|x-x_0|^2 + (y-y_0)^2 \leq t_0^2,\ y \geq 0\}$$

and

$$\operatorname{supp} u_0(x,y,t_0) \subset \{|x-x_0|^2 + (y-y_0)^2 \leq 4t_0^2,\ y \geq 0\},$$

change the integration domains in the last relation:

$$\int\limits_{\Omega_2^0} y^k \left[p(x,y)\left(\frac{\partial u_0}{\partial t}\right)^2 + \sum_{j=1}^n \left(\frac{\partial u_0}{\partial x_j}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2\right]\Bigg|_{t=t_0} dxdy$$
$$= \int\limits_{\Omega_1^0} y^k \left[p(x,y)\left(\frac{\partial u_0}{\partial t}\right)^2 + \sum_{j=1}^n \left(\frac{\partial u_0}{\partial x_j}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2\right]\Bigg|_{t=\delta_0} dxdy,$$

where $\Omega_j^0 = \{|x-x_0|^2 + (y-y_0)^2 \leq (jt)^2,\ y \geq 0\}$, $j = 1, 2$.

Assuming that $n+k \geq 1$, introduce the following functions for $m = 1, \dots, \left[\dfrac{n+k+1}{2}\right]$:

$$u_m(x,y,t) = \frac{\partial^m u_0}{\partial t^m}(x,y,t).$$

Then the function u_m, $m = \overline{1, \left[\dfrac{n+k+1}{2}\right]}$, satisfies Eq. (5.26) and the following initial-value conditions:

$$u_m\big|_{t=\delta_0} = 0 \text{ and } \frac{\partial u_m}{\partial t}\big|_{t=\delta_0} = \left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m}{2}} \psi_0(x,y) \text{ if } m \text{ is even}$$

$$u_m\big|_{t=\delta_0} = \left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m-1}{2}} \psi_0(x,y) \text{ and } \frac{\partial u_m}{\partial t}\big|_{t=\delta_0} = 0 \text{ if } m \text{ is odd .}$$

Obviously, $\dfrac{\partial u_m}{\partial y}\big|_{y=0} = 0$, $m = \overline{1, \left[\dfrac{n+k+1}{2}\right]}$.

Thus, the following energy integral identity holds for the functions $u_m(x,y,t)$:

$$\begin{aligned}
&\int\limits_{\Omega_2^0} y^k \left[p(x,y)\left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{j=1}^{n} (\nabla u_m)^2\right]\Bigg|_{t=t_0} dxdy \\
&= \int\limits_{\Omega_1^0} y^k \left[p(x,y)\left(\frac{\partial u_m}{\partial t}\right)^2 + \sum_{j=1}^{n} (\nabla u_m)^2\right]\Bigg|_{t=\delta_0} dxdy \\
&= \begin{cases}
\displaystyle\int\limits_{\Omega_1^0} y^k p(x,y) \left[\left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m}{2}} \psi_0(x,y)\right]^2 dxdy \\
\text{if } m \text{ is even (including the case where } m=0) \\
\displaystyle\int\limits_{\Omega_1^0} y^k p(x,y) \left(\nabla\left[\left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m-1}{2}} \psi_0(x,y)\right]\right)^2 dxdy \\
\text{if } m \text{ is odd (including the case where } m=0).
\end{cases}
\end{aligned} \tag{5.30}$$

Here $\nabla = \left(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}, \dfrac{\partial}{\partial y}\right)$ and $\nabla^2(\cdot) = (\nabla\cdot, \nabla\cdot)$.

Let $R(x,y) = \dfrac{1}{p(x,y)}$. Then $\dfrac{\partial^m R}{\partial y^m}\big|_{y=0} = 0$, $m = \overline{1, \dots, \left[\dfrac{n+k+1}{2}\right]}$. Then the following Leibnitz formula holds (see [3]):

$$B^m(Ru) = RB^m u + \sum_{\substack{i_1+i_2+2j_1+2j_2+s=2m \\ i_1+i_2+j_1 \geq 1 \\ i_1=0,1;\ i_2=0,1}} C_{i_1,j_1}^{i_2,j_2} \frac{1}{y^s} \frac{\partial^{i_1} R}{\partial y^{i_1}} B^{j_1} R \frac{\partial^{i_2} u}{\partial y^{i_2}} B^{j_2} u,$$

where $C_{i_1,j_1}^{i_2,j_2}$ depends only on m, i_1, j_1, i_2, and j_2.

Use this formula and consider the operators acting on $\psi_0(x,y)$ and $p(x,y)$ at the right-hand part of inequality (5.30): the order of the operator $\widetilde{D}$ acting on $\psi_0(x,y)$ does not exceed m, while the order of the operator $\widetilde{D}$ acting on $R(x,y)$ (i.e., on $p(x,y)$)does not exceed $m-2$. This yields that the following inequality is valid provided that $m \le \left[\dfrac{n+k+1}{2}\right]$:

$$\int\limits_{\Omega_2^0} y^k \left[p(x,y)\left(\frac{\partial u_m}{\partial t}\right)^2 + (\nabla u_m)^2\right]\Bigg|_{t=t_0} dxdy \le C_m t_0^{n+1}(t_0+y_0)^k, \tag{5.31}$$

where C_m does not depend on x_0, y_0, and t_0.

Further,

$$\frac{\partial u_m}{\partial t}\Big|_{t=t_0} = \left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m+1}{2}} u_0\big|_{t=t_0}$$

if m is odd, while

$$\frac{\partial u_m}{\partial x_j}\Big|_{t=t_0} = \frac{\partial}{\partial x_j}\left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m}{2}} u_0\big|_{t=t_0} \quad \text{and} \quad \frac{\partial u_m}{\partial y}\Big|_{t=t_0} = \frac{\partial}{\partial y}\left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m}{2}} u_0\big|_{t=t_0}$$

if m is even.

Further, if m is even, then we use the nonnegativity of $\left(\dfrac{\partial u_m}{\partial t}\right)^2$; if m is odd, then we use the nonnegativity of $(\nabla u_m)^2$. Then inequality (5.31) yields the estimate

$$\begin{cases} \displaystyle\int\limits_{\Omega_2^0} y^k \left[\nabla\left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m}{2}} u_0(x,y,t_0)\right]^2 dxdy \\ \text{if } m \text{ is even (including the case where } m=0) \\ \displaystyle\int\limits_{\Omega_2^0} y^k \left[\left(\frac{1}{p(x,y)}\Delta_B\right)^{\frac{m+1}{2}} u_0(x,y,t_0)\right]^2 dxdy \\ \text{if } m \text{ is odd (including the case where } m=1) \end{cases} \le C_m^* t_0^{n+1}(t_0+y_0)^k, \tag{5.32}$$

where C_m^* does not depend on x_0, y_0, and t_0.

Introduce the denotation

$$\sum_{|\beta|=l} [\widetilde{D}^\beta f(x,y)]^2 \overset{\text{def}}{=} [\widetilde{D}^l f(x,y)]^2 \text{ and } \sum_{|\beta|=l} [D_x^\beta f(x,y)]^2 \overset{\text{def}}{=} [D_x^l f(x,y)]^2.$$

Let $m=0$. Then inequality (5.32) implies that

$$\int\limits_{\Omega_2^0} y^k [\nabla u_0(x,y,t_0)]^2 dxdy \le C_0^* t_0^{n+1}(t_0+y_0)^k.$$

Therefore,

$$\int\limits_{\Omega_2^0} y^k \left[\widetilde{D}^1 u_0(x,y,t_0)\right]^2 dxdy \le C_0^* t_0^{n+1}(t_0+y_0)^k.$$

This means that

$$\|u_0(x,y,t_0)\|_1^2 \le \widetilde{C}_0 t_0^{n+1}(t_0+y_0)^k,$$

where $\widetilde{C}_0 = C_0^*$.

Let $m=1$. Then inequality (5.32) implies that

$$\int\limits_{\Omega_2^0} y^k [\Delta_B u_0(x,y,t_0)]^2 dxdy \le C_1^* \|p\|^2 t_0^{n+1}(t_0+y_0)^k.$$

Since the support of the function $u_0(x,y,t_0)$ is compact, it follows that there exists an absolute constant C such that the inequality

$$\|u_0(x,y,t_0)\|_2^2 \le C \int\limits_{\Omega_1^0} y^k \left[\Delta_B u_0(x,y,t_0)\right]^2 dxdy$$

holds. Then

$$\|u_0(x,y,t_0)\|_2^2 \le \widetilde{C}_1 t_0^{n+1}(t_0+y_0)^k.$$

Let $m=2$. Then inequality (5.32) implies that

$$\int\limits_{\Omega_2^0} y^k \left[\nabla\left(\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right)\right]^2 dxdy \le C_2^* t_0^{n+1}(t_0+y_0)^k.$$

We must estimate $\|u_0(x,y,t_0)\|_3^2$, i.e.,

$$\int\limits_{\Omega_2^0} y^k \left(\left[\frac{\partial}{\partial y}Bu_0(x,y,t_0)\right]^2 + \left[D_x^3 u_0(x,y,t_0)\right]^2\right) dxdy.$$

For $j=\overline{1,n}$, consider the following integral:

$$\int\limits_{\Omega_2^0} y^k \left[\Delta_B\left(\frac{\partial u_0}{\partial x_j}(x,y,t_0)\right)\right]^2 dxdy = \int\limits_{\Omega_2^0} y^k \left[\frac{\partial}{\partial x_j}\Delta_B u_0(x,y,t_0)\right]^2 dxdy$$

$$= \int\limits_{\Omega_2^0} y^k \left[\frac{\partial}{\partial x_j}\left(p(x,y)\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right)\right]^2 dxdy \le 2\int\limits_{\Omega_2^0} y^k \left[\frac{\partial p}{\partial x_j}\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right]^2 dxdy$$

$$+ 2\int\limits_{\Omega_2^0} y^k \left[p(x,y)\frac{\partial}{\partial x_j}\left(\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right)\right]^2 dxdy \le \widetilde{C}_2^* t_0^{n+1}(t_0+y_0)^k,$$

where $\widetilde{C}_2^* = 2\left(\left\|\frac{\partial p}{\partial x_j}\right\|^2 C_1^* + \|p\|^2 C_2^*\right)$.

Let $m=3$. Then inequality (5.32) implies that

$$\int\limits_{\Omega_2^0} y^k \left[\Delta_B\left(\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right)\right]^2 dxdy \le C_3^*\|p\|^2 t_0^{n+1}(t_0+y_0)^k.$$

We must estimate $\|u_0(x,y,t_0)\|_4^2$. Consider the integral

$$\int\limits_{\Omega_2^0} y^k \left[\Delta_B^2 u_0(x,y,t_0)\right]^2 dxdy = \int\limits_{\Omega_2^0} y^k \left[\Delta_B\left(p(x,y)\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right)\right]^2 dxdy.$$

Applying the Leibnitz formula to the integrand, we see that the last integral is equal to

$$\int\limits_{\Omega_2^0} y^k \left[p(x,y)\Delta_B\left(\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right) + 2\left(2\nabla p(x,y), \nabla\left(\frac{1}{p(x,y)}\Delta_B u_0(x,y,t_0)\right)\right)\right.$$

$$\left. + \frac{1}{p(x,y)}\Delta_B p(x,y)\Delta_B u_0(x,y,t_0)\right]^2 dxdy \le \widetilde{C}_3^* t_0^{n+1}(t_0+y_0)^k,$$

where $\widetilde{C}_3^* = 4(C_3^*\|p\|^4 + 2\|\widetilde{D}^1 p\| C_2^* + C_1^*\|p\|^2\|\Delta_B p\|^2)$.

Since

$$\|u_0(x,y,t_0)\|_4^2 \le \text{const} \int\limits_{\Omega_2^0} y^k \left[\Delta_B^2 u_0(x,y,t_0)\right]^2 dxdy$$

(because the support of $u_0(x,y,t_0)$ is compact), we have the estimate

$$\|u_0(x,y,t_0)\|_4^2 \le \widetilde{C}_3 t_0^{n+1}(t_0+y_0)^k.$$

Note that at the mth step, the finiteness of the norms $\|\widetilde{D}^j p\|$, $j \le m-1$, is used. By the condition, those norms are finite provided that $j \le \left[\dfrac{n+k+1}{2}\right]$. Therefore, the procedure described above can be continued until the $\left[\dfrac{n+k+1}{2}\right]$th step (inclusively). At the specified step, we apply the Leibnitz formula and take into account that $\dfrac{\partial^j p}{\partial y^j}\Big|_{y=0}= 0$ for $j = 1,\dots, \left[\dfrac{n+k+1}{2}\right]$. This yields the estimate

$$\|u_0(x,y,t_0)\|^2_{\left[\frac{n+k+1}{2}\right]+1} \le \widetilde{C}_{\left[\frac{n+k+1}{2}\right]} t_0^{n+1}(t_0+y_0)^k, \tag{5.33}$$

where the constant $C_{\left[\frac{n+k+1}{2}\right]}$ does not depend on x_0, y_0, and t_0.

Since the function $u_0(x,y,t)$ belongs to $W_{2,k,loc}^{\left[\frac{n+k+1}{2}\right]+2}(\overline{\mathbb{R}^{n+2}_{++}})$, it follows that the function $u_0(x,y,T)$ belongs to $W_{2,k,loc}^{\left[\frac{n+k+1}{2}\right]+\frac{3}{2}}(\overline{\mathbb{R}^{n+1}_{+}})$ for any fixed T from $(\delta_0,+\infty)$ (see [33]). Therefore, the function $u_0(x,y,t_0)$ belongs to $W_{2,k,loc}^{\left[\frac{n+k+1}{2}\right]+1}(\overline{\mathbb{R}^{n+1}_{+}})$.

Now, we apply the following embedding theorem (see [32]): if $f(x,y) \in W_{2,k}^{\left[\frac{n+k+1}{2}\right]+1}(\overline{\mathbb{R}^{n+1}_{+}})$ and $\operatorname{supp} f \subset \left\{|x|^2+y^2 \le 1, y \ge 0\right\}$, then $f(x,y) \in C(\overline{\mathbb{R}^{n+1}_{+}})$ and

$$|f(0)| \le \text{const}\,\|f\|_{\left[\frac{n+k+1}{2}\right]+1} = \text{const}\left(\int\limits_{\mathbb{R}^{n+1}_{+}} \eta^k \left[\widetilde{D}_{\xi,\eta}^{\left[\frac{n+k+1}{2}\right]+1} f(\xi,\eta)\right]^2 d\xi d\eta\right)^{\frac{1}{2}}.$$

Now, let $g \in W_{2,k}^{\left[\frac{n+k+1}{2}\right]+1}(\overline{\mathbb{R}^{n+1}_{+}})$ and $\operatorname{supp} g \subset \left\{|x|^2+y^2 \le 4(t_0+y_0)^2,\, y \ge 0\right\}$.

Introduce the function $f(x,y) \stackrel{\text{def}}{=} g[2(t_0+y_0)x, 2(t_0+y_0)y]$. The specified embedding theorem is applicable to this function; it yields

$$|g(0)| = |f(0)| \le \text{const}\Bigg(\int\limits_{\mathbb{R}^{n+1}_{+}} \eta^k \Bigg[(2(t_0+y_0))^{\left[\frac{n+k+1}{2}\right]+1}$$
$$\times \widetilde{D}_{2(t_0+y_0)\xi,2(t_0+y_0)\eta}^{\left[\frac{n+k+1}{2}\right]+1} g\Big(2(t_0+y_0)\xi, 2(t_0+y_0)\eta\Big)\Bigg]^2 d\xi d\eta\Bigg)^{\frac{1}{2}}.$$

Change the variables in the last integral: $x_j = 2(t_0+y_0)\xi_j, j=\overline{1,n}$, and $y = 2(t_0+y_0)\eta$. We obtain that $|g(0)| \le \widetilde{C}(t_0+y_0)^{\left[\frac{n+k+1}{2}\right]+1-\frac{n+k+1}{2}}\|g\|_{\left[\frac{n+k+1}{2}\right]+1}$, where $\widetilde{C}$ depends only on n and k.

Taking into account that $\operatorname{supp} u_0(x,y,t_0) \subset \left\{|x-x_0|^2+y^2 \le (2t_0+2y_0)^2,\, y\ge 0\right\}$, treat the point $(x_0,0) = (x_1^0,\dots,x_n^0,0)$ as the origin in $\mathbb{R}^{n+1}$ and assign $g(x,y) = u_0(x,y,t_0)$. This yields the relation

$$|Z(x_0,y_0,t_0)| = |u_0(x_0,y_0,t_0)| \le \widetilde{C}(t_0+y_0)^{\left[\frac{n+k+1}{2}\right]+1-\frac{n+k+1}{2}}\|u_0(x,y,t_0)\|_{\left[\frac{n+k+1}{2}\right]+1}.$$

It follows from (5.13) that

$$|Z(x_0,y_0,t_0)| \le \text{const}\, t_0^{\frac{n+1}{2}} C(t_0+y_0)^{\frac{k}{2}+1-\{\frac{n+k+1}{2}\}},$$

which completes the proof of Lemma 5.2.2 because x_0, y_0, and t_0 are selected arbitrarily and the constant does not depend on x_0, y_0, and t_0.

Remark. If $n+k<1$, then it suffices to apply inequality (5.32) for $m=0$ (note that it is valid only for $m=0$ then) and the embedding theorem.

Lemma 5.2.2 is proved. □

Let us show that

$$\widetilde{Z}(x,y,\lambda) \overset{\text{def}}{=} \int\limits_{\delta_0}^{\infty} e^{-(t-\delta_0)\lambda} Z(x,y,t)dt$$

has at most a power growth as $y\to\infty$, i.e., there exists $\varkappa_0$ such that $\widetilde{Z}(x,y,\lambda) = O(y^{\varkappa_0})$ as $y\to\infty$.

To do that, we note that

$$|\widetilde{Z}(x,y,\lambda)| \le C|e^{\delta_0\lambda}| \int\limits_0^{\infty} |e^{-t\lambda}| t^{\frac{n+1}{2}} C(t+y)^{\frac{k}{2}+1-\{\frac{n+k+1}{2}\}}dt$$

$$= C\Gamma\left(\frac{n+3}{2}\right) e^{\lambda \mathrm{Re}\delta_0} y^{[\frac{n+k+1}{2}]+2} G\left(\frac{n+3}{2}, \left[\frac{n+k+1}{2}\right]+3, y\mathrm{Re}\lambda\right),$$

where $G(\alpha_1,\alpha_2,z)$ is the confluent hypergeometric function of the second type (see [79, p. 246]). From the asymptotic representation of this function as $z\to\infty$ (see [79, p. 283]), it follows that $\widetilde{Z}(x,y,\lambda)$ has at most a power growth with respect to y at infinity.

Further, for positive $\mathrm{Re}\lambda$, the function $\widetilde{Z}(x,y,\lambda)$ satisfies the equation

$$-\Delta_B\widetilde{Z} + \lambda^2 p(x,y)\widetilde{Z} = p(x,y)\psi(x,y).$$

Indeed, let $g(x,y)$ belong to $C_0^{\infty}(\mathbb{R}^{n+1})$. Then

$$\int\limits_{\mathbb{R}^{n+1}_+} y^k \Delta_B Z(x,y,t) g(x,y)dxdy = \int\limits_{\mathbb{R}^{n+1}_+} y^k Z(x,y,t)\Delta_B g(x,y)dxdy$$

and

$$0 = \int\limits_{\delta_0}^{T}\int\limits_{\mathbb{R}^{n+1}_+} y^k g(x,y)(T-t)\left[p(x,y)\frac{\partial^2 Z}{\partial t^2} - \Delta_B Z\right]dxdydt$$

$$= \int\limits_{\mathbb{R}^{n+1}_+} y^k \left(g(x,y)p(x,y)\Big[-(T-\delta_0)\psi(x,y) + Z(x,y,T)\Big] - \Delta_B g(x,y)\int\limits_{\delta_0}^{T}(T-t)Z(x,y,t)dt\right)dxdy.$$

Multiply the last identity by $e^{-(T-\delta_0)\lambda}$, where $\mathrm{Re}\lambda>0$, and integrate with respect to T from δ_0 to ∞. We obtain that

$$0 = \int\limits_{\mathbb{R}^{n+1}_+} y^k \Big[g(x,y)p(x,y)\int\limits_0^{\infty} e^{-\tau\lambda} Z(x,y,\tau+\delta_0)d\tau - g(x,y)p(x,y)\psi(x,y)\int\limits_0^{\infty}\tau e^{-\tau\lambda}d\tau$$

$$-\Delta_B g(x,y) \int_0^\infty e^{-\tau\lambda} \int_0^\tau (\tau-\theta) Z(x,y,\theta+\delta_0) d\theta d\tau\Big] dxdy$$

$$= \int_{\overline{\mathbb{R}^{n+1}_+}} y^k \left[g(x,y)p(x,y)\widetilde{Z}(x,y,\lambda) - \frac{g(x,y)p(x,y)\psi(x,y)}{\lambda^2} - \Delta_B g(x,y) \frac{\widetilde{Z}(x,y,\lambda)}{\lambda^2} \right] dxdy.$$

Since $g(x,y)$ is selected arbitrarily, it follows that the function $\widetilde{Z}(x,y,\lambda)$ belongs to $W^2_{2,k,loc}(\overline{\mathbb{R}^{n+1}_+})$ and satisfies the equation

$$-\Delta_B \widetilde{Z} + \lambda^2 p(x,y)\widetilde{Z} = p(x,y)\psi(x,y).$$

Let λ be real and positive. Consider the following problem:

$$-\Delta_B \widetilde{Z} + \lambda^2 p(x,y)\widetilde{Z} = p(x,y)\psi(x,y), \ \ x \in \mathbb{R}^n, y > 0, \tag{5.34}$$

$$\frac{\partial \widetilde{Z}}{\partial y}\Big|_{y=0} = 0. \tag{5.35}$$

The function $\widetilde{Z}(x,y,\lambda)$ satisfies problem (5.34), (5.35), it is analytic with respect to λ from $(0,+\infty)$ and bounded with respect to x from $\mathbb{R}^n$, and it has at most a power growth with respect to y at infinity. Such a solution of problem (5.34), (5.35) is unique. On the other hand, it is proved in the previous section that the function $\widetilde{w}(x,y,\lambda)$ is a solution analytic with respect to λ from $(0,+\infty)$ and bounded with respect to x from $x \in \overline{\mathbb{R}^{n+1}_+}$ of the problem

$$-\Delta_B \widetilde{w} + \lambda p(x,y)\widetilde{w} = p(x,y)\psi(x,y), \ \ x \in \mathbb{R}^n, y > 0; \tag{5.36}$$

$$\frac{\partial \widetilde{w}}{\partial y}\Big|_{y=0} = 0. \tag{5.37}$$

Such a solution of problem (5.36), (5.37) is unique as well. Hence,

$$\widetilde{w}(x,y,\lambda) = \widetilde{Z}(x,y,\sqrt{\lambda}) \tag{5.38}$$

for positive λ. The function $\widetilde{Z}(x,y,\lambda)$ is analytic for $|\arg\lambda| < \dfrac{\pi}{2}$. Therefore, the function $\widetilde{Z}(x,y,\sqrt{\lambda})$ is analytic with respect to λ for $|\arg\lambda| < \pi$. Hence, using (5.38), one can extend the function $\widetilde{w}(x,y,\lambda)$ to the domain $|\arg\lambda| < \pi$ such that the extended function is analytic with respect to λ in the specified domain and bounded with respect to x from $\mathbb{R}^n$ and it has at most a power growth with respect to y at infinity.

Lemma 5.2.3. *For any x from $\mathbb{R}^n$, any nonnegative y, and any positive σ, the limit relation*

$$|\widetilde{Z}(x,y,\lambda)| \xrightarrow{|\lambda|\to\infty} 0$$

holds uniformly with respect to $|\arg\lambda| \le \dfrac{\pi}{2}$.

Proof. We have

$$|\widetilde{Z}(x,y,\lambda)| = \left| \int_{\delta_0}^\infty e^{-(t-\delta_0)\lambda} Z(x,y,t)dt \right| \le C \int_0^\infty e^{-\tau \mathrm{Re}\lambda} (\tau+\delta_0)^{[\frac{n+k+1}{2}]+1} (\tau+\delta_0+y)^k d\tau$$

$$\le \frac{C}{|\lambda|\sin\frac{\sigma}{2}} \int_0^\infty e^{-\rho} \left(\frac{\rho}{|\lambda|\sin\frac{\sigma}{2}} + \delta_0 \right)^{[\frac{n+k+1}{2}]+1} \left(\frac{\rho}{|\lambda|\sin\frac{\sigma}{2}} + \delta_0 + y \right)^k d\rho.$$

If $|\lambda|$ is sufficiently large, then $|\lambda|\sin\dfrac{\sigma}{2} > 1$; therefore, $|\widetilde{Z}(x,y,\lambda)| \le \dfrac{\text{const}}{|\lambda|}$, where the constant does not depend on λ.

This completes the proof of Lemma 5.2.3. □

Lemma 5.2.4. *For any positive σ there exists a positive δ such that the function $\|\widetilde{w}(x,y,\lambda)\|$ is bounded for* $\lambda \in \left\{|\lambda| < \delta \middle| |\arg\lambda| \le \pi - \sigma\right\}$.

Proof. We have

$$\widetilde{w}(x,y,\lambda) = \int\limits_{\delta_0}^{\infty} e^{-(t-\delta_0)\lambda} w(x,y,t)dt = \int\limits_{\delta_0}^{\infty} e^{-(t-\delta_0)\lambda} \frac{\partial u}{\partial t}(x,y,t)dt$$

$$= \lambda \int\limits_{0}^{\infty} e^{-\lambda t} u(x,y,t+\delta_0)dt + e^{-\lambda t} u(x,y,t+\delta_0)\Big|_0^{\infty}$$

$$= \lambda e^{\lambda\delta_0}\widetilde{u}(x,y,\lambda) - u(x,y,\delta_0) - \lambda \int\limits_{0}^{\delta_0} e^{-(t-\delta_0)\lambda} u(x,y,t)dt.$$

It is proved in the previous section that for any positive σ there exists a positive δ such that $\widetilde{u}(x,y,\lambda)$ is analytic for $|\arg\lambda| \le \pi - \sigma$ and $|\lambda| < \delta$. Hence, the last relation is valid (at least) for $|\arg\lambda| \le \pi - \sigma$ and $|\lambda| < \delta$. Now, fix σ from $\left(0, \frac{\pi}{4}\right)$. It is proved in the previous section that the inequality $\|\widetilde{u}(x,y,\lambda)\| \le \frac{\text{const}}{|\lambda|}$ is valid provided that $|\arg\lambda| \le \pi - \sigma$ and $|\lambda| < \delta(\sigma)$. Therefore, the inequality $\|\lambda e^{\lambda\delta_0}\widetilde{u}(x,y,\lambda)\| \le \text{const}\, e^{\delta_0 \mathrm{Re}\lambda} \le e^{\delta\delta_0}$ is valid for $|\arg\lambda| \le \pi - \sigma$ and $|\lambda| < \delta$. The function $u(x,y,\delta_0)$ does not depend on λ and is bounded with respect to $(x,y) \in \mathbb{R}^n \times [0,+\infty)$. It remains to estimate the third term:

$$\left|e^{-(t-\delta_0)\lambda}\right| = e^{(\delta_0 - t)\mathrm{Re}\lambda} \le e^{|\lambda|\delta_0} < e^{\delta\delta_0} \text{ for } |\lambda| < \delta.$$

This implies that

$$\left|\lambda \int\limits_{0}^{\delta_0} e^{-(t-\delta_0)\lambda} u(x,y,t)dt\right| \le |\lambda|\delta_0 e^{\delta\delta_0}\|u(x,y,t)\| < \delta\delta_0 e^{\delta\delta_0}\|\varphi(x,y)\| \text{ for } |\lambda| < \delta.$$

This completes the proof of Lemma 5.2.4. □

It follows from Lemma 5.2.3 that for any x from $\mathbb{R}^n$ and any nonnegative y, the function $\widetilde{w}(x,y,\lambda)$ is bounded on $\Gamma = \left\{|\arg\lambda| \le \frac{3\pi}{4}\right\}$, and the integration contour in the inverse Laplace transformation

$$w(x,y,t+\delta_0) = \int\limits_{\mathrm{Re}\lambda = \sigma_0 > 0} e^{\lambda t}\widetilde{w}(x,y,\lambda)d\lambda$$

can be replaced by the contour Γ.

By virtue of Lemma 5.2.4, there exists a positive δ such that the function $\widetilde{w}(x,y,\lambda)$ is bounded in the domain $\{0 < |\lambda| < \delta\} \cap G$, where $G = \{|\arg\lambda| < \pi - \sigma\}$. Moreover, the function $\widetilde{w}(x,y,\lambda)$ is analytic in the domain G. Thus, either the origin is a regular point of the function $\widetilde{w}(x,y,\lambda)$ or the specified function has a removable singularity at the origin. Thus, the function $\widetilde{w}(x,y,\lambda)$ can be defined for $\lambda = 0$ to preserve its continuity; now, the function $\widetilde{w}(x,y,\lambda)$ is analytic in the domain G and continuous in its closure.

By virtue of Lemma 5.2.3, $\lim\limits_{\substack{|\lambda|\to\infty \\ \lambda\in\Gamma}} \widetilde{w}(x,y,\lambda) = 0$; therefore, the function $\widetilde{w}(x,y,\lambda)$ is bounded on Γ indeed.

Now, on the complex plane $\lambda = \lambda_1 + i\lambda_2$, consider the contour bounded by the following five segments:

the segment of the line $\lambda_2 = R$ such that $-R \le \lambda_1 \le \sigma_0$;
the segment of the line $\lambda_2 = -R$ such that $-R \le \lambda_1 \le \sigma_0$;
the segment of the line $\lambda_2 = \lambda_1$ such that $-R \le \lambda_1 \le 0$;
the segment of the line $\lambda_2 = -\lambda_1$ such that $-R \le \lambda_1 \le 0$;
the segment of the line $\lambda_1 = \sigma_0$ such that $-R \le \lambda_2 \le R$.

Denote this contour by Γ_R.

The Cauchy theorem is applicable to the integral $\int\limits_{\Gamma_R} e^{\lambda t}\widetilde{w}(x,y,\lambda)d\lambda$. Passing to the limit as $R \to \infty$, we obtain the relation

$$\int\limits_{\mathrm{Re}\lambda=\sigma_0>0} e^{\lambda t}\widetilde{w}(x,y,\lambda)d\lambda = \int\limits_{\Gamma} e^{\lambda t}\widetilde{w}(x,y,\lambda)d\lambda.$$

Fix an arbitrary x from $\mathbb{R}^n$ and an arbitrary nonnegative y. Taking into account the boundedness of the function $\widetilde{w}(x,y,\lambda)$ on Γ, we see that

$$\left|\int\limits_{\Gamma} e^{\lambda t}\widetilde{w}(x,y,\lambda)d\lambda\right| \le C\int\limits_{\Gamma} e^{t\mathrm{Re}\lambda}d\lambda = 2C\int\limits_0^\infty e^{-\frac{\rho t}{\sqrt{2}}}d\rho = \frac{\mathrm{const}}{t}.$$

Thus, $\left|\frac{\partial u}{\partial t}\right| \le \frac{\mathrm{const}}{t}$ for $t \ge \delta_0$. Since the function $v(x,y,t)$ satisfies problem (5.12), (5.2) for $p(x,y) \equiv 1$, it follows that the inequality $\left|\frac{\partial v}{\partial t}\right| \le \frac{\mathrm{const}}{t}$ is valid provided that $t \ge \delta_0$. Therefore, the relation

$$f''(t) = O\left(\frac{1}{t}\right) \text{ as } t \to \infty$$

holds for any x from $\mathbb{R}^n$ and any nonnegative y.

This completes the proof of Theorem 5.2.3.

5.2.4. Proof of the main theorem. Now, we can pass directly to the proof of Theorem 5.2.1. Fix a point (x,y) from $\overline{\mathbb{R}^{n+1}_+}$ and denote $u(x,y,t) - v(x,y,t)$ by $g(t)$. Obviously, the function $g(t)$ is bounded. Denote $g(t)+tg'(t)$ by $h(t)$. It follows from Theorem 5.2.3 that the function $h(t)$ is bounded at least for $t \ge 1$. Since we are investigating the long-time behavior of $g(t)$, it follows that, without loss of generality, one can assume that $g(t) = 0$ for $0 \le t \le 1$. Therefore, the function $h(t)$ is bounded on the positive semiaxis.

Further, it is easy to check that $g(t) = \frac{1}{t}\int\limits_0^t h(\tau)d\tau$. By virtue of Theorem 5.2.2, the limit relation $\frac{1}{t}\int\limits_0^t g(\tau)d\tau \xrightarrow{t\to\infty} 0$ is valid, i.e., $\frac{1}{t}\int\limits_0^t \frac{1}{\tau}\int\limits_0^\tau h(\rho)d\rho d\tau \xrightarrow{t\to\infty} 0$. Since the function $h(t)$ is bounded, it follows that the limit relation $\frac{1}{t}\int\limits_0^t h(\tau)d\tau \xrightarrow{t\to\infty} 0$ is valid as well (by virtue of [96, Lemma 1]); therefore, $\lim\limits_{t\to\infty} g(t) = 0$. This means that for any x from $\mathbb{R}^n$, any nonnegative y, and any real l, the limit relation $u(x,y,t) \xrightarrow{t\to\infty} l$ is valid if and only if the limit relation $v(x,y,t) \xrightarrow{t\to\infty} l$ is valid.

By virtue of Theorem 5.1.1, the last relation is equivalent to the relation

$$\lim_{r\to\infty}\frac{n+k+1}{r^{n+k+1}}\int\limits_{B_+(r)} y^k p(x,y)\varphi(x,y)dxdy = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+k+1}{2}\right)}\,l.$$

The following inequality holds:

$$\frac{1}{r^{n+k+1}}\left|\int\limits_{B_+(r)} y^k q(x,y)\varphi(x,y)dxdy\right| \le \frac{\|\varphi\|}{r^{n+k+1}}\int\limits_{B_+(r)} y^k|q(x,y)|dxdy.$$

The right-hand part tends to zero as $r\to\infty$ (because the function $q(x,y)$ satisfies condition (5.16)) and the function $\varphi(x,y)$ is bounded. This completes the proof of Theorem 5.2.1 because $p(x,y) = q(x,y)+1$.

Theorem 5.2.1 is proved completely.

Theorem 5.2.1, Theorem 5.1.5, and Theorem 5.1.6 imply the following assertions.

Theorem 5.2.4. *Let $n>0$, $\alpha\ge 0$, $\beta\ge 0$, and $\alpha\ne\beta$. Let $u_k(x,y,t)$ be the classical bounded solution of problem* (5.12), (5.2). *Let the conditions of Theorem* 5.2.1 *be satisfied for $k=\alpha$ and $k=\beta$. Then there exists a bounded function φ from $C^\infty(\overline{\mathbb{R}^{n+1}_+})$ such that for any x from $\mathbb{R}^n$ and any nonnegative y, the limit $\lim\limits_{t\to\infty} u_\alpha(x,y,t)$ exists, while the limit $\lim\limits_{t\to\infty} u_\beta(x,y,t)$ does not exist.*

Theorem 5.2.5. *Let $n=0$, $\alpha\ge 0$, $\beta\ge 0$, and $\alpha\ne\beta$. Let $u_k(y,t)$ be the classical bounded solution of problem* (5.12), (5.2). *Let the conditions of Theorem* 5.2.1 *be satisfied for $k=\alpha$ and $k=\beta$. Then for any continuous and bounded function φ and any nonnegative y, the existence of $\lim\limits_{t\to\infty} u_\alpha(y,t)$ is equivalent to the existence of $\lim\limits_{t\to\infty} u_\beta(x,y,t)$. If the specified limits exist, then $\lim\limits_{t\to\infty} u_\alpha(y,t) = \dfrac{\beta+1}{\alpha+1}\lim\limits_{t\to\infty} u_\beta(x,y,t)$.*

Remark. Consider the $(n+2)$-dimensional integral

$$\int\limits_{D_{a,0,0}} \xi^{k-1}\left|q\left(x,\sqrt{\xi^2+(\eta-d)^2}\right)\right|dxd\eta d\xi$$

depending on parameters a from $\mathbb{R}^n$ and d from $[0,+\infty)$; here $D_{a,b,c}$ denotes the $(n+2)$-dimensional semiball $\{x\in\mathbb{R}^n,\ \eta\ge 0,\ \xi\in\mathbb{R}^1\big||x-a|^2+(\eta-b)^2+(\xi-c)^2\le r^2\}$.

Change the variables: $\eta = y\cos\theta$ and $\xi = y\sin\theta$. We obtain that this integral is equal to

$$\int\limits_{D_{a,0}}\int\limits_0^\pi y^k\sin^{k-1}\theta\left|q\left(x,\sqrt{y^2\sin^2\theta+y^2\cos^2\theta+2dy\cos\theta+d^2}\right)\right|d\theta dxdy,$$

where $D_{a,b}$ denotes the $(n+1)$-dimensional semiball $\left\{x\in\mathbb{R}^n, y\ge 0\,\big||x-a|^2+(y-b)^2\le r^2\right\}$. The last integral is equal to

$$\frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}\int\limits_{D_{a,0}} y^k T_y^d|q(x,y)|dxdy = \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}\int\limits_{B_+(r)} \eta^k T_\eta^y|q(\xi+a,\eta)|d\xi d\eta.$$

Thus, condition (5.15) is equivalent to the following condition: there exists a constant b such that

$$\lim_{r\to\infty}\frac{1}{r^{n+k+1}}\int\limits_{D_{x,-y,0}} \rho^{k-1}\left|p\left(\xi,\sqrt{\eta^2+\rho^2}\right)-b\right|d\xi d\eta d\rho = 0$$

uniformly with respect to $(x,y)\in\overline{\mathbb{R}^{n+1}}$.

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