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RADIATIVE TRANSFER EQUATION WITH DIFFUSE REFLECTION AND REFRACTION CONDITIONS IN A SYSTEM OF BODIES WITH PIECEWISE SMOOTH BOUNDARIES

A. A. Amosov
National Research University "Moscow Power Engineering Institute"
14, Krasnokazarmennaya St., Moscow 111250, Russia
AmosovAA@mpei.ru

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We consider the boundary value problem for the radiative transfer equation with diffuse reflection and refraction conditions in a system of semitransparent bodies with piecewise smooth boundaries. For the problem with data in the complete scale of Lebesgue spaces we establish the existence and uniqueness of a solution. We obtain a priori estimates for the solution and show the continuous dependence of the solution on the data. The conjugate problem is also studied. Bibliography: 8 titles.

The boundary value problem with diffuse reflection and refraction conditions describing the monochromatic radiation transfer in a system $G = \bigcup_{j=1}^{m} G_j$ of semitransparent bodies G_j separated by the vacuum was studied by the author in [1, 2] under rather restrictive assumptions. First, it is assumed that every body G_j has smooth boundary ∂G_j of class $C^{1+\lambda}$, $0 < \lambda < 1$. However, in practice, it often happens that bodies have only piecewise smooth boundaries. Second, the majority of the obtained results are valid for data in the Lebesgue spaces with exponents $p \in (1 + (2\lambda)^{-1}, \infty]$. Thereby the obtained results do not cover the important case of data with finite energy, i.e., data in the Lebesgue space with exponent $p = 1$.

In this paper, we show that the results of [1, 2] are extended to the case of a system of bodies with Lipschitz piecewise smooth boundary and remain valid for the problems with data in the complete scale of Lebesgue spaces with exponents $p \in [1, \infty]$.

Assume that every body G_j of the system G is a domain in $\mathbb{R}^3$ with Lipschitz piecewise smooth boundary (in Subsection 1.1, we explain how to understand the piecewise smoothness of boundary). We also assume that domains G_i and G_j are pairwise disjoint, but their boundaries can intersect for some $i \neq j$.

Let $\Omega = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$ be the unit sphere in $\mathbb{R}^3$ (the sphere of directions).

The sought function $I(\omega, x)$ is defined on the set $D = \Omega \times G$ and is interpreted as the

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radiation intensity at the point $x \in G$ when the radiation propagates in direction $\omega \in \Omega$.

Assume that each body G_j is occupied by a semitransparent medium with constant absorption $\varkappa_j > 0$, scattering coefficient $s_j \geqslant 0$, and refraction exponent $k_j > 1$. We set $\varkappa(x) = \varkappa_j$, $s(x) = s_j$, and $k(x) = k_j$ for $x \in G_j$, $1 \leqslant j \leqslant m$.

To describe the radiation propagation in G, we use the radiative transfer equation

$$\omega \cdot \nabla I + \beta I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D,$$

where

$$\omega \cdot \nabla I = \sum_{i=1}^{3} \omega_i \frac{\partial}{\partial x_i} I$$

denotes the derivative of I along the direction ω. We denote by $\mathscr{S}$ the scattering operator

$$\mathscr{S}(I)(\omega, x) = \frac{1}{4\pi} \int\limits_{\Omega} \theta_j(\omega' \cdot \omega) I(\omega', x)\, d\omega', \quad (\omega, x) \in D_j = \Omega \times G_j, \quad 1 \leqslant j \leqslant m,$$

with the scattering indicatrix possessing the following properties:

$$\theta_j \in L^1(-1,1), \quad \theta_j \geqslant 0, \quad \frac{1}{2}\int\limits_{-1}^{1} \theta_j(\mu)\, d\mu = 1, \quad 1 \leqslant j \leqslant m.$$

Furthermore, $\beta(x) = \varkappa(x) + s(x)$ is the extinction coefficient and $F(\omega, x)$ characterizes the density of radiation of volume sources.

The paper is organized as follows. In Section 1, we introduce the notation and prove a number of auxiliary assertions of geometric character. In Section 2, we introduce the function spaces and study their properties. In Section 3, we formulate the boundary value problem under consideration and study its properties.

Basically, the logic of reasoning follows the logic of the papers [1, 2]. Some assertions in this paper are counterparts of the corresponding assertions in [1]–[3] and can be proved in a similar way. In such cases, we restrict ourselves to mention the corresponding references without proof in order to avoid repetitions. However, if the proof should be essentially modified or the assertion has no analogs, we provide a detailed proof.

1 Auxiliaries

1.1. Notation and assumptions on the boundaries ∂G_j

We consider $\mathbb{R}^3$ as an Euclidean space of elements $x = (x_1, x_2, x_3)$ equipped with the inner product $x \cdot y = \sum\limits_{i=1}^{3} x_i y_i$. We denote by $]x, y[$ the interval joining the points $x, y \in \mathbb{R}^3$, $x \neq y$: $]x, y[= \{\, \alpha x + (1-\alpha) y \mid 0 < \alpha < 1\}$. We denote by $B_r(x_0)$ an open ball in $\mathbb{R}^3$ with radius r and center $x_0 \in \mathbb{R}^3$. Let V_r and $\overline{V}_r$ be an open and closed disks in $\mathbb{R}^2$ with radius r centered at the origin. Assume that $\omega_0 \in \Omega$ and $x_0 \in \mathbb{R}^3$. In $\mathbb{R}^3$, we introduce the coordinates with center x_0 and the orthonormal basis $\mathbf{e}_1(\omega_0)$, $\mathbf{e}_2(\omega_0)$, $\mathbf{e}_3(\omega_0) = \omega_0$.

We consider the plane $\pi_{\omega_0} = \{\mathbf{y} \in \mathbb{R}^3 \mid \omega_0 \cdot \mathbf{y} = 0\}$ passing through the origin and having the normal vector ω_0. We note that the pair $\mathbf{e}_1(\omega_0)$, $\mathbf{e}_2(\omega_0)$ forms an orthonormal basis in π_{ω_0}.

We denote by P_{ω_0} the operator of orthogonal projection onto the plane π_{ω_0}. Denote by $\operatorname{meas}_2 E$ the plane measure of a measurable set $E \subset \pi_{\omega_0}$. We introduce the notation by

$$\begin{aligned}
\ell(\omega_0, x_0) &= \{x = x_0 + t\,\omega_0 \mid t \in \mathbb{R}\},\\
\ell^-(\omega_0, x_0) &= \{x = x_0 - t\,\omega_0 \mid t > 0\},\\
\ell^+(\omega_0, x_0) &= \{x = x_0 + t\,\omega_0 \mid t > 0\}
\end{aligned}$$

for the line with directed vector ω_0 passing through the point x_0 and for the corresponding rays outgoing from the point x_0. We set

$$\begin{aligned}
V_{r,\omega_0} &= \{\mathbf{y} \in \pi_{\omega_0} \mid |\mathbf{y}| < r\},\\
\overline{V}_{r,\omega_0} &= \{\mathbf{y} \in \pi_{\omega_0} \mid |\mathbf{y}| \leqslant r\}.
\end{aligned}$$

Let γ be a function defined on $\overline{V}_{r,\omega_0}$, and let $\widetilde{\gamma}(y') = \gamma(\mathbf{y})$, where $y' = (y_1, y_2)$, $\mathbf{y} = y_1\mathbf{e}_1(\omega_0) + y_2\mathbf{e}_2(\omega_0)$. We write $\gamma \in C^k(\overline{V}_{r,\omega_0})$ if $\widetilde{\gamma} \in C^k(\overline{V}_r)$, $k = 0, 1$. Similarly, we write $\gamma \in \operatorname{Lip}(\overline{V}_{r,\omega_0})$ if $\widetilde{\gamma} \in \operatorname{Lip}(\overline{V}_r)$, i.e., if $\widetilde{\gamma}$ is defined on $\overline{V}_r$ and satisfies the Lipschitz condition with some constant L.

Let $\alpha, \beta, \gamma \in C^0(\overline{V}_{r,\omega_0})$, $\alpha < \beta$. We introduce the surface

$$\Pi_{r,\gamma}(\omega_0, x_0) = \{x = x_0 + \mathbf{y} + \gamma(\mathbf{y})\omega_0 \mid \mathbf{y} \in V_{r,\omega_0}\}$$

and the curvilinear cylinder

$$C_{r,\alpha,\beta}(\omega_0, x_0) = \{x = x_0 + \mathbf{y} + t\,\omega_0 \mid \mathbf{y} \in V_{r,\omega_0},\ \alpha(\mathbf{y}) < t < \beta(\mathbf{y})\}$$

with the axis $\ell(\omega_0, x_0)$ and two lateral surfaces $\Pi_{r,\alpha}(\omega_0, x_0)$ and $\Pi_{r,\beta}(\omega_0, x_0)$. Assume that the domains G_j for all $1 \leqslant j \leqslant m$ are bounded and Lipschitz. The latter means that for each point $x_0 \in \partial G_j$ there exist a direction $\omega_0 \in \Omega$, numbers $r_0 > 0$, $h_0 > 0$ and a function $\gamma \in \operatorname{Lip}(\overline{V}_{r,\omega_0})$, $-h_0 < \gamma < h_0$, such that

$$G_j \cap C_{r_0,-h_0,h_0}(\omega_0, x_0) = C_{r_0,-h_0,\gamma}(\omega_0, x_0), \tag{1.1}$$

$$(\mathbb{R}^3 \setminus \overline{G}_j) \cap C_{r_0,-h_0,h_0}(\omega_0, x_0) = C_{r_0,\gamma,h_0}(\omega_0, x_0), \tag{1.2}$$

$$\partial G_j \cap C_{r_0,-h_0,h_0}(\omega_0, x_0) = \Pi_{r_0,\gamma}(\omega_0, x_0). \tag{1.3}$$

We denote by $d\omega$ and $d\sigma(x)$ the measures induced by the Lebesgue measure in $\mathbb{R}^3$ on Ω and ∂G respectively.

Remark 1.1. Since $\widetilde{\gamma}$ satisfies the Lipschitz condition with constant L, it is differentiable almost everywhere on V_{r_0}; moreover, the gradient of this function

$$\nabla'\widetilde{\gamma}(y') = \left(\frac{\partial\widetilde{\gamma}}{\partial y_1}(y'), \frac{\partial\widetilde{\gamma}}{\partial y_2}(y')\right)$$

satisfies the inequality $|\nabla\widetilde{\gamma}| \leqslant L$. Consequently, the outward normal $n_j(x)$ to the surface ∂G_j exists for almost all $x \in \Pi_{r_0,\gamma}(\omega_0, x_0)$, which can be written in the form

$$n_j(x) = \frac{1}{\sqrt{1 + |\nabla'\widetilde{\gamma}(y')|^2}}\left(-\frac{\partial\widetilde{\gamma}}{\partial y_1}(y')\,\mathbf{e}_1(\omega_0) - \frac{\partial\widetilde{\gamma}}{\partial y_2}(y')\mathbf{e}_2(\omega_0) + \omega_0\right). \tag{1.4}$$

In addition, we assume that for every $1 \leqslant j \leqslant m$ the boundary ∂G_j is piecewise smooth in the following sense. There exists a closed set $\mathscr{G}_j \subset \partial G_j$ such that $\operatorname{meas}(\mathscr{G}_j; d\sigma) = 0$; moreover, the outward normal exists for each point $x_0 \in \partial' G_j = \partial G_j \setminus \mathscr{G}_j$ and there are numbers $r_0 > 0$, $h_0 > 0$ and a function $\gamma \in C^1(\overline{V}_{r_0,\omega_0})$, $-h_0 < \gamma < h_0$ such that for $\omega_0 = n_j(x_0)$ the properties (1.1)–(1.3) hold; moreover, $\widetilde{\gamma}(\mathbf{0}) = 0$ and $|\nabla' \widetilde{\gamma}(\mathbf{0})| = 0$. It is clear that $\Pi_{r_0,\gamma}(n_j(x_0), x_0) \subset \partial' G_j$ and the outward normal is continuous on $\partial' G_j$.

Thus, by assumption, ∂G_j consists of two parts: the "smooth" part $\partial' G_j$, where the outward normal exists and is continuous, and the set $\mathscr{G}_j$, where the boundary smoothness fails. We recall that the set $\mathscr{G}_j$ is closed and has measure zero. Hence the set $\partial' G_j$ is open (in the topology of ∂G_j) and $\operatorname{meas}(\partial' G_j; d\sigma) = \operatorname{meas}(\partial G_j; d\sigma)$. Naturally, the case $\partial G_j \in C^1$ is not excluded. Then $\mathscr{G}_j = \varnothing$ and $\partial' G_j = \partial G_j$.

Remark 1.2. In this paper we do not require the condition of generalized convexity of the set G, which is often used in the mathematical theory of the radiative transfer equation [4, 5].

1.2. The sets $\Gamma_j^{\pm}$, $\Gamma^{\pm}$, Γ_j^0, Γ^0 and their properties

We set

$$\partial G = \bigcup_{j=1}^{m} \partial G_j, \quad \Gamma = \Omega \times \partial G = \bigcup_{j=1}^{m} \Gamma_j, \quad \Gamma_j = \Omega \times \partial G_j, \quad 1 \leqslant j \leqslant m,$$

$$\Gamma^- = \bigcup_{j=1}^{m} \Gamma_j^-, \quad \Gamma_j^- = \{(\omega, x) \in \Omega \times \partial' G_j \mid \omega \cdot n_j(x) < 0\}, \quad 1 \leqslant j \leqslant m,$$

$$\Gamma^+ = \bigcup_{j=1}^{m} \Gamma_j^+, \quad \Gamma_j^+ = \{(\omega, x) \in \Omega \times \partial' G_j \mid \omega \cdot n_j(x) > 0\}, \quad 1 \leqslant j \leqslant m,$$

$$\Gamma^0 = \bigcup_{j=1}^{m} \Gamma_j^0, \quad \Gamma_j^0 = \{(\omega, x) \in \Omega \times \partial' G_j \mid \omega \cdot n_j(x) = 0\}, \quad 1 \leqslant j \leqslant m.$$

We note that $\Gamma_j^{\pm}$ and $\Gamma^{\pm}$ are open sets (in the topology of set Γ). It is easy to see that $(\omega, x) \in \Gamma_j^-$ if and only if $(-\omega, x) \in \Gamma_j^+$. On Γ, we introduce the measure $d\Gamma(\omega, x) = d\omega\, d\sigma(x)$. On Γ^- and Γ^+, we introduce the measures

$$\widehat{d}\Gamma^-(\omega, x) = |\omega \cdot n_j(x)|\, d\omega d\sigma(x), \quad (\omega, x) \in \Gamma_j^-, \quad 1 \leqslant j \leqslant m,$$
$$\widehat{d}\Gamma^+(\omega, x) = \omega \cdot n_j(x)\, d\omega d\sigma(x), \quad (\omega, x) \in \Gamma_j^+, \quad 1 \leqslant j \leqslant m.$$

We emphasize that a set $E^{\pm} \subset \Gamma^{\pm}$ is measurable with respect to the measure $\widehat{d}\Gamma^{\pm}$ if and only if it is measurable with respect to the measure $d\Gamma$. We also note that $\operatorname{meas}(E^{\pm}; \widehat{d}\Gamma^{\pm}) = 0$ if and only if $\operatorname{meas}(E^{\pm}; d\Gamma) = 0$.

Assume that $E \subset \Gamma$ and $\omega \in \Omega$. We set $E(\omega) = \{x \in \partial G \mid (\omega, x) \in E\}$. Thus, for example,

$$\Gamma_j^-(\omega) = \{x \in \partial' G_j \mid \omega \cdot n_j(x) < 0\},$$
$$\Gamma_j^+(\omega) = \{x \in \partial' G_j \mid \omega \cdot n_j(x) > 0\}.$$

Lemma 1.1. *Assume that $x_0 \in \partial' G_j$ and $\varepsilon \in (0,1)$. Then there exists $r_1 = r_1(x_0, \varepsilon) > 0$ such that for all $\omega \in \Omega$, $\omega \cdot n_j(x_0) \geqslant \varepsilon$, and all $r \in (0, r_1]$, $h = 4r/\varepsilon$,*

$$G_j \cap C_{r,-h,h}(\omega, x_0) = C_{r,-h,\gamma_\omega}(\omega, x_0),$$
$$(\mathbb{R}^3 \setminus \overline{G}_j) \cap C_{r,-h,h}(\omega, x_0) = C_{r,\gamma_\omega,h}(\omega, x_0),$$
$$\partial G_j \cap C_{r,-h,h}(\omega, x_0) = \Pi_{r,\gamma_\omega}(\omega, x_0) \subset \Gamma_j^+(\omega),$$
$$\Pi_{r,-h}(\omega, x_0) \subset G_j, \quad \Pi_{r,h}(\omega, x_0) \subset \mathbb{R}^3 \setminus \overline{G}_j,$$

where $\gamma_\omega \in C^1(\overline{V}_{r_1,\omega})$; moreover, $|\gamma_\omega(\mathbf{y})| \leqslant 2|\mathbf{y}|/\varepsilon$ for all $\mathbf{y} \in \overline{V}_{r_1,\omega}$.

The proof of this lemma repeats the proof of Lemma 1.1 in [3] with the only difference that $x_0 \in \partial G_j$ in [3].

Remark 1.3. By Lemma 1.1, for $(\omega, x_0) \in \Gamma_j^+$ the following property holds: the line $\ell(\omega, x_0)$ comes out from G_j to the vacuum. (i.e., to the set $\mathbb{R}^3 \setminus \overline{G}_j$), intersecting ∂G_j at the point x_0. In other words, there exists $\delta = \delta(\omega, x_0) > 0$ such that $]x_0, x_0 - \delta\omega[\subset G_j$ and $]x_0, x_0 + \delta\omega[\subset \mathbb{R}^3 \setminus \overline{G}_j$.

Similarly, for $(\omega, x_0) \in \Gamma_j^-$ the following property holds: the line $\ell(\omega, x_0)$ comes in G_j from the vacuum, intersecting ∂G_j at the point x_0. In other words, there exists $\delta = \delta(\omega, x_0) > 0$ such that $]x_0, x_0 + \delta\omega[\subset G_j$ and $]x_0, x_0 - \delta\omega[\subset \mathbb{R}^3 \setminus \overline{G}_j$.

We note that the sets Γ_j^+ and Γ_j^- can be represented as countable unions

$$\Gamma_j^+ = \bigcup_{\ell=1}^{\infty} K_{j,\ell}^+, \quad \Gamma_j^- = \bigcup_{\ell=1}^{\infty} K_{j,\ell}^- \tag{1.5}$$

of sequences of expanding compact sets. If $\mathscr{G}_j = \varnothing$, then

$$K_{j,\ell}^+ = \{(\omega, x) \in \Gamma_j^+ \mid \omega \cdot n_j(x) \geqslant 1/\ell\}, \tag{1.6}$$
$$K_{j,\ell}^- = \{(\omega, x) \in \Gamma_j^- \mid \omega \cdot n_j(x) \leqslant -1/\ell\}, \tag{1.7}$$

and if $\mathscr{G}_j \neq \varnothing$, then

$$K_{j,\ell}^+ = \{(\omega, x) \in \Gamma_j^+ \mid \omega \cdot n_j(x) \geqslant 1/\ell, \ \operatorname{dist}(x, \mathscr{G}_j) \geqslant 1/\ell\}, \tag{1.8}$$
$$K_{j,\ell}^- = \{(\omega, x) \in \Gamma_j^- \mid \omega \cdot n_j(x) \leqslant -1/\ell, \ \operatorname{dist}(x, \mathscr{G}_j) \geqslant 1/\ell\}. \tag{1.9}$$

Lemma 1.2. *Let E be a measurable subset of $\Gamma_j^\pm$. If $\operatorname{meas}_2(P_\omega E(\omega)) = 0$ for almost all $\omega \in \Omega$, then $\operatorname{meas}(E; d\Gamma) = 0$.*

The proof of this lemma repeats that of Lemma 1.2 in [3] and is based on the representations (1.5). The only difference is that $K_{j,\ell}^\pm$ are not necessarily of the form (1.6), (1.7), but can have the form (1.8), (1.9).

Lemma 1.3. $\operatorname{meas}_2(P_\omega \Gamma_j^0(\omega)) = 0$ *for all $\omega \in \Omega$ and $1 \leqslant j \leqslant m$.*

Proof. In the case $\mathscr{G}_j = \varnothing$, this assertion is established in [3, Lemma 1.3].

Let $\mathscr{G}_j \neq \varnothing$. We represent Γ_j^0 as the countable union of expanding compact sets $K_{j,\ell}^0 = \{(\omega, x) \in \Gamma_j^0 \mid \operatorname{dist}(x, \mathscr{G}_j) \geqslant 1/\ell\}$. Then the set $\Gamma_j^0(\omega)$ is represented as the countable union

$\Gamma_j^0(\omega) = \bigcup_{\ell=1}^{\infty} K_{j,\ell}^0(\omega)$ of sequences of expanding compact sets $K_{j,\ell}^0(\omega) = \{x \in \partial' G_j \mid \omega \cdot n_j(x) = 0,\ \operatorname{dist}(x, \mathscr{G}_j) \geqslant 1/\ell\}$. Therefore, from the open (in the topology of the set ∂G_j) covering $\{\Pi_{r_0,\gamma}(n_j(x_0), x_0)\}_{x_0 \in \Gamma_j^0(\omega)}$ of the set $\Gamma_j^0(\omega)$ by surfaces $\Pi_{r_0,\gamma}(n_j(x_0), x_0) \subset \partial' G_j$ one can extract a countable subcovering $\{\Pi_{r_k,\gamma_k}(n_j(x_k), x_k)\}_{k=1}^{\infty}$. Arguing as in [3, Lemma 1.3], we verify that $\operatorname{meas}_2 P_\omega(\Gamma_j^0(\omega) \cap \Pi_{r_k,\gamma_k}(n_j(x_k), x_k)) = 0$ for all $k \geqslant 1$. Hence $\operatorname{meas}_2(P_\omega \Gamma_j^0(\omega)) = 0$. □

Lemma 1.4. $\operatorname{meas}_2(P_\omega \mathscr{G}_j) = 0$ *for all* $\omega \in \Omega$ *and* $1 \leqslant j \leqslant m$.

Proof. Since the boundary ∂G_j is closed and Lipschitz, there exists a covering of ∂G_j consisting of Lipschitz surfaces $\Pi_{r_i,\gamma_i}(\omega_i, x_i) \subset \partial G_j$ such that each surface is uniquely projected onto the corresponding plane π_{ω_i}. To prove the lemma, it suffices to show that $\operatorname{meas}_2(P_\omega \mathscr{G}_{ji}) = 0$ for all $\omega \in \Omega$ provided that the set $\mathscr{G}_{ji} = \mathscr{G}_j \cap \Pi_{r_i,\gamma_i}(\omega_i, x_i)$ is not empty. Since $\operatorname{meas}(\mathscr{G}_j; d\sigma) = 0$, we have $\operatorname{meas}_2(P_{\omega_i} \mathscr{G}_{ji}) \leqslant \operatorname{meas}(\mathscr{G}_{ji}; d\sigma) = 0$. Therefore, for any $\varepsilon > 0$ there exists an open set $O_\varepsilon \subset V_{r_i,\omega_i}$ such that $P_{\omega_i} \mathscr{G}_{ji} \subset O_\varepsilon$ and $\operatorname{meas}_2 O_\varepsilon < \varepsilon$. We cover each point $x \in \mathscr{G}_{ji}$ by a ball $B_{r(x)}(x)$ of a sufficiently small radius so that $P_{\omega_i} B_{r(x)}(x) \subset O_\varepsilon$. By the $5r$-covering theorem [6], from the obtained covering of the set $\mathscr{G}_{ji}$ one can extract at most countable system of pairwise disjoint balls $\{B_{r_k}(x_k)\}$ with $r_k = r(x_k)$ such that $\mathscr{G}_{ji} \subset \bigcup_k B_{5r_k}(x_k)$. It is clear that

$$\sum_k \operatorname{meas}_2(P_{\omega_i} B_{r_k}(x_k)) = \sum_k \pi r_k^2 \leqslant \operatorname{meas}_2 O_\varepsilon < \varepsilon.$$

As a consequence, for every $\omega \neq \omega_i$

$$\operatorname{meas}_2^*(P_\omega \mathscr{G}_{ji}) \leqslant \sum_k \operatorname{meas}_2(P_\omega B_{5r_k}(x_k)) = \sum_k 25\pi r_k^2 < 25\varepsilon.$$

Since the obtained inequality holds for all $\varepsilon > 0$, we have $\operatorname{meas}_2(P_\omega \mathscr{G}_{ji}) = 0$. □

1.3. The sets $\widehat{S}_j^{\pm}$, $\widehat{S}^{\pm}$ and their properties

We set

$$\widehat{\tau}^+(\omega, x) = \sup\{t > 0 \mid x + s\,\omega \in G_j\ \forall s \in (0, t)\},$$
$$\widehat{X}^+(\omega, x) = x + \widehat{\tau}^+(\omega, x)\omega$$

for $(\omega, x) \in D_j \cup \Gamma_j^-$ and

$$\begin{aligned} &\widehat{\tau}^-(\omega, x) = \sup\{t > 0 \mid x - s\,\omega \in G_j\ \forall s \in (0, t)\}, \\ &\widehat{X}^-(\omega, x) = x - \widehat{\tau}^-(\omega, x)\omega \end{aligned} \tag{1.10}$$

for $(\omega, x) \in D_j \cup \Gamma_j^+$. We note that $\widehat{X}^\pm(\omega, x) \in \partial G_j$; moreover, $(\omega, \widehat{X}^\pm(\omega, x)) \in \Gamma_j^\pm \cup \Gamma_j^0 \cup (\Omega \times \mathscr{G}_j)$. We introduce the sets

$$\widehat{S}_j^- = \{(\omega, x) \in \Gamma_j^- \mid (\omega, \widehat{X}^+(\omega, x)) \in \Gamma_j^+\},$$
$$\widehat{S}_j^+ = \{(\omega, x) \in \Gamma_j^+ \mid (\omega, \widehat{X}^-(\omega, x)) \in \Gamma_j^-\}.$$

Lemma 1.5. *The sets* $\widehat{S}_j^-$ *and* $\widehat{S}_j^+$ *are open* (*in the topology of the set* Γ_j), *and the mapping* $(\omega, x) \to (\omega, \widehat{X}^+(\omega, x))$ *is a homeomorphism from* $\widehat{S}_j^-$ *onto* $\widehat{S}_j^+$ *with the inverse* $(\omega, x) \to (\omega, \widehat{X}^-(\omega, x))$.

Lemma 1.6. *A set* $\widehat{E}^{-} \subset \widehat{S}_j^{-}$ *is measurable with respect to the measure* $\widehat{d\Gamma}^{-}$ *if and only if its image* $\widehat{E}^{+} \subset \widehat{S}_j^{+}$ *under the mapping* $(\omega, x) \to (\omega, \widehat{X}^{+}(\omega, x))$ *is measurable with respect to the measure* $\widehat{d\Gamma}^{+}$; *moreover,* $\operatorname{meas}(\widehat{E}^{+}; \widehat{d\Gamma}^{+}) = \operatorname{meas}(\widehat{E}^{-}; \widehat{d\Gamma}^{-})$.

The proof of Lemmas 1.5 and 1.6 repeats that of Lemmas 1.4 and 1.5 in [3].

Lemma 1.7. $\operatorname{meas}(\Gamma_j^{\pm} \setminus \widehat{S}_j^{\pm},\ \widehat{d\Gamma}^{\pm}) = 0$.

Proof. Assume that $\omega \in \Omega$ and $x \in \Gamma_j^{\pm}(\omega) \setminus \widehat{S}_j^{\pm}(\omega)$. Then $\widehat{X}^{\pm}(\omega, x) \in \Gamma_j^0(\omega) \cup \mathscr{G}_j$ and, consequently, $P_\omega x \in P_\omega(\Gamma_j^0(\omega) \cup \mathscr{G}_j)$. By Lemmas 1.3 and 1.4,

$$\operatorname{meas}_2 P_\omega(\Gamma_j^{\pm}(\omega) \setminus \widehat{S}_j^{\pm}(\omega)) \leqslant \operatorname{meas}_2 P_\omega(\Gamma^0(\omega) \cup \mathscr{G}) = 0.$$

To complete the proof, it remains to apply Lemma 1.2. □

1.4. The sets $S_j^{\pm}$, $S^{\pm}$, $\overset{*}{S}{}_j^{\pm}$, $\overset{*}{S}{}^{\pm}$ and their properties

We recall that ∂G_i and ∂G_j can intersect for some $i \neq j$. We set

$$\partial G_{ij} = \partial G_i \cap \partial G_j, \quad \Sigma = \bigcup_{j=1}^{m} \Sigma_j, \quad \Sigma_j = \bigcup_{i=1, i \neq j}^{m} \partial G_{ij}, \quad 1 \leqslant j \leqslant m.$$

We introduce the sets

$$S^{\pm} = \bigcup_{j=1}^{m} S_j^{\pm}, \quad S_j^{\pm} = \{(\omega, x) \in \Gamma_j^{\pm} \mid x \in \partial' G_j \setminus \Sigma_j\}, \quad 1 \leqslant j \leqslant m,$$

$$\overset{*}{S}{}^{\pm} = \bigcup_{j=1}^{m} \overset{*}{S}{}_j^{\pm}, \quad \overset{*}{S}{}_j^{\pm} = \{(\omega, x) \in S_j^{\pm} \mid \ell^{\pm}(\omega, x) \cap \overline{G} = \varnothing\}, \quad 1 \leqslant j \leqslant m.$$

It is clear that the sets $S_j^{\pm}$ and $S^{\pm}$ are open (in the topology of the set Γ). We note that the set $\overset{*}{S}{}^{\pm}$ consists of $(\omega, x) \in S^{\pm}$ such that the ray $\ell^{\pm}(\omega, x)$ does not intersect $\overline{G}$.

Lemma 1.8. *The sets* $\overset{*}{S}{}_j^{\pm}$ *are open* (*in the topology of the set* Γ). *As a consequence, the sets* $\overset{*}{S}{}^{\pm}$ *are open.*

Proof. Let $(\omega_0, x_0) \in \overset{*}{S}{}_j^{+}$. From Lemma 1.1 and the fact that S_j^{+} is open it follows that there exist $\varepsilon > 0$ and $h > 0$ such that for $(\omega, x) \in \Gamma_j$ the inequalities $|\omega - \omega_0| < \varepsilon$ and $|x - x_0| < \varepsilon$ imply $(\omega, x) \in S_j^{+}$ and $x + t\omega \notin \overline{G}$ for all $t \in (0, h)$.

We show that there exists $\varepsilon_1 \in (0, \varepsilon]$ such that for $(\omega, x) \in \Gamma_j^{+}$ the inequalities $|\omega - \omega_0| < \varepsilon_1$ and $|x - x_0| < \varepsilon_1$ imply $(\omega, x) \in \overset{*}{S}{}_j^{+}$. Assume the contrary. Then for every $k \geqslant 1$ there exist $(\omega_k, x_k) \in S_j^{+}$ and $t_k \geqslant h$ such that $|\omega_k - \omega_0| < \varepsilon/k$, $|x_k - x_0| < \varepsilon/k$, and $x_k + t_k\omega_k \in \overline{G}$. The boundedness of the set G implies the boundedness of the sequence $\{t_k\}_{k=1}^{\infty}$. Therefore, there exists a converging subsequence $\{t_{k_s}\}_{s=1}^{\infty}$ such that $\lim\limits_{s \to \infty} t_{k_s} = t_0 \geqslant h$. As a consequence, $x_0 + t_0\omega_0 \in \overline{G}$, which is impossible since $(\omega_0, x_0) \in \overset{*}{S}{}_j^{+}$. Thus, the set $\overset{*}{S}{}_j^{+}$ is open. Since $\overset{*}{S}{}_j^{-} = \{(\omega, x) \in S_j^{-} \mid (-\omega, x) \in \overset{*}{S}{}_j^{+}\}$, the set $\overset{*}{S}{}_j^{-}$ is also open. □

Lemma 1.9. $\operatorname{meas}(\overset{*}{S}{}^+; \widehat{d\Gamma}{}^+) = \operatorname{meas}(\overset{*}{S}{}^-; \widehat{d\Gamma}{}^-) > 0.$

Proof. We fix $\omega \in \Omega$ and consider the orthogonal projection of the set G onto the plane π_ω. It is clear that $\operatorname{meas}_2 P_\omega G > 0$. At the same time, $\operatorname{meas}_2(P_\omega \Gamma^0(\omega)) = 0$ and $\operatorname{meas}_2(P_\omega \mathscr{G}) = 0$ in view of Lemmas 1.3 and 1.4. Consequently, there exists a point $x_0 \in G$ such that $P_\omega x_0 \notin P_\omega(\Gamma^0(\omega) \cup \mathscr{G})$. We set $t_0 = \sup\{t > 0 \mid x_0 + t\omega \in \overline{G}\}$. It is clear that $x = x_0 + t_0\omega \in \partial G$ and $\ell^+(\omega, x) \cap \overline{G} = \varnothing$. Since $P_\omega x = P_\omega x_0 \notin P_\omega(\Gamma^0(\omega) \cup \mathscr{G})$, we have $(\omega, x) \in \overset{*}{S}{}^+$. By Lemma 1.8, the set $\overset{*}{S}{}^+$ is open. Therefore, $\operatorname{meas}(\overset{*}{S}{}^+; \widehat{d\Gamma}{}^+) > 0$. Since $\overset{*}{S}{}^- = \{(\omega, x) \in \Gamma^- \mid (-\omega, x) \in \overset{*}{S}{}^+\}$, we have $\operatorname{meas}(\overset{*}{S}{}^-; \widehat{d\Gamma}{}^-) = \operatorname{meas}(\overset{*}{S}{}^+; \widehat{d\Gamma}{}^+) > 0$. □

1.5. The sets $\widetilde{S}^\pm$ and their properties

Let $(\omega, x) \in S^+ \setminus \overset{*}{S}{}^+$. Then the ray $\ell^+(\omega, x)$ intersects $\overline{G}$. We set

$$\tau^+(\omega, x) = \inf\{t > 0 \mid x + t\omega \in \overline{G}\},$$
$$X^+(\omega, x) = x + \tau^+(\omega, x)\omega.$$

It is clear that $\tau^+(\omega, x) > 0$, $X^+(\omega, x) \in \partial G$, and $(\omega, X^+(\omega, x)) \in \Gamma^- \cup \Gamma^0 \cap (\Omega \times \mathscr{G})$. We set

$$\widetilde{S}^+ = \{(\omega, x) \in S^+ \setminus \overset{*}{S}{}^+ \mid (\omega, X^+(\omega, x)) \in \Gamma^-\}.$$

Similarly, let $(\omega, x) \in S^- \setminus \overset{*}{S}{}^-$. Then the ray $\ell^-(\omega, x)$ intersects $\overline{G}$. We set

$$\tau^-(\omega, x) = \inf\{t > 0 \mid x - t\omega \in \overline{G}\},$$
$$X^-(\omega, x) = x - \tau^-(\omega, x)\omega.$$

It is clear that $\tau^-(\omega, x) > 0$, $X^-(\omega, x) \in \partial G$, and $(\omega, X^-(\omega, x)) \in \Gamma^+ \cup \Gamma^0 \cup (\Omega \times \mathscr{G})$. We set

$$\widetilde{S}^- = \{(\omega, x) \in S^- \setminus \overset{*}{S}{}^- \mid (\omega, X^-(\omega, x)) \in \Gamma^+\}.$$

Lemma 1.10. *The sets $\widetilde{S}^+$ and $\widetilde{S}^-$ are open (in the topology of the set Γ), and the mapping $(\omega, x) \to (\omega, X^+(\omega, x))$ is a homeomorphism from $\widetilde{S}^+$ onto $\widetilde{S}^-$ with the inverse $(\omega, x) \to (\omega, X^-(\omega, x))$.*

Lemma 1.11. *A set $E^+ \subset \widetilde{S}^+$ is measurable with respect to the measure $\widehat{d\Gamma}{}^+$ if and only if its image $E^- \subset \widetilde{S}^-$ under the mapping $(\omega, x) \to (\omega, X^+(\omega, x))$ is measurable with respect to the measure $\widehat{d\Gamma}{}^-$; moreover,* $\operatorname{meas}(E^+; \widehat{d\Gamma}{}^+) = \operatorname{meas}(E^-; \widehat{d\Gamma}{}^-)$.

The proof of Lemmas 1.10 and 1.11 repeats the proof of Lemmas 1.7 and 1.8 in [3].

Lemma 1.12. $\operatorname{meas}(S^\pm \setminus (\widetilde{S}^\pm \cup \overset{*}{S}{}^\pm); \widehat{d\Gamma}{}^\pm) = 0.$

Proof. Assume that $\omega \in \Omega$ and $x \in S^\pm(\omega) \setminus (\widetilde{S}^\pm(\omega) \cup \overset{*}{S}{}^\pm(\omega))$. Then $X^\pm(\omega, x) \in \Gamma^0(\omega) \cup \mathscr{G}$. Therefore, $P_\omega x \in P_\omega(\Gamma^0(\omega) \cup \mathscr{G})$. By Lemmas 1.3 and 1.4, we have

$$\begin{aligned}\operatorname{meas}_2 P_\omega(S^\pm(\omega) \setminus (\widetilde{S}^\pm(\omega) \cup \overset{*}{S}{}^\pm(\omega))) &\leqslant \operatorname{meas}_2 P_\omega(\Gamma^0(\omega) \cup \mathscr{G}) \\ &\leqslant \sum_{j=1}^m \operatorname{meas}_2 P_\omega(\Gamma_j^0(\omega)) + \sum_{j=1}^m \operatorname{meas}_2 P_\omega \mathscr{G}_j = 0.\end{aligned}$$

To complete the proof, we apply Lemma 1.2. □

2 Function Spaces and Their Properties

2.1. Spaces of functions defined on Γ^+ and Γ^-

Let $E^{\pm}$ be a measurable subset of $\Gamma^{\pm}$ with respect to the measure $d\Gamma$. We denote by $\mathfrak{M}(E^{\pm})$ the set of functions defined on $E^{\pm}$ and measurable with respect to the measure $d\Gamma$. We denote by $L^p(E^{\pm})$ and $\widehat{L}^p(E^{\pm})$, $1 \leqslant p \leqslant \infty$, the Banach spaces of functions $g \in \mathfrak{M}(E^{\pm})$ equipped with the norms

$$\|g\|_{L^p(E^{\pm})} = \begin{cases} \Big(\int\limits_{E^{\pm}} |g(\omega,x)|^p \, d\Gamma(\omega,x) \Big)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}\limits_{(\omega,x)\in E^{\pm}} |g(\omega,x)|, & p = \infty, \end{cases}$$

$$\|g\|_{\widehat{L}^p(E^{\pm})} = \begin{cases} \Big(\int\limits_{E^{\pm}} |g(\omega,x)|^p \, d\widehat{\Gamma}^{\pm}(\omega,x) \Big)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}\limits_{(\omega,x)\in E^{\pm}} |g(\omega,x)|, & p = \infty, \end{cases}$$

respectively. It is clear that $\widehat{L}^{\infty}(E^{\pm}) = L^{\infty}(E^{\pm})$.

We introduce the space $L^p_{\text{loc}}(\Gamma^{\pm}_j)$ as the set of all functions $g \in \mathfrak{M}(\Gamma^{\pm}_j)$ such that $g \in L^p(K)$ for any compact subset $K \subset \Gamma^{\pm}_j$. It is clear that $L^p(\Gamma^{\pm}_j) \subset \widehat{L}^p(\Gamma^{\pm}_j) \subset L^p_{\text{loc}}(\Gamma^{\pm}_j)$ for all $1 \leqslant p < \infty$. The space $L^p_{\text{loc}}(\Gamma^{\pm})$ is defined in a similar way.

We will use the spaces $\widehat{L}^{1,p}(E^{\pm})$ of functions $g \in \mathfrak{M}(E^{\pm})$ that, defined by zero on $\Gamma^{\pm} \setminus E^{\pm}$, have the finite norms

$$\|g\|_{\widehat{L}^{1,p}(E^{\pm})} = \begin{cases} \Big(\sum\limits_{j=1}^{m} \int\limits_{\partial' G_j} \Big[\int\limits_{\Omega^{\pm}_j(x)} |g(\omega,x)| |\omega \cdot n_j(x)| \, d\omega \Big]^p d\sigma(x) \Big)^{1/p}, & 1 \leqslant p < \infty, \\ \max\limits_{1\leqslant j\leqslant m} \ \operatorname*{ess\,sup}\limits_{x\in\partial' G_j} \int\limits_{\Omega^{\pm}_j(x)} |g(\omega,x)| |\omega \cdot n_j(x)| \, d\omega, & p = \infty. \end{cases}$$

Hereinafter,

$$\Omega^+_j(x) = \{\omega \in \Omega \mid \omega \cdot n_j(x) > 0\},$$

$$\Omega^-_j(x) = \{\omega \in \Omega \mid \omega \cdot n_j(x) < 0\}.$$

It is clear that $\widehat{L}^{1,1}(E^{\pm}) = \widehat{L}^1(E^{\pm})$.

Let $E^{\pm} = \bigcup\limits_{j=1}^{m} E^{\pm}_j$, where $E^{\pm}_j = \{(\omega,x) \in \Gamma^{\pm}_j \mid x \in M_j\}$, and the sets $M_j \subset \partial' G_j$, $1 \leqslant j \leqslant m$ are measurable. We denote by $L^p_{\text{const}}(E^{\pm})$ the space of functions $g \in L^p(E^{\pm})$ such that $g(\omega,x)$ is independent of $\omega \in \Omega^{\pm}_j(x)$ (i.e., $g(\omega,x) = g_{\pm}(x)$) for almost all $x \in M_j$ and all $1 \leqslant j \leqslant m$. We equip the space $L^p_{\text{const}}(E^{\pm})$ with the norm

$$\|g\|_{L^p_{\text{const}}(E^{\pm})} = \begin{cases} \Big(\sum\limits_{j=1}^{m} \|g_{\pm}\|^p_{L^p(M_j)} \Big)^{1/p}, & 1 \leqslant p < \infty, \\ \max\limits_{1\leqslant j\leqslant m} \|g_{\pm}\|_{L^{\infty}(M_j)}, & p = \infty. \end{cases}$$

2.2. The spaces $L^p(D_j)$, $L^p(D)$ and the change of variables $(\omega, x) \to (\omega, x', t)$

We recall that $D = \Omega \times G = \bigcup\limits_{j=1}^{m} D_j$, where $D_j = \Omega \times G_j$, $1 \leqslant j \leqslant m$.

Let $1 \leqslant p \leqslant \infty$. We denote by $L^p(D_j)$ and $L^p(D)$ the Banach spaces of functions f on D_j and D respectively that are measurable with respect to the measure $d\omega\, dx$ and have the finite norms

$$\|f\|_{L^p(D_j)} = \begin{cases} \Big(\int\limits_{D_j} |f(\omega, x)|^p \, d\omega dx \Big)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}\limits_{(\omega,x)\in D_j} |f(\omega, x)|, & p = \infty, \end{cases}$$

$$\|f\|_{L^p(D)} = \begin{cases} \Big(\int\limits_{D} |f(\omega, x)|^p \, d\omega dx \Big)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}\limits_{(\omega,x)\in D} |f(\omega, x)|, & p = \infty. \end{cases}$$

We emphasize that for all $\omega \in \Omega$ an arbitrary point $x \in G_j$ is uniquely represented in the form $x = x' + t\,\omega$, where $(\omega, x') \in \Gamma_j^- \cup \Gamma_j^0 \cup (\Omega \times \mathscr{G})$, $x' = \widehat{X}^-(\omega, x)$, $t = \widehat{\tau}^-(\omega, x) \in (0, \widehat{\tau}^+(\omega, x'))$.

We introduce the sets

$$Q_j^- = \{ (\omega, x', t) \in \widehat{S}_j^- \times \mathbb{R} \mid t \in (0, \widehat{\tau}^+(\omega, x')) \},$$
$$\widehat{D}_j = \{ (\omega, x) \in D_j \mid (\omega, \widehat{X}^-(\omega, x)) \in \widehat{S}_j^-, \ (\omega, \widehat{X}^+(\omega, x)) \in \widehat{S}_j^+ \}.$$

Lemma 2.1. *The set Q_j^- is open in the topology of the set $\Gamma_j \times \mathbb{R}$, the set $\widehat{D}_j$ is open in the topology of the set $\Omega \times \mathbb{R}^3$, and the mapping $(\omega, x', t) \to (\omega, x' + t\omega)$ is a homeomorphism from Q_j^- onto $\widehat{D}_j$ with the inverse $(\omega, x) \to (\omega, \widehat{X}^-(\omega, x), \widehat{\tau}^-(\omega, x))$.*

Lemma 2.2. *A set $\widehat{E} \subset \widehat{D}_j$ is measurable if and only if its image $E \subset Q_j^-$ is measurable under the mapping $(\omega, x) \to (\omega, \widehat{X}^-(\omega, x), \widehat{\tau}^-(\omega, x))$; moreover,*

$$\operatorname{meas}(\widehat{E}; d\omega dx) = \operatorname{meas}(E; d\widehat{\Gamma}^- dt)$$

The proof of Lemmas 2.1 and 2.2 repeats the proof of the corresponding assertions in [3].

Lemma 2.3. $\operatorname{meas}(D_j \setminus \widehat{D}_j;\ d\omega dx) = 0$.

Proof. Let $(\omega, x) \in D_j \setminus \widehat{D}_j$. Then $\widehat{X}^+(\omega, x) \in \Gamma_j^0(\omega) \cup \mathscr{G}_j$ or $\widehat{X}^-(\omega, x) \in \Gamma_j^0(\omega) \cup \mathscr{G}_j$. As a consequence, $P_\omega x \in P_\omega(\Gamma_j^0(\omega) \cup \mathscr{G}_j)$.

We denote by χ the characteristic function of the set $D_j \setminus \widehat{D}_j$ and by d_j the diameter of G_j. By Lemmas 1.3 and 1.4,

$$\operatorname{meas}(D_j \setminus \widehat{D}_j;\ d\omega dx) = \int\limits_{\Omega\times\mathbb{R}^3} \chi(\omega, x)\, d\omega\, dx = \int\limits_{\Omega} \Big[\int\limits_{\pi_\omega} \Big(\int\limits_{\mathbb{R}} \chi(\omega, \mathbf{y} + t\omega)\, dt \Big) d\mathbf{y} \Big] d\omega$$

$$\leqslant \int\limits_{\Omega} \Big[\int\limits_{P_\omega(\Gamma_j^0(\omega)\cup\mathscr{G}_j)} \Big(\int\limits_{\mathbb{R}} \chi(\omega, \mathbf{y} + t\omega) dt \Big) d\mathbf{y} \Big]\, d\omega \leqslant \int\limits_{\Omega} d_j [\operatorname{meas}_2(P_\omega \Gamma_j^0(\omega)) + \operatorname{meas}_2(P_\omega \mathscr{G}_j)]\, d\omega = 0.$$

The lemma is proved. $\square$

Lemma 2.4. *Assume that f defined in D_j and $\widetilde{f}$ defined in Q_j^- are such that*

$$\widetilde{f}(\omega, x', t) = f(\omega, x' + t\omega) \quad \forall\, (\omega, x', t) \in Q_j^-.$$

Then the following assertions hold:

1) *f is measurable in D_j if and only if $\widetilde{f}$ is measurable in Q_j^-,*

2) *$f \in L^1(D_j)$ if and only if $\widetilde{f} \in L^1(Q_j^-)$; moreover,*

$$\int\limits_{D_j} f(\omega, x)\, d\omega\, dx = \int\limits_{Q_j^-} \widetilde{f}(\omega, x', t)\, \widehat{d\Gamma}^-(\omega, x')\, dt. \tag{2.1}$$

The proof follows from Lemmas 2.3 and 2.2.

Theorem 2.1. *Let $f \in L^1(D_j)$. Then*

$$\int\limits_{D_j} f(\omega, x)\, d\omega\, dx = \int\limits_{\Gamma_j^-} \Bigg[\int\limits_0^{\widehat{\tau}^+(\omega,x')} f(\omega, x' + t\omega)\, dt \Bigg] \widehat{d\Gamma}^-(\omega, x'), \tag{2.2}$$

$$\int\limits_{D_j} f(\omega, x)\, d\omega\, dx = \int\limits_{\Gamma_j^+} \Bigg[\int\limits_0^{\widehat{\tau}^-(\omega,x')} f(\omega, x' - t\omega)\, dt \Bigg] \widehat{d\Gamma}^+(\omega, x'). \tag{2.3}$$

Proof. Formula (2.2) is obtained from (2.1) since from the Fubini theorem and the equality $\operatorname{meas}(\Gamma_j^- \setminus \widehat{S}_j^-;\ \widehat{d\Gamma}^-) = 0$ valid in view of Lemma 1.7 it follows that

$$\int\limits_{Q_j^-} \widetilde{f}(\omega, x', t)\, \widehat{d\Gamma}^-(\omega, x')\, dt = \int\limits_{\widehat{S}_j^-} \Bigg[\int\limits_0^{\widehat{\tau}^+(\omega,x')} \widetilde{f}(\omega, x', t)\, dt \Bigg] \widehat{d\Gamma}^-(\omega, x') = \int\limits_{\Gamma_j^-} \Bigg[\int\limits_0^{\widehat{\tau}^+(\omega,x')} f(\omega, x' + t\omega)\, dt \Bigg] \widehat{d\Gamma}^-(\omega, x').$$

Formula (2.3) is obtained from (2.2) with ω replaced by $-\omega$. ☐

2.3. The spaces $\mathscr{W}^p(D_j)$ and $\mathscr{W}^p(D)$

By the *weak directional derivative* of a function $f \in L^1(D_j)$ along a direction ω we understand a function $w \in L^1(D_j)$ satisfying the integral identity

$$\int\limits_{D_j} \big(f(\omega, x)\, \omega \cdot \nabla \varphi(\omega, x) + w(\omega, x) \varphi(\omega, x) \big)\, d\omega\, dx = 0$$

for all functions $\varphi \in C(\overline{D}_j)$ such that $\varphi(\omega, \cdot) \in C_0^\infty(G_j)$ for almost all $\omega \in \Omega$. For this function we use the notation $w = \omega \cdot \nabla f$.

We denote by $\mathscr{W}^p(D_j)$ the linear space of functions $f \in L^p(D_j)$ possessing the weak directional derivatives $\omega \cdot \nabla f \in L^p(D_j)$. The space $\mathscr{W}^p(D_j)$ equipped with the norm

$$\|f\|_{\mathscr{W}^p(D_j)} = \begin{cases} \Big(\|f\|^p_{L^p(D_j)} + \|\,\omega \cdot \nabla f\|^p_{L^p(D_j)} \Big)^{1/p}, & 1 \leqslant p < \infty, \\ \max\{\|f\|_{L^\infty(D_j)}, \|\,\omega \cdot \nabla f\|_{L^\infty(D_j)}\}, & p = \infty, \end{cases}$$

becomes a Banach space.

We denote by $\mathscr{W}^p(D)$ the Banach space of functions $f \in L^p(D)$ such that $f \in \mathscr{W}^p(D_j)$ for all $1 \leqslant j \leqslant m$, equipped with the norm

$$\|f\|_{\mathscr{W}^p(D)} = \begin{cases} \left(\|f\|^p_{L^p(D)} + \|\,\omega \cdot \nabla f\|^p_{L^p(D)}\right)^{1/p}, & 1 \leqslant p < \infty, \\ \max\{\|f\|_{L^\infty(D)}, \|\,\omega \cdot \nabla f\|_{L^\infty(D)}\}, & p = \infty. \end{cases}$$

Let $C^{(0,1)}(\overline{D}_j)$ be the set of continuous functions φ on $\overline{D}_j$ possessing continuous partial derivatives $\dfrac{\partial \varphi}{\partial x_i}$, $1 \leqslant i \leqslant 3$, in $\overline{D}_j$. It is known [5] that for Lipschitz domains and $1 \leqslant p < \infty$ the set $C^{(0,1)}(\overline{D}_j)$ is dense in $\mathscr{W}^p(D_j)$.

We recall how to introduce the traces $f|_{\Gamma^+}$ and $f|_{\Gamma^-}$ of a function $f \in \mathscr{W}^p(D)$, $1 \leqslant p \leqslant \infty$, on Γ^+ and Γ^-. Since $\Gamma^+ = \bigcup\limits_{j=1}^{m} \Gamma^+_j$, $\Gamma^- = \bigcup\limits_{j=1}^{m} \Gamma^-_j$, it suffices to define the traces $f|_{\Gamma^+_j}$ and $f|_{\Gamma^-_j}$ on Γ^+_j and Γ^-_j for every $1 \leqslant j \leqslant m$.

For $f \in C^{(0,1)}(\overline{D}_j)$ the traces $f|_{\Gamma^\pm_j}$ are naturally defined as the restrictions of f on $\Gamma^\pm_j$. Let $1 \leqslant p < \infty$, and let $K^\pm$ be arbitrary compact subsets of $\Gamma^\pm_j$. Then [7]

$$\|f|_{\Gamma^\pm_j}\|_{L^p(K^+)} \leqslant c_{K^\pm} \|f\|_{\mathscr{W}^p(D_j)} \quad \forall f \in C^{(0,1)}(\overline{D}_j), \tag{2.4}$$

where the constants $c_{K^\pm}$ depend only on $K^\pm$, G_j, and p. Although these estimates are proved in [7] only for domains G_j with smooth boundaries, the proof remains true for domains with piecewise smooth boundaries.

Since the set $C^{(0,1)}(\overline{D}_j)$ is dense in $\mathscr{W}^p(D_j)$, the estimate (2.4) allows us to extend the linear operators $f \to f|_{\Gamma^\pm_j}$ to linear continuous operators acting from $\mathscr{W}^p(D_j)$ to $L^p(K^\pm)$ and satisfying the estimates

$$\|f|_{\Gamma^+_j}\|_{L^p(K^+)} \leqslant c_{K^+} \|f\|_{\mathscr{W}^p(D_j)} \quad \forall f \in \mathscr{W}^p(D_j).$$

Since the sets $\Gamma^\pm_j$ can be represented as the countable unions (1.5) of expanding compact sets, functions $f \in \mathscr{W}^p(D_j)$, $1 \leqslant p < \infty$. have the traces $f|_{\Gamma^\pm_j} \in L^p_{\text{loc}}(\Gamma^\pm_j)$. As a consequence, all functions $f \in \mathscr{W}^p(D)$, $1 \leqslant p < \infty$, have the traces $f|_{\Gamma^\pm} \in L^p_{\text{loc}}(\Gamma^\pm)$.

Remark 2.1. It is easy to see that for $1 < p < \infty$ the linear trace operators $f \to f|_{\Gamma^\pm}$, regarded as operators from $\mathscr{W}^p(D)$ to $L^p_{\text{loc}}(\Gamma^\pm)$, are restrictions of the same trace operators, regarded as operators from $\mathscr{W}^1(D)$ to $L^1_{\text{loc}}(\Gamma^\pm)$.

Remark 2.2. Since $\mathscr{W}^\infty(D) \subset \mathscr{W}^p(D)$ for all $1 \leqslant p < \infty$, the traces are also defined for $f \in \mathscr{W}^\infty(D)$; moreover, $f|_{\Gamma^\pm} \in L^\infty(\Gamma^\pm)$ in view of Theorem 2.3.

2.4. Additional information on properties of functions in the space $\mathscr{W}(D_j)$

Theorems 2.2 and 2.3 below provide an additional information about the properties of functions in the space $\mathscr{W}^p(D_j)$. In fact, these theorems contain different from, but equivalent to the above definitions of the directional derivative $\omega \cdot \nabla f$ and traces $f|_{\Gamma^+_j}$, $f|_{\Gamma^-_j}$.

Theorem 2.2. *A function $w \in L^1(D_j)$ is the weak directional derivative of $f \in L^1(D_j)$ along a direction ω (i.e., $w = \omega \cdot \nabla f$) if and only if the following property holds: for almost all*

$(\omega, x') \in \Gamma_j^-$ *the function* $f(\omega, x' + t\omega)$, *regarded as a function of the variable* t, *belongs to the space* $W^{1,1}(0, \widehat{\tau}^+(\omega, x'))$ *and*

$$\frac{d}{dt} f(\omega, x' + t\omega) = w(\omega, x' + t\omega) \quad \text{for a.e. } t \in \big(0, \widehat{\tau}^+(\omega, x')\big).$$

In the case of domains with smooth boundaries, the proof of this theorem can be found in [3], where the smoothness assumption was used only to have the possibility to apply formula (2.2). By Theorem 2.1, formula (2.2) is also valid for domains with Lipschitz piecewise smooth boundaries. Therefore, the reasoning remains valid in the case under consideration.

Corollary 2.1. *A function* $w \in L^1(D_j)$ *is the weak directional derivative of a function* $f \in L^1(D_j)$ *in a direction* ω *if and only if the following property holds*: *for almost all* $(\omega, x') \in \Gamma_j^+$ *the function* $f(\omega, x' - t\omega)$, *regarded as a function of the variable* t, *belongs to the space* $W^{1,1}(0, \widehat{\tau}^-(\omega, x'))$ *and*

$$\frac{d}{dt} f(\omega, x' - t\omega) = -w(\omega, x' - t\omega) \quad \text{for a.e. } t \in \big(0, \widehat{\tau}^-(\omega, x')\big).$$

Corollary 2.2. *Assume that* $f \in \mathscr{W}^p(D_j)$, $1 \leqslant p < \infty$, *and* $f^{[M,N]} = \max\{\min\{f, N\}, M\}$, *where* M *and* N *are constant*, $-\infty \leqslant M < N \leqslant \infty$. *Then* $f^{[M,N]} \in \mathscr{W}^p(D_j)$, $\omega \cdot \nabla f^{[M,N]} = 0$ *on* $E_{M,N} = \{(\omega, x) \in D_j \mid f \leqslant M \text{ or } f \geqslant N\}$, *and* $\omega \cdot \nabla f^{[M,N]} = \omega \cdot \nabla f$ *in* $D_j \setminus E_{M,N}$.

The proof of this corollary is based on Theorem 2.2 and actually repeats the proof of the corresponding assertion in [3].

Theorem 2.3. *Let* $f \in \mathscr{W}^p(D_j)$, $1 \leqslant p \leqslant \infty$. *Then for the traces* $f|_{\Gamma_j^-}$ *and* $f|_{\Gamma_j^+}$ *the following formulas hold*:

$$f|_{\Gamma_j^-}(\omega, x) = \lim_{t \to 0^+} \operatorname{ap} f(\omega, x + t\,\omega) \quad \text{for a.e. } (\omega, x) \in \Gamma_j^-, \tag{2.5}$$

$$f|_{\Gamma_j^+}(\omega, x) = \lim_{t \to 0^+} \operatorname{ap} f(\omega, x - t\,\omega) \quad \text{for a.e. } (\omega, x) \in \Gamma_j^+. \tag{2.6}$$

In the case of domains with smooth boundaries, this theorem was proved in [3]. Repeating the proof in [3] and taking into account Theorem 2.2, we obtain Theorem 2.3.

Remark 2.3. Formulas (2.5) and (2.6) are convenient by the possibility to compute the traces of a function f via its values without modifying f on a set of measure zero or constructing an approximate sequence of smooth functions.

Corollary 2.3. *If* $f \in \mathscr{W}^\infty(D_j)$, *then* $f|_{\Gamma_j^\pm} \in L^\infty(\Gamma_j^\pm)$ *and* $\|f|_{\Gamma_j^\pm}\|_{L^\infty(\Gamma_j^+)} \leqslant \|f\|_{L^\infty(D_j)}$.

Corollary 2.4. *If* $f \in \mathscr{W}^p(D_j)$, $1 \leqslant p \leqslant \infty$ *and* $f \geqslant 0$, *then* $f|_{\Gamma_j^+} \geqslant 0$, $f|_{\Gamma_j^-} \geqslant 0$.

The following important theorem asserts that any function $f \in \mathscr{W}^1(D_j)$ can be modified on a set of measure zero in such a way that becomes a function that, regarded as a function of the variable x, for almost all $(\omega, x') \in \Gamma_j^-$ is absolutely continuous on the interval $]x', \widehat{X}^+(\omega, x')[$.

Theorem 2.4. *For any function* $f \in \mathscr{W}^1(D_j)$ *there exists an equivalent function* $\widetilde{f} \in \mathscr{W}^1(D_j)$ *such that for almost all* $(\omega, x') \in \Gamma_j^-$ *the function* $\widetilde{f}(\omega, x' + t\omega)$, *regarded as a function of the variable* t, *is absolutely continuous on the interval* $(0, \widehat{\tau}^+(\omega, x'))$.

Proof. We consider the function

$$\widehat{f}(\omega, x', t) = \frac{1}{\widehat{\tau}^+(\omega, x')} \int\limits_0^{\widehat{\tau}^+(\omega,x')} \left[f(\omega, x' + s\,\omega) - (\widehat{\tau}^+(\omega, x') - s)\,\omega \cdot \nabla I(\omega, x' + s\,\omega)\right] ds$$
$$+ \int\limits_0^t \omega \cdot \nabla I(\omega, x' + s\,\omega)\, ds. \tag{2.7}$$

on Q_j^-. We note that this function is measurable on Q_j^- (recall that $\widehat{\tau}^+$ is continuous on $\widehat{S}_j^-$).

We also note that for almost all $(\omega, x') \in \widehat{S}_j^-$ the function (2.7), regarded as a function of the variable t, is absolutely continuous on the interval $(0, \widehat{\tau}^+(\omega, x))$ and $\dfrac{d}{dt}\widehat{f}(\omega, x', t) = \omega \cdot \nabla I(\omega, x' + t\,\omega)$. It is easy to see that

$$\int\limits_0^{\widehat{\tau}^+(\omega,x')} \widehat{f}(\omega, x', s)\, ds = \int\limits_0^{\widehat{\tau}^+(\omega,x')} f(\omega, x' + s\omega)\, ds.$$

By Theorem 2.2, for almost all $(\omega, x') \in \widehat{S}_j^-$ the functions $\widehat{f}(\omega, x', t)$ and $f(\omega, x' + t\omega)$ are equivalent on $(0, \widehat{\tau}^-(\omega, x'))$. Since these functions are measurable on Q_j^-, we conclude that $\widehat{f}(\omega, x', t)$ and $f(\omega, x' + t\omega)$ are equivalent on Q_j^-. By Lemma 2.2, the function $f(\omega, x)$ is equivalent to the function $\widetilde{f}(\omega, x) = \widehat{f}(\omega, \widehat{X}^-(\omega, x), \widehat{\tau}^-(\omega, x))$ on $\widehat{D}_j$. To complete the proof, it remains to note that for almost all $(\omega, x') \in \widehat{S}_j^-$ the function $\widetilde{f}(\omega, x' + t\omega) = \widehat{f}(\omega, x', t)$, regarded as a function of the variable t, is absolutely continuous on the interval $(0, \widehat{\tau}^+(\omega, x'))$. $\square$

Remark 2.4. By Theorem 2.4, the function $f \in \mathscr{W}^1(D_j)$ can be modified on a set of measure zero in such a way that formulas (2.5) and (2.6) take the following simpler form:

$$f|_{\Gamma_j^-}(\omega, x) = \lim_{t \to 0^+} f(\omega, x + t\,\omega) \quad \text{for a.e. } (\omega, x) \in \Gamma_j^-,$$
$$f|_{\Gamma_j^+}(\omega, x) = \lim_{t \to 0^+} f(\omega, x - t\,\omega) \quad \text{for a.e. } (\omega, x) \in \Gamma_j^+.$$

2.5. The spaces $\widehat{\mathscr{W}^p}(D_j)$ and $\widehat{\mathscr{W}^p}(D)$

We introduce the spaces

$$\widehat{\mathscr{W}^p}(D_j) = \{f \in \mathscr{W}^p(D_j) \mid f|_{\Gamma_j^-} \in \widehat{L}^p(\Gamma_j^-)\},$$
$$\widehat{\mathscr{W}^p}(D) = \{f \in \mathscr{W}^p(D) |\ f|_{\Gamma^-} \in \widehat{L}^p(\Gamma^-)\},$$

where $1 \leqslant p \leqslant \infty$. We note that $\widehat{\mathscr{W}^\infty}(D_j) = \mathscr{W}^\infty(D_j)$ and $\widehat{\mathscr{W}^\infty}(D) = \mathscr{W}^\infty(D)$.

Theorem 2.5. *Let $f \in \widehat{\mathscr{W}^p}(D_j)$, $1 \leqslant p < \infty$. Then $f|_{\Gamma_j^+} \in \widehat{L}^p(\Gamma_j^+)$ and the Green formula holds*:

$$\int\limits_{D_j} \omega \cdot \nabla f\, d\omega dx = \int\limits_{\Gamma_j^+} f|_{\Gamma_j^+} d\widehat{\Gamma}^+ - \int\limits_{\Gamma_j^-} f|_{\Gamma_j^-} d\widehat{\Gamma}^-.$$

Furthermore,

$$|f|^p \in \widehat{\mathscr{W}}^1(D_j), \quad \omega\cdot\nabla|f|^p = p|f|^{p-1}\operatorname{sgn}(f)\,\omega\cdot\nabla f, \quad |f^p|\big|_{\Gamma_j^\pm} = |f|_{\Gamma_j^\pm}|^p,$$

$$\int\limits_{D_j} \omega\cdot\nabla|f|^p\,d\omega\,dx = \|f|_{\Gamma_j^+}\|^p_{\widehat{L}^p(\Gamma_j^+)} - \|f|_{\Gamma_j^-}\|^p_{\widehat{L}^p(\Gamma_j^-)}.$$

The proof of this theorem repeats the proof of Corollaries 2.8, 2.9, and 2.10 in [3] and is based on Lemmas 1.5, 1.6, 1.7, 2.3 and Theorems 2.2, 2.3 of the present paper.

Corollary 2.5. *The following estimate holds*:

$$\|f|_{\Gamma_j^+}\|^p_{\widehat{L}^p(\Gamma_j^+)} \leqslant p\|f\|^{p-1}_{L^p(D_j)}\|\omega\cdot\nabla f\|_{L^p(D_j)} + \|f|_{\Gamma_j^-}\|^p_{\widehat{L}^p(\Gamma_j^-)} \quad \forall\, f\in\widehat{\mathscr{W}}^p(D_j). \tag{2.8}$$

Corollary 2.6. *Let $f\in\widehat{\mathscr{W}}^p(D_j)$, $g\in\widehat{\mathscr{W}}^{p'}(D_j)$, $1\leqslant p\leqslant\infty$. Then*

$$\int\limits_{D_j}(\omega\cdot\nabla f)g\,d\omega dx + \int\limits_{D_j} f(\omega\cdot\nabla g)\,d\omega dx = \int\limits_{\Gamma_j^+} f|_{\Gamma_j^+}g|_{\Gamma_j^+}\widehat{d}\Gamma^+ - \int\limits_{\Gamma_j^-} f|_{\Gamma_j^-}g|_{\Gamma_j^-}\,\widehat{d}\Gamma^-.$$

Throughout the paper, p' denotes the Hölder conjugate exponent of p.

Remark 2.5. It is natural that for functions in the space $\widehat{\mathscr{W}}^p(D)$ we have analogs of Theorem 2.5 and Corollaries 2.5, 2.6 with replacements of D_j by D and $\Gamma_j^\pm$ by $\Gamma^\pm$.

2.6. The spaces $\widetilde{\mathscr{W}}^p_\pm(D_j)$ and $\widetilde{\mathscr{W}}^p_\pm(D)$

Let $1\leqslant p<\infty$. We denote by $\widetilde{\mathscr{W}}^p_\pm(D_j)$ the space of functions $f\in\mathscr{W}^p(D_j)$ whose traces $f|_{\Gamma_j^\pm}$ are independent of $\omega\in\Omega_j^\pm(x)$ (i.e., $f|_{\Gamma_j^\pm}(\omega,x) = f|_{\Gamma_j^\pm}(x)$) for almost all $x\in\partial' G_j$.

Theorem 2.6. *Let $1\leqslant p\leqslant\infty$. The traces of a function $f\in\widetilde{\mathscr{W}}^p_-(D_j)$ possess the properties $f|_{\Gamma_j^-}\in L^p_{\rm const}(\Gamma_j^-)$, $f|_{\Gamma_j^+}\in\widehat{L}^p(\Gamma_j^+)$, and the following estimates hold*:

$$\|f|_{\Gamma_j^-}\|_{L^p_{\rm const}(\Gamma_j^-)} \leqslant C_{1,p}\|f\|_{\mathscr{W}^p(D_j)} \quad \forall\, f\in\widetilde{\mathscr{W}}^p_-(D_j), \tag{2.9}$$

$$\|f|_{\Gamma_j^+}\|_{\widehat{L}^p(\Gamma_j^+)} \leqslant C_{2,p}\|f\|_{\mathscr{W}^p(D_j)} \quad \forall\, f\in\widetilde{\mathscr{W}}^p_-(D_j), \tag{2.10}$$

where the constants $C_{1,p}$ and $C_{2,p}$ are independent of f.

Proof. In the case $p=\infty$, the assertions of the theorem are obvious. Let $1\leqslant p<\infty$. Since the domain G_j is Lipschitz, for every points $x_0\in\partial G_j$ there exist a direction $\omega_0\in\Omega$, numbers $r_0>0$, $h_0>0$, and a function $\gamma\in Lip(\overline{V}_{r,\omega_0})$, $-h_0<\gamma<h_0$ such that (1.1)–(1.3) hold.

We set $\Omega_{\varepsilon,\omega_0} = \{\omega\in\Omega \mid |\omega+\omega_0|<\varepsilon\}$, where $\varepsilon = \frac{1}{2\sqrt{1+L^2}}$ and L is the Lipschitz constant of the function γ. From (1.4) it follows that for almost all $x\in\Pi_{r_0,\gamma}(\omega_0,x_0)$

$$\omega_0\cdot n_j(x) \geqslant \frac{1}{\sqrt{1+L^2}} = 2\varepsilon.$$

Therefore, $\omega\cdot n_j(x)\leqslant-\varepsilon$ for all $\omega\in\Omega_{\varepsilon,\omega_0}$ and almost all $x\in\Pi_{r_0,\gamma}(\omega_0,x_0)$.

Let $(\omega, x) \in U_{\varepsilon,r}(\omega_0, x_0) = \Omega_{\varepsilon,\omega_0} \times \Pi_{r,\gamma}(\omega_0, x_0)$, where $r < \frac{1}{2}\min\{r_0, h_0/L\}$. Let $t \in (0, \delta)$, where $\delta = \frac{1}{2}\min\{r_0, h_0\}$, and let $\mathbf{y} = P_{\omega_0}(x - x_0)$. Then

$$x + t\omega = x_0 + \mathbf{y} + tP_{\omega_0}\omega + (\gamma(\mathbf{y}) + t\omega \cdot \omega_0)\omega_0$$

and it is clear that $|\mathbf{y} + tP_{\omega_0}\omega| \leqslant r + \delta < r_0$ and $\gamma(\mathbf{y}) + t\omega \cdot \omega_0 > -L|\mathbf{y}| - \delta > -h_0$. Since $\varepsilon = \frac{1}{2\sqrt{1+L^2}} \leqslant 1/2$ and $|\omega + \omega_0| < \varepsilon$, we have

$$L < (2\varepsilon)^{-1}, \quad \omega \cdot \omega_0 < \varepsilon^2/2 - 1 \leqslant -7/8, \quad |P_{\omega_0}\omega| = \sqrt{1 - |\omega \cdot \omega_0|^2} < \varepsilon.$$

By the Lipschitz condition,

$$\gamma(\mathbf{y}) + t\omega \cdot \omega_0 - \gamma(\mathbf{y} + tP_{\omega_0}\omega) \leqslant Lt|P_{\omega_0}\omega| + t\omega \cdot \omega_0 < t(L\varepsilon - 7/8) < -3/8t < 0.$$

Thus, $x + t\omega \in C_{r,-h_0,\gamma}(\omega_0, x_0) \subset G_j$ for all $(\omega, x) \subset U_{\varepsilon,r}(\omega_0, x_0)$ and $t \in (0, \delta)$.

From Theorems 2.2 and 2.3 it follows that for almost all $(\omega, x) \in U_{\varepsilon,r}(\omega_0, x_0)$

$$f|_{\Gamma_j^-}(x) = f(\omega, x + t\omega) + \int\limits_0^t \omega \cdot \nabla f(\omega, x + s\,\omega)\,ds \quad \text{for a.e. } t \in (0, \delta).$$

Taking into account that $\delta < \widehat{\tau}^+(\omega, x)$, from the last inequality we find

$$\begin{aligned}|f|_{\Gamma_j^-}(x)|^p &\leqslant \left(\frac{1}{\delta}\int\limits_0^\delta |f(\omega, x + t\omega)|\,dt + \int\limits_0^\delta |\omega \cdot \nabla f(\omega, x + t\omega)|\,dt\right)^p \\ &\leqslant 2^{p-1}\delta^{-1}\int\limits_0^{\widehat{\tau}^+(\omega,x)} |f(\omega, x + t\omega)|^p\,dt + 2^{p-1}\delta^{p-1}\int\limits_0^{\widehat{\tau}^+(\omega,x)} |\omega \cdot \nabla f(\omega, x + t\omega)|^p\,dt.\end{aligned}$$

Using (2.2), we obtain the estimate

$$\int\limits_{U_{\varepsilon,r}(\omega_0,x_0)} |f|_{\Gamma_j^-}(x)|^p d\widehat{\Gamma}^-(\omega, x) \leqslant 2^{p-1}\delta^{-1}\int\limits_{\Gamma_j^-}\left[\int\limits_0^{\widehat{\tau}^+(\omega,x)} |f(\omega, x + t\omega)|^p\,dt\right]d\widehat{\Gamma}^-(\omega, x)$$

$$+ 2^{p-1}\delta^{p-1}\int\limits_{\Gamma_j^-}\left[\int\limits_0^\delta |\omega \cdot \nabla f(\omega, x + t\omega)|^p dt\right]d\widehat{\Gamma}^-(\omega, x) = 2^{p-1}\delta^{-1}(\|f\|^p_{L^p(D_j)} + \delta^p\,\|\omega \cdot \nabla f\|^p_{L^p(D_j)}).$$

Since $|\omega \cdot n_j(x)| \geqslant \varepsilon$ for almost all $(\omega, x) \in U_{\varepsilon,r}(\omega_0, x_0)$, we have the inequality

$$\varepsilon \cdot \operatorname{meas}(\Omega_{\varepsilon,\omega_0}; d\omega)\|f|_{\Gamma_j^-}\|^p_{L^p(\Pi_{r,\gamma}(\omega_0,x_0))} \leqslant 2^{p-1}\delta^{-1}\big(\|f\|^p_{L^p(D_j)} + \delta^p\,\|\omega \cdot \nabla f\|^p_{L^p(D_j)}\big).$$

Thus, the following estimate holds:

$$\|f|_{\Gamma_j^-}\|^p_{L^p(\Pi_{r,\gamma}(\omega_0,x_0))} \leqslant C_p(x_0)\|f\|^p_{\mathscr{W}^p(D_j)}. \tag{2.11}$$

We cover each point $x_0 \in \partial G_j$ by surfaces of the form $\Pi_{r,\gamma}(\omega_0, x_0)$. From the obtained covering we extract a finite subcovering $\{\Pi_{r_k,\gamma_k}(\omega_k, x_k)\}_{k=1}^N$ and arrive at the estimate

$$\|f|_{\Gamma_j^-}\|^p_{L^p(\partial G_j)} \leqslant \sum_{k=1}^N \|f|_{\Gamma_j^-}\|^p_{L^p(\Pi_{r_k,\gamma_k}(\omega_k,x_k))} \leqslant \sum_{k=1}^N C_p(x_k)\|f\|^p_{\mathscr{W}^p(D_j)} = C^p_{1,p}\|f\|^p_{\mathscr{W}^p(D_j)}.$$

The estimate (2.9) is proved. Taking into account (2.8), we obtain the estimate (2.10) from the estimate (2.9). □

Corollary 2.7. *Let* $1 \leqslant p \leqslant \infty$. *The traces of a function* $f \in \widetilde{\mathscr{W}}{}^p_+(D_j)$ *possess the properties* $f|_{\Gamma_j^+} \in L^p_{\text{const}}(\Gamma_j^+)$, $f|_{\Gamma_j^-} \in \widehat{L}^p(\Gamma_j^-)$, *and the following estimates hold*:

$$\|f|_{\Gamma_j^+}\|_{L^p_{\text{const}}(\Gamma_j^+)} \leqslant C_{1,p}\|f\|_{\mathscr{W}^p(D_j)} \quad \forall\, f \in \widetilde{\mathscr{W}}{}^p_+(D_j),$$

$$\|f|_{\Gamma_j^-}\|_{\widehat{L}^p(\Gamma_j^-)} \leqslant C_{2,p}\|f\|_{\mathscr{W}^p(D_j)} \quad \forall\, f \in \widetilde{\mathscr{W}}{}^p_+(D_j),$$

where the constants $C_{1,p}$ *and* $C_{2,p}$ *are independence of* f.

To prove the corollary, it suffices to note that the mapping $f(\omega, x) \to f(-\omega, x)$ is a mutually one-to-one isometric mapping from $\widetilde{\mathscr{W}}{}^p_-(D_j)$ onto $\widetilde{\mathscr{W}}{}^p_+(D_j)$.

Corollary 2.8. $\widetilde{\mathscr{W}}{}^p_\pm(D_j) \subset \widehat{\mathscr{W}}{}^p(D_j)$ *for all* $1 \leqslant p \leqslant \infty$; *moreover,* $\widetilde{\mathscr{W}}{}^p_\pm(D_j)$ *is a closed subspace in* $\mathscr{W}^p(D_j)$.

We denote by $\widetilde{\mathscr{W}}{}^p_\pm(D)$ the space of functions $f \in \mathscr{W}^p(D)$ such that $f \in \widetilde{\mathscr{W}}{}^p_\pm(D_j)$ for all $1 \leqslant j \leqslant m$. It is clear that for $\widetilde{\mathscr{W}}{}^p_\pm(D)$ analogs of Theorem 2.6 and Corollaries 2.7, 2.8 are valid with the only difference that D_j and $\Gamma_j^\pm$ should be replaced with D and $\Gamma^\pm$ respectively.

3 Boundary Value Problem for the Radiative Transfer Equation with Diffuse Reflection and Refraction Conditions in a System of Bodies with Piecewise Smooth Boundaries and Its Properties

We proceed with the main objective of the study in this paper: the boundary value problem for the radiative transfer equation with diffuse reflection and refraction conditions in a system of bodies with piecewise smooth boundaries.

3.1. Statement of the problem

We briefly describe the statement of the problem under consideration (cf. details in [1]).

The unknowns are a function $I(\omega, x)$ defined in $D = \Omega \times G$ and interpreted as the radiation intensity at a point $x \in G$, when the radiation propagates in a system of bodies in direction ω and a function $J(\omega, x)$ defined on S^- and interpreted as the intensity of the radiation falling from the vacuum on Σ at a point x in direction ω. This pair of functions is a solution to the following problem:

$$\omega \cdot \nabla I + \beta I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D, \tag{3.1}$$

$$I|_{\Gamma^-} = \mathscr{R}_d^-(I|_{\Gamma^+}) + \mathscr{P}_d^-(J), \quad (\omega, x) \in S^-, \tag{3.2}$$

$$I|_{\Gamma_i^-} = \mathscr{R}_{d,ij}^-(I|_{\Gamma_i^+}) + \mathscr{P}_{d,ij}^-(I|_{\Gamma_j^+}), \quad (\omega, x) \in \Gamma_{ij}^- = \Gamma_i^- \cap \Gamma_j^+, \quad i \neq j, \tag{3.3}$$

$$J = T\mathscr{R}_d^+(J) + T\mathscr{P}_d^+(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-, \tag{3.4}$$

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}^-. \tag{3.5}$$

It is assumed that we are given the functions $F \in L^p(G)$ and $J_* \in \widehat{L}^{1,p}(\overset{*}{S}^-)$, $1 \leqslant p \leqslant \infty$, interpreted as the volume radiation source and the intensity of the radiation falling on a system of bodies from the vacuum and coming from outside. The condition (3.3) is imposed for those $i \neq j$ for which $\operatorname{meas}(\partial G_{ij}; d\sigma) > 0$.

3.2. The diffuse reflection operator and the diffuse refraction operator

The operators of diffuse reflection $\mathscr{R}_d^-$, $\mathscr{R}_d^+$ and operators of diffuse refraction $\mathscr{P}_d^-$, $\mathscr{P}_d^+$ are introduced as follows. Assume that $\varphi \in \widehat{L}^1(S^+)$ and $\psi \in \widehat{L}^1(S^-)$. We set

$$\mathscr{R}_d^-(\varphi)(\omega, x) = \frac{\rho_j^-(x)}{\pi} \int\limits_{\Omega_j^+(x)} \varphi(\omega', x)\, \omega' \cdot n_j(x)\, d\omega', \quad (\omega, x) \in S_j^-, \quad 1 \leqslant j \leqslant m,$$

$$\mathscr{R}_d^+(\psi)(\omega, x) = \frac{\rho_j^+(x)}{\pi} \int\limits_{\Omega_j^-(x)} \psi(\omega', x)\, |\omega' \cdot n_j(x)|\, d\omega', \quad (\omega, x) \in S_j^+, \quad 1 \leqslant j \leqslant m,$$

$$\mathscr{P}_d^-(\psi)(\omega, x) = \frac{1 - \rho_j^+(x)}{\pi} \int\limits_{\Omega_j^-(x)} \psi(\omega', x) |\omega' \cdot n_j(x)|\, d\omega', \ (\omega, x) \in S_j^-, \quad 1 \leqslant j \leqslant m,$$

$$\mathscr{P}_d^+(\varphi)(\omega, x) = \frac{1 - \rho_j^-(x)}{\pi} \int\limits_{\Omega_j^+(x)} \varphi(\omega', x)\, \omega' \cdot n_j(x)\, d\omega', \ (\omega, x) \in S_j^+, \quad 1 \leqslant j \leqslant m.$$

The quantity $\rho_j^\pm$ characterizing the reflective ability of the surfaces ∂G_j are connected by the equality

$$1 - \rho_j^- = \frac{1}{k_j^2}(1 - \rho_j^+);$$

moreover, $\rho_j^\pm \in L^\infty(\partial G_j)$ and $0 < \rho_j^\pm < 1$. We assume that

$$\overline{\rho}^+ = \max_{1 \leqslant j \leqslant m} \|\rho_j^+\|_{L^\infty(\partial G_j)} < 1.$$

Remark 3.1. Applying the operators $\mathscr{R}_d^\pm$ and $\mathscr{P}_d^\pm$, we obtain functions defined on $S^\pm$, but independent of ω.

The operators $\mathscr{R}^-_{d,ij}$ and $\mathscr{P}^-_{d,ij}$ are defined as follows. Let $i \neq j$, and let meas$(\partial G_{ij}; d\sigma) > 0$. We set $\Gamma^-_{ij} = \Gamma^+_{ji} = \Gamma^-_i \cap \Gamma^+_j$. Assume that $\varphi \in \widehat{L}^1(\Gamma^+_{ij})$ and $\psi \in \widehat{L}^1(\Gamma^+_{ji})$. Then

$$\mathscr{R}^-_{d,ij}(\varphi)(\omega, x) = \frac{\rho^-_{ij}(x)}{\pi} \int\limits_{\Omega^+_i(x)} \varphi(\omega', x)\, \omega' \cdot n_i(x)\, d\omega', \quad (\omega, x) \in \Gamma^-_{ij}, \tag{3.6}$$

$$\mathscr{P}^-_{d,ij}(\psi)(\omega, x) = \frac{1 - \rho^-_{ji}(x)}{\pi} \int\limits_{\Omega^+_j(x)} \psi(\omega', x)\, \omega' \cdot n_j(x)\, d\omega', \quad (\omega, x) \in \Gamma^-_{ij}. \tag{3.7}$$

The quantities ρ^-_{ij} characterizing the reflective ability of the surfaces $\partial G_{ij} = \partial G_i \cap \partial G_j$ are connected by the equality

$$1 - \rho^-_{ij} = (1 - \rho^-_{ji}) \frac{k_j^2}{k_i^2};$$

moreover, $\rho^-_{ij} \in L^\infty(\partial G_{ij})$ and $0 < \rho^-_{ij} < 1$. If bodies G_i and G_j are separated by an infinitely thin vacuum layer, then the following equality holds [1]:

$$\rho^-_{ij} = 1 - \frac{(1 - \rho^-_i)(1 - \rho^+_j)}{1 - \rho^+_i \rho^+_j}.$$

Remark 3.2. Applying the operators $\mathscr{R}^-_{d,ij}$ and $\mathscr{P}^-_{d,ij}$, we obtain functions defined on Γ^-_{ij}, but independent of $\omega \in \Omega^-_i(x)$.

It is easy to see that the following assertion holds.

Lemma 3.1. 1. *For all $1 \leqslant p \leqslant \infty$ the operators $\mathscr{R}^-_d$ and $\mathscr{R}^+_d$ are linear bounded operators acting from $\widehat{L}^{1,p}(S^+)$ to $L^p_{\mathrm{const}}(S^-)$ and from $\widehat{L}^{1,p}(S^-)$ i to $L^p_{\mathrm{const}}(S^+)$ respectively.*

2. *For all $1 \leqslant p \leqslant \infty$ the operators $\mathscr{P}^-_d$ and $\mathscr{P}^+_d$ are linear bounded operators acting from $\widehat{L}^{1,p}(S^-)$ to $L^p_{\mathrm{const}}(S^-)$ and from $\widehat{L}^{1,p}(S^+)$ to $L^p_{\mathrm{const}}(S^+)$ respectively.*

3. *For all $1 \leqslant p \leqslant \infty$ the operators $\mathscr{R}^-_{d,ij}$ and $\mathscr{P}^-_{d,ij}$ are linear bounded operators acting from $\widehat{L}^{1,p}(\Gamma^+_{ij})$ to $L^p_{\mathrm{const}}(\Gamma^-_{ij})$ and from $\widehat{L}^{1,p}(\Gamma^+_{ji})$ to $L^p_{\mathrm{const}}(\Gamma^-_{ij})$ respectively.*

3.3. The translation operator T

The translation operator T is defined by

$$T\varphi(\omega, x) = \begin{cases} \varphi(\omega, X^-(\omega, x)), & (\omega, x) \in \widetilde{S}^-, \\ 0, & (\omega, x) \in S^- \setminus \widetilde{S}^-. \end{cases}$$

From the properties of the sets $\widetilde{S}^-$, $\widetilde{S}^+$ and the mapping $(\omega, x) \to (\omega, X^-(\omega, x))$ indicated in Lemmas 1.10 and 1.11 we obtain the following assertion.

Lemma 3.2. *For all $1 \leqslant p \leqslant \infty$ the operator T is a linear bounded operator acting from $\widehat{L}^p(\widetilde{S}^+)$ to $\widehat{L}^p(\widetilde{S}^-)$; moreover, $\|T\|_{\widehat{L}^p(\widetilde{S}^+) \to \widehat{L}^p(\widetilde{S}^-)} = 1$ and*

$$\int\limits_{\widetilde{S}^-} T\varphi\, d\widehat{\Gamma}^- = \int\limits_{\widetilde{S}^+} \varphi\, d\widehat{\Gamma}^+ \quad \forall\, \varphi \in \widehat{L}^1(\widetilde{S}^+).$$

3.4. The operators $\mathscr{B}_d$ and $\mathscr{C}_d$

We recall that the function J in the statement of the problem (3.1)–(3.5) is such that

$$J = T\mathscr{R}_d^+(J) + T\mathscr{P}_d^+(I|_{\Gamma^+}), \quad (\omega, x) \in \widetilde{S}^-, \tag{3.8}$$

$$J = J_*, \quad (\omega, x) \in \overset{*}{S}{}^-. \tag{3.9}$$

We also recall that

$$\operatorname{meas}(S^- \setminus (\widetilde{S}^- \cup \overset{*}{S}{}^-); d\Gamma) = 0.$$

Let $I|_{\Gamma^+} \in \widehat{L}^{1,p}(S^+)$, and let $J_* \in \widehat{L}^{1,p}(\overset{*}{S}{}^-)$. We set $J_* = 0$ on $S^- \setminus \overset{*}{S}{}^-$ and reduce the system (3.8), (3.9) to the equivalent equation

$$J = T\mathscr{R}_d^+(J) + T\mathscr{P}_d^+(I|_{\Gamma^+}) + J_*, \quad (\omega, x) \in S^-. \tag{3.10}$$

From [1, Lemma 4.6] it follows that

$$\|T\mathscr{R}_d^+\|_{\widehat{L}^{1,p}(S^-)\to\widehat{L}^{1,p}(S^-)} \leqslant \overline{\rho}^+ < 1.$$

Hence a solution $J \in \widehat{L}^{1,p}(S^-)$ to Equation (3.10) exists, is unique, and is represented as

$$J = \mathscr{B}_d(I|_{\Gamma^+}) + \mathscr{C}_d(J_*),$$

where the linear bounded operators $\mathscr{B}_d : \widehat{L}^{1,p}(S^+) \to \widehat{L}^{1,p}(S^-)$ and $\mathscr{C}_d : \widehat{L}^{1,p}(\overset{*}{S}{}^-) \to \widehat{L}^{1,p}(S^-)$ are defined by

$$\mathscr{B}_d(I|_{\Gamma^+}) = \sum_{\ell=0}^{\infty} (T\mathscr{R}_d^+)^\ell T\mathscr{P}_d^+(I|_{\Gamma^+}),$$

$$\mathscr{C}_d(J_*) = \sum_{\ell=0}^{\infty} (T\mathscr{R}_d^+)^\ell J_*.$$

3.5. Statement of the problem

Excluding the function $J = \mathscr{B}_d(I|_{\Gamma^+}) + \mathscr{C}_d(J_*)$ from the problem (3.1)–(3.5), we obtain the boundary value problem

$$\omega \cdot \nabla I + \beta I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D, \tag{3.11}$$

$$I|_{\Gamma^-} = \mathfrak{B}_d(I|_{\Gamma^+}) + \mathfrak{C}_d(J_*), \quad (\omega, x) \in \Gamma^-, \tag{3.12}$$

where $F \in L^p(D)$, $J_* \in \widehat{L}^{1,p}(\overset{*}{S}{}^-)$, $1 \leqslant p \leqslant \infty$, whereas the linear bounded operators $\mathfrak{B}_d : \widehat{L}^p(\Gamma^+) \to L^p_{\text{const}}(\Gamma^-)$ and $\mathfrak{C}_d : \widehat{L}^{1,p}(\overset{*}{S}{}^-) \to L^p_{\text{const}}(\Gamma^-)$ are defined by

$$\mathfrak{B}_d(I|_{\Gamma^+})(\omega, x) = \begin{cases} \mathscr{R}_d^-(I|_{\Gamma^+})(\omega, x) + \mathscr{P}_d^- \mathscr{B}_d(I|_{\Gamma^+})(\omega, x), & (\omega, x) \in S^-, \\ \mathscr{R}_{d,ij}^-(I|_{\Gamma_i^+})(\omega, x) + \mathscr{P}_{d,ij}^-(I|_{\Gamma_j^+})(\omega, x), & (\omega, x) \in \Gamma_{ij}^-,\ i \neq j, \end{cases}$$

$$\mathfrak{C}_d(J_*)(\omega, x) = \begin{cases} \mathscr{P}_d^- \mathscr{C}_d(J_*)(\omega, x), & (\omega, x) \in S^-, \\ 0, & (\omega, x) \in \Gamma^- \setminus S^-. \end{cases}$$

By a solution to the problem (3.11),(3.12) we mean a function $I \in \widetilde{\mathscr{W}}{}^p_-(D)$ that satisfies Equation (3.11) almost everywhere in D and the condition (3.12) almost everywhere on Γ^-.

Remark 3.3. The following important fact has not been paid due attention in [1], where the solution I was understood as a function in the space $\mathscr{W}^p(D)$. As a result of the action of the boundary operators $\mathfrak{B}$ and $\mathfrak{C}$ on $I|_{\Gamma^+}$ and J_*, we obtain functions independent of ω. By (3.12), $I|_{\Gamma^-}$ is independent of ω and, consequently, any solution $I \in \mathscr{W}^p(D)$, automatically belongs to the space $\widetilde{\mathscr{W}^p_-}(D)$ (cf. Theorem 2.6).

3.6. Auxiliary problem

In the proof of the existence of a solution to the problem (3.11), (3.12), the following auxiliary problem was essentially used:

$$\omega \cdot \nabla I + \beta_j I = f, \quad (\omega, x) \in D_j, \tag{3.13}$$

$$I|_{\Gamma_j^-} = g, \quad (\omega, x) \in \Gamma_j^-, \tag{3.14}$$

where $f \in L^p(D_j)$, $g \in \widehat{L}^p(\Gamma_j^-)$, $1 \leqslant p \leqslant \infty$, $\beta_j = \text{const} > 0$.

By a solution to the problem (3.13), (3.14) we mean a function $I \in \mathscr{W}^p(D_j)$ that satisfies Equation (3.13) almost everywhere in D_j and the condition (3.14) almost everywhere on Γ_j^-. It is clear that $I \in \widehat{\mathscr{W}^p}(D_j)$.

The proof of the following theorem in the particular case where the domain G_j has smooth boundary and $p = \infty$ can be found, for example, in [7, 8]).

Theorem 3.1. *A solution to the problem* (3.13), (3.14) *exists, is unique, and is represented in the form*

$$I(\omega, x) = e^{-\beta_j \widehat{\tau}^-(\omega,x)} g(\omega, \widehat{X}^-(\omega, x)) + \int\limits_0^{\widehat{\tau}^-(\omega,x)} e^{-\beta_j s} f(\omega, x - s\omega)\, ds, \quad (\omega, x) \in D_j, \tag{3.15}$$

where $\widehat{\tau}^-(\omega, x)$ and $\widehat{X}^-(\omega, x)$ are defined by (1.10). *For $1 \leqslant p < \infty$ the solution satisfies the estimates*

$$\|I\|_{L^p(D_j)} \leqslant \big(\beta_j^{-p} \|f\|^p_{L^p(D_j)} + \beta_j^{-1} \|g\|^p_{\widehat{L}^p(\Gamma_j^-)}\big)^{1/p}, \tag{3.16}$$

$$\|\omega \cdot \nabla I\|_{L^p(D_j)} \leqslant 2\big(\|f\|^p_{L^p(D_j)} + \beta_j^{p-1} \|g\|^p_{\widehat{L}^p(\Gamma_j^-)}\big)^{1/p}, \tag{3.17}$$

and for $p = \infty$

$$\|I\|_{L^\infty(D_j)} \leqslant \max\big\{\beta_j^{-1} \|f\|_{L^\infty(D_j)},\, \|g\|_{L^\infty(\Gamma_j^-)}\big\}, \tag{3.18}$$

$$\|\omega \cdot \nabla I\|_{L^\infty(D_j)} \leqslant 2 \max\big\{\|f\|_{L^\infty(D_j)},\, \beta_j \|g\|_{L^\infty(\Gamma_j^-)}\big\}. \tag{3.19}$$

Proof. We first assume that $f \in L^1(D_j)$ and $g \in \widehat{L}^1(\Gamma_j^-)$. We pass from the variables $(\omega, x) \in \widehat{D}_j$ to the variables $(\omega, x', t) \in Q_j^-$, where $x = x' + \omega t$, $x' = \widehat{X}^-(\omega, x)$, $t = \widehat{\tau}^-(\omega.x)$, $(\omega, x') \in \widehat{S}_j^-$. We recall that $\text{meas}\,(\Gamma_j^- \setminus \widehat{S}_j^-; \widehat{d}\Gamma^-) = 0$ and $\text{meas}\,(D_j \setminus \widehat{D}_j; d\omega dx) = 0$.

By Theorem 2.4 and Corollary 2.4, the sought function $I \in \mathscr{W}^1(D_j)$ can be modified on a set of measure zero in such a way that the obtained function is a solution to the problem (3.13), (3.14) if and only if for almost all $(\omega, x') \in \widehat{S}_j^-$ the function $I(\omega, x' + t\omega)$, regarded as a function of the variable t, is absolutely continuous on $(0, \widehat{\tau}^+(\omega, x))$; moreover,

$$\frac{d}{dt} I(\omega, x' + t\omega) + \beta_j I(\omega, x' + t\omega) = f(\omega, x' + t\omega) \quad \text{for a.e. } t \in (0, \widehat{\tau}^+(\omega, x)), \tag{3.20}$$

$$\lim_{t \to 0^+} I(\omega, x' + t\omega) = g(\omega, x'). \tag{3.21}$$

Thus, a function $I \in \mathscr{W}^1(D_j)$ is a solution to the problem (3.13), (3.14) if and only if for almost all $(\omega, x') \in \widehat{S}_j^-$ and all $t \in (0, \widehat{\tau}^+(\omega, x'))$

$$I(\omega, x' + t\omega) = e^{-\beta_j t} g(\omega, x') + \int_0^t e^{-\beta_j (t-\tau)} f(\omega, x' + \tau\omega) \, d\tau. \tag{3.22}$$

The existence and uniqueness of a solution $I \in \mathscr{W}^1(D_j)$ are established. Formula (3.15) is obtained from (3.22) by passing to the variables (x, t).

Now, $f \in L^p(D_j)$ and $g \in \widehat{L}^p(\Gamma_j^-)$, $1 \leqslant p < \infty$. Let $I^{[N]} = \max\{\min\{I, N\}, -N\}$, where $N > 0$ is a parameter. We denote by χ_N the characteristic function of the set $\{(\omega, x) \in D_j \mid |I(x.\omega)| < N\}$. Multiplying Equation (3.20) by $p\,\chi_N |I^{[N]}|^{p-1} \operatorname{sgn}(I^{[N]})$, we get

$$\omega \cdot \nabla |I^{[N]}|^p + \beta_j p\, \chi_N |I^{[N]}|^p = p\, \chi_N |I^{[N]}|^{p-1} \operatorname{sgn}(I^{[N]}) f \leqslant \beta_j (p-1) \chi_N |I^{[N]}|^p + \beta_j^{1-p} |f|^p$$

which implies

$$\omega \cdot \nabla |I^{[N]}|^p + \beta_j \chi_N |I^{[N]}|^p \leqslant \beta_j^{1-p} |f|^p,$$

Integrating the obtained inequality over D_j, we find

$$\begin{aligned} \|I^{[N]}|_{\Gamma_j^+}\|^p_{\widehat{L}^p(\Gamma_j^+)} + \beta_j \|\chi_N I^{[N]}\|^p_{L^p(D_j)} &\leqslant \|I^{[N]}|_{\Gamma_j^-}\|^p_{\widehat{L}^p(\Gamma_j^-)} + \beta_j^{1-p} \|f\|^p_{L^p(D_j)} \\ &\leqslant \|g\|^p_{\widehat{L}^p(\Gamma_j^-)} + \beta_j^{1-p} \|f\|^p_{L^p(D_j)}. \end{aligned}$$

Removing the first term on the left-hand side of the obtained inequality and passing to the limit as $N \to \infty$, we arrive at the inequality

$$\beta_j \|I\|^p_{L^p(D_j)} \leqslant \|g\|^p_{\widehat{L}^p(\Gamma_j^-)} + \beta_j^{1-p} \|f\|^p_{L^p(D_j)}.$$

Thus, $I \in L^p(D_j)$ and the estimate (3.16) holds. If $f \in L^\infty(D)$ and $g \in L^\infty(\Gamma_j^-)$, then, passing to the limit as $p \to \infty$, we obtain the estimate (3.18).

From the above-obtained estimates and Equation (3.20) we conclude that $\omega \cdot \nabla I \in L^p(D_j)$ and the estimates (3.17), (3.19) hold. □

Owing to formula (3.15), the following assertion is obvious.

Corollary 3.1. *Let I be a solution to the problem* (3.13), (3.14).

1. *Let $f \geqslant 0$ and $g \geqslant 0$. Then $I \geqslant 0$.*
2. *Let $f \leqslant \beta_j M_j$ in D_j, and let $g \leqslant M_j$ in Γ_j^-, where $M_j = \text{const}$. Then $I \leqslant M_j$ in D_j.*

3.7. Existence, uniqueness, and a priori estimates

We set

$$F_+ = \max\{F, 0\},\ J_{*+} = \max\{J_*, 0\},\ I_+ = \max\{I, 0\},$$
$$F_- = \min\{F, 0\},\ J_{*-} = \min\{J_*, 0\},\ I_- = \min\{I, 0\}.$$

Theorem 3.2. *Let* $1 \leqslant p \leqslant \infty$, *and let* $I \in \widetilde{\mathscr{W}}^p_-(D)$ *be a solution to the problem* (3.11), (3.12). *Then for* $1 \leqslant p < \infty$

$$\|\varkappa^{1/p} k^{-2/q} I_\pm\|_{L^p(D)} \leqslant \Big(\|\varkappa^{1/p} k^{2/p} F_\pm\|^p_{L^p(D)} + \frac{1}{\pi^{p-1}} \|J_{*\pm}\|^p_{\widehat{L}^{1,p}(\overset{*}{S}{}^-)}\Big)^{1/p}, \tag{3.23}$$

$$\|\varkappa^{1/p} k^{-2/q} I\|_{L^p(D)} \leqslant \Big(\|\varkappa^{1/p} k^{2/p} F\|^p_{L^p(D)} + \frac{1}{\pi^{p-1}} \|J_*\|^p_{\widehat{L}^{1,p}(\overset{*}{S}{}^-)}\Big)^{1/p}, \tag{3.24}$$

$$\|\varkappa^{-1/q} k^{-2/q}\, \omega \cdot \nabla I\|_{L^p(D)} \leqslant \frac{2}{1 - \varpi_{\max}} \Big(\|\varkappa^{1/p} k^{2/p} F\|_{L^p(D)} + \frac{1}{\pi^{p-1}} \|J_*\|^p_{\widehat{L}^{1,p}(\overset{*}{S}{}^-)}\Big) \tag{3.25}$$

and for $p = \infty$

$$\|k^{-2} I_\pm\|_{L^\infty(D)} \leqslant \max\Big\{\|F_\pm\|_{L^\infty(D)}, \frac{1}{\pi}\|J_{*\pm}\|_{\widehat{L}^{1,\infty}(\overset{*}{S}{}^-)}\Big\}, \tag{3.26}$$

$$\|k^{-2} I\|_{L^\infty(D)} \leqslant \max\Big\{\|F\|_{L^\infty(D)}, \frac{1}{\pi}\|J_*\|_{\widehat{L}^{1,\infty}(\overset{*}{S}{}^-)}\Big\}, \tag{3.27}$$

$$\|\varkappa^{-1} k^{-2}\, \omega \cdot \nabla I\|_{L^\infty(D)} \leqslant \frac{2}{1 - \varpi_{\max}} \max\Big\{\|F\|_{L^\infty(D)}, \frac{1}{\pi}\|J_*\|_{\widehat{L}^{1,\infty}(\overset{*}{S}{}^-)}\Big\}. \tag{3.28}$$

Here, $q = p'$ *and* $\varpi_{\max} = \max\limits_{1 \leqslant j \leqslant m} \dfrac{s_j}{\varkappa_j + s_j} < 1$.

The proof of this theorem literally repeats the proof of Theorem 5.2 in [1]. Naturally, one should take into account that $\widetilde{\mathscr{W}}^p_-(D) \subset \widehat{\mathscr{W}}^p(D)$ and the required properties of functions in the space $\mathscr{W}^p(D)$ are established for domains with Lipschitz piecewise smooth boundaries in Section 2 of the present paper.

Corollary 3.2. *Let* I *be a solution to the problem* (3.11), (3.12).

1. *If* $F \leqslant 0$, $J_* \leqslant 0$, *then* $I \leqslant 0$.
2. *If* $F \geqslant 0$, $J_* \geqslant 0$, *then* $I \geqslant 0$.
3. *If* $F = 0$, $J_* = 0$, *then* $I = 0$.

Corollary 3.3. *If a solution to the problem* (3.11), (3.12) *exists, then it is unique.*

Theorem 3.3. *Let* $F \in L^\infty(D)$, $J_* \in L^{1,\infty}(\overset{*}{S}{}^-)$. *Then the problem* (3.11), (3.12) *has a unique solution* $I \in \widetilde{\mathscr{W}}^\infty_-(D)$.

The proof of this theorem repeats the proof of Theorem 5.3 in [1]. One also should use Theorem 3.1 and the properties of functions in the space $\mathscr{W}^\infty(D)$ which are established for domains with Lipschitz piecewise smooth boundaries in Section 2 of the present paper.

Theorem 3.4. *Let $F \in L^p(D)$, $J_* \in \widehat{L}^{1,p}(\overset{*}{S}{}^-)$, $1 \leqslant p < \infty$. Then the problem* (3.11), (3.12) *has a unique solution $I \in \widetilde{\mathscr{W}}{}^p_-(D)$.*

Proof. We set $F_n = \max\{-n, \min\{F, n\}\}$, $J_{*,n} = \max\{-n, \min\{J_*, n\}\}$, $n \geqslant 1$. Since $F_n \in L^\infty(D)$, $J_{*,n} \in \widehat{L}^{1,\infty}(\overset{*}{S}{}^-)$, from Theorem 3.3 it follows that for every $n \geqslant 1$ the problem

$$\omega \cdot \nabla I_n + \beta I_n = s\mathscr{S}(I_n) + \varkappa k^2 F_n, \quad (x, \omega) \in D, \tag{3.29}$$

$$I_n|_{\Gamma^-} = \mathfrak{B}_d(I_n|_{\Gamma^+}) + \mathfrak{C}_d(J_{*,n}), \quad (\omega, x) \in \Gamma^- \tag{3.30}$$

has a unique solution $I_n \in \widetilde{\mathscr{W}}{}^\infty_-(D) \subset \widetilde{\mathscr{W}}{}^p_-(D)$. Since the problem is linear, the estimates of Theorem 3.2 imply

$$\|I_n - I_\ell\|_{\mathscr{W}^p(D)} \leqslant C(\|F_n - F_\ell\|_{L^p(D)} + \|J_{*,n} - J_{*,\ell}\|_{\widehat{L}^{1,p}(\overset{*}{S}{}^-)}) \quad \forall\, n \geqslant 1,\ \forall \ell \geqslant 1.$$

Since $F_n \to F$ in $L^p(D)$ and $J_{*,n} \to J_*$ in $\widehat{L}^{1,p}(\overset{*}{S}{}^-)$ as $n \to \infty$, the sequence $\{I_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{W}^p(D)$. Since $\widetilde{\mathscr{W}}{}^p_-(D)$ is closed in $\mathscr{W}^p(D)$, there exists a function $I \in \widetilde{\mathscr{W}}{}^p_-(D)$ such that $I_n \to I$ in $\widetilde{\mathscr{W}}{}^p_-(D)$ as $n \to \infty$.

By Theorem 2.6, the space $\widetilde{\mathscr{W}}{}^p_-(D)$ is continuously embedded into $L^p_{\text{const}}(\Gamma^-)$ and into $\widehat{L}^p(\Gamma^+)$. Therefore, $I_n|_{\Gamma^-} \to I|_{\Gamma^-}$ in $L^p_{\text{const}}(\Gamma^-)$ and $I_n|_{\Gamma^+} \to I|_{\Gamma^+}$ in $\widehat{L}^p(\Gamma^+)$ as $n \to \infty$. As a consequence, $\mathfrak{B}(I_n|_{\Gamma^+}) \to \mathfrak{B}(I|_{\Gamma^+})$ in $L^p_{\text{const}}(\Gamma^-)$. Furthermore, $\mathfrak{C}(J_{*,n}) \to \mathfrak{C}(J_*)$ in $L^p_{\text{const}}(\Gamma^-)$.

The limit passage in (3.29), (3.30) as $n \to \infty$ leads to the equalities (3.11) and (3.12).

The existence of solutions $I \in \widetilde{\mathscr{W}}{}^p_-(D)$ is proved. The uniqueness follows from Corollary 3.3. The theorem is proved. ☐

3.8. Continuous dependence of the solution to the problem (3.11), (3.12) on data

We consider the sequence of problems

$$\omega \cdot \nabla I^{(n)} + \beta^{(n)} I^{(n)} = s^{(n)} \mathscr{S}^{(n)}(I^{(n)}) + \varkappa^{(n)} (k^{(n)})^2 F^{(n)}, \quad (\omega, x) \in D, \tag{3.31}$$

$$I^{(n)}|_{\Gamma^-} = \mathfrak{B}^{(n)}_d(I^{(n)}|_{\Gamma^+}) + \mathfrak{C}^{(n)}_d(J^{(n)}_*), \quad (\omega, x) \in \Gamma^-, \tag{3.32}$$

corresponding to the sequences of data $\{\varkappa^{(n)}_j\}_{n=1}^\infty$, $\{s^{(n)}_j\}_{n=1}^\infty$, $\{k^{(n)}_j\}_{n=1}^\infty$, $\{\theta^{(n)}_j\}_{n=1}^\infty$, $\{\rho^{\pm,(n)}_j\}_{n=1}^\infty$ for $1 \leqslant j \leqslant m$, $\{\rho^{-,(n)}_{ij}\}_{n=1}^\infty$ for $i \neq j$ and $\{F^{(n)}\}_{n=1}^\infty \subset L^1(D)$, $\{J^{(n)}_*\}_{n=1}^\infty \subset \widehat{L}^1(\overset{*}{S}{}^-)$, $1 \leqslant p \leqslant \infty$.

In Equation (3.31), $\varkappa^{(n)}(x) = \varkappa^{(n)}_j > 0$, $s^{(n)}(x) = s^{(n)}_j \geqslant 0$, $k^{(n)}(x) = k^{(n)}_j > 1$ for $x \in G_j$, $1 \leqslant j \leqslant m$; moreover, $\beta^{(n)} = \varkappa^{(n)} + s^{(n)}$. The operator $\mathscr{S}^{(n)}$ is defined by

$$\mathscr{S}^{(n)}(I)(\omega, x) = \frac{1}{4\pi} \int\limits_\Omega \theta^{(n)}_j(\omega' \cdot \omega) I(\omega', x)\, d\omega', \quad (\omega, x) \in D_j, \quad 1 \leqslant j \leqslant m;$$

moreover, it is assumed that

$$\theta^{(n)}_j \in L^1(-1, 1), \quad \theta^{(n)}_j \geqslant 0, \quad \frac{1}{2} \int\limits_{-1}^{1} \theta^{(n)}_j(\mu)\, d\mu = 1, \quad 1 \leqslant j \leqslant m.$$

It is also assumed that $\rho_j^{\pm,(n)} \in L^\infty(\partial G_j)$, $0 < \rho_j^{\pm,(n)} < 1$, $1 - \rho_j^{-,(n)} = \dfrac{1}{(k_j^{(n)})^2}(1 - \rho_j^{+,(n)})$, $1 \leqslant j \leqslant m$, $\overline{\rho}^{+,(n)} = \max\limits_{1\leqslant j\leqslant m} \|\rho_j^{+,(n)}\|_{L^\infty(\partial G_j)} < 1$. Furthermore, $\rho_{ij}^{-,(n)} \in L^\infty(\partial G_{ij})$, $0 < \rho_{ij}^{-,(n)} < 1$, $1 - \rho_{ji}^{-,(n)} = (1 - \rho_{ij}^{-,(n)})\dfrac{(k_i^{(n)})^2}{(k_j^{(n)})^2}$. The only difference of the operators $\mathfrak{B}_d^{(n)}$ and $\mathfrak{C}_d^{(n)}$ from the corresponding operators $\mathfrak{B}_d$ and $\mathfrak{C}_d$ is that, in their definitions, the functions ρ^+, ρ_{ij}^- are replaced with $\rho^{+,(n)}$ and $\rho_{ij}^{-,(n)}$, whereas the refraction exponents k_j are replaced with $k_j^{(n)}$.

Theorem 3.5. *Assume that* $\{I^{(n)}\}_{n=1}^\infty$ *is a sequence solutions to the problems* (3.31), (3.32) *and* I *is a solution to the problem* (3.11), (3.12). *Let the following limit relations hold as* $n \to \infty :$ $\varkappa_j^{(n)} \to \varkappa_j$, $s_j^{(n)} \to s_j$, $k_j^{(n)} \to k_j$, $\theta_j^{(n)} \to \theta_j$ *in* $L^1(-1,1)$, $\rho_j^{+,(n)} \to \rho_j^+$ *in* $L^\infty(\partial G_j)$ *for all* $1 \leqslant j \leqslant m$, $\rho_{ij}^{-,(n)} \to \rho_{ij}^-$ *in* $L^\infty(\partial G_{ij})$ *for all* $i \neq j$ *such that* $\operatorname{meas}(\partial G_{ij}; d\sigma) > 0$, $F^{(n)} \to F$ *in* $L^1(D)$ *and* $J_*^{(n)} \to J_*$ *in* $\widehat{L}^1(\overset{*}{S}{}^-)$. *Then* $I^{(n)} \to I$ *in* $\mathscr{W}^1(D)$ *as* $n \to \infty$.

Assume, in addition, that $F \in L^p(D)$, $J_* \in \widehat{L}^{1,p}(\overset{*}{S}{}^-)$, $\{F^{(n)}\}_{n=1}^\infty \subset L^p(D)$, $\{J_*^{(n)}\}_{n=1}^\infty \subset \widehat{L}^p(\overset{*}{S}{}^-)$ *with some* $p \in (1,\infty]$, $\sup\limits_{n\geqslant 1} \|F^{(n)}\|_{L^p(D)} < \infty$, $\sup\limits_{n\geqslant 1} \|J_*^{(n)}\|_{\widehat{L}^{1,p}(\overset{*}{S}{}^-)} < \infty$. *Then* $I^{(n)} \to I$ *in* $\mathscr{W}^q(D)$ *for all* $q \in [1,p)$ *as* $n \to \infty$.

The proof of this theorem is the same as that of Theorem 4.2 in [2].

3.9. The conjugate boundary value problem

The boundary value problem

$$\omega \cdot \nabla I + \beta I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D, \tag{3.33}$$

$$I|_{\Gamma^-} = \mathfrak{B}_d(I|_{\Gamma^+}), \quad (\omega, x) \in \Gamma^-, \tag{3.34}$$

i.e., the problem (3.11), (3.12) with the homogeneous boundary condition will be referred to as the *main problem*. We denote by $\mathscr{A}_d$ the resolving operator for the main problem which with a function $F \in L^p(D)$ associates the solution $I = \mathscr{A}_d(F)$. By Theorems 3.2–3.4, this operator is a linear bounded operator acting from $L^p(D)$ to $\widetilde{\mathscr{W}}_-^p(D)$ for all $1 \leqslant p \leqslant \infty$.

Following [4], we introduce the operator U by the formula

$$Uf(\omega, x) = f(-\omega, x).$$

For the boundary valued problem (3.33), (3.34) we consider the conjugate problem

$$-\,\omega \cdot \nabla I^* + \beta I^* = s\mathscr{S}(I^*) + \varkappa k^2 F^*, \quad (\omega, x) \in D, \tag{3.35}$$

$$I^*|_{\Gamma^+} = \overline{\mathfrak{B}}_d^*(I^*|_{\Gamma^-}), \quad (\omega, x) \in \Gamma^+, \tag{3.36}$$

where $F^* \in L^q(D)$, $1 \leqslant q \leqslant \infty$, and $\overline{\mathfrak{B}}_d^* = U\mathfrak{B}_d U$ is a linear bounded operator acting from $\widehat{L}^q(\Gamma^-)$ to $L^q_{\text{const}}(\Gamma^+)$, $1 \leqslant q \leqslant \infty$.

By a *solution* to the conjugate problem we mean a function $I^* \in \widetilde{\mathscr{W}}_+^q(D)$ that satisfies Equation (3.35) almost everywhere in D and the condition (3.36) almost everywhere on Γ^+.

It is easy to see that a function I^* is a solution to the conjugate problem if and only if $I = UI^*$ is a solution to the main problem with $F = UF^*$. Thus, for any $F^* \in L^q(D)$, $1 \leqslant q \leqslant \infty$, the conjugate problem has a unique solution $I^* \in \widetilde{\mathscr{W}}^q_+(D)$; moreover, the resolving operator $\mathscr{A}^*_d$ of the conjugate problem associating with F^* the solution I^* is connected with the resolving operator of the main problem by the equality

$$\mathscr{A}^*_d = U\mathscr{A}_d U. \tag{3.37}$$

Remark 3.4. Formula (3.37) is similar to the formula due to Vladimirov [4] for the problem with the "shooting" boundary condition.

From formula (3.37) and properties of the operator $\mathscr{A}_d$ it follows that the operator $\mathscr{A}^*_d$ is a linear bounded operator acting from $L^q(D)$ to $\widetilde{\mathscr{W}}^q_+(D)$ for all $1 \leqslant q \leqslant \infty$.

Theorem 3.6. *The operator* $\varkappa\mathscr{A}^*_d$ *is the adjoint of the operator* $\varkappa\mathscr{A}_d$ *in the following sense*:

$$(\varkappa\mathscr{A}_d(F), F^*)_D = (F, \varkappa\mathscr{A}^*_d(F^*))_D \quad \forall\, F \in L^p(D),\ \forall\, F^* \in L^q(D) \tag{3.38}$$

for all $1 \leqslant p \leqslant \infty$, $q = p'$.

The proof of this theorem repeats the proof of Theorem 5.1 in [2] with the only difference that the arguments are now valid for all $1 \leqslant p \leqslant \infty$.

We consider the main problem with an isotropic radiation source $F = F(x)$. We introduce the operator $\langle\mathscr{A}_d\rangle_\Omega : L^p(G) \to L^p(G)$ by the formula

$$\langle\mathscr{A}_d\rangle_\Omega(F)(x) = \frac{1}{4\pi}\int\limits_\Omega \mathscr{A}_d(F)(\omega, x)\, d\omega.$$

As in [2], from Theorem 3.6 we obtain the following assertion.

Theorem 3.7. *The operator* $\varkappa\langle\mathscr{A}_d\rangle_\Omega$ *is selfadjoint in the following sense*:

$$(\varkappa\langle\mathscr{A}_d\rangle_\Omega(F), F^*)_G = (F, \varkappa\langle\mathscr{A}_d\rangle_\Omega(F^*))_G \quad \forall\, F \in L^p(G),\ \forall\, F^* \in L^q(G) \tag{3.39}$$

for all $1 \leqslant p \leqslant \infty$, $q = p'$.

We emphasize that, unlike [2], the identity (3.39) holds for all $1 \leqslant p \leqslant \infty$.

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