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ALGEBRAIC-ANALYTIC METHODS FOR CONSTRUCTING SOLUTIONS TO DIFFERENTIAL EQUATIONS AND INVERSE PROBLEMS

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We present new algebraic-analytic methods for constructing solutions to differential equations and inverse problems. In particular, we develop a new approach based on the ray method for inverse problems in mathematical physics. Bibliography: 23 *titles.*

In this paper, we continue to develop algebraic-analytic methods for studying problems in mathematical physics (cf. [1]–[9]). We consider an abstract equation including some classical differential equations. For such equations with special sources it is possible to construct partial solutions. In this paper, we use algebraic tools for reproducing the partial solutions. The formulas for solutions obtained in this paper can be used for studying problems in mathematical physics and, in particular, inverse problems. Furthermore, we develop a new approach to the use of the ray method [10]–[13] for studying inverse problems in mathematical physics. This approach consists in searching not only amplitudes, but also a function defining the Riemann metric under the assumption that the ray series is an exact solution to differential equations in finite and infinite cases. Owing to this result, it is possible to study particular inverse problems.

In fact, we use the method of generalized separation of variables, i.e., the representation of solutions, coefficients, and other information in the form of sums (possibly, infinite) in the tensor product of vector spaces. Solutions to the abstract equation in Sections 1 and 2 and the ray series in Section 3 are represented in this form. Section 3 contains a criterion of representation of an analytic function as a finite sum in the tensor product of function spaces.

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1 Construction of Particular Solutions

In this section, we deal with constructive methods for obtaining particular solutions to abstract equations with rather large arbitrariness. Assume that V and U are vector spaces over a field K, A and B are linear operators in the spaces V and U sending elements of these spaces to themselves. We denote by $F(\alpha, \beta)$ a polynomial in variables α and β with coefficients in K and by $V \otimes U$ the tensor product of the spaces V and U. We consider the following equation for $w \in V \otimes U$:

$$F(A, B)w = R, \quad R \in V \otimes U. \tag{1}$$

Our main goal is to construct partial solutions to Equation (1). Assume that $a_p \in V$, $b_q \in U$, $p, q \geqslant 0$, are elements of the spaces V and U. Let $F_{pq}(\alpha, \beta)$ and $G_{pq}(\alpha, \beta)$ be polynomials in variables α and β over the field K.

The following general result will be used below.

Theorem 1. *If*

$$R = \sum_{p,q \geqslant 0} G_{pq}(A, B)(a_p \otimes b_q), \quad F(\alpha, \beta)F_{pq}(\alpha, \beta) = G_{pq}(\alpha, \beta),$$

then the function

$$w = \sum_{p,q \geqslant 0} F_{pq}(A, B)(a_p \otimes b_q)$$

satisfies Equation (1).

We consider an example connected with the geometric progression and used below for constructing partial solutions to evolution systems. Setting

$$F(\alpha, \beta) = \beta - \alpha, \quad F_{pq}(\alpha, \beta) = \sum_{k=0}^{p+q-1} \alpha^{p+q-1-k}\beta^k, \quad G_{pq}(\alpha, \beta) = \beta^{p+q} - \alpha^{p+q}$$

in Theorem 1, we get $F(\alpha, \beta)F_{pq}(\alpha, \beta) = G_{pq}(\alpha, \beta)$. By Theorem 1, the element

$$w = \sum_{p+q \geqslant 1} \sum_{k=0}^{p+q-1} A^{p+q-1-k}B^k(a_p \otimes b_q)$$

is a partial solution to the equation

$$(B - A)w = R, \quad R = \sum_{p+q \geqslant 1} (B^{p+q} - A^{p+q})(a_p \otimes b_q).$$

From Theorem 1 and the above example we obtain a representation of a partial solution $w(x, y)$ to the system of differential equations with a special vector-valued source function $R(x, y)$. We consider the system of linear differential equations

$$B_y w(x, y) = A_x w(x, y) + R(x, y), \tag{2}$$

where $(x, y) \in \widetilde{D} \subset \mathbb{R}^{n+m}$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_m)$, $n \geqslant 1$, $m \geqslant 1$, $\widetilde{D}$ is a domain in the real Euclidean space $\mathbb{R}^{n+m}$, $w(x, y) = (w_1(x, y), \ldots, w_N(x, y))$, $R(x, y) =$

$(R_1(x,y),\dots,R_N(x,y))$ are complex vector-valued functions of dimension $N \geqslant 1$, and B_y, A_x are linear differential operators defined by the relations

$$B_y = \sum_{|\beta|\leqslant m_2} B_\beta(y) D_y^\beta, \quad D_y^\beta = \frac{\partial^{\beta_1+\ldots+\beta_m}}{\partial y_1^{\beta_1}\ldots \partial y_m^{\beta_m}},$$

where $B_\beta(y)$ are infinitely differentiable functions of the variable y,

$$A_x = \sum_{|\alpha|\leqslant m_1} A_\alpha(x) D_x^\alpha, \quad D_x^\alpha = \frac{\partial^{\alpha_1+\ldots+\alpha_m}}{\partial x_1^{\alpha_1}\ldots \partial x_n^{\alpha_n}},$$

where $A_\alpha(x)$ are quadratic matrices of order N with infinitely differentiable entries depending on the variable x.

If $m = 1$ and $y = t$, then the system (1) is a system of evolution equations with time-independent coefficients on the right-hand side, for example, the Lamé or Maxwell equations.

Let $b_0(y)$, $b_1(y)$,…,$b_{N_2}(y)$ be arbitrary infinitely differentiable complex-valued functions of the variable y, and let $a_0(x)$, $a_1(x)$,…,$a_{N_1}(x)$ be arbitrary infinitely differentiable complex vector-valued functions of the variable x and dimension N, $N_1 \geqslant 1$, $N_2 \geqslant 1$.

We denote by A_x^k and B_y^j the powers of operators A_x and B_y of degree k and j respectively. The operators A_x^0 and B_y^0 are the identity operators.

We also note that the operators B_y^j act on functions depending only on y, whereas the operators A_x^k act on vector-valued functions of dimension N depending only on x.

Theorem 2. *Let*

$$R(x,y) = \sum_{j=1}^{N_2} B_y^j b_j(y) a_0(x) - b_0(y) \sum_{k=1}^{N_1} A_x^k a_k(x)$$
$$+ \sum_{k=1}^{N_1} B_y^k\Big(b_0(y) + \sum_{j=1}^{N_2} B_y^j b_j(y)\Big) a_k(x) - \sum_{j=1}^{N_2} b_j(y) A_x^j\Big(a_0(x) + \sum_{k=1}^{N_1} A_x^k a_k(x)\Big).$$

Then the function

$$w(x,y) = \sum_{\substack{0\leqslant k,s;\\ k+s+1\leqslant N_1}} B_y^k\Big(b_0(y) + \sum_{j=1}^{N_2} B_y^j b_j(y)\Big) A_x^s a_{k+s+1}(x)$$
$$+ \sum_{\substack{0\leqslant i,j;\\ i+j+1\leqslant N_2}} B_y^j b_{i+j+1}(y) A_x^i\Big(a_0(x) + \sum_{k=1}^{N_1} A_x^k a_k(x)\Big) + \widetilde{w}(x,y)$$

satisfies the equation

$$B_y w(x,y) = A_x w(x,y) + R(x,y),$$

where $\widetilde{w}(x,y)$ is any solution to the homogeneous system (2) with the coefficients defined by $a_k(x)$, $b_l(y)$, $k = 0,\dots,N_1$, $l = 0,\dots,N_2$.

Proof. We set $A_x = A$ and $B_y = B$ in Theorem 1. We fix a pair (p, q) and find a term in the expression for w that corresponds to the product $a_p b_q$. Let

$$w_1 = \sum_{\substack{0\leqslant k,s;\\ k+s+1\leqslant N_1}} B^k\Big(\sum_{j=0}^{N_2} B^j b_j\Big) A^s a_{k+s+1}.$$

Then for $j = q$ we have

$$w_{1q} = \sum_{\substack{0\leqslant i,j;\\ k+s+1\leqslant N_1}} B^{k+q} b_q A^s a_{k+s+1}.$$

Setting $k + s + 1 = p$, we obtain

$$w_{1pq} = \sum_{k=0}^{p-1} B^{k+q} b_q A^{p-k-1} a_p.$$

Similarly, if

$$w_2 = \sum_{\substack{0\leqslant i+j,\\ i+j+1\leqslant N_2}} B^j b_{i+j+1} A^i \sum_{k=0}^{N_1} A^k a_k,$$

then

$$w_{2pq} = \sum_{i=0}^{q-1} B^{q-i-1} b_q A^{p+i} a_p.$$

Thus,

$$\begin{aligned} w = w_1 + w_2 &= \sum_{p=0}^{N_1}\sum_{q=0}^{N_2}(w_{1pq} + w_{2pq}) \\ &= \sum_{p=0}^{N_1}\sum_{q=0}^{N_2}\sum_{k=0}^{p+q-1} A^{p+q-k-1} a_p B^k b_q = \sum_{p=0}^{N_1}\sum_{q=0}^{N_2}\Big(\sum_{\substack{0\leqslant k,l;\\ k+l=p+q-1}} A^l B^k\Big)(a_p b_q). \end{aligned}$$

According to the example, we find

$$(B - A)w = \sum_{p=0}^{N_1}\sum_{q=0}^{N_2}\left(B^{p+q} - A^{p+q}\right)(a_p b_q) = \sum_{p=0}^{N_1} a_p \sum_{q=0}^{N_2} B^{p+q} b_q - \sum_{q=0}^{N_2} b_q \sum_{p=0}^{N_1} A^{p+q} a_p = R.$$

The theorem is proved. □

Remark 1. If $R(x, y)$ is a known or partially known function, then the expression for $R(x, y)$ in Theorem 2 can be regarded as a system of equations for coefficients of the operators A_x and B_y, for the coefficients $a_k(x)$, $b_j(y)$ and so on, which leads to a new method of studying multidimensional inverse problems. In particular, we set $b_j(y) = 0$, $j = 1, \ldots, N_2$, $b_0(y) \neq 0$ and

$$R(x, y) = b_0(y)a_0(x) + \sum_{k=1}^{N_1} B_y^k b_0(y) a_k(x)$$

in Theorem 2. By Theorem 2, from the conditions $b_j(y) = 0$, $j = 1, \ldots, N_2$, we have

$$R(x,y) = -b_0(y)\sum_{k=1}^{N_1} A_x^k a_k(x) + \sum_{k=1}^{N_1} B_y^k b_0(y) a_k(x) - b_0(y)a_0(x) + b_0(y)a_0(x).$$

Consequently,

$$a_0(x) + \sum_{k=1}^{N_1} A_x^k a_k(x) = 0.$$

For given $a_k(x)$ this equation is an equation for the operator A_x or, for a given operator A_x, it is an equation for some coefficient $a_k(x)$ [9].

2 Algebraic Methods for Reproducing Solutions to Inverse Problems

According to Theorem 1, by a *solution* to the inverse problem for the equation

$$F(A,B)w = R,$$

where

$$w = \sum_{p,q\geqslant 0} F_{pq}(A,B)(a_p \otimes b_q), \quad R = \sum_{p,q\geqslant 0} G_{pq}(A,B)(a_p \otimes b_q),$$

we mean $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_p, b_q)$ such that $F(A,B)F_{pq}(A,B) = G_{pq}(A,B)$. We consider the following problem: for a given solution $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_p, b_q)$ to construct another solution $(\widetilde{F}(\widetilde{A},\widetilde{B}), \widetilde{F}_{pq}(\widetilde{A},\widetilde{B}), \widetilde{G}_{pq}(\widetilde{A},\widetilde{B}), \widetilde{a}_p, \widetilde{b}_q)$ by algebraic-analytic operations. In other words, the problem is to reproduce solutions. We indicate some methods for solving this problem.

1. *Multiplication by an operator.* Let $H(\alpha,\beta)$ be a polynomial in α and β. Then the solution has the form $(H(A,B)F(A,B), F_{pq}(A,B), H(A,B)G_{pq}(A,B), a_p, b_q)$.

2. *Linearity in argument.* If $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_{1p}, b_q)$ and $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_{2p}, b_q)$ are solutions, then $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_{1p} + a_{2p}, b_q)$ is also a solution.

3. *Transformation in operator I.* Let $L : V \longrightarrow V$ be an invertible operator in the space V, and let $a_p = L\widetilde{a}_p$. Then $(F(L^{-1}AL, B), F_{pq}(L^{-1}AL, B), G_{pq}(L^{-1}AL, B), \widetilde{a}_p, b_q)$ is a solution.

4. *Transformation in operator II.* Let $L : V \longrightarrow V$ be an invertible operator in the space V, and let $A = L^{-1}\widetilde{A}L$. Then $(F(\widetilde{A}, B), F_{pq}(\widetilde{A}, B), G_{pq}(\widetilde{A}, B), La_p, b_q)$ is a solution.

5. *Differentiation in parameter.* Let $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_p, b_q)$ depend on the parameter s. Differentiating the equality

$$F(A,B)\sum_{p,q\geqslant 0} F_{pq}(A,B)(a_p \otimes b_q) = \sum_{p,q\geqslant 0} G_{pq}(A,B)(a_p \otimes b_q)$$

with respect to s, we find

$$F'(A,B)\sum_{p,q\geqslant 0} F_{pq}(A,B)(a_p \otimes b_q) + F(A,B)\sum_{p,q\geqslant 0} F'_{pq}(A,B)(a_p \otimes b_q)$$

$$+ F(A,B)\sum_{p,q\geqslant 0} F_{pq}(A,B)(a'_p\otimes b_q + a_p\otimes b'_q)$$

$$= \sum_{p,q\geqslant 0} G'_{pq}(A,B)(a_p\otimes b_q) + \sum_{p,q\geqslant 0} G_{pq}(A,B)(a'_p\otimes b_q + a_p\otimes b'_q).$$

Here, the prime denotes $\dfrac{d}{ds}$. In particular, if $a'_p = b'_q = 0$, then

$$\sum_{p,q\geqslant 0}\big(F'(A,B)F_{pq}(A,B) + F(A,B)F'_{pq}(A,B)\big)(a_p\otimes b_q) = \sum_{p,q\geqslant 0} G'_{pq}(A,B)(a_p\otimes b_q).$$

For example, if $F = F_0^n$, $n\geqslant 2$, then

$$F'(A,B)F_{pq}(A,B) + F(A,B)F'_{pq}(A,B) = F_0^{n-1}(A,B)(nF_{pq}(A,B) + F_0(A,B)F'_{pq}(A,B)),$$

i.e., $(F_0^{n-1}(A,B), nF'_0(A,B)F_{pq}(A,B) + F_0(A,B)F'_{pq}(A,B), G'_{pq}(A,B), a_p, b_q)$ is a solution. If, in addition, only the operator A depends on the parameter s and $AA' = A'A$, then

$$\Big(F_0^{n-1}(A,B), n\frac{\partial F_0}{\partial A}(A,B)F_{pq}(A,B) + F_0(A,B)\frac{\partial F_{pq}}{\partial A}(A,B), \frac{\partial G_{pq}}{\partial A}(A,B), A'a_p, b_q\Big)$$

is a solution. To illustrate this transformation, we formulate the following assertion.

Proposition 1. *Let*

$$R = \sum_{p,q\geqslant 0}\big(A^{p+q+1} - (p+q+1)AB^{p+q} + (p+q)B^{p+q+1}\big)\, a_p\otimes b_q.$$

Then the element

$$w = \sum_{p=0}^{\infty}\Big(\sum_{q=0}^{p}\sum_{k=0}^{p+q}(k+1)A^{p+q-k}B^k(a_{p+1}\otimes b_q + a_q\otimes b_{p+1}) + \sum_{k=0}^{2p-1}(k+1)A^{2p-1-k}B^k a_p\otimes b_p\Big)$$

satisfies the equation $(A-B)^2 w = R$.

Corollary 1. *If only the operator A depends on the parameter s, $AA' = A'A$, and*

$$R = \sum_{p,q\geqslant 0}(p+q+1)\Big(A^{p+q} - B^{p+q}\Big)A'a_p\otimes b_q,$$

then the element

$$\begin{aligned} w = 2\sum_{p=0}^{\infty}\Big(&\sum_{q=0}^{p}\sum_{k=0}^{p+q}(k+1)A^{p+q-k}B^k(A'a_{p+1}\otimes b_q + A'a_q\otimes b_{p+1}) \\ &+ \sum_{k=0}^{2p-1}(k+1)A^{2p-1-k}B^kA'a_p\otimes b_p\Big) \\ &+ (A-B)\sum_{p=0}^{\infty}\Big(\sum_{q=0}^{p}\sum_{k=0}^{p+q}(k+1)(p+q-k)A^{p+q-k-1}B^k(A'a_{p+1}\otimes b_q + A'a_q\otimes b_{p+1}) \\ &+ \sum_{k=0}^{2p-1}(k+1)(2p-1-k)A^{2p-2-k}B^kA'a_p\otimes b_p\Big), \end{aligned}$$

satisfies the equation $(A-B)w = R$.

6. *The Abelian transformation in operator.*

Proposition 2.

$$\sum_{p,q=0}^{\infty} F_{pq}a_p \otimes b_q = \sum_{p,q=0}^{\infty} \Big(\sum_{l=0}^{p}\sum_{k=0}^{q} F_{lk}\Big)(a_p - a_{p+1}) \otimes (b_q - b_{q+1}).$$

Proof. Applying the Abelian transformation to the series

$$\sum_{k=0}^{\infty} x_k y_k = \sum_{k=0}^{\infty} X_k(y_k - y_{k+1}), \quad X_k = \sum_{l=0}^{k} x_l,$$

we find

$$\sum_{p,q=0}^{\infty} F_{pq}a_p \otimes b_q = \sum_{q=0}^{\infty}\Big(\sum_{p=0}^{\infty} F_{pq}a_p\Big) \otimes b_q = \sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\Big(\sum_{l=0}^{p} F_{lq}\Big)(a_p - a_{p+1}) \otimes b_q$$

$$= \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{q=0}^{\infty} F_{lq}(a_p - a_{p+1}) \otimes b_q = \sum_{p=0}^{\infty}\sum_{l=0}^{p}\sum_{q=0}^{\infty}\Big(\sum_{k=0}^{q} F_{lk}\Big)(a_p - a_{p+1}) \otimes (b_q - b_{q+1})$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big(\sum_{l=0}^{p}\sum_{k=0}^{q} F_{lk}\Big)(a_p - a_{p+1}) \otimes (b_q - b_{q+1}).$$

The proposition is proved. □

Corollary 2. *If $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_p, b_q)$ is a solution, then*

$$(F(A,B), \widetilde{F}_{pq}(A,B), \widetilde{G}_{pq}(A,B), a_p - a_{p+1}, b_q - b_{q+1}),$$

is also a solution, where

$$\widetilde{F}_{pq} = \sum_{l=0}^{p}\sum_{k=0}^{q} F_{lk}, \quad \widetilde{G}_{pq}(A,B) = \sum_{l=0}^{p}\sum_{k=0}^{q} G_{lk}.$$

Remark 2. If $\widetilde{F}_{pq} = \sum\limits_{l=0}^{p}\sum\limits_{k=0}^{q} F_{lk}$, then $F_{pq} = \widetilde{F}_{pq} - \widetilde{F}_{p-1,q} - \widetilde{F}_{p,q-1} + \widetilde{F}_{p-1,q-1}$.

Corollary 3. *If $\widetilde{F}_{pq} = \sum\limits_{l+k=p+q} A^l B^k$, then*

$$F_{pq} = A^{p+q} + A^{p+q-1}(B-2) + (B-1)^2 \sum_{l+k=p+q-2} A^l B^k$$

and for

$$R = \sum_{p,q=0}^{\infty} A^{p+q+1}(a_p - a_{p+1}) \otimes (b_q - b_{q+1}) - \sum_{p,q=0}^{\infty} B^{p+q+1}(a_p - a_{p+1}) \otimes (b_q - b_{q+1})$$

the element $w = \sum\limits_{p,q=0}^{\infty} F_{pq}a_p \otimes b_q$ is a solution to the equation $(A-B)w = R$.

Remark 3. If $a_0 = \ldots = a_{N_1} = a$, $a_{N_1+1} = a_{N_1+2} = \ldots = 0$, $b_0 = \ldots = b_{N_2} = b$, and $b_{N_2+1} = b_{N_2+2} = \ldots = 0$ in Corollary 1, then $R = (A^{N_1+1} - B^{N_2+1})(a \otimes b)$.

7. *The Abelian transformation in argument.*

Proposition 3.

$$\sum_{p,q=0}^{\infty} F_{pq} a_p \otimes b_q = \sum_{p,q=0}^{\infty} \Delta_{pq} F\Big(\sum_{l=0}^{p}\sum_{k=0}^{q} a_l \otimes b_k\Big),$$

where $\Delta_{pq}F = F_{pq} - F_{p+1,q} - F_{p,q+1} + F_{p+1,q+1}$.

Proof. Using the Abelian transformation, we find

$$\sum_{p,q=0}^{\infty} F_{pq} a_p \otimes b_q = \sum_{q=0}^{\infty}\Big(\sum_{p=0}^{\infty} F_{pq} a_p\Big) \otimes b_q = \sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\Big(F_{pq} - F_{p+1,q}\Big)\Big(\sum_{l=0}^{p} a_l\Big) \otimes b_q$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big(F_{pq} - F_{p+1,q}\Big)\Big(\sum_{l=0}^{p} a_l\Big) \otimes b_q$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big(F_{pq} - F_{p+1,q} - F_{p,q+1} + F_{p+1,q+1}\Big)\Big(\sum_{l=0}^{p} a_l\Big) \otimes \Big(\sum_{k=0}^{q} b_k\Big)$$

$$= \sum_{p,q=0}^{\infty} \Delta_{pq} F\Big(\sum_{l=0}^{p}\sum_{k=0}^{q} a_l \otimes b_k\Big).$$

The proposition is proved. □

Corollary 4. *If* $(F(A,B), F_{pq}(A,B), G_{pq}(A,B), a_p, b_q)$ *is a solution, then*

$$(F(A,B), \Delta_{pq}F(A,B), \Delta_{pq}G(A,B), \widetilde{a}_p, \widetilde{b}_q)$$

is also a solution, where $\Delta_{pq}F = F_{pq} - F_{p+1,q} - F_{p,q+1} + F_{p+1,q+1}$, $\Delta_{pq}G = G_{pq} - G_{p+1,q} - G_{p,q+1} + G_{p+1,q+1}$, $\widetilde{a}_p = \sum\limits_{l=0}^{p} a_l$, *and* $\widetilde{b}_q = \sum\limits_{k=0}^{q} b_k$.

3 The Ray Method and Inverse Problems

We consider the second order equation

$$\lambda^2(x)\frac{\partial^2 w}{\partial t^2} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} \equiv Lw, \tag{3}$$

where $t_0 \leqslant t \leqslant t_1$, $x \in D \subset \mathbb{R}^n$, D is a domain in $\mathbb{R}^n$, $n \geqslant 1$, and $a_{ij}(x) = a_{ji}(x)$ are continuous functions. According to the ray method [10]–[13], we look for a solution to Equation (3) in the form of the formal series

$$w(x,t) = \sum_{k=0}^{\infty} w_k(x) f_k(t - \tau(x)), \tag{4}$$

where $f_k'(s) = f_{k-1}(s)$. Substituting (4) into (3), we obtain the following system of differential equations for $\tau(x)$ and $w_k(x)$:

$$
\begin{aligned}
&L\tau w_0(x) + 2\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial w_0}{\partial x_i}\frac{\partial \tau}{\partial x_j} = 0,\\
&Lw_{k-1} = L\tau w_k(x) + 2\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial w_k}{\partial x_i}\frac{\partial \tau}{\partial x_j}, \quad k = 1, 2, \ldots,
\end{aligned} \tag{5}
$$

with the overdetermination condition with a given function $\lambda(x)$:

$$
\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_j} = \lambda^2(x). \tag{6}
$$

For $w_0(x) \neq 0$, $x \in D$, we can write the system (5) with the resolved higher order derivatives of $\tau(x)$ and $w_k(x)$:

$$
\begin{aligned}
&L\tau = -\frac{2}{w_0}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial w_0}{\partial x_i}\frac{\partial \tau}{\partial x_j},\\
&Lw_{k-1} = \frac{2}{w_0}\sum_{i,j=1}^{n} a_{ij}(x)\left(w_0\frac{\partial w_k}{\partial x_i} - w_k\frac{\partial w_0}{\partial x_i}\right)\frac{\partial \tau}{\partial x_j}, \quad k = 1, 2, \ldots .
\end{aligned} \tag{7}
$$

For given $a_{ij}(x)$ the system (7) for $\tau(x)$, $w_k(x)$, $k = 1, 2, \ldots$, is well defined, which is the content of our approach: find $\tau(x)$, $w_k(x)$, $k = 1, 2, \ldots$, from (7) and then compute $\lambda(x)$ by formula (6). Certainly, to find particular solutions to the nonlinear system (7), we need to impose boundary conditions on $\tau(x)$ and $w_k(x)$, for example, $\tau(x)|_{\partial D}$ and $w_k(x)|_{\partial D}$, which allows us to solve a particular inverse problem for $w(x,t)$ and $\lambda(x)$.

We emphasize that, according to the ray method for Equation (3), the study of the inverse problem for $w(x,t)$ and $\lambda(x)$ is reduced to the study of the well-defined system (7) for $w_k(x)$ and $\tau(x)$ in a finite or infinite version. This approach is applicable to other equations and systems in mathematical physics, for example, the Lamé equations, Maxwell equations, and soliton type equations. We note that soliton type equations are studied by the Kovalevskii–Painleve method (known as the Wentzel–Kramers–Brillouin method for equations of quantum mechanics [11]), where solutions are represented as power series with respect to some functions [13]–[16].

For an example we consider the system of ordinary differential equations with one-dimensional wave equation

$$
\lambda^2(x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2},
$$

where $t_0 \leqslant t \leqslant t_1$, $x_0 \leqslant x \leqslant x_1$. In this case, we can exclude the derivative $\tau'(x)$ and obtain an equation only for $w_k(x)$, $k = 0, 1, \ldots$ The following assertion holds.

Proposition 4. *Let $w_k(x)$, $k = 0, 1, \ldots$, $w_0(x) \neq 0$, be a solution to the system*

$$
w_k'' = \frac{2C}{w_0}\left(\frac{w_{k+1}}{w_0}\right)', \quad k = 0, 1, \ldots
$$

Then the function

$$w(x,t)=\sum_{k=0}^{\infty} w_k(x) f_k(t-\tau(x)),$$

where $f_k'(s) = f_{k-1}(s)$, $k = 1, 2, \ldots$, *and* $\tau = C\int \dfrac{dx}{w_0^2}$, C *is a constant, is a solution to the equation*

$$\left(\frac{\partial\tau}{\partial x}\right)^2 \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}.$$

In what follows, we consider only exact results for finite series

$$w(x,t)=\sum_{k=0}^{m} w_k(x) f^{(m-k)}(t-\tau(x)),$$

where $f(s)$ is an infinitely differentiable function (possibly, a distribution) in a domain of $\mathbb{R}$.

A counterpart of Proposition 4 for finite representations is formulated as follows.

Proposition 5. *Let* $w_0(x), \ldots, w_{m-1}(x)$ *be a solution to the system*

$$w_k'' = \frac{2C}{w_0}\left(\frac{w_{k+1}}{w_0}\right)', \quad k = 0, \ldots, m-1,$$

where $w_m = ax + b$, $a, b \in \mathbb{R}$. *Then the function*

$$w(x,t)=\sum_{k=0}^{m} w_k(x) f^{(m-k)}(t-\tau(x)),$$

where $\tau = C\int \dfrac{dx}{w_0^2}$ *and* C *is a constant, is a solution to the equation*

$$\left(\frac{\partial\tau}{\partial x}\right)^2 \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}.$$

For the general second order equation (3) the following assertion holds.

Proposition 6. *Suppose that* $x \in D \subset \mathbb{R}^n$, D *is a domain in the real Euclidean space* $\mathbb{R}^n$, $n \geqslant 1$, *and* $m \geqslant 0$ *is an integer. If* $w_k(x)$, $\tau(x)$, $k = 0, 1, \ldots, m$, *solve the system of* $m+2$ *differential equations for* $w_0(x) \neq 0$, $x \in D$,

$$L\tau = -\frac{2}{w_0}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial w_0}{\partial x_i}\frac{\partial \tau}{\partial x_j},$$

$$Lw_{k-1} = \frac{2}{w_0}\sum_{i,j=1}^{n} a_{ij}(x)\left(w_0\frac{\partial w_k}{\partial x_i} - w_k\frac{\partial w_0}{\partial x_i}\right)\frac{\partial \tau}{\partial x_j}, \quad k = 1, \ldots, m,$$

$$Lw_m = 0,$$

then the function

$$w(x,t)=\sum_{k=0}^{m} w_k(x) f^{(m-k)}(t-\tau(x))$$

is a solution to the equation

$$\Big(\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\tau}{\partial x_i}\frac{\partial\tau}{\partial x_j}\Big)\frac{\partial^2 w}{\partial t^2} = Lw.$$

Remark 4. Since the system (7) is resolved with respect to the higher order derivatives, we obtain a Cauchy–Kovalevskaya type system under additional assumptions on $a_{ij}(x)$ [17]. Therefore, we can formulate the existence and uniqueness result for the class of analytic functions.

We consider the inverse problem and formulate an exact assertion for the Laplace operator

$$L = \Delta = \sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2}.$$

The inverse problem is to find functions $w(x,y,t)$, $\lambda(x,y) > 0$ for $|x| < r$, $y \geqslant 0$, $t \geqslant 0$ such that

$$\lambda^2(x,y)\frac{\partial^2 w}{\partial t^2} = \sum_{j=1}^{n-1}\frac{\partial^2 w}{\partial x_j^2} + \frac{\partial^2 w}{\partial y^2}, \quad n \geqslant 1,\ r > 0, \tag{8}$$

$$w|_{y=0} = \varphi(x,t) = \sum_{k=0}^{m} a_k(x) f_k(t - \tau_0(x)), \tag{9}$$

$$\frac{\partial w}{\partial y}\Big|_{y=0} = \psi(x,t) = \sum_{k=-1}^{m} b_k(x) f_k(t - \tau_0(x)), \tag{10}$$

where $m > 1$ is a fixed integer, the functions $f_k(s)$, $f_k'(s) = f_{k-1}(s)$ are fixed, $\tau_0(x)$, $a_k(x)$, $k = 0,\dots,m$, $b_k(x)$, $k = -1,\dots,m$, are known functions for $|x| < r$.

Theorem 3. *If $\tau_0(x)$, $a_k(x)$, $k = 0,\dots,m$, $b_k(x)$, $k = -1,\dots,m$, are analytic functions for $|x| < r$ and $a_0(x) \neq 0$, then there exists a neighborhood of the origin in the space $\mathbb{R}^n$ of variables (x,y) such that there is a unique solution to the inverse problem (8)–(10) represented as a finite ray series*

$$w(x,y,t) = \sum_{k=0}^{m} w_k(x,y) f_k(t - \tau(x,y)) \tag{11}$$

with analytic functions $w_k(x,y)$ and $\tau(x,y)$; moreover,

$$\lambda(x,y) = \sqrt{\sum_{j=1}^{n-1}\Big(\frac{\partial\tau}{\partial x_j}\Big)^2 + \Big(\frac{\partial\tau}{\partial y}\Big)^2}.$$

Proof. Substituting the representation (11) of $w(x,y,t)$ with analytic functions $w_k(x,y)$ and $\tau(x,y)$ into Equation (8) and assuming that a given function exactly satisfies Equation (8), we obtain a finite system of $m+2$ equations for $w_k(x,y)$ and $\tau(x,y)$

$$\begin{aligned}
&2(\nabla\tau, \nabla w_0) + w_0\Delta\tau = 0,\\
&2(\nabla\tau, \nabla w_k) + w_k\Delta\tau = \Delta w_{k-1}, \quad k = 1,\dots,m,\\
&\Delta w_m = 0.
\end{aligned} \tag{12}$$

Moreover, the functions $\lambda(x, y)$ and $\tau(x, y)$ are connected by the equality

$$\lambda^2(x,y) = \sum_{j=1}^{n-1} \left(\frac{\partial \tau}{\partial x_j}\right)^2 + \left(\frac{\partial \tau}{\partial y}\right)^2.$$

Let us consider the boundary conditions in the inverse problem. From the finite representation of a solution (11) and the condition (9) it follows that

$$\tau(x,0) = \tau_0(x), \quad w_k(x,0) = a_k(x), \quad k = 0, \dots, m. \tag{13}$$

By (8), the condition (10) takes the form

$$\sum_{k=1}^{m} \frac{\partial w_k}{\partial y}\bigg|_{y=0} f_k(t - \tau_0(x)) - \sum_{k=-1}^{m-1} a_{k+1}(x,0) \frac{\partial \tau}{\partial y}\bigg|_{y=0} f_k(t - \tau_0(x)) = \sum_{k=-1}^{m} b_k(x) f_k(t - \tau_0(x)).$$

Since $a_0(x) \neq 0$, we have

$$\begin{aligned} \frac{\partial \tau}{\partial y}\bigg|_{y=0} &= -\frac{b_{-1}(x)}{a_0(x)}, \\ \frac{\partial w_k}{\partial y}\bigg|_{y=0} &= -\frac{a_{k+1}(x) b_{-1}(x)}{a_0(x)} + b_k(x), \quad k = 0, 1, \dots, m. \end{aligned} \tag{14}$$

Thus, based on the finite representation (11), we compute the Cauchy data for the system (12).

We show that the system (10) can be reduced to a Cauchy–Kovalevskaya type system. Since $a_0(x) \neq 0$ and $w_0(x, 0) = a_0(x)$, there exists a neighborhood of the origin in $\mathbb{R}^n$ such that $w_0(x, y) \neq 0$ in this neighborhood. From the first equation in (12) we have

$$\Delta \tau = -\frac{2}{w_0} (\nabla \tau, \nabla w_0)$$

in this neighborhood. Therefore,

$$\begin{aligned} \frac{\partial^2 w_{k-1}}{\partial y^2} &= -\sum_{j=1}^{n-1} \frac{\partial^2 w_{k-1}}{\partial x_j^2} + 2(\nabla \tau, \nabla w_k) - 2\frac{w_k}{w_0}(\nabla \tau, \nabla w_0), \quad k = 1, \dots, m, \\ \frac{\partial^2 w_m}{\partial y^2} &= -\sum_{j=1}^{n-1} \frac{\partial^2 w_k}{\partial x_j^2}, \\ \frac{\partial^2 \tau}{\partial y^2} &= -\sum_{j=1}^{n-1} \frac{\partial^2 \tau}{\partial x_j^2} - \frac{2}{w_0}(\nabla \tau, \nabla w_0). \end{aligned} \tag{15}$$

The system (15) is a Cauchy–Kovalevskaya type system with analytic Cauchy data (13) and (14). By the Cauchy–Kovalevskaya theorem, there exists a neighborhood of the origin in $\mathbb{R}^n$ such that the system (15) with the conditions (13), (14) has a unique analytic solution $\tau(x, y)$, $w_k(x, y)$, $k = 0, \dots, m$. □

According to Theorem 2 with the functions $a_k(x)$ and $b_l(y)$, we have the following finite expansion of the solution $w(x, y)$ to Equation (4):

$$w(x, y) = \sum_{k=0}^{m} \widetilde{a}_k(x) \widetilde{b}_k(y).$$

If the ray series is finite, we have an usual finite expansion. Indeed, let

$$\widehat{w}(x, \omega) = \int_{-\infty}^{\infty} w(x, t) e^{i\omega t} dt = e^{i\omega\tau(x)} \sum_{k=0}^{m} w_k(x) \widehat{f}_k(\omega)$$

be the Fourier-image of the function

$$w(x, t) = \sum_{k=0}^{m} w_k(x) f_k(t - \tau(x)).$$

Then in the sum

$$e^{-i\omega\tau(x)} \widehat{w}(x, \omega) = \sum_{k=0}^{m} w_k(x) \widehat{f}_k(\omega),$$

the variables ω and x are separated.

In inverse and ill-posed problems, the question arises to find a finite-dimensional expansion based on different basis functions, solutions and so on. We formulate and prove a criterion for finite representation of an arbitrary function of many variables.

Let $\alpha_k(x)$, $\beta_k(y)$, $(x, y) \in D \subseteq \mathbb{R}^{n+m}$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$, $n \geqslant 1$, $m \geqslant 1$, be analytic functions, $k = 1, \ldots, M - 1$, $M \geqslant 2$.

Theorem 4. *An analytic function $f(x, y)$ admits the finite representation*

$$f(x, y) = \sum_{k=1}^{M-1} \alpha_k(x) \beta_k(y), \quad (x, y) \in D, \tag{16}$$

if and only if

$$\det(f(\xi_i, \eta_j)) = 0 \tag{17}$$

at any points $(\xi_i, \eta_j) \in D$, $i, j = 1, \ldots, M$.

Proof. If the equality (16) holds, then an elementary computation shows that the equality (17) holds. To prove the converse assertion, we proceed by induction on M. For $M = 2$ we have

$$\begin{vmatrix} f(\xi_1, \eta_1) & f(\xi_1, \eta_2) \\ f(\xi_2, \eta_1) & f(\xi_2, \eta_2) \end{vmatrix} = 0.$$

Consequently,

$$f(\xi_1, \eta_1) = A f(\xi_1, \eta_2),$$
$$f(\xi_2, \eta_1) = A f(\xi_2, \eta_2)$$

for some function $A = A(\xi_1, \xi_2, \eta_1, \eta_2)$. From the second equation it follows that A depends only on (ξ_2, η_1, η_2). Then, in the first equality $f(\xi_1, \eta_1) = A(\xi_2, \eta_1, \eta_2) f(\xi_1, \eta_2)$, we fix $\xi_2 = \xi_2^0$, $\eta_2 = \eta_2^0$

and denote $\alpha(\xi_1){=}f(\xi_1,\eta_2^0)$, $\beta(\eta_1){=}A(\xi_2^0,\eta_1,\eta_2^0)$. Then we obtain the required representation $f(\xi_1,\eta_1)=\alpha(\xi_1)\beta(\eta_1)$.

We assume that the assertion holds for M and prove it for $M+1$. Denote $f(\xi_i,\eta_j)=f_{ij}$. The relation (17) is equivalent to the following system for some functions $A_2,\dots,A_{M+1}$ of $(\xi_1,\dots,\xi_n;\eta_1,\dots,\eta_m)$:

$$f_{i1}=\sum_{k=2}^{M+1}A_kf_{ik},\quad i=1,\dots,M+1.$$

Taking the subsystem with $i=2,\dots,M+1$, we find

$$A_k=\sum_{k=2}^{M+1}g_{ki}f_{i1},\quad k=2,\dots,M+1,$$

where g_{ki} are entries of the inverse matrix $(f_{ij})^{-1}_{i,j=2,\dots,M+1}$ and, consequently, depend only on variables $(\xi_2,\dots,\xi_n;\eta_2,\dots,\eta_m)$. In turn, A_k depend on $(\xi_2,\dots,\xi_n;\eta_1,\dots,\eta_m)$. (Note that, assuming the nonsingularity of the matrix $(f_{ij})_{i,j=2,\dots,M+1}$, we obtain the relation (17) for M and, by the inductive assumption, we obtain the representation (16).)

We fix $\xi_i=\xi_i^0$, $\eta_j=\eta_j^0$, $i,j=2,\dots,M+1$, and set $\beta_k(\eta_1)=A_k|_{\xi^0,\eta^0}$, $\alpha_k(\xi_1)=f(\xi_1,\eta_k^0)$, $k=2,\dots,M+1$. Substituting $\xi_i=\xi_i^0$, $\eta_j=\eta_j^0$, $i,j=2,\dots,M+1$ into the first relation of the system

$$f_{11}=\sum_{k=2}^{M+1}A_kf_{1k},$$

we find

$$f(\xi_1,\eta_1)=\sum_{k=2}^{M+1}\alpha_k(\xi_1)\beta_k(\eta_1).$$

The theorem is proved. □

We note that the method of physical structures [18, 19] is based on relations of the form (17).

Remark 5. Theorem 4 holds not only for analytic functions, but also for continuously differentiable functions under certain restrictions.

To conclude the paper, we formulate a counterpart of Proposition 6 for systems of second order equations

$$\lambda(x)\frac{\partial^2w}{\partial y^2}=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2w}{\partial x_i\partial x_j}+\sum_{j=1}^{n}a_j(x)\frac{\partial w}{\partial x_j}+A(x)w+F(x,y),\tag{18}$$

where $A(x)$ is an $N\times N$-matrix, $w=(w_1,\dots,w_N)^T$, $F=(F_1,\dots,F_N)^T$ are vector-valued functions, and $\lambda(x)$, $a_{ij}(x)$, $a_j(x)$ are functions.

Proposition 7. *If*

$$w(x,y)=\sum_{k=0}^{m}v_k(x)f^{(m-k)}(y-\tau(x)),$$

$$F(x,y)=\sum_{k=0}^{m}\Phi_k(x)f^{(m-k)}(y-\tau(x)),$$

where $\tau(x)$, $v_k(x)$, *and* $\Phi_k(x)$ *are connected by*

$$2\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\tau}{\partial x_i}\frac{\partial v_0}{\partial x_j} + L\tau v_0 + \sum_{j=1}^{n} a_j(x)\frac{\partial\tau}{\partial x_j}v_0 = 0,$$

$$Lv_{k-1} + A(x)v_{k-1} + \Phi_{k-1} = 2\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\tau}{\partial x_i}\frac{\partial v_k}{\partial x_j} + L\tau v_k + \sum_{j=1}^{n} a_j(x)\frac{\partial\tau}{\partial x_j}v_k, \tag{19}$$

$$Lv_m + A(x)v_m + \Phi_m = 0,$$

where

$$L = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j},$$

then

$$\lambda(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\tau}{\partial x_i}\frac{\partial\tau}{\partial x_j}$$

and the vector-valued functions $w(x,y)$ *and* $F(x,y)$ *satisfy the system* (18).

It is of interest to study the system (5), (12), (15), (19) by methods of group analysis [20, 21] or, more generally, by the method of differential connections [22, 23] and find classes of exact solutions.

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