DOI 10.1007/s10958-016-2879-x
Journal of Mathematical Sciences, Vol. 215, No. 6, June, 2016

A HIGHER ORDER ASYMPTOTIC EXPANSION OF THE KRAWTCHOUK POLYNOMIALS

A. R. Minabutdinov*

UDC 517.987, 519.214.2

The paper extends a classical result on the convergence of the Krawtchouk polynomials to the Hermite polynomials. We provide a uniform asymptotic expansion in terms of Hermite polynomials and obtain explicit expressions for a few first terms of this expansion. The research is motivated by the study of ergodic sums of the Pascal adic transformation. Bibliography: 10 titles.

1. Introduction

Let $0 < p < 1$, $q = 1 - p$, and let N be a positive integer. The nonnormalized Krawtchouk polynomials of a variable x can be defined by the identity

$$K_n(x,p,N) = {}_2F_1\left[\begin{matrix}-x,\ -n\\ -N\end{matrix};\frac{1}{p}\right], \tag{1}$$

where x and n are in $\{0,1,\dots,N\}$ and ${}_2F_1$ is the Gauss hypergeometric function. The normalized Krawtchouk polynomials are usually defined as follows:

$$k_n^{(p)}(x,N) = (-p)^n\binom{N}{n}K_n(x,p,N). \tag{2}$$

The second argument N is usually omitted, so we write $k_n^{(p)}(x)$ instead of $k_n^{(p)}(x,N)$. The Krawtchouk polynomials have multiple applications in probability theory, classical coding theory, cryptography, stochastic processes, and other fields, see, e.g., [2] and references therein. These polynomials form an orthogonal system on the discrete set $\{0,1,2,\dots,N\}$ with the weight function

$$\rho(x) = \frac{N!\,p^x q^{N-x}}{\Gamma(1+x)\Gamma(N+1-x)}$$

and the orthogonality relation

$$\sum_{x=0}^{N} k_i^{(p)}(x)k_j^{(p)}(x)\rho(x) = \binom{N}{j}(pq)^j\delta_{ij}, \quad i,j = 0,1,\dots,N.$$

The Krawtchouk polynomials satisfy the following Rodrigues-type formula (see [5, Sec 2, (22a)]):

$$k_n^{(p)}(x) = \frac{(-q)^n}{n!}\frac{\Delta^n\big(\rho(x)x^{\underline{n}}\big)}{\rho(x)}, \tag{3}$$

where $\Delta f(x) = f(x+1) - f(x)$ and $y^{\underline{k}} = y(y-1)\dots(y-k+1)$. Formula (3) is often taken as a definition of the Krawtchouk polynomials, see, e.g., [7]. It also shows that the Krawtchouk polynomial $k_n^{(p)}(\,\cdot\,)$ can be regarded as an analytic function on $[0,N]$.

Finally, it was recently found in [4] that the Krawtchouk polynomial $(-2p)^nK_n(k,p,N)$ has a natural interpretation as the ergodic sum along the tower $\tau_{N,k}$ of the Pascal adic transformation

*National Research University Higher School of Economics, St.Petersburg, Russia, e-mail: aminabutdinov@gmail.com.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 436, 2015, pp. 174–188. Original article submitted September 21,2015.

for the *orthogonalized*[1] *Walsh–Paley function* w_t^p where the sum of the binary digits in the binary representation of $t \in \mathbb{N}$ equals n (see [4] for details).

There is a classical result on the convergence of the (properly renormalized) Krawtchouk polynomials to the Hermite polynomials (see [3]):

$$\lim_{N\to\infty} \left(\frac{2}{Npq}\right)^{n/2} n!\, k_n^{(p)}(\widehat{x}) = H_n(x), \tag{4}$$

with $\widehat{x} = Np + (2Npq)^{1/2}x$ and $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. This result was extended by Sharapudinov in [7], where he obtained the asymptotic formula

$$(2Npq\pi n!)^{1/2}(Npq)^{-n/2}\rho(\widehat{x})e^{x^2/2}k_n^{(p)}(\widehat{x}) = e^{-x^2/2}(2^n n!)^{-1/2}H_n(x) + O(n^{7/4}N^{-1/2}), \tag{5}$$

with $\widehat{x} = Np + (2Npq)^{1/2}x$, $n = O(N^{1/3})$, $x = O(n^{1/2})$.

Let A be a positive real number. In this paper, we are interested in a higher order, uniform in $v \in [-A\sqrt{N}, A\sqrt{N}]$, asymptotic expansion of the Krawtchouk polynomials of the form

$$k_n^{(p)}(\widehat{x}) = \sum_{j=0}^{M} c_{j+1}(v)N^{[n/2]-j} + o(N^{[n/2]-M}), \tag{6}$$

where $\widehat{x} = Np + v$, $n = O(1)$, $M \in \mathbb{N} \cup \{0\}$, and $[t]$ is the integer part of a real number t. Our main result is a uniform asymptotic expansion of $\rho(\widehat{x})k_n^{(p)}(\widehat{x})$ in terms of Hermite polynomials stated in Theorem 2. Our approach is based on a result by V. V. Petrov ([6]) extending the local limit theorem (LLT).

We are especially interested in an expression for the first nonconstant term $c_j(v)$ in the expansion (6), in connection with the study of ergodic sums of the Pascal adic transformation in [4]. It turns out that the value of j depends on the parity of the index n. In Corollary 1, under the additional assumption $v = o(N^{1/3})$, we obtain elegant explicit expressions for $c_1(v)$ and $c_2(v)$ for odd and even values of n, respectively.

Recently, considerable interest has been focused on the asymptotics of the Krawtchouk polynomials as the parameter N goes to infinity (see, e.g., [1] and references therein). Dai and Wong [1] considered (among many others) the case where $x = O(1)$ as $N \to \infty$. However, they excluded the case $n \approx Np$. Note that the self-duality relation $K_x(n,p,N) = K_n(x,p,N)$ implies that the local one-term asymptotic expansion in this case follows already from (4).

2. The main results

Using the identity $\Delta^s x^{\underline{n}} = n^s x^{\underline{n-s}}$, we rewrite the Rodrigues formula (3) as follows:

$$\begin{aligned}\rho(x)k_n^{(p)}(x) &= \frac{(-q)^n}{n!}\Delta^n\left[\rho(x)x^{\underline{n}}\right] = \frac{(-q)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\rho(x)\,\Delta^k(x+n-k)^{\underline{n}} \\ &= \frac{(-q)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}n^{\underline{k}}\,(x+n-k)^{\underline{(n-k)}}\Delta^{n-k}\rho(x).\end{aligned} \tag{7}$$

Further, we consider the term $\Delta^s\rho(x)$, $s \geq 0$, separately. We introduce the h-step forward difference operator $\Delta_h f(x) = f(x+h) - f(x)$, $\Delta_h^n f(x) = \Delta_h(\Delta_h^{n-1}f(x))$, $n \geq 2$. Following [7], we set h equal to $\frac{1}{\sqrt{2Npq}}$, which allows us to write[2]

$$\Delta^s\rho(\widehat{x}) = \Delta_h^s\,\rho(\widehat{x}(x)), \tag{8}$$

[1]With respect to the Bernoulli (p,q)-measure.

[2]Note that Δ is the forward difference in $\widehat{x}$, while Δ_h is the h-step forward difference in x.

where $\widehat{x} = Np + (2Npq)^{1/2}x$. In fact, the asymptotic relation (4), as well as (5), follows[3] from the LLT applied to $\rho(\widehat{x})$ and the mean value theorem $\Delta_h^n f(x) = h^n \frac{d^n}{dx^n} f(x + nh\theta)$, $\theta \in (0,1)$, combined with a proper estimate on the residual term (see the details in [7]). In order to obtain a higher order approximation, we regard the function $\rho(x)$ as the probability of x successes in a sequence of N independent (p,q)-Bernoulli trials and use Theorem 16 from [6, Sec. 3] for the Bernoulli distribution.[4]

Theorem 1. *Let M be a nonnegative integer and $\sigma^2 = pq$. The following asymptotic expansion holds uniformly in x such that $Np + (2Npq)^{1/2}x \in \mathbb{Z}$:*

$$(1 + |x|^{M+2})\left(\sqrt{N}\rho(\widehat{x}) - \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2} \sum_{\nu=0}^{M} \frac{\widetilde{q}_\nu(x)}{N^{\nu/2}}\right) = o\left(\frac{1}{N^{M/2}}\right), \tag{9}$$

where $\widehat{t} = Np + (2Npq)^{1/2}t$ and functions $\widetilde{q}_\nu$ are defined as follows:

$$\widetilde{q}_\nu(x) = \sum \frac{1}{2^{(\nu/2+s)}} H_{\nu+2s}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!}\left(\frac{\gamma_{m+2}}{(m+2)!\sigma^{m+2}}\right)^{k_m}, \tag{10}$$

with γ_i, $i \geq 0$, being the cumulants of the Bernoulli (p,q)-distribution, and the summation in the right-hand side being over all nonnegative solutions $(k_1, k_2, \ldots, k_\nu)$ of the equation $k_1 + 2k_2 + \cdots + \nu k_\nu = \nu$ such that $s = k_1 + k_2 + \cdots + k_\nu$.

In order to obtain a higher order approximation for Δ_h^s, we use the formal representation $\Delta_h = e^{hD} - 1$ where $D = \frac{d}{dx}$ is the difference operator, which yields

$$\Delta_h^s = (e^{hD} - 1)^s = \sum_{i=s}^{\infty} a_{s,i-s}(hD)^i. \tag{11}$$

For any nonnegative integer K and any analytic real-valued function f, we can truncate the series and write simply

$$\Delta_h^s f(x) = \sum_{i=0}^{K} a_{s,i}\, D^{s+i} f(x) h^{s+i} + O(h^{K+s+1}). \tag{12}$$

The coefficients $a_{s,j}$ can be found by the multinomial theorem as follows:

$$a_{s,j} = \sum s! \prod_{r=1}^{j+1} \frac{1}{k_r!}\left(\frac{1}{r!}\right)^{k_r}, \tag{13}$$

where s and j are nonnegative integers and the summation in the right-hand side of (13) is over all nonnegative solutions $(k_1, k_2, \ldots, k_j)$ of the equation $k_1 + 2k_2 + \cdots + jk_j = j$ satisfying $k_1 + k_2 + \cdots + k_j = s$. In particular, we have

$$a_{s,0} = 1, \quad a_{s,1} = \frac{s}{2}, \quad a_{s,2} = \frac{s(3s+1)}{24}, \quad \text{and so on.} \tag{14}$$

[3]Other approaches are based on the convergence of the difference equation whose polynomial solutions define the Krawtchouk polynomials to the differential equation defining the Hermite polynomials (see the details, e.g., in [5]) or on the convergence of generating functions (see [8]).

[4]In [6], a slightly different definition of the Hermite polynomials was used: $\mathrm{He}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$. These polynomials are related to the polynomials $H_n(x)$ by the equality $\mathrm{He}_n(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right)$.

Theorem 1 suggests that in order to obtain an asymptotic expansion for $\Delta_h^s \rho(\widehat{x})$, we need asymptotic expansions for $\Delta_h^s \big(e^{-x^2}\widetilde{q}_\nu(x)\big)$, $\nu \geq 0$. Denote by $b_{\nu,s}$ the coefficients

$$\frac{1}{2^{(\nu/2+s)}} \prod_{m=1}^{\nu} \frac{1}{k_m!}\Big(\frac{\gamma_{m+2}}{(m+2)!\sigma^{m+2}}\Big)^{k_m}$$

arising in the right-hand side of formula (10). Thus we have $\widetilde{q}_\nu(x) = \sum b_{\nu,s} H_{\nu+2s}(x)$. Denote $\widetilde{g}_{\nu,r}(x) = e^{x^2}\frac{d^r}{dx^r}e^{-x^2}\widetilde{q}_\nu(x)$. For $e^{x^2}\frac{d^s}{dx^s}\big(e^{-x^2}H_n(x)\big)$, we have the identity

$$e^{x^2}\frac{d^s}{dx^s}\big(e^{-x^2}H_n(x)\big) = (-1)^n e^{x^2}\frac{d^s}{dx^s}\big(\frac{d^n}{dx^n}e^{-x^2}\big) = (-1)^s H_{n+s}(x), \tag{15}$$

which implies

$$\widetilde{g}_{\nu,r}(x) = e^{x^2}\frac{d^r}{dx^r}e^{-x^2}\widetilde{q}_\nu(x) = \sum (-1)^r b_{\nu,s} H_{\nu+2s+r}(x),$$

where the summation limits in the right-hand side are the same as in (10). Let A be a positive real number and r be a nonnegative integer. In the appendix, it is shown that we can differentiate[5] the asymptotic expansion (9) preserving the uniform estimate on the residual term for *any*[6] x from the set $[-A, A]$:

$$\frac{d^r}{dx^r}\sqrt{N}\rho(\widehat{x}(x)) = \frac{1}{\sqrt{2\pi}\sigma}e^{-x^2}\sum_{\nu=0}^{M}\frac{\widetilde{g}_{\nu,r}(x)}{N^{\nu/2}} + o\Big(\frac{1}{N^{M/2}}\Big). \tag{16}$$

It is instructive to compare the above result with Theorem 7 from [6, Sec. 6].

Remark 1. Since the parameter M in Theorem 1 and formula (16) is an arbitrary positive integer, we can write the residual term as $O\Big(\frac{1}{N^{(M+1)/2}}\Big)$ instead of $o\Big(\frac{1}{N^{M/2}}\Big)$.

Using the expansions (16) and (12) and taking into account that $h \sim N^{-1/2}$, we obtain the following asymptotic expansion for $\Delta^s \rho(\widehat{x})$:

$$\begin{aligned}\sqrt{N}\Delta_h^s \rho(\widehat{x}(x)) &= \sum_{i=0}^{K} a_{s,i}\frac{d^{s+i}}{dx^{s+i}}\sqrt{N}\rho(\widehat{x})h^{s+i} + O(N^{-(K+s+1)/2}) \\ &= \frac{e^{-x^2}}{\sqrt{2\pi}\sigma}\Big(\sum_{i=0}^{K} a_{s,i}\sum_{\nu=0}^{M}\frac{\widetilde{g}_{\nu,r}(x)}{N^{\nu/2}}h^{s+i}\Big) + O(N^{-(K+s+1)/2}) \\ &= \frac{e^{-x^2}}{\sqrt{2\pi}\sigma}\Big(\sum_{i=0}^{K} a_{s,i}\sum_{\nu=0}^{M-i}\frac{\widetilde{g}_{\nu,r}(x)}{N^{\nu/2}}h^{s+i}\Big) + O(N^{-(K+s+1)/2}),\end{aligned} \tag{17}$$

where $K = M+1$, $x = O(1)$, and $\widehat{x}(x) = Np + (2Npq)^{1/2}x$. This expression is written, in fact, in terms of Hermite polynomials, but it can also be rewritten using the parabolic cylinder functions $D_n(x)$, due to their close relation to the Hermite polynomials:

$$D_n(x) = 2^{-n/2}e^{-x^2/4}H_n\big(\frac{x}{\sqrt{2}}\big), \quad n \in \mathbb{N}\cup\{0\}.$$

Denote

$$\psi_s^K(x) = \sum_{i=0}^{K} a_{s,i}\sum_{\nu=0}^{K-1-i}\frac{\widetilde{g}_{\nu,s+i}(x)}{N^{\nu/2}}h^{s+i}$$

where the coefficients $a_{s,j}$, with $j,\ s \in \mathbb{N}\cup\{0\}$, are defined by (13).

[5]For $s = 0$, we use the convention $\frac{d^s}{dx^s}f(x) \equiv f(x)$.
[6]Note that Theorem 1 states this only for $r = 0$ and values of x from a discrete set.

Using the representation (7) together with (17), we obtain our main result providing an asymptotic expansion of the Krawtchouk polynomials in terms of Hermite polynomials.

Theorem 2. *Let M and n be nonnegative integers and A be a positive real number. Let $\widehat{x} - Np = (2Npq)^{1/2}x$. Then the following asymptotic expansion holds uniformly in $|x| \le A$:*

$$\rho(\widehat{x})k_n^{(p)}(\widehat{x}) = \frac{e^{-x^2}}{\sqrt{2\pi N}\sigma}\frac{(-q)^n}{n!}\sum_{k=0}^{M}\binom{n}{k}n^{\underline{k}}\,\overline{(\widehat{x}+n-k)}^{(n-k)}\psi_{n-k}^{M}(x) + O(N^{\frac{n-M-2}{2}}). \tag{18}$$

Proof. Since $h = \frac{1}{\sqrt{2pq}}N^{-1/2}$, we see that $\frac{e^{-x^2}}{\sqrt{2\pi N}\sigma}\psi_s^M(x) = O(N^{-(s+1)/2})$ and $(\widehat{x}+s)^{\underline{s}} = O(N^s)$, whence

$$(\widehat{x}+s)^{\underline{s}}\frac{e^{-x^2}}{\sqrt{2\pi N}\sigma}\psi_s^M(x) = O(N^{(s-1)/2}).$$

Formula (18) follows from the Rodrigues formula (7) by replacing the terms $\Delta_h^s\rho(\widehat{x}(x))$ with their approximations $\frac{e^{-x^2}}{\sqrt{2\pi N}\sigma}\psi_s^M(x)$ obtained in (17). □

In the special case $M = 0$, Theorem 2 reduces to the well-known result

$$k_n^{(p)}(\widehat{x}) = \Big(\frac{Npq}{2}\Big)^{n/2}\frac{H_n(x)}{n!} + o(N^{\frac{n}{2}}),$$

stated above in (4) (here we have used the expansion $\rho(\widehat{x}) = \frac{1}{\sqrt{2\pi N}\sigma}e^{-x^2} + o(1)$ from Theorem 1).

In the general case, formula (18), in fact, yields the expansion[7] (6) (i.e., its right-hand side comprises only powers of N and $v = x\sqrt{2Npq}$, but not of $\sqrt{N} \sim h^{-1}$).

In the special case where $M = 2$ and $v = \widehat{x} - Np = o(N^{1/3})$, we are going to obtain a more natural expression than the (slightly cumbersome) expression (18) suggested by Theorem 2.

First of all, note that for the Hermite polynomials we have the following representations (see, e.g., [5]):

$$H_{2l}(x) = (-1)^l 2^l (2l-1)!!\Big(1 + \sum_{j=1}^{l}\frac{4^j(-l)^{\bar{j}}}{(2j)!}x^{2j}\Big),$$

$$H_{2l+1}(x) = (-1)^l 2^{l+1}(2l+1)!!\Big(x + \sum_{j=1}^{l}\frac{4^j(-l)^{\bar{j}}}{(2j+1)!}x^{2j+1}\Big),$$

where l is a nonnegative integer. The additional assumption $x \to 0$ provides the asymptotic expansions

$$\begin{aligned} H_{2l}(x) &= (-1)^l 2^l (2l-1)!!\big(1 - 2lx^2\big) + o(x^2), \quad && l = 1, 2\dots, \\ H_{2l+1}(x) &= (-1)^l 2^{l+1}(2l+1)!!x + o(x^2), && l = 0, 1, \dots. \end{aligned} \tag{19}$$

For any nonnegative integers l, m, and C, we have the following asymptotic expansion for the falling factorial $(m+C)^{\underline{l}}$ as $m \to \infty$:

$$(m+C)^{\underline{l}} = m^l + \Big(lC - \frac{l(l-1)}{2}\Big)m^{l-1} + O(m^{l-2}). \tag{20}$$

[7]Of course, it makes sense to take $M \le n$.

Corollary 1. *For any sequence $\varepsilon(N)$ such that $\lim_{N\to\infty} \varepsilon(N) = 0$, the following asymptotic expansions hold uniformly in v satisfying $|v| \le \varepsilon(N)N^{1/3}$:*

$$k_{2l}^{(p)}(Np+v) = (-1)^l \frac{(2l-1)!!\big(pqN\big)^l}{(2l)!}\bigg(1 - \frac{9v^2 + t_1 v + t_2}{9pqN} l\bigg) + o(N^{l-1}),$$

where $t_1 = 6(p - \frac{1}{2})(4l-1)$, $t_2 = (l-1)\big(1 + 4l + (16l-5)pq\big)$, and $l \in \mathbb{N}$;

$$k_{2l+1}^{(p)}(Np+v) = (-1)^l \frac{(2l-1)!!\big(pqN\big)^l}{(2l)!} \frac{4l(p-\frac{1}{2}) + 3v}{3} + o(N^l),$$

where $l \in \mathbb{N} \cup \{0\}$.

Proof. Denote $\psi(x) = \frac{\sqrt{2\pi N}\sigma}{e^{-x^2}}$. Let k be a nonnegative integer. Applying Theorem 1 for $M = 2$ and identity (15), we obtain

$$\psi(x)\frac{d^k}{dx^k}\rho(\widehat{x}) = (-1)^k\Big(H_k(x) + \frac{\gamma_3}{2^{3/2}\sigma^3}\frac{H_{3+k}(x)}{3!\sqrt{N}} + \frac{\frac{\gamma_4}{\sigma^4}H_{4+k}(x) + \frac{1}{3!}\big(\frac{\gamma_3}{\sigma^3}\big)^2 H_{6+k}(x)}{4\cdot 4!N}\Big) + o\Big(\frac{1}{N}\Big),$$

where $\gamma_3 = pq(1-2p)$ and $\gamma_4 = -pq(6pq-1)$. Under the additional assumption[8] that $v := \widehat{x} - Np = o(N^{1/3})$, we get a simpler expression:

$$\psi(x)\rho\Big(\frac{v}{\sqrt{2N}\sigma}\Big) = 1 - \frac{1 - pq - 6v(p-q)}{12pqN} + o\Big(\frac{1}{N}\Big). \tag{21}$$

In the same manner, for the odd-order derivatives we have

$$\psi(x)\frac{d^{2l+1}}{dx^{2l+1}}\rho(\widehat{x}) = (2l-1)!!(-2)^l\Big(1 + \frac{36lv^2 + \tau_1 v + \tau_2}{36pqN}\Big) + o\Big(\frac{1}{N}\Big),$$

where $\tau_1 = 6(1-2p)(2l+3)(2l+1)$, $\tau_2 = (2l+1)(2l+3)(1 + l - (1+4l)pq)$; and for the even-order derivatives we obtain

$$\psi(x)\frac{d^{2l}}{dx^{2l}}\rho(\widehat{x}) = (-1)^{l+1}(2l+1)!!2^l\frac{\sqrt{2}}{\sqrt{Npq}}(v + \frac{(1-2p)(2l+3)}{6}) + o\Big(\frac{1}{N}\Big),$$

with $l \in \mathbb{N}$. With some algebra, using the expansions (19) and (20) and ignoring terms of order lower than $[\frac{n-1}{2}]$, we obtain from (18) the required asymptotic expansion (and it naturally depends on the parity of n). □

Taking higher order asymptotic expansions in (19) and (20) and an appropriately high value of M, one can easily obtain other values of c_j, $j \ge 1$, from the expansion (6).

We conclude this paper with a slight modification of Corollary 1 for the function $K_n(x,p,N_1)$ with $N_1 = N - i$, $v = x - Np = o(N^{1/3})$, and $i = O(1)$.

Corollary 2. *For any sequence $\varepsilon(N)$ such that $\lim_{N\to\infty} \varepsilon(N) = 0$, the following asymptotic expansions hold uniformly in v satisfying $|v| \le \varepsilon(N)N^{1/3}$:*

$$K_{2l}(x,p,N-i) = \Big(-\frac{q}{p}\Big)^l\frac{(2l-1)!!}{N^l}\bigg(1 - \frac{9(v+ip)^2 + \widetilde{t}_1(v+ip) + \widetilde{t}_2}{9pqN}l\bigg) + o\big(N^{-l-1}\big),$$

where $\widetilde{t}_1 = 6(p-\frac{1}{2})(4l-1)$, $\widetilde{t}_2 = (l-1)\big(1 + 4l + (16l-5)pq\big) - 9pq(i + 2l - 1)$, and $l \in \mathbb{N}$;

$$K_{2l+1}(x,p,N-i) = \Big(-\frac{q}{p}\Big)^l\frac{(2l+1)(2l-1)!!}{N^{l+1}} \cdot \frac{4l(p-\frac{1}{2}) + 3(v+ip)}{3p} + o(N^{-l-1}),$$

[8] Under the slightly weaker assumption $v = O(N^{1/3})$, we would already need the additional term $\frac{1}{6}\frac{v^3(q-p)}{(pq)^2N^2}$ in the right-hand side of (21).

where $l \in \mathbb{N} \cup \{0\}$.

Proof. We have

$$K_n(x,p,N_1) = K_n(Np+v,p,N_1) = K_n(N_1p+v_1,p,N_1) = (-p)^n \binom{N_1}{n} k_n^{(p)}(N_1p+v_1,N_1),$$

where $v_1 = v + ip$. Using (20) with the expression $(N-i)^{-l} = (N)^{-l} + il\,(N)^{-l-1} + O(N^{-l-2})$, we easily obtain the required asymptotic expansions from Corollary 1. ☐

The uniform asymptotic expansions from Corollaries 1 and 2 are obtained under the assumption $|v| \leq \varepsilon(N)N^{1/3}$. We can also use the same expansions for $|v| \leq \varepsilon(N)N^{1/2}$, $\varepsilon(N) \to 0$. However, if $N^{1/3} = O(v)$, we must multiply the estimates of the residual terms of these asymptotic expansions by $\sqrt{N}$.

3. Appendix

In this section, we prove formula (16). We denote by $S_N^M(\xi)$ a function such that

$$\sqrt{N}\rho(\widehat{\xi}) = e^{S_N^M(\xi)}\phi^M(\xi),$$

where

$$\phi^M(\xi) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\xi^2} \sum_{\nu=0}^{M} \frac{\widetilde{q}_\nu(\xi)}{N^{\nu/2}},$$

the polynomials $\widetilde{q}_\nu$ are defined in Theorem 1, and, as above, $\widehat{s} = Np + \sqrt{2Npq}s$. We denote by $\phi_s^M(\xi)$ the derivative

$$\frac{d^s}{d\xi^s}\phi^M(\xi) = (-1)^s \frac{1}{\sqrt{2\pi}\sigma} e^{-\xi^2} \sum_{\nu=0}^{M} \frac{\widetilde{q}_{\nu+s}(\xi)}{N^{\nu/2}}.$$

Using Cauchy's integral formula, we obtain

$$\begin{aligned}\sqrt{N}\frac{d^s}{dx^s}\rho(\widehat{x}(x)) &= \frac{\sqrt{N}s!}{2\pi i}\int_{C_x} \frac{\rho(\widehat{\xi})d\xi}{(x-\xi)^{s+1}} = \frac{s!}{2\pi i}\Bigg[\int_{C_x} \frac{\phi^M(\xi)d\xi}{(x-\xi)^{s+1}} + \int_{C_x} \frac{\big(\sqrt{N}\rho(\widehat{\xi}) - \phi^M(\xi)\big)d\xi}{(x-\xi)^{s+1}}\Bigg] \\ &= \phi_s^M(x) + \frac{s!}{2\pi i}\int_{C_x} \frac{\phi^M(\xi)\big(e^{S_N^M(\xi)} - 1\big)d\xi}{(x-\xi)^{s+1}}\end{aligned}$$

where C_x is a closed contour around the point x. Somewhat arbitrarily, we set C_x to be the unit circle around x:

$$C_x = \{\xi \mid \xi = x + e^{i\varphi}, \varphi \in [0, 2\pi]\}.$$

Lemma 1. *Let M be a nonnegative integer and A be a positive real number. Then for all $x \in [-A, A]$ and $\xi \in C_x$, we have the uniform estimate*

$$\big|e^{S_N^M(\xi)} - 1\big| = o(N^{-M/2}).$$

Proof. The Stirling expansion for the gamma function (see, e.g., [9, p. 34, Example 1.2] or [10, p. 83]) can be written as follows:

$$\log\Gamma(z) = \big(z - \frac{1}{2}\big)\log(z) - z + \frac{\log(2\pi)}{2} + F_m(z) + O(|z|^{-2m-1}), \qquad (22)$$

where $|z| \to \infty$, $|\arg z| \le \pi - \varepsilon < \pi$, $F_m(z) = \sum\limits_{k=1}^{m} \frac{B_{2k}}{2k(2k-1)z^{2k-1}}$, and B_k, $k \ge 1$, are the Bernoulli numbers. With the help of this expression, it was shown in [7, formula (17)] that

$$\log \rho(z) = -\frac{N}{2pq}\Big(\frac{z}{N} - p\Big)^2 - \frac{1}{2}\log(2\pi Npq) + S_N^0\Big(\frac{z - Np}{\sqrt{2Npq}}\Big), \; z \to \infty,$$

where for any integer $m \ge 0$ we have[9] $S_N^0(\xi) = \Phi_m(\xi) + O(N^{-2m-1})$ and the function $\Phi_m(\xi)$ is defined by the identity

$$\Phi_m(\xi) = F_m(N) - F_m(\widehat{\xi}) - F_m(N - \widehat{\xi}) - Nr\Big(\frac{\widehat{\xi}}{N}\Big) - \frac{1}{2}D\Big(\frac{\widehat{\xi}}{N}\Big)$$

with

$$r(\tau) = \tau \log\frac{\tau}{p} + (1-\tau)\log\frac{1-\tau}{q} - \frac{1}{2pq}(\tau - p)^2, \quad D(\tau) = \log(1 + \frac{(p-\tau)(\tau-q)}{pq}).$$

The function $\Phi_m(\xi)$ is analytic in the union $\bigcup_x B_x$ of the unit balls B_x bounded by the circles C_x, $x \in [-A, A]$. For the function $S_N^0(\xi)$, it was shown in [7, formula (27)] that $|S_N^0(\xi)| \le \frac{C}{\sqrt{N}}$ for $\xi \in C_x$ and $x \in [-A, A]$. Since we know from Theorem 1 that at any point x such that $\widehat{x} = Np + (\sqrt{2Npq}x) \in \mathbb{Z}$,

$$\big|e^{S_N^0(x)} - \sum_{\nu=0}^{M} \frac{\widetilde{q}_\nu(x)}{N^{\nu/2}}\big| = o(N^{-M/2}),$$

it suffices to show that

$$\big|e^{\Phi_m(\xi)} - \sum_{\nu=0}^{M} \frac{\widetilde{p}_\nu(\xi)}{N^{\nu/2}}\big| = o(N^{-M/2})$$

for some chosen $m = m(M)$, $\xi \in C_x$, and $x \in [-A, A]$ and *some* polynomials $\{\widetilde{p}_\nu(\xi)\}_{\nu=0}^{M}$. If this is already shown, then, making N large enough, we see that $\widetilde{p}_\nu(\xi) = \widetilde{q}_\nu(\xi)$, $0 \le \nu \le M$, for all ξ, due to the coincidence of these polynomials in at least $[\sqrt{N}]$ points. Analyzing the functions $F_m(\widehat{\xi}), r(\frac{\widehat{\xi}}{N}), D(\frac{\widehat{\xi}}{N})$ as functions of the variable $\frac{1}{\sqrt{N}}$ with the parameter x, we see that for each of these functions we can write the Taylor series[10] as $\frac{1}{\sqrt{N}} \to 0$ of the form $\sum\limits_{j=0}^{\infty} \frac{c_j}{(\sqrt{N})^j}\xi^{j+h}$ for some integer h. Since $x \in [-A, A]$, we can truncate each series and write $\sum\limits_{j=0}^{m} \frac{c_j}{(\sqrt{N})^j}x^{j+h} + f(N)$, where $|f(N)| \le \frac{C}{N^m}$ uniformly for $\xi \in \bigcup_x B_x$. It suffices to set m equal to $2M$. In the same manner, truncating the Taylor series for the function $e^{\Phi_m(\xi)}$, we see that $|e^{\Phi_m(\xi)} - \sum\limits_{\nu=0}^{m} \frac{\widetilde{p}_\nu(\xi)}{N^{\nu/2}}| = o(\frac{1}{N^{-M/2}})$ for $p_0(\xi) \equiv 1$ and some polynomials $p_\nu(\xi)$, where $\xi \in C_x$ and $x \in [-A, A]$.

Since $S_N^M(\xi)$ equals $S_N^0(\xi) - \Phi_m(\xi) + o(N^{-m})$, we see that $|S_N^M(\xi)| \le o(N^{-M/2})$ and, using the Lagrange theorem, obtain that $\big|e^{S_N^M(\xi)} - 1\big| = o(N^{-M/2})$ for $\xi \in C_x$ and $x \in [-A, A]$. □

Remark 2. One can obtain the estimate $|e^{\Phi^M(\xi)} - \sum\limits_{\nu=0}^{M} \frac{\widetilde{q}_\nu(\xi)}{N^{\nu/2}}| = o(N^{-M/2})$ without using Theorem 1, but directly from the analysis of the function S_N^0 at least for small values of M.

[9]This follows, of course, from the estimate on the residual term in (22); see the details, e.g., in [10].

[10]We present initial terms of these series in Remark 2 below.

For example, we can obtain the following asymptotic expansions as $N \to \infty$:

$$r\Big(\frac{\widehat{x}}{N}\Big) = -\frac{\sqrt{2}}{3}\frac{(2\,p-1)}{\sqrt{(1-p)\,p}}\frac{x^3}{N^{3/2}} + O\left(N^{-2}\right), \quad D\Big(\frac{\widehat{x}}{N}\Big) = \frac{\sqrt{2}\,(2p-1)}{\sqrt{(1-p)\,p}}\frac{x}{N^{1/2}} + O\left(N^{-1}\right).$$

For the function F_m with $m = 2$, we have $F_2(z) = 1/(12z)$. Analogously, we obtain the expansions

$$F_2(\widehat{x}) = \frac{1}{12Np} + O\left(N^{-3/2}\right), \quad F_2(N-\widehat{x}) = \frac{1}{12\,(1-p)\,N} + O\left(N^{-3/2}\right),$$
$$F_2(N) = 1/12\,N^{-1} + O(N^{-3/2}).$$

Therefore, we can write the first term of the asymptotic expansion of the function $e^{\Phi_M(x)}$ taking only the initial terms of the above expressions:

$$e^{\Phi_M(x)} = \frac{(1-2p)}{2^{3/2}(pq)^{1/2}}\frac{8x^3-12x}{3!\sqrt{N}} + O\Big(\frac{1}{N}\Big) = \frac{\widetilde{q}_1(x)}{N^{1/2}} + O\Big(\frac{1}{N}\Big).$$

This gives an explicit proof of Lemma 1 for $M = 1$. It is an interesting question whether there is any short and explicit proof of Lemma 1 for all values of the parameter M that does not use Theorem 1.

Lemma 2. *Let M be a nonnegative integer and A be a positive real number. Then for all $x \in [-A, A]$ and N large enough, we have the uniform estimate*

$$\frac{s!}{2\pi i}\int\limits_{C_x}\Big|\frac{\phi^M(\xi)}{(x-\xi)^{s+1}}\Big||d\xi| = O(1).$$

Proof. For a given value of M, all $x \in [-A, A]$, and $N = N(A, M)$ large enough, we have $|\sum\limits_{\nu=0}^{M}\frac{\widetilde{q}_\nu(x+e^{i\varphi})}{N^{\nu/2}}| \le 2$. Then we can write the estimate

$$\int\limits_{C_x}\Big|\frac{\phi^M(\xi)}{(x-\xi)^{s+1}}\Big||d\xi| \le 2\int\limits_0^{2\pi} e^{-(x+2\cos(\varphi))^2/2}d\varphi < Ce^{-x^2/4},$$

and the constant C does not depend on x. □

Combining Lemmas 1 and 2, we can estimate the second integral as

$$\Big|\frac{s!}{2\pi i}\int\limits_{C_x}\frac{\phi^M(\xi)\big(e^{S_N^M(\xi)}-1\big)d\xi}{(x-\xi)^{s+1}}\Big| = o(N^{-M/2}),$$

for $\xi \in C_x$, $x \in [-A, A]$, which finishes the proof.

Acknowledgments. The author is grateful to Professors A. A. Lodkin and A. M. Vershik for their advice and kind support.

Supported by the RFBR (grant 14-01-00373).

Translated by the author.

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