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FURTHER GENERALIZATIONS OF RESULTS ON STRUCTURES OF CONTINUOUS FUNCTIONS

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We consider new applications of the generalized interpretation method for studying the decidability of theories of some structures in analysis. We study the algebraic structure of continuous functions over a perfectly normal space and prove the decidability of the theory of this structure. Bibliography: 20 *titles.*

The connection between the decidability of a theory of some structure and the topology of the space determining the structure was first established in [1] and further studied in [2]. The logical approach to analysis of topological spaces was applied in [3]. Based on these results, the lattice $(C(\mathbb{R}), \leqslant)$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ equipped with the pointwise order was studied in [4]. The choice of this structure is reasonable because, first, some results concerning the undecidability of elementary theories of lattices and semilattices in classical structures were obtained in [5] and, second, a similar structure was used in [6]. However, the classical methods (cf., for example, [2] and [7]–[14]) are not suitable to prove whether or not the elementary theory $(C(\mathbb{R}), \leqslant)$ is decidable. The decidability of $(C(\mathbb{R}), \leqslant)$ was proved by the generalized interpretation method proposed by O. Kudinov and described in [4]. It is natural to try to generalize this result. We note that the structure of continuous functions over $\mathbb{R}$ can be generalized in various ways, whereas the proof of reducibility in [4] can be extended with slight modifications to the case of the lattice of continuous functions over $\mathbb{R}^n$, $n > 1$.

One can consider an arbitrary metric space instead of $\mathbb{R}^n$. In turn, metric spaces can be replaced with perfectly normal spaces [15] and the generalized interpretation method can be used to establish the m-reducibility of the theory of the structure of continuous functions over a perfectly normal space to the theory of the structure of open subsets of this space.

The class of perfectly normal spaces, regarded as the class of topological spaces, is of a special interest [16, 17]. These spaces are the most general among the spaces where the theory of continuous functions is m-reducible to the theory of open subsets. Using the generalized interpretation method (Section 1) and some properties of perfectly normal spaces, we establish (Section 2) the m-reducibility. Using the classical interpretation method, we prove (Section 3) the inverse m-reducibility of the theory of open subsets to the theory of continuous functions over

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a perfectly normal space. In this case, we extend the signature of the structure of continuous functions with the relation $<$ defined by the rule

$$f < g \Leftrightarrow f(x) < g(x) \quad \forall\, x \in X,$$

which leads to slight modifications in the proof of the m-reducibility in Section 2. At the end of the paper, we describe an important example. All necessary facts of mathematical analysis can be found in [18, 19].

1 Generalized Interpretation Method

Assume that $\mathfrak{A} = (A, \sigma)$ is a structure of a signature σ, $\mathfrak{B} = (B, \sigma_1)$ is a structure of a signature σ_1, and $g : A^{<\omega} \mapsto B^{<\omega}$ is a mapping from a finite set of elements of A to some finite set of elements of B. We write $\Phi(x_1, \ldots, x_n)$ if all free variables in Φ are contained among $x_1, \ldots, x_n$. We write $\langle a_1, \ldots, a_k\rangle \sqsubseteq \langle a_1, \ldots, a_n\rangle$ if $\langle a_1, \ldots, a_k\rangle$, $k \leqslant n$, is the initial segment. We denote by $\overline{x} \frown \{u\}$ the concatenation of a tuple $\overline{x}$ and a letter u. Under the above notation, we assume that there is a monotone computable function $m : \mathbb{N} \to \mathbb{N}$ such that

1) $m(0) = 0$ and $g(\varnothing) = \varnothing$,
2) $lh(g(\overline{a})) = m(n)$, where $\overline{a} = \langle a_1, \ldots, a_n\rangle$, i.e., the length of the tuple $g(\langle a_1, \ldots, a_n\rangle)$ is equal to the value of the function m of variable n,
3) if $k \leqslant n$, then $g(\langle a_1, \ldots, a_k\rangle) \sqsubseteq g(\langle a_1, \ldots, a_n\rangle)$,
4) there exists an effective procedure of transformation of atomic formulas $\Phi(x_1, \ldots, x_n)$ of a signature σ to some formulas $\Phi_g(y_1, \ldots, y_{m(n)})$ of a signature σ_1 such that $\mathfrak{A} \models \Phi(\overline{a}) \Leftrightarrow \mathfrak{B} \models \Phi_g(g(\overline{a}))$ for any $\overline{a} = \langle a_1, \ldots, a_n\rangle \in A^n$,
5) there exists a computable sequence of formulas $\{\Psi_n(\overline{z})\}_{n\in\omega}$, where $\overline{z} = \langle z_1, \ldots, z_{m(n)}\rangle$ is a tuple of variables of the signature σ_1, such that
   (a) for any $\overline{b} = \langle b_1, \ldots, b_{m(n)}\rangle \in B^{m(n)}$
   $$\mathfrak{B} \models \Psi_n(\overline{b}) \Leftrightarrow \exists \overline{a}\ g(\overline{a}) = \overline{b}, \quad \overline{a} = \langle a_1, \ldots, a_n\rangle \in A^n,$$
   (b) for any $\overline{b} = \langle b_1, \ldots, b_{m(n+1)}\rangle \in B^{m(n+1)}$ from the condition $g(\overline{a}) \sqsubseteq \overline{b}$ (with some $\overline{a} = \langle a_1, \ldots, a_n\rangle$) and $\mathfrak{B} \models \Psi_{n+1}(\overline{b})$ it follows that $\exists u \in A\ g(\overline{a} \frown \{u\}) = \overline{b}$.

Conditions 1)–5) are sufficient for the reducibility of the theory of the structure $\mathfrak{A}$ to the theory of the structure $\mathfrak{B}$ because of the following lemma proved in [4].

Lemma 1. *If Conditions* 1)–5) *are satisfied, then there is an effective procedure of transformation of arbitrary formulas* $\Phi(x_1, \ldots, x_n)$ *of a signature* σ *to some formulas* $\Phi_g(y_1, \ldots, y_{m(n)})$ *of a signature* σ_1 *such that* $\mathfrak{A} \models \Phi(\overline{a}) \Leftrightarrow \mathfrak{B} \models \Phi_g(g(\overline{a}))$ *for any formula* $\Phi(x_1, \ldots, x_n)$ *of the signature* σ *and any* $\overline{a} = \langle a_1, \ldots, a_n\rangle \in A^n$

2 Reducibility of Theory of Continuous Functions to Theory of Open Sets

For a set X we denote by $\mathfrak{A}_X \rightleftharpoons (C(X), \leqslant, <)$ the algebraic structure of continuous functions from X to $\mathbb{R}$ equipped with the pointwise order $\leqslant$ and the relation $<$ defined by

$$f < g \Leftrightarrow f(x) < g(x) \quad \forall\, x \in X.$$

We denote by $\mathfrak{B}_X$ the structure $(O(X), \subseteq)$, where $O(X)$ is the set of open subsets of X.

We recall some definitions from [15].

A subset A of a topological space X is called *functionally closed* if $A = f^{-1}(0)$ for some continuous mapping $f : X \to \mathbb{R}$.

A topological space X is called a *T_1-space* if for each pair of distinct points $x_1, x_2 \in X$ there exists an open set $U \subset X$ such that $x_1 \in U$ and $x_2 \notin U$.

We formulate one of the equivalent definitions of a perfectly normal space (cf. [15]). A topological T_1-space X is called a *perfectly normal space* if closed subsets of X are functionally closed.

As is noted in [15], the class of perfectly normal spaces is narrower than the class of normal spaces. In other words, any perfectly normal space is normal. The converse assertion is false.

Finally, we recall the following fact [15] which will be used below: for each pair of disjoint closed sets A and B of a perfectly normal space X there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

To prove the main result of this section, we need the following assertion.

Lemma 2. *Assume that X is a perfectly normal space, A and B are closed subsets of X, and $h_1, h_2 : X \to \mathbb{R}$ are two continuous functions such that $h_1(x) \leqslant h_2(x)$ for $x \in X$ and $h_1(x) = h_2(x)$ for $x \in A \cap B$. Then there exists a continuous function $h : X \to \mathbb{R}$ such that*

1) *$h(x) = h_1(x)$ if and only if $x \in A$,*

2) *$h(x) = h_2(x)$ if and only if $x \in B$,*

3) *$h_1(x) < h(x) < h_2(x)$ for all $x \in X \setminus (A \cup B)$.*

Proof. By the definition of a perfectly normal space, for closed sets A and B there exist continuous functions χ_A and χ_B such that $\chi_A(x) = 0$ for $x \in A$, $\chi_A(x) > 0$ for the remaining $x \in X$, $\chi_B(x) = 0$ for $x \in B$, and $\chi_B(x) > 0$ for the remaining $x \in X$. We verify that

$$f(x) = \begin{cases} \dfrac{\chi_B(x)h_1(x) + \chi_A(x)h_2(x)}{\chi_A(x) + \chi_B(x)}, & x \notin A \cap B, \\[2ex] \dfrac{h_1(x) + h_2(x)}{2}, & x \in A \cap B, \end{cases}$$

is continuous and satisfies the required conditions. Indeed, let $x_0 \in X$ be an arbitrary fixed point in $A \cap B$ such that any neighborhood of x contains points $y \in X \setminus (A \cap B)$. We fix $\varepsilon > 0$. By the continuity of h_1 and h_2, we can choose a neighborhood of the point x_0 such that for all z in this neighborhood we have $|h_1(x_0) - h_1(z)| < \varepsilon/2$ and $|h_2(x_0) - h_2(z)| < \varepsilon/2$. For any point $y_0 \in X \setminus (A \cap B)$ in this neighborhood

$$\Big|\frac{h_1(x_0) + h_2(x_0)}{2} - \frac{\chi_B(y_0)h_1(y_0) + \chi_A(y_0)h_2(y_0)}{\chi_A(y_0) + \chi_B(y_0)}\Big|$$

$$= \Big|\frac{1}{2(\chi_A(y_0) + \chi_B(y_0))}\Big(\chi_A(y_0)(h_1(x_0) - h_2(y_0)) + \chi_A(y_0)(h_2(x_0) - h_2(y_0))$$

$$+ \chi_B(y_0)(h_1(x_0) - h_1(y_0)) + \chi_B(y_0)(h_2(x_0) - h_1(y_0))\Big)\Big|$$

$$\leqslant \Big|\frac{\chi_A(y_0)}{2(\chi_A(y_0) + \chi_B(y_0))}\Big| * (|h_1(x_0) - h_2(y_0)| + |h_2(x_0) - h_2(y_0)|)$$

$$+\left|\frac{\chi_B(y_0)}{2(\chi_A(y_0)+\chi_B(y_0))}\right| * (|h_1(x_0)-h_1(y_0)|+|h_2(x_0)-h_1(y_0)|) < \frac{1}{2}\Big(4\frac{\varepsilon}{2}\Big) = \varepsilon.$$

Thus, we proved the continuity at the point x_0. The continuity at interior points of the set $A \cap B$ and at points of the open set $X \setminus (A \cap B)$ follows from the formula for constructing the function h (we use the composition of continuous functions and the fact that the denominator does not vanish). Thus, h is continuous.

We verify that $h(x) = h_1(x)$ for $x \in A$ and $h(x) = h_2(x)$ for $x \in B$. We note that $\chi_A(x) = 0$ for all $x \in A$ and $\chi_B(x) = 0$ for all $x \in B$. For points $x \in A \setminus B$ we use the upper formula for computing h and get $h(x) = h_1(x)$. The equality $h(x) = h_2(x)$ for $x \in B \setminus A$ is proved as above. For $x \in A \cap B$ we use the lower formula for h and the assumption $h_1(x) = h_2(x)$.

Let us prove that $h_1(x) < h(x) < h_2(x)$ for $x \notin A \cup B$. We have

$$h(x) = \frac{\chi_B(x)h_1(x)+\chi_A(x)h_2(x)}{\chi_A(x)+\chi_B(x)} = h_1(x) + \frac{\chi_A(x)(h_2(x)-h_1(x))}{\chi_A(x)+\chi_B(x)} > h_1(x), \quad x \notin A \cup B,$$

$$h(x) = \frac{\chi_B(x)h_1(x)+\chi_A(x)h_2(x)}{\chi_A(x)+\chi_B(x)} = h_2(x) - \frac{\chi_B(x)(h_2(x)-h_1(x))}{\chi_A(x)+\chi_B(x)} < h_2(x), \quad x \notin A \cup B.$$

Thus, the lemma is proved. □

The following assertion is proved in Sections 2 and 3.

Theorem 1. *Let X be a perfectly normal space. The theories of the structures $\mathfrak{A}_X$ and $\mathfrak{B}_X$ are m-equivalent.*

In this section, we prove the following lemma.

Lemma 3. *The theory of the structure $\mathfrak{A}_X$ is m-reduced to the theory of the structure $\mathfrak{B}_X$.*

Proof. Let us verify that all the conditions of the generalized interpretation method are satisfied. Then the required result follows from Lemma 1.

According to the generalized interpretation method, we need to construct a mapping $g : C(X)^{<\omega} \mapsto O(X)^{<\omega}$ sending a finite tuple of elements of the set $C(X)$ to some finite tuple of elements of the set $O(X)$ in such a way that certain conditions are satisfied. We set

$g(\varnothing) = \varnothing$,

$g(\langle f_1, \ldots, f_n\rangle) = \langle\{G_{ij} : 1 \leqslant i, j \leqslant n\}\rangle$, $n \geqslant 1$, where $G_{ij} = \{x \in X | f_i(x) < f_j(x)\}$, and the number of position G_{ij} in this tuple (denoted by $\mathrm{Num}\,(i, j)$) is computed as follows:

$$\mathrm{Num}\,(i,j) = \begin{cases} j^2 - i + 1, & i \leqslant j, \\ (i-1)^2 + j, & j < i. \end{cases} \tag{1}$$

A tuple of open sets G_{ij} ordered by $\mathrm{Num}\,(i, j)$ is written as $\langle\{G_{ij} : 1 \leqslant i, j \leqslant n\}\rangle$.

We show that for the monotone computable function $m(n) = n^2$ the conditions of the generalized interpretation method are satisfied.

Indeed, $m(0) = 0$ and $g(\varnothing) = \varnothing$. The cardinality of the set $\{G_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n\}$ is equal to n^2, i.e., Condition 2 holds. The position G_{ij} is computed by formula (1). Then

Condition 3 holds. Condition 4 is satisfied since it is possible to construct $\Phi_g(y_1, \ldots, y_{n^2})$ from the atomic formula $\Phi(x_1, \ldots, x_n)$ as follows:

$$[x_i \leqslant x_j]_g \rightleftharpoons y_{\mathrm{Num}(j,i)} = \varnothing,$$

$$[x_i < x_j]_g \rightleftharpoons y_{\mathrm{Num}(i,j)} = X.$$

These effective construction procedures preserve the truth of the corresponding formulas. Thus, Condition 4 is satisfied.

It remains to construct a computable sequence $\{\Psi_n(\overline{y}^n)\}_{n\in\omega}$ of the signature $\mathfrak{B}_X$ such that Condition 5 holds. Let $\overline{a} = \langle a_1, \ldots, a_n\rangle \in A_X^n$. Then g maps $\overline{a}$ to $\overline{b} = \langle b_1, \ldots, b_{n^2}\rangle \in B_X^{n^2}$. We extend $\overline{b}$ to $\overline{b'}$ by adding the sets $G_{n+1\ 1}, \ldots, G_{n+1\ n}, G_{n+1\ n+1}, G_{n\ n+1}, \ldots, G_{1\ n+1}$ to the tuple. Thus, $lh(\overline{b'}) = (n+1)^2$. We see that it is the value of the function m of $n+1$, i.e., $m(n+1) = (n+1)^2$. Thereby $\overline{b'} = \langle\{G_{ij} \mid 1 \leqslant i, j \leqslant n+1\}\rangle$.

For the sake of brevity we denote by T_{ij} the set $\{x \mid f_i(x) \leqslant f_j(x)\}$. We note that $T_{ij} = X \setminus G_{ji}$. We introduce the sequence $\{\Psi_n\}_{n\in\omega}$ as follows:

$\Psi_0(\overline{y})$ is identically true,

$\Psi_n(\langle\{G_{ij} : 1 \leqslant i, j \leqslant n\}\rangle)$

$\rightleftharpoons (\bigwedge_{i=1}^n G_{ii} = \varnothing) \wedge (\bigwedge_{i,j,k=1}^n G_{ij} \cap G_{jk} \subseteq G_{ik}) \wedge (\bigwedge_{i,j,k=1}^n T_{ij} \cap T_{jk} \subseteq T_{ik})$.

We note that the formula Ψ_n is a formula of the signature $\mathfrak{B}_X$. We show the validity of Condition 5 which can be written as follows.

(a) $\mathfrak{B}_X \models \Psi_n(\overline{b}) \Leftrightarrow \exists\overline{a}\ g(\overline{a}) = \overline{b}$, $\overline{a} = \langle a_1, \ldots, a_n\rangle \in A_X^n$ for any $\overline{b} = \langle b_1, \ldots, b_{m(n)}\rangle \in B_X^{m(n)}$,

(b) for any $\overline{b} = \langle b_1, \ldots, b_{m(n+1)}\rangle \in B_X^{m(n+1)}$ from the conditions $g(\overline{a}) \sqsubseteq \overline{b}$ for some $\overline{a} = \langle a_1, \ldots, a_n\rangle$ and $\mathfrak{B}_X \models \Psi_{n+1}(\overline{b})$ it follows that $\exists u \in A_X\ g(\overline{a} \frown \{u\}) = \overline{b}$.

It is easy to verify condition (a) from right to left. Using (a), we can prove (b).

Assume that G_{ij}, $1 \leqslant i, j \leqslant n+1$, are given open subsets of $\mathfrak{B}_X \models \Psi_{n+1}(\langle\{G_{ij} | 1 \leqslant i, j \leqslant n+1\}\rangle)$ and $f_1, \ldots, f_n$ are continuous functions such that

$$x \in G_{ij} \Leftrightarrow f_i(x) < f_j(x) \quad \forall\, 1 \leqslant i, j \leqslant n.$$

We construct a continuous function f_{n+1} such that

$$g(\langle f_1, \ldots, f_{n+1}\rangle) = \langle\{G_{ij} : 1 \leqslant i, j \leqslant n+1\}\rangle,$$

i.e.,

$$x \in G_{i\ n+1} \Leftrightarrow f_i(x) < f_{n+1}(x) \quad \forall\, 1 \leqslant i \leqslant n,$$

$$x \in G_{n+1\ j} \Leftrightarrow f_{n+1}(x) < f_j(x) \quad \forall\, 1 \leqslant j \leqslant n.$$

1. Let $x \notin G_{i\ n+1} \cup G_{n+1\ i}$ for some $i \leqslant n$. We write $x \in A_i$, where $A_i \rightleftharpoons \{x \in X \mid x \notin G_{i\ n+1} \cup G_{n+1\ i}\}$. We set $f_{n+1}(x) \rightleftharpoons f_i(x)$ at such points x, i.e., for $f_{n+1}(x)$ we take the known value $f_i(x)$.

2. Let A and B be fixed disjoint number sets such that $A \cup B = \{1, \ldots, n\}$ (the sets A and B are assumed to be nonempty; the case where one of these sets is empty will be considered independently). We denote

$$C \rightleftharpoons \bigcap_{i\in A} T_{i\ n+1} \cap \bigcap_{j\in B} T_{n+1\ j}.$$

We note that C contains only those points $x \in X$ where $\max\limits_{i\in A} f_i(x) \leqslant f_{n+1}(x) \leqslant \min\limits_{j\in B} f_j(x)$. For $x \in C$ we introduce the closed sets

$$A^* \rightleftharpoons \bigcup_{i_0\in A} A_{i_0}, \quad B^* \rightleftharpoons \bigcup_{j_0\in B} A_{j_0}.$$

We note that

$$A^* = \bigl\{x \in X \mid f_{n+1}(x) = \max_{i\in A} f_i(x)\bigr\}, \quad B^* = \bigl\{x \in X \mid f_{n+1}(x) = \min_{j\in B} f_j(x)\bigr\}.$$

We use Lemma 2. We take $\max_{i\in A} f_i$ for h_1 and $\min_{j\in B} f_j$ for h_2. We note that C is a closed subset of X. Hence all closed sets in C are also closed in X and, consequently, they are functionally closed. Thus, C is a perfectly normal space. We have $h_1(x) \leqslant h_2(x)$ for all $x \in C$. It is easy to see that $h_1(x) = h_2(x) \Leftrightarrow x \in A^* \cap B^*$ for $x \in C$. By Lemma 2, there exists a continuous function h such that

$$h(x) = h_1(x) \Leftrightarrow x \in A^*;\ h(x) = h_2(x) \Leftrightarrow x \in B^*;\ h_1(x) < h(x) < h_2(x)$$

for all x in the set $T_{A,B} \rightleftharpoons C \setminus (A^* \cup B^*)$. We note that this set can be written as

$$T_{A,B} = \bigcap_{i\in A} G_{i\ n+1} \cap \bigcap_{j\in B} G_{n+1\ j}.$$

Thus, f_{n+1} for all x in $T_{A,B}$ satisfies the inequalities

$$\max_{i\in A} f_i(x) < f_{n+1}(x) < \min_{j\in B} f_j(x).$$

For $x \in Cl(T_{A,B})$, where $Cl(T_{A,B})$ is the closure of the set $T_{A,B}$, we set $f_{n+1}(x) = h(x)$.

If we consider $A' \cup B' = \{1, \dots, n\}$ instead of A and B, then $f_{n+1}(x)$ for $x \in Cl(T_{A',B'})$ is constructed in a similar way.

Assume that A or B is empty. We begin with the case $A = \{1, \dots, n\}$; $B = \varnothing$. We denote by $f_{\max}(x)$ the continuous function $\max\limits_{i\in A} f_i(x) = \max\limits_{i\in\{1,\dots,n\}} f_i(x)$. As above, A^* denotes the set where $\{x \in X \mid f_{n+1}(x) = \max\limits_{i\in A} f_i(x) = f_{\max}(x)\}$. For given A and B we denote

$$T_{A,B} \rightleftharpoons \bigcap_{i=\{1,\dots,n\}} G_{i\ n+1}.$$

We construct f_{n+1} on $Cl(T_{A,B})$ by the rule

$$f_{n+1}(x) = f_{\max}(x) + \chi_{A^*}(x).$$

If $A = \varnothing$ and $B = \{1, \dots, n\}$, then we set $f_{\min}(x) \rightleftharpoons \min\limits_{j\in\{1,\dots,n\}} f_j(x)$ and $B^* \rightleftharpoons \{x \in X \mid f_{n+1}(x) = f_{\min}(x)\}$. In this case,

$$T_{A,B} \rightleftharpoons \bigcap_{j=\{1,\dots,n\}} G_{n+1\ j}.$$

We construct f_{n+1} on $Cl(T_{A,B})$ by the rule

$$f_{n+1}(x) = f_{\min}(x) - \chi_{B^*}(x).$$

It is easy to see that

$$Cl(T_{A,B}) \setminus T_{A,B} \subseteq \bigcup_{i=1}^{n} A_i$$

for any A and B such that $A \cup B = \{1, \dots, n\}$.

To prove 5 (b), it remains to verify the following conditions:

(a) $f_{n+1}(x)$ is defined for any $x \in X$,

(b) the function f_{n+1} is well defined, i.e. if $f_{n+1}(x)$ for some $x \in X$ is defined more than once, then these values coincide,

(c) for f_{n+1}

$$x \in G_{i\ n+1} \Leftrightarrow f_i(x) < f_{n+1}(x) \quad \forall\ i = 1, \dots, n,$$
$$x \in G_{n+1\ j} \Leftrightarrow f_{n+1}(x) < f_j(x) \quad \forall\ j = 1, \dots, n,$$

(d) the constructed function f_{n+1} is continuous.

Indeed if $x \in A_i$ for some $i \leqslant n$, then $f_{n+1}(x) = f_i(x)$. If $x \notin A_i$ for all $i \leqslant n$, then for every $i \leqslant n$ either $x \in G_{i\ n+1}$ or $x \in G_{n+1\ i}$. Thus, there are A and B such that $A \cup B = \{1, \dots, n\}$ and $x \in \bigcap_{i \in A} G_{i\ n+1} \cap \bigcap_{j \in B} G_{n+1\ j}$ (A or B can be empty) and for such x it is indicated how to construct $f_{n+1}(x)$. Thus, we have verified Condition (a).

Let us check that if $f_{n+1}(x_0)$ is constructed more than once for some point $x_0 \in X$, then these values coincide. We consider the case where $x_0 \in A_i \cap A_j$ for some $i, j = 1, \dots, n$. In this case, $f_{n+1}(x_0) = f_i(x_0)$. On the other hand, $f_{n+1}(x_0) = f_j(x_0)$. However, as was already mentioned, from the condition $\Psi_{n+1}(\langle\{G_{ij} \mid 1 \leqslant i, j \leqslant n+1\}\rangle)$; namely, from $T_{i\ n+1} \cap T_{n+1\ j} \subseteq T_{ij}$ and $T_{j\ n+1} \cap T_{n+1\ i} \subseteq T_{ji}$ it follows that $f_i(x_0) = f_j(x_0)$.

Let $x_0 \in A_i$ for some $i \leqslant n$ and, at the same time, $x_0 \in Cl(T_{A,B})$ for some number sets A and B. We note that $x_0 \notin T_{A,B}$; otherwise, at least one of the following inequalities holds: $f_{n+1}(x_0) > f_i(x_0)$ if $i \in A$ or $f_{n+1}(x_0) < f_i(x_0)$ if $i \in B$. However, $f_{n+1}(x_0) = f_i(x_0)$ since $x_0 \in A_i$. Hence $x_0 \in Cl(T_{A,B}) \setminus T_{A,B}$. Then $x_0 \in A^*$ or $x_0 \in B^*$, i.e., $x_0 \in \bigcap_{k \in M} A_k$, where M is a nonempty set of indices in $1, \dots, n$. It is known that $i \in M$. If M consists of more than one i, then we argue in the same way as in the previous case.

It can happen that $x_0 \in Cl(T_{A,B}) \cap Cl(T_{A',B'})$ for some $A \neq A'$ and $B \neq B'$. We note that $x_0 \in Cl(T_{A,B}) \setminus T_{A,B}$ and $x_0 \in Cl(T_{A',B'}) \setminus T_{A',B'}$ since for every $x \in T_{A,B}$ the sets A and B are uniquely defined. Then $x_0 \in A_i$ and $x_0 \in A_{i'}$ for some $i, i' \leqslant n$, i.e., $x_0 \in A_i \cap A_{i'}$. The fact that the values of f_{n+1} coincide was already shown.

Let us verify condition (c). We first show that the conditions $\Psi_{n+1}(\langle\{G_{ij}|1 \leqslant i, j \leqslant n+1\}\rangle)$ imply the inclusion $G_{i\ n+1} \cap T_{n+1\ j} \subseteq G_{ij}$ for some $i, j \leqslant n$. Indeed, the following equivalent transformations hold:

$$T_{n+1\ j} \cap T_{ji} \subseteq T_{n+1\ i}, \quad T_{ji} \cap T_{n+1\ j} \subseteq T_{n+1\ i}, \quad T_{ji} \subseteq (X \setminus T_{n+1\ j}) \cup T_{n+1\ i},$$
$$T_{n+1\ j} \cap G_{i\ n+1} \subseteq G_{ij}, \quad G_{i\ n+1} \cap T_{n+1\ j} \subseteq G_{ij}.$$

Let $x \in G_{i\ n+1}$. Two cases are possible: either $x \in A_j$ for some $j \neq i$, $j = 1, \dots, n$, or there are no $j = 1, \dots, n$ such that $x \in A_j$. In the first case, without loss of generality we can assume that $i < j$. We note that the conditions $\Psi_{n+1}(\langle\{G_{ij} \mid 1 \leqslant i, j \leqslant n+1\}\rangle)$ imply $G_{i\ n+1} \cap T_{n+1\ j} \subseteq G_{ij}$. Consequently, $x \in G_{ij}$, i.e., $f_i(x) < f_j(x) = f_{n+1}(x)$. In the second case, $x \in T_{A,B}$ for some uniquely defined number sets A and B; moreover, $i \in A$. By the construction of f_{n+1}, we have $f_{n+1}(x) > \max_{k \in A} f_k(x) \geqslant f_i(x)$. Thus, in both cases, the required inequality is proved.

Further, $f_i(x) \geqslant f_{n+1}(x)$ for $x \notin G_{i\ n+1}$ Thereby we have probed the first required equivalence. The inequality $f_{n+1}(x) < f_i(x)$ for $x \in G_{n+1\ i}$ can be proved as above. Thus, $x \notin G_{i\ n+1}$, i.e., $x \in T_{n+1\ i}$. Two cases are possible: $x \in A_i$ and $x \in G_{n+1\ i}$. In the first case, $f_{n+1}(x) = f_i(x)$, whereas $f_{n+1}(x) < f_i(x)$ in the second one. Therefore, in both cases, $f_i(x) \geqslant f_{n+1}(x)$. The first equivalence is proved. The second equivalence is proved in a similar way.

It remains to verify Condition (d), i.e., the continuity of f_{n+1}. For this purpose we use the following assertion which can be found in [20].

Proposition 1. *Let F_1 and F_2 be two closed sets in X that constitute the entire space X, and let continuous mappings $f_1 : F_1 \to Y$ and $f_2 : F_2 \to Y$ from F_1 and F_2 to a space Y coincide on $F_1 \cap F_2$. Then the following mapping $f : X \to Y$ is continuous:*

$$f(x) = \begin{cases} f_1(x), & x \in F_1, \\ f(x) = f_2(x), & x \in F_2. \end{cases}$$

This assertion can be generalized to the case where the space X is the union of finitely many closed sets.

Proposition 2. *Let $F_1, \dots, F_p$ be closed sets in X such that $X = F_1 \cup \dots \cup F_p$, and let continuous mappings $f_1 : F_1 \to Y, \dots, f_p : F_p \to Y$ from $F_1, \dots, F_p$ to a space Y coincide on all possible pairwise intersections $F_i \cap F_j$, $i, j \leqslant p$. Then the following mapping $f : X \to Y$ is continuous:*

$$f(x) = \begin{cases} f_1(x), & x \in F_1, \\ f_p(x), & x \in F_p. \end{cases}$$

Proof. It suffices to prove that the preimage $f^{-1}\Phi$ of any set Φ closed in Y is closed in X. However, it is easy to verify that $f^{-1}\Phi = f_1^{-1}\Phi \cup \dots \cup f_p^{-1}\Phi$. The sets $f_1^{-1}\Phi, \dots, f_p^{-1}\Phi$ are closed in the closed sets $F_1, \dots, F_p$ respectively and, consequently, in the entire space X. Hence the set $f^{-1}\Phi = f_1^{-1}\Phi \cup \dots \cup f_p^{-1}\Phi$ is also closed. □

In our case, a perfectly normal space X can be represented as the union of finitely many closed sets

$$X = A_1 \cup \dots \cup A_n \cup Cl(T_{A^1,B^1}) \cup \dots \cup Cl(T_{A^l,B^l}),$$

where A^i, B^i, $i = 1, \dots, l$, are all possible disjoint number sets (one of these sets can be empty) such that $A^i \cup B^i = \{1, \dots, n\}$. On each closed set, the required continuous functions are well constructed (they coincide on the corresponding intersections). By assumption, the function f_{n+1} coinciding with the constructed functions on the corresponding closed sets is continuous.

It remains to prove Condition 5(a) from left to right, i.e.,

$$\mathfrak{B}_X \models \Psi_n(\langle b_1, \dots, b_{m(n)} \rangle) \Rightarrow \exists \langle a_1, \dots, a_n \rangle\ g(\langle a_1, \dots, a_n \rangle) = \langle b_1, \dots, b_{m(n)} \rangle.$$

We proceed by induction on n. It is easy to prove the base $n = 0$. We assume that the required assertion is proved for $n = k$. Let $n = k + 1$. By the construction of $\{\Psi_n\}_{n \in \omega}$, it is clear that $\mathfrak{B}_X \models \Psi_{k+1}(\langle b_1, \dots, b_{m(k+1)} \rangle)$ implies $\mathfrak{B}_X \models \Psi_k(\langle b_1, \dots, b_{m(k)} \rangle)$. By the induction assumption,

$$\exists \langle a_1, \dots, a_k \rangle\ g(\langle a_1, \dots, a_k \rangle) = \langle b_1, \dots, b_{m(k)} \rangle.$$

By Condition (b),

$$\exists u \in C(X)\ g(\overline{a}^k \frown \{u\}) = \langle b_1, \ldots, b_{m(k+1)} \rangle.$$

We denote by $\overline{a}^{k+1}$ the concatenation of $\overline{a}^k \frown \{u\}$. Thus, we have shown that

$$\mathfrak{B}_X \models \Psi_{k+1}(\langle b_1, \ldots, b_{m(k+1)} \rangle) \Rightarrow \exists \overline{a}^{k+1}\ g(\overline{a}^{k+1}) = \langle b_1, \ldots, b_{m(k+1)} \rangle.$$

Condition 5(a) is proved.

Thus, all the assumptions of Lemma 1 are satisfied. Hence $\mathrm{Th}\,(\mathfrak{A}_X) \leqslant_m \mathrm{Th}\,(\mathfrak{B}_X)$. □

3 Reducibility of Theory of Open Subsets to Theory of Continuous Functions

In this section, we preserve the notation $\mathfrak{A}_X$ and $\mathfrak{B}_X$ introduced in Section 2. To complete the proof of Theorem 1, it remains to prove the following assertion.

Lemma 4. $\mathrm{Th}\,(\mathfrak{B}_X) \leqslant_m \mathrm{Th}\,(\mathfrak{A}_X)$.

Proof. We use the method of relative elementary definability [7]. We note that the theory $\mathfrak{B}_X$ is undecidable if and only if the theory of the structure $\mathfrak{D}_X$ of closed subsets of X is undecidable. In particular, $\mathrm{Th}\,(\mathfrak{D}_X) \leqslant_m \mathrm{Th}\,(\mathfrak{B}_X)$. We show that $\mathrm{Th}\,(\mathfrak{A}_X) \leqslant_m \mathrm{Th}\,(\mathfrak{D}_X)$.

Since $\mathfrak{A}_X$ is the lattice of continuous functions, for any two continuous functions there are the greatest lower bound $f \cap g$ and the least upper bound $f \cup g$.

We define a predicate Ψ^2 such that $\Psi(f, g)$ is true if and only if $f \leqslant g$ and the set $C \rightleftharpoons \{x \in X | f(x) = g(x)\}$ is nonepmty and connected. It is convenient to construct a formula $\neg\Psi(f, g)$ that is true if either $\neg g \geqslant f$, or $f < g$, or the nonempty set C is not connected, i.e., there exist two closed disjoint sets $A \subseteq X$ and $B \subseteq X$ such that

$$\begin{aligned} &f(x) = g(x) \ \Leftrightarrow\ x \in A \cup B, \\ &\neg\Psi(f, g) \rightleftharpoons \neg g \geqslant f \vee g > f \vee \exists f_A \exists f_B \exists g_A \exists g_B (f_A \cup f_B = f \wedge g_A \cap g_B = g \\ &\qquad \wedge \neg g > f_A \wedge \neg g > f_B \wedge g_A > f_B \wedge g_B > f_A). \end{aligned} \tag{1}$$

If there are functions f_A, f_B, g_A, and g_B satisfying all the conjunctions in the above formulas, then we consider the sets $A' = \{x | g(x) = f_A(x)\}$ and $B' = \{x | g(x) = f_B(x)\}$. Since $\neg g > f_A$ and $\neg g > f_B$, both sets are nonempty.

We prove that $C = A' \cup B'$. Indeed, if $x \in C$, then $g(x) = f(x)$. Consequently, at least one of the equalities $g(x) = f_A(x)$ or $g(x) = f_B(x)$ holds since $f_A \cup f_B = f$. Thus, $x \in A' \cup B'$. Let $x \in A' \cup B'$, i.e., $g(x) = f_A(x)$ or $g(x) = f_B(x)$. Then $g(x) \geqslant f(x) \geqslant f_A(x)$ and $g(x) \geqslant f(x) \geqslant f_B(x)$ imply $g(x) = f(x)$, i.e., $x \in C$.

We prove that $A' \cap B' = \varnothing$. Indeed, if there is $x_0 \in A' \cap B'$, then for such x_0 we have $g(x_0) = f_A(x_0) = f_B(x_0) = y_0$ with some y_0. On the other hand, $g_A(x_0) > f_B(x_0) = y_0$ and $g_B(x_0) > f_A(x_0) = y_0$, i.e., $g(x_0) > y_0$. We obtain a contradiction.

Assume that the set $\{x \in X | f(x) = g(x)\}$ is not connected, i.e., there are disjoint closed sets A and B such that $f(x) = g(x)$ if and only if $x \in A \cup B$. We construct functions f_A, f_B, g_A, and g_B satisfying all the conjunctions in formula (1).

Since X is a perfectly normal space, there exists a continuous function $h_{A,B}(x)$ from X to $[0,1]$ such that

$$h_{A,B}(x) = \begin{cases} 0, & x \in A, \\ 1, & x \in B. \end{cases}$$

The property of perfectly normal spaces was already mentioned at the beginning of Section 2.

The following continuous functions satisfy all the conjunctions in the formula $\neg\Psi(f,g)$:

$$f_A(x) = \begin{cases} f(x), & h_{A,B}(x) \geqslant 1/2, \\ f(x) + 2h_{A,B}(x) - 1, & h_{A,B}(x) \leqslant 1/2, \end{cases}$$

$$f_B(x) = \begin{cases} f(x), & h_{A,B}(x) \leqslant 1/2, \\ f(x) - 2h_{A,B}(x) + 1, & h_{A,B}(x) \geqslant 1/2, \end{cases}$$

$$g_A(x) = \begin{cases} g(x), & h_{A,B}(x) \geqslant 1/2, \\ g(x) - 2h_{A,B}(x) + 1, & h_{A,B}(x) \leqslant 1/2, \end{cases}$$

$$g_B(x) = \begin{cases} g(x), & h_{A,B}(x) \leqslant 1/2, \\ g(x) + 2h_{A,B}(x) - 1, & h_{A,B}(x) \geqslant 1/2. \end{cases}$$

Indeed, $f_A(x) \leqslant f(x)$, $f_B(x) \leqslant f(x)$, $g_A(x) \geqslant g(x)$, and $g_B(x) \geqslant g(x)$. All these functions are continuous, $g_A \cap g_B = g$, and $f_A \cup f_B = f$. It remains to prove that $g_A(x) > f_B(x)$ and $g_B(x) > f_A(x)$.

Case 1. $h_{A,B}(x) > 1/2$. Then for any x

$$g_A(x) = g(x) > f(x) - 2h_{A,B}(x) + 1 = f_B(x),$$
$$g_B(x) = g(x) + 2h_{A,B}(x) - 1 > g(x) \geqslant f(x) = f_A(x).$$

Case 2. $h_{A,B}(x) < 1/2$. For such x

$$g_A(x) = g(x) - 2h_{A,B}(x) + 1 > g(x) \geqslant f(x) = f_B(x),$$
$$g_B(x) = g(x) > f(x) + 2h_{A,B}(x) - 1 = f_A(x).$$

Case 3. $h_{A,B}(x) = 1/2$. Then $f(x) < g(x)$ for all such x since $f(x) = g(x)$ only for $x \in A \cup B$, i.e., such that $h_{A,B}(x) = 0$ or $h_{A,B}(x) = 1$. Thus, for all x such that $h_{A,B}(x) = 1/2$

$$g_A(x) = g(x) > f(x) = f_B(x),$$
$$g_B(x) = g(x) > f(x) = f_A(x).$$

The predicate $\Psi(f,g)$ is satisfied for continuous functions f and g such that $f \leqslant g$. However, it is more convenient to use the predicate $\Psi'(f,g)$ of the same sense as $\Psi(f,g)$, but the functions f and g do not necessarily satisfy the condition $f \leqslant g$:

$$\Psi'(f,g) \rightleftharpoons \Psi(f \cap g, f \cup g).$$

This formula is correct since for any two functions in the lattice $\mathfrak{A}_X$ we can consider their supremum and infimum.

Now, we consider the predicate $\Psi^*(f,g)$ that is true if and only if the intersection of the graphs of the continuous functions f and g consists of a single point:

$$\Psi^*(f,g) \rightleftharpoons \Psi'(f,g) \wedge (\forall g' \geqslant f \cup g)(g' > f \cap g \vee \Psi'(f \cap g, g'))).$$

Let us assume that there are two distinct points $x_0 \in X$ and $x_1 \in X$ such that $f(x_0) = g(x_0)$ and $f(x_1) = g(x_1)$. We show that there exists a function $g' \geqslant f \cup g$ such that $\neg g' > f \cap g$ and $\neg\Psi'(g', f \cap g)$. For this purpose we consider the sets $A = \{x_0\}$ and $B = \{x_1\}$. Since X is perfectly normal, there exists a continuous function $h_{A,B}(x)$ such that $0 \leqslant h_{A,B}(x) \leqslant 1$ for all $x \in X$, $h^{-1}_{A,B}(0) = x_0$, and $h^{-1}_{A,B}(1) = x_1$. Then the required function g' can be defined as follows:

$$g'(x) = (f \cup g)(x) + h_{A,B}(x)(1 - h_{A,B}(x)).$$

It is easy to verify that this function satisfies all the required conditions.

Conversely, let (x_0, y_0) be a unique common point of the continuous functions f and g. We consider an arbitrary continuous function $g' \geqslant f \cup g$. Two cases are possible: $g' > f \cap g$ or $(\neg g' > f \cap g) \wedge (g' \geqslant f \cup g)$. In the second case, $\neg g' > f \cap g$ implies the existence of at least $x^* \in X$ such that $g'(x^*) = (f \cap g)(x^*)$. The inequalities $g'(x) \geqslant (f \cup g)(x) \geqslant (f \cap g)(x)$ also hold for all $x \in X$. Thus, $g'(x^*) = (f \cup g)(x^*) = (f \cap g)(x^*)$ if and only if $x^* = x_0$. Hence $\Psi'(f \cap g, g')$ is true and the formula holds.

We introduce a predicate that is true if and only if the intersection of the graphs of the three continuous functions consists of a single point, i.e., the graphs of the three functions are pairwise intersect at unique points and the points of the pairwise intersections coincide:

$$R(g_1, g_2, g_3) \rightleftharpoons \Psi^*(g_1, g_2) \wedge \Psi^*(g_2, g_3) \wedge \Psi^*(g_1, g_3) \wedge \Psi^*(g_1 \cap g_2 \cap g_3, g_1 \cup g_2 \cup g_3).$$

We denote $l_1 = g_1 \cap g_2 \cap g_3$ and $l_2 = g_1 \cup g_2 \cup g_3$. Assume that the graphs of g_1, g_2, and g_3 intersect at a unique point (x_0, y_0), i.e., the graphs of g_1, g_2 (g_1, g_3 and g_2, g_3) intersect at a single point (x_0, y_0). Hence the graphs of any two functions in the set $\{g_1, g_2, g_3\}$ have a unique common point (x_0, y_0). It is easy to see that the graphs of l_1 and l_2 also have a unique common point (x_0, y_0). Conversely if the graphs of any two functions have a unique common point and there is only one point (x_0, y_0) such that $l_1(x_0) = l_2(x_0)$, then $g_1(x_0) = g_2(x_0) = g_3(x_0)$ and (x_0, y_0) is a unique common point of the graphs of the three functions in the above sense.

Let the predicate θ^3 be such that the predicate $\theta(f_1, f_2, f_3)$ is true if and only if there exists a unique element x_0 such that $f_1(x_0) = f_2(x_0)$ and for this x_0 we have $f_3(x_0) = f_1(x_0) = f_2(x_0)$ (in general, the graph of f_3 can intersect the graphs of f_1 and f_2 at more than one point):

$$\theta(f_1, f_2, f_3) \rightleftharpoons \Psi^*(f_1, f_2) \wedge (\forall h > f_3)\neg R(f_1, f_2, h) \wedge (\forall h < f_3)\neg R(f_1, f_2, h).$$

Indeed, assume that there exists a unique element x_0 such that $f_1(x_0) = f_2(x_0)$ and $f_3(x_0) = f_1(x_0) = f_2(x_0)$. By definition, from the first conditions we obtain the predicate $\Psi^*(f_1, f_2)$. Let $h > f_3$ be an arbitrary continuous function. Then $h(x_0) > f_1(x_0) = f_2(x_0)$. Since the graphs of f_1 and f_2 intersect at a single point, there are no points where the graphs of the functions f_1, f_2, and h intersect, i.e., $\neg R(f_1, f_2, h)$. A similar argument is valid for $h < f_3$.

Assume that the intersection of the graphs of f_1 and f_2 consists of more than one point or there exists a unique element $x_0 \in X$ such that $f_1(x_0) = f_2(x_0)$, but $f_3(x_0) \neq f_1(x_0)$. In the first case, the predicate $\Psi^*(f_1, f_2)$ fails. In the second case, without loss of generality we can assume that $f_3(x_0) < f_1(x_0)$. Since X is perfectly normal and $\{x_0\}$ is closed, there exists a continuous function χ such that $\chi_{x_0}(x_0) = 0$ and $\chi_{x_0}(x) \neq 0$ for $x \neq x_0$. We set $c_{x_0}(x) = |\chi_{x_0}(x)|$ and construct a continuous function h such that $h(x) \rightleftharpoons \max(f_1(x), f_2(x), f_3(x)) + c_{x_0}(x)$. By assumption, $\max(f_1(x_0), f_2(x_0), f_3(x_0)) = f_1(x_0) = f_2(x_0)$. By construction, $c_{x_0}(x_0) = 0$. Hence $h(x_0) = f_1(x_0) = f_2(x_0)$. If $x \neq x_0$, then $h(x) > f_1(x)$ and $h(x) > f_2(x)$. We note that $h(x) > f_3(x)$ for all $x \in X$. Thus, $(\exists h > f_3)R(f_1, f_2, h)$ and the predicate θ is true.

Let the predicate N^4 be true if and only if the intersection of the graphs of f_1 and f_2 consists of a single point. The graphs of f_3 and f_4 also have a single intersection point. The abscissas of these intersection points are equal, whereas the ordinates are different:

$$N(f_1, f_2, f_3, f_4) \rightleftharpoons \Psi^*(f_1, f_2) \wedge \Psi^*(f_3, f_4) \wedge \neg\exists g(\theta(f_1, f_2, g) \wedge \theta(f_3, f_4, g)).$$

We assume that the graphs of f_1 and f_2 have only one intersection point. The graphs of f_3 and f_4 also have a unique intersection point. The abscissas of these intersection points are equal (denoted by x_0) and $f_1(x_0) = f_2(x_0) \neq f_3(x_0) = f_4(x_0)$. The first conditions imply $\Psi^*(f_1, f_2)$ and $\Psi^*(f_3, f_4)$. Let us verify the last conjunction. Assume that there is a function g such that $\theta(f_1, f_2, g) \wedge \theta(f_3, f_4, g)$. By the definition of θ, we have $g(x_0) = f_1(x_0) = f_2(x_0) \neq f_3(x_0) = f_4(x_0) = g(x_0)$. We arrive at a contradiction which proves the last conjunction.

Let f_1, f_2, f_3, and f_4 satisfy one of the following conditions:

1) the intersection of the graphs of f_1 and f_2 consists of more than one point,

2) the intersection of the graphs of f_3 and f_4 consists of more than one point,

3) the graphs of f_1, f_2 and f_3, f_4 intersect at unique points (x_0, y_0) and (x_1, y_1) respectively; moreover, $x_0 \neq x_1$,

4) the graphs of f_1, f_2 and f_3, f_4 intersect at unique points (x_0, y_0) and (x_1, y_1) respectively; moreover, $x_0 = x_1$ and $y_0 = y_1$.

We have listed all the cases where the predicate $N(f_1, f_2, f_3, f_4)$ fails. In cases 1) and 2), the first conjunctions of the predicate N do not hold. In case 4), for any continuous function g with the graph passing through the point $(x_0, y_0) = (x_1, y_1)$ we have $\theta(f_1, f_2, g)$ and $\theta(f_3, f_4, g)$. In the remaining case 3), we show that there is a continuous function g such that $\theta(f_1, f_2, g)$ and $\theta(f_3, f_4, g)$ hold. Thus, we assume that the graphs of f_1 and f_2 intersect at a unique point (x_0, y_0), the graphs of f_3 and f_4 intersect at a unique point (x_1, y_1), and $x_0 \neq x_1$. We first define the function g in the closed set $\{x_0, x_1\}$ by the formula

$$g(x) = \begin{cases} y_0, & x = x_0, \\ y_1, & x = x_1. \end{cases}$$

To define g in the whole space X, we use the Tietze–Urysohn theorem on normal spaces.

Theorem 2 (cf. [15]). *Every continuous function on a closed subspace of some normal space X with the range in $\mathbb{R}$ is continuously extended to X.*

Thus, we proved the existence of the required function g and justified the predicate N.

We note that the predicate N can be easily modified to a predicate N^* such that N^* is true for the same collections of functions f_1, f_2, f_3, and f_4 as for the predicate N, and for the same continuous functions f_1, f_2, f_3, and f_4 as the graphs of f_1 and f_2 have a unique intersection point (x_0, y_0), the graphs of f_3 and f_4 have a unique intersection point (x_1, y_1); $x_0 = x_1$; $y_0 = y_1$, i.e., the graphs of all functions intersect at the point (x_0, y_0):

$$N^*(f_1, f_2, f_3, f_4) \rightleftharpoons N(f_1, f_2, f_3, f_4) \vee (\theta(f_1, f_2, f_3) \wedge \theta(f_1, f_2, f_4) \wedge \Psi^*(f_3, f_4)).$$

Indeed if for functions f_1, f_2, f_3, f_4 we have a description of the predicate N^*, and it is known that $\neg N(f_1, f_2, f_3, f_4)$, then the graphs of f_1 and f_2 have a unique intersection point x_0, y_0; the graphs of f_3 and f_4 also have a unique intersection point x_0, y_0, i.e., the predicate $\Psi^*(f_3, f_4)$ holds. Since $f_3(x_0) = y_0$ and $f_4(x_0) = y_0$, we have $\theta(f_1, f_2, f_3) \wedge \theta(f_1, f_2, f_4)$. It is easy to verify the converse assertion.

With each closed subset of V we associate a pair pari of continuous functions in the following sense. We say that two continuous functions f and g represent a closed subset V of the perfectly normal space X if $f \leqslant g$ and $V = \{x \in X \mid f(x) = g(x)\}$. We note that for every closed subset $V \subseteq X$ there are continuous functions f and g such that $f \leqslant g$ and $f(x) = g(x) \Leftrightarrow x \in V$. Indeed, since V is functionally closed, there exists a continuous function χ_V such that $\chi_V(x) = 0 \Leftrightarrow x \in V$. Then for f and g we can take $f(x) = -|\chi_V(x)|$ and $g(x) = |\chi_V(x)|$. These representations are not unique. Therefore, we will write the congruence relation on the set of pairs of continuous functions.

Thus, we introduce the following formulas of the signature $\mathfrak{A}_X$:

$$\mathfrak{U}(f, g) \rightleftharpoons f \leqslant g,$$

$$S(f_1, f_2, f_3, f_4) \rightleftharpoons (\forall h \geqslant f_2)(\Psi^*(f_1, h) \to (\exists g \geqslant f_4)(\Psi^*(f_3, g) \wedge N^*(f_1, h, f_3, g))),$$

$$\mathfrak{W}(f_1, f_2, f_3, f_4) \rightleftharpoons S(f_1, f_2, f_3, f_4) \wedge S(f_3, f_4, f_1, f_2).$$

It is easy to verify that the set $L \leftrightharpoons \{(f, g) | f,\ g \in \mathfrak{A}_X, \mathfrak{A}_X \models \mathfrak{U}(f, g)\}$ is not empty.

The formula $\mathfrak{W}$ defines the congruence relation η on the structure $\mathfrak{L}$ of the signature $\mathfrak{A}_X$ with the universe L, whereas the predicate $\subseteq$ is defined by the formula $S(f_1, f_2, f_3, f_4)$. Indeed, we show that if f_1 and f_2 are closed subsets of V, whereas f_3 and f_4 are closed subsets of T, then

$$\mathfrak{D}_X \models V \subseteq T \Leftrightarrow \mathfrak{A}_X \models S(f_1, f_2, f_3, f_4).$$

Suppose that $f_1 \leqslant f_2$ and $\{x \in X | f_1(x) = f_2(x)\} = V$, $f_3 \leqslant f_4$ and $\{x \in X | f_3(x) = f_4(x)\} = T$, $\mathfrak{D}_X \models V \subseteq T$. Assume that h is an arbitrary continuous function such that $h \geqslant f_2$ and $\Psi^*(f_1, h)$. By condition there exists a unique point x_0 such that $f_1(x_0) = f_2(x_0) = h(x_0)$. We construct the required function g as follows: $g(x) = f_4(x) + |\chi_{x_0}(x)|$, where, as usual, $\chi(x)$ is a continuous function such that $\chi_{x_0}(x_0) = 0$, $\chi_{x_0}(x) \neq 0$ if $x \neq x_0$. We note that $g(x_0) = f_4(x_0) = f_3(x_0)$ since $x_0 \in T$; $g(x) > f_4(x) \geqslant f_3(x)$ for $x \neq x_0$. Hence the predicate $\Psi^*(f_3, g)$ si true. Further, from the conditions $\Psi^*(f_1, h)$, $f_1(x_0) = h(x_0)$, $\Psi^*(f_3, g)$ $f_3(x_0) = g(x_0)$ we obtain the predicate $N^*(f_1, h, f_3, g)$. Consequently, $\mathfrak{A}_X \models S(f_1, f_2, f_3, f_4)$.

Let $\mathfrak{D}_X \models \neg V \subseteq T$, i.e., there is $x_0 \in V$ such that $x_0 \notin T$. We show that

$$\mathfrak{A}_X \models (\exists h \geqslant f_2)(\Psi^*(f_1, h) \wedge (\forall g \geqslant f_4)(\neg\Psi^*(f_3, g) \vee \neg N^*(f_1, h, f_3, g))).$$

We define $h(x) = f_2(x) + |\chi_{x_0}(x)|$. It is easy to see that $\Psi^*(f_1, h)$. Let g be an arbitrary continuous function such that $g \geqslant f_4$ and $\Psi^*(f_3, g)$. Then $g(x^*) = f_3(x^*)$ for unique $x^* \in X$. For f_1 and h we have $f_1(x_0) = h(x_0)$. We note that $x_0 \neq x^*$ since $x_0 \notin T$. Thus, $\neg N^*(f_1, h, f_3, g)$.

Two pairs of continuous functions are congruent if they represent the same closed subset. For such pairs of continuous functions the formula $\mathfrak{W}$ holds.

It is easy to verify that the quotient structure $\mathfrak{D}_X$ is isomorphic to the structure $\mathfrak{L}/_\sim$. Thereby all the assumptions of the method of relative elementary definability [7] hold, which, in particular, means $\mathrm{Th}(\mathfrak{D}_X) \leqslant_m \mathrm{Th}(\mathfrak{A}_X)$. The lemma is proved. □

Thus, the theorem is proved. In fact, we proved a more general result: $Th_2(\mathfrak{D}_X) \leqslant_m \mathrm{Th}(\mathfrak{A}_X)$, where $Th_2(\mathfrak{D}_X)$ is the theory of the second order of closed subsets of X such that the values of the variables over sets are closed subsets of X. An element $x_0 \in X$, regarded as a closed set, can be represented by a pair of continuous functions f and g such that $\{x \in X | f(x) = g(x)\} = \{x_0\}$. Let functions f_1 and f_2 present an element x, and let f_3 and f_4 present a closed set in V. Then with a formula $x \in V$ of the signature $\mathfrak{D}_X$ we associate the formula

$$P(f_1, f_2, f_3, f_4) \rightleftharpoons \exists h_1 \exists h_2 (h_1 \leqslant f_3 \wedge h_2 \geqslant f_4 \wedge \Psi^*(h_1, h_2) \wedge N^*(f_1, f_2, h_1, h_2))$$

of the signature $\mathfrak{A}_X$. It is easy to verify that

$$\mathfrak{A}_X \models x \in V \Leftrightarrow \mathfrak{D}_X \models P(f_1, f_2, f_3, f_4).$$

The further proof of the m-reducibility is the same as in the theorem.

Thus, the (un)decidability of the theory of open subsets in theories of open subsets of some spaces in the class of perfectly normal spaces implies the (un)decidability of the theory of continuous functions over this space.

For an important example one can consider $\mathbb{R}^n$. The decidability of the theory of open sets in $\mathbb{R}^n$ is established in [4]. Consequently, the following assertion holds.

Corollary 1. *The theory of lattices of continuous functions over* $\mathbb{R}^n$*,* $n > 1$*, is decidable.*

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